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Modified Inertial Projection Method for Solving Pseudomonotone Variational Inequalities with Non-Lipschitz in Hilbert Spaces

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Abstract This paper deals with a class of inertial gradient projection methods for solving a variational inequality problem involving pseudomonotone and non-Lipschitz mappings in Hilbert spaces. The proposed algorithm incorporates inertial techniques and the projection and contraction method. The weak convergence is proved without the condition of the Lipschitz continuity of the mappings. Meanwhile, the linear convergence of the algorithm is established under strong pseudomonotonicity and Lipschitz continuity assumptions. The main results obtained in this paper extend and improve some related works in the literature.

Keywords Inertial method, projection and contraction method, variational inequality problem, pseudomonotone mapping, convergence rate

MR(2010) Subject Classification 47H09, 47J20, 47J05, 47J25

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. Let $F : H \to H$ be a single-valued continuous mapping. We consider classical variational inequality (VI) in the sense of Fichera [13] and Stampacchia [24] (see also Kinderlehrer and Stampacchia [19], Facchinei and Pang [12]) which is formulated as follows: Find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We denote by $\mathrm{Sol}(C, F)$ the solution set of the VI (1.1), which is assumed to be nonempty.

From the characterization of the projection, it follows that x^* is a solution of (1.1) if and only if it solves the following fixed point equation:

$$x^* = P_C(x^* - \tau F x^*), \tag{1.2}$$

where τ is any positive real number and P_C denotes the metric projection onto C. Using (1.2), one can easily construct the following iterative scheme which is generally called as the gradient projection method:

$$x_{n+1} = P_C(x_n - \tau F x_n),$$

where the positive number τ is the stepsize. The projected gradient method converges provided that the mapping F is Lipschitz continuous and strongly monotone.

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To avoid the hypotheses of the strong monotonicity in the gradient projection method, Korpelevich [20] (also by Antipin [2] independently) introduced the extragradient method for solving monotone VIs, which requires two projections onto the feasible set in each iteration. Let $F : C \to H$ be monotone and L-Lipschitz continuous operator. The extragradient method has the following form:

$$\begin{cases} y_n = P_C(x_n - \tau_n F x_n), \\ x_{n+1} = P_C(x_n - \tau_n F y_n), \end{cases} \tag{1.3}$$

where $\tau_n \in (0, 1/L)$ or τ_n is updated by an adaptive rule such that

$$\tau_n \|F x_n - F y_n\| \le \mu \|x_n - y_n\|, \quad \mu \in (0, 1). \tag{1.4}$$

Observe that the extragradient method requires the evaluation of two orthogonal projections onto C per iteration. The first method which overcomes this obstacle is the *projection and contraction method* (PC) of He [16] and Sun [25]. For each iteration $n \in \mathbb{N}$ generates point y_n in the spirit of (1.3):

$$y_n = P_C(x_n - \tau_n F x_n),$$

and then the next iteration x_{n+1} is generated via the following

$$x_{n+1} = x_n - \gamma \eta_n d(x_n, y_n),$$

where $\gamma \in (0, 2)$,

$$\eta_n := \frac{\langle x_n - y_n, d(x_n, y_n) \rangle}{\|d(x_n, y_n)\|^2},$$

and

$$d(x_n, y_n) := x_n - y_n - \tau_n (F x_n - F y_n),$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous operator and $\tau_n \in (0, 1/L)$ or τ_n is updated by some adaptive rule like (1.4). Recently, projection and contraction type methods for solving VI have received great attention from many authors, see, e.g., [4, 10]. A combination of these extensions has been recently considered in [10], which takes advantage of both the projection contraction method and the inertial method [23]. However, a drawback of this method is that, to determine stepsizes, it requires line-search procedures containing many additional projections, and its convergence analysis is performed under the assumptions of Lipschitz continuity as well as the monotonicity of the cost operator. Motivated and inspired by [6, 7, 10, 11, 29, 30], and by the ongoing research in these directions, in the present paper, we revisit the algorithm in [10] for solving variational inequalities with uniformly continuous pseudomonotone operators. In particular, we use an Armijo-type line search in order to relax both of these assumptions, Lipschitz continuity as well as the monotonicity. This improvement allows the algorithm to be applied to a wider class of nonlinear mappings. Moreover, the linear convergence rate of the algorithm is presented under strong pseudomonotonicity and Lipschitz continuity assumptions of the variational inequality mapping, which is not known in the literature. The article is organized as follows: in Section 2, we recall some concepts and lemmas which will be used in the proof of main results and, in Section 3, a general inertial projection and contraction method with a line-search procedure is introduced and its weak convergence is proved. In Section 4, the linear convergence of the proposed algorithm is proved. In Section 5, a numerical example is reported to illustrate the performance of the proposed algorithm. In Section 6, the final conclusions are given.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n\to\infty$, while the strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n\to\infty$. For all $x,y\in H$ we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle.$$

Definition 2.1 ([5]) *Let $T: H\to H$ be an operator. Then*

(1) *T is called L-Lipschitz continuous with constant $L>0$ if*

$$\|Tx-Ty\| \le L\|x-y\|, \quad \forall x,y\in H,$$

if $L=1$ then the operator T is called nonexpansive and if $L\in(0,1)$, T is called a contraction;

(2) *T is called monotone if*

$$\langle Tx-Ty, x-y\rangle \ge 0, \quad \forall x,y\in H;$$

(3) *T is called pseudomonotone in the sense of Karamardian* [17] *if*

$$\langle Tx, y-x\rangle \ge 0 \Longrightarrow \langle Ty, y-x\rangle \ge 0, \quad \forall x,y\in H; \tag{2.1}$$

(4) *T is called α-strongly monotone if there exists a constant $\alpha>0$ such that*

$$\langle Tx-Ty, x-y\rangle \ge \alpha\|x-y\|^2, \quad \forall x,y\in H;$$

(5) *T is called α-strongly pseudomonotone if there exists a constant $\alpha>0$ such that*

$$\langle Tx, y-x\rangle \ge 0 \Longrightarrow \langle Ty, y-x\rangle \ge \alpha\|x-y\|^2, \quad \forall x,y\in H;$$

(6) *The operator T is called sequentially weakly continuous if for each sequence $\{x_n\}$ we have: x_n converges weakly to x implies Tx_n converges weakly to Tx.*

Definition 2.2 *Let (X,d) and (Y,ρ) be metric spaces and let B be a subset of X. A function $f: X\to Y$ is said to be uniformly continuous on B if $\forall\ \epsilon>0$, there exists $\delta>0$ such that if $x,y\in B$ and $d(x,y)<\delta$ then $\rho(f(x),f(y))<\epsilon$.*

We note that (2.1) is only one of the definitions of pseudomonotonicity which can be found in the literature. For every point $x\in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x-P_C x\| \le \|x-y\|$, $\forall y\in C$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive. For properties of the metric projection, the interested reader could be referred to Section 3 in [15].

We need to recall the following lemmas, which are useful for the later convergence analysis.

Lemma 2.3 ([15]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x\in H$ and $z\in C$, then $z=P_C x \Longleftrightarrow \langle x-z, z-y\rangle \ge 0$, $\forall y\in C$. Moreover,*

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x-y\rangle, \quad \forall x,y\in C.$$

Lemma 2.4 ([8]) *Consider the problem* $\mathrm{Sol}(C, F)$ *with* C *being a nonempty, closed, convex subset of a real Hilbert space* H *and* $F : C \to H$ *being pseudomonotone and continuous. Then,* x^* *is a solution of* $\mathrm{Sol}(C, F)$ *if and only if*

$$\langle Fx, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.5 ([9]) *Let* $F : C \to H$ *be a mapping. For* $x \in H$ *and* $\alpha \geq \beta > 0$ *the following inequalities hold*:

$$\frac{\|x - P_C(x - \alpha Fx)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Fx)\|}{\beta},$$
$$\|x - P_C(x - \beta Fx)\| \leq \|x - P_C(x - \alpha Fx)\|.$$

Lemma 2.6 ([1]) *Let* $\{\varphi_n\}$, $\{\delta_n\}$ *and* $\{\alpha_n\}$ *be sequences in* $[0, +\infty)$ *such that*

$$\varphi_{n+1} \leq \varphi_n + \alpha_n(\varphi_n - \varphi_{n-1}) + \delta_n, \quad \forall n \geq 1, \sum_{n=1}^{+\infty} \delta_n < +\infty,$$

and there exists a real number α *with* $0 \leq \alpha_n \leq \alpha < 1$ *for all* $n \in \mathbb{N}$. *Then the following hold:*

(i) $\sum_{n=1}^{+\infty} [\varphi_n - \varphi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$;

(ii) *there exists* $\varphi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \varphi_n = \varphi^*$.

Lemma 2.7 ([21]) *Let* C *be a nonempty set of* H *and* $\{x_n\}$ *be a sequence in* H *such that the following two conditions hold*:

(i) *for every* $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ *exists*;

(ii) *every sequential weak cluster point of* $\{x_n\}$ *is in* C.

Then $\{x_n\}$ *converges weakly to a point in* C.

Definition 2.8 ([22]) *Let* $\{x_n\}$ *be a sequence in* H.

(i) $\{x_n\}$ *is said to converge R-linearly to* x^* *with rate* $\rho \in [0, 1)$ *if there is a constant* $c > 0$ *such that*

$$\|x_n - x^*\| \leq c\rho^n, \quad \forall n \in \mathbb{N}.$$

(ii) $\{x_n\}$ *is said to converge Q-linearly to* x^* *with rate* $\rho \in [0, 1)$ *if*

$$\|x_{n+1} - x^*\| \leq \rho \|x_n - x^*\|, \quad \forall n \in \mathbb{N}.$$

3 Weak Convergence Analysis

In this section, we propose a modified gradient projection method for solving VIs. We assume that the following conditions hold:

Condition 1 The solution set $\mathrm{Sol}(C, F)$ is nonempty.

Condition 2 The mapping $F : H \to H$ is pseudomonotone on H, that is,

$$\langle Fx, y - x \rangle \geq 0 \Longrightarrow \langle Fy, y - x \rangle \geq 0, \quad \forall x, y \in H.$$

In addition, the mapping $F : H \to H$ satisfies the condition

$$\{z_n\} \subset C, \ z_n \rightharpoonup z \Longrightarrow \|Fz\| \leq \liminf_{n\to\infty} \|Fz_n\|. \tag{3.1}$$

Condition 3 $F : H \to H$ is uniformly continuous on bounded subsets of H.

Algorithm 3.1

Initialization: *Given* $\rho, l \in (0,1), \mu \in (0,1),\ \gamma \in (0,2), \theta \in [0,1), \alpha \in (0,1)$. *Let* $s_0, s_1 \in H$ *be arbitrary.*

Iterative Steps: *Given the current iterate* s_n, *calculate* s_{n+1} *as follows*:

Step 1 *Compute* $w_n = s_n + \theta(s_n - s_{n-1})$,

$$y_n = P_C(w_n - \tau_n F w_n),$$

where τ_n *is chosen to be the largest* $\tau \in \{\rho, \rho l, \rho l^2, \dots\}$ *satisfying*

$$\tau \|Fy_n - Fw_n\| \le \mu \|y_n - w_n\|^2. \tag{3.2}$$

If $w_n = y_n$ *or* $Fw_n = 0$ *then stop and* w_n *is a solution of* (1.1). *Otherwise*

Step 2 *Compute*

$$s_{n+1} = w_n - \gamma \eta_n d_n,$$

where

$$\eta_n := \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} & \text{if } d_n \neq 0, \\ 0 & \text{if } d_n = 0, \end{cases}$$

and

$$d_n := w_n - y_n - \tau_n (Fw_n - Fy_n).$$

Set $n := n+1$ *and go to* **Step 1**.

Remark 3.2 It notes that we can use the technique of the subgradient extragradient method to compute $\{s_{n+1}\}$ as follows:

$$s_{n+1} = P_{T_n}(w_n - \gamma \tau_n \eta_n F y_n),$$

where

$$T_n = \{x \in H \mid \langle w_n - \tau_n F w_n - y_n, x - y_n\rangle \le 0\}.$$

In this case, Algorithm 3.1 is called the inertial subgradient extragradient method [26]. The weak convergence was studied in [26] under the assumptions that the cost operator $F : H \to H$ is monotone and L-Lipschitz continuous. However, here we only need to assume that the operator F is pseudomonotone and non-Lipschitz continuous. Moreover, the inertial technique also applies to accelerate the convergence of the algorithm.

Lemma 3.3 *Assume that the mapping* $F : H \to H$ *is uniformly continuous on bounded subsets of* H. *The Armijo-line search rule* (3.2) *is well defined. In addition, we have* $\tau_n \le \gamma$.

Proof If $w_n \in \mathrm{Sol}(C,F)$ then $w_n = P_C(w_n - \gamma F w_n)$ and $m_n = 0$. We consider the situation $w_n \notin \mathrm{Sol}(C,F)$ and assume the contrary that for all m we have

$$\gamma l^m \|FP_C(w_n - \gamma l^m F w_n) - Fw_n\| > \mu \|P_C(w_n - \gamma l^m F w_n) - w_n\|. \tag{3.3}$$

This implies that

$$\|FP_C(w_n - \gamma l^m Fw_n) - Fw_n\| > \mu \frac{\|P_C(w_n - \gamma l^m Fw_n) - w_n\|}{\gamma l^m}. \tag{3.4}$$

Now, we consider the two cases of w_n. First, if $w_n \in C$ then from P_C is continuous, we have $\lim_{m\to\infty} \|P_C(w_n - \gamma l^m Fw_n) - w_n\| = 0$. From the uniform continuity of the operator F on bounded subsets of C it implies that

$$\lim_{m\to\infty} \|FP_C(w_n - \gamma l^m Fw_n) - Fw_n\| = 0. \tag{3.5}$$

Combining (3.4) and (3.5) we get

$$\lim_{m\to\infty} \frac{\|P_C(w_n - \gamma l^m Fw_n) - w_n\|}{\gamma l^m} = 0. \tag{3.6}$$

Assume that $z_m = P_C(w_n - \gamma l^m Fw_n)$ we have

$$\langle z_m - w_n + \gamma l^m Fw_n, x - z_m \rangle \geq 0, \quad \forall x \in C.$$

This implies that

$$\left\langle \frac{z_m - w_n}{\gamma l^m}, x - z_m \right\rangle + \langle Fw_n, x - z_m \rangle \geq 0, \quad \forall x \in C. \tag{3.7}$$

Taking the limit $m \to \infty$ in (3.7) and using (3.6) we obtain

$$\langle Fw_n, x - w_n \rangle \geq 0, \quad \forall x \in C,$$

which implies that $w_n \in \mathrm{Sol}(C, F)$ this is a contraction.

If $w_n \notin C$, then we have

$$\lim_{m\to\infty} \|w_n - P_C(w_n - \gamma l^m Fw_n)\| = \|w_n - P_C w_n\| > 0,$$

and

$$\lim_{m\to\infty} \gamma l^m \|Fw_n - FP_C(w_n - \gamma l^m Fw_n)\| = 0,$$

which, in view of (3.3), also leads to a contradiction. This completes the proof. □

Lemma 3.4 *Assume that Conditions* 1 *and* 2 *hold. Let* $\{s_n\}$ *be a sequence generated by Algorithm* 3.1. *Then*

$$\|s_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n - s_{n+1}\|^2, \quad \forall x^* \in \mathrm{Sol}(C, F). \tag{3.8}$$

Proof From the formula $y_n = P_C(w_n - \tau_n Fw_n)$ we get

$$\langle w_n - y_n - \tau_n Fw_n, y_n - x^* \rangle \geq 0, \tag{3.9}$$

By the monotonicity of F and $x^* \in \mathrm{Sol}(C, F)$ we have

$$\langle Fy_n, y_n - x^* \rangle \geq \langle Fx^*, y_n - x^* \rangle \geq 0. \tag{3.10}$$

Adding (3.9) and (3.10) we get

$$\langle y_n - x^*, w_n - y_n - \tau_n(Fw_n - Fy_n) \rangle \geq 0. \tag{3.11}$$

That is

$$\langle y_n - x^*, d_n \rangle \geq 0. \tag{3.12}$$

Using the inequality (3.12) we get

$$\langle w_n - x^*, d_n\rangle = \langle w_n - y_n, d_n\rangle + \langle y_n - x^*, d_n\rangle \geq \langle w_n - y_n, d_n\rangle. \tag{3.13}$$

On the other hand, we have

$$\|s_{n+1} - x^*\|^2 = \|w_n - \gamma\eta_n d_n - x^*\|^2 = \|w_n - x^*\|^2 - 2\gamma\eta_n\langle w_n - x^*, d_n\rangle + \gamma^2\eta_n^2\|d_n\|^2. \tag{3.14}$$

It implies from (3.13) and (3.14) that

$$\|s_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - 2\gamma\eta_n(1-\mu)\langle w_n - y_n, d_n\rangle + \gamma^2\eta_n^2\|d_n\|^2. \tag{3.15}$$

Since $\eta_n = \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}$, we get

$$\eta_n\|d_n\|^2 = \langle w_n - y_n, d_n\rangle. \tag{3.16}$$

Substituting (3.16) into (3.15), we get

$$\begin{aligned}
\|s_{n+1} - x^*\|^2 &\leq \|w_n - x^*\|^2 - 2\gamma\eta_n\langle w_n - y_n, d_n\rangle + \gamma^2\eta_n\langle w_n - y_n, d_n\rangle \\
&= \|w_n - x^*\|^2 - (2-\gamma)\gamma\eta_n\langle w_n - y_n, d_n\rangle \\
&= \|w_n - x^*\|^2 - \gamma(2-\gamma)\|\eta_n d_n\|^2 \\
&= \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|\gamma\eta_n d_n\|^2 \\
&= \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n - s_{n+1}\|^2.
\end{aligned}$$

□

Lemma 3.5 *Assume that Conditions* 1 *and* 2 *hold and let the sequence* $\{s_n\}$ *be generated by Algorithm* 3.1. *Then*

$$\|w_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|w_n - s_{n+1}\|^2.$$

Proof First, we show that

$$(1-\mu)\|w_n - y_n\| \leq \|d_n\| \leq (1+\mu)\|w_n - y_n\|. \tag{3.17}$$

Indeed, we have

$$\begin{aligned}
\|d_n\| &= \|w_n - y_n - \tau_n(Fw_n - Fy_n)\| \\
&\geq \|w_n - y_n\| - \tau_n\|Fw_n - Fy_n\| \\
&\geq \|w_n - y_n\| - \mu\|w_n - y_n\| \\
&= (1-\mu)\,\|w_n - y_n\|,
\end{aligned}$$

and it is easy to see that $\|d_n\| \leq (1+\mu)\|w_n - y_n\|$. Now, we will prove the lemma. We have

$$\begin{aligned}
\langle w_n - y_n, d_n\rangle &= \langle w_n - y_n, w_n - y_n - \tau_n(Fw_n - Fy_n)\rangle \\
&= \|w_n - y_n\|^2 - \tau_n\langle w_n - y_n, Fw_n - Fy_n\rangle \\
&\geq (1-\mu)\|w_n - y_n\|^2.
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
\|w_n - y_n\|^2 &\leq \frac{1}{1-\mu}\langle w_n - y_n, d_n\rangle = \frac{1}{1-\mu}\eta_n\|d_n\|^2 \\
&= \frac{1}{\eta_n(1-\mu)}\|\eta_n d_n\|^2 = \frac{1}{\eta_n(1-\mu)\gamma^2}\|w_n - z_n\|^2.
\end{aligned} \tag{3.19}$$

Using the definition of $\{\eta_n\}$, (3.18) and (3.17) we get

$$\frac{1}{\eta_n} = \frac{\|d_n\|^2}{\langle w_n - y_n, d_n\rangle} \leq \frac{1}{1-\mu}\frac{\|d_n\|^2}{\|w_n - y_n\|^2} \leq \frac{(1+\mu)^2}{(1-\mu)}. \tag{3.20}$$

Substituting (3.20) into (3.19) we obtain

$$\|w_n - y_n\|^2 \leq \frac{(1+\mu)^2}{[(1-\mu)\gamma]^2}\|w_n - s_{n+1}\|^2. \qquad \square$$

Lemma 3.6 *Assume that Conditions* 1–3 *hold. Let* $\{w_n\}$ *be any sequence generated by Algorithm* 3.1. *If there exists a subsequence* $\{w_{n_k}\}$ *of* $\{w_n\}$ *such that* $\{w_{n_k}\}$ *converges weakly to* $z \in C$ *and* $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$ *then* $z \in \mathrm{Sol}(C, F)$.

Proof We have $y_{n_k} = P_C(w_{n_k} - \tau_{n_k} F w_{n_k})$ thus,

$$\langle w_{n_k} - \tau_{n_k} F w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0, \quad \forall x \in C.$$

or equivalently

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle F w_{n_k}, x - y_{n_k}\rangle, \quad \forall x \in C.$$

This implies that

$$\frac{1}{\tau_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle F w_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle F w_{n_k}, x - w_{n_k}\rangle, \quad \forall x \in C. \tag{3.21}$$

Now, we show that

$$\liminf_{k\to\infty}\langle F w_{n_k}, x - w_{n_k}\rangle \geq 0. \tag{3.22}$$

For showing this, we consider two possible cases. Suppose first that $\liminf_{k\to\infty}\tau_{n_k} > 0$. We have $\{w_{n_k}\}$ is a bounded sequence, F is uniformly continuous on bounded subsets of H. By Lemma 2.4, we get that $\{F w_{n_k}\}$ is bounded. Taking $k \to \infty$ in (3.21) since $\|w_{n_k} - y_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty}\langle F w_{n_k}, x - w_{n_k}\rangle \geq 0.$$

Now, we assume that $\liminf_{k\to\infty}\tau_{n_k} = 0$. Assume $z_{n_k} = P_C(w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k})$, we have $\tau_{n_k} l^{-1} > \tau_{n_k}$. Applying Lemma 2.5, we obtain

$$\|w_{n_k} - z_{n_k}\| \leq \frac{1}{l}\|w_{n_k} - y_{n_k}\| \to 0 \quad \text{as } k \to \infty.$$

Consequently, $z_{n_k} \rightharpoonup z \in C$. This implies that $\{z_{n_k}\}$ is bounded, from which and the uniformly continuity of the operator F on bounded subsets of C it follows that

$$\|F w_{n_k} - F z_{n_k}\| \to 0 \quad \text{as } k \to \infty. \tag{3.23}$$

By the Armijo line-search rule (3.2) we have

$$\tau_{n_k}.l^{-1}\|F w_{n_k} - F P_C(w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k})\| > \mu\|w_{n_k} - P_C(v n_k - \tau_{n_k} l^{-1} F w_{n_k})\|.$$

That is,

$$\frac{1}{\mu}\|F w_{n_k} - F z_{n_k}\| > \frac{\|w_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}}. \tag{3.24}$$

Combining (3.23) and (3.24) we obtain

$$\lim_{k\to\infty}\frac{\|w_{n_k} - z_{n_k}\|}{\tau_{n_k} l^{-1}} = 0.$$

Furthermore, we have

$$\langle w_{n_k} - \tau_{n_k} l^{-1} F w_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0, \quad \forall x \in C.$$

This implies that

$$\frac{1}{\tau_{n_k} l^{-1}}\langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle F w_{n_k}, z_{n_k} - w_{n_k}\rangle \leq \langle F w_{n_k}, x - w_{n_k}\rangle, \quad \forall x \in C. \tag{3.25}$$

Taking the limit $k \to \infty$ in (3.25) we get

$$\liminf_{k\to\infty}\langle F w_{n_k}, x - w_{n_k}\rangle \geq 0.$$

Therefore, the inequality (3.22) is proved. Next, we show that $z \in \mathrm{Sol}(C, F)$.

Now we choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle F w_{n_j}, x - w_{n_j}\rangle + \epsilon_k \geq 0, \quad \forall j \geq N_k, \tag{3.26}$$

where the existence of N_k follows from (3.22). Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{w_{N_k}\} \subset C$ we have $F w_{N_k} \neq 0$ and, setting

$$q_{N_k} = \frac{F w_{N_k}}{\|F w_{N_k}\|^2},$$

we have $\langle F w_{N_k}, w_{N_k}\rangle = 1$ for each k. Now, we can deduce from (3.26) that for each k

$$\langle F w_{N_k}, x + \epsilon_k q_{N_k} - w_{N_k}\rangle \geq 0.$$

By the fact that F is pseudo-monotone, we get

$$\langle F(x + \epsilon_k q_{N_k}), x + \epsilon_k q_{N_k} - w_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Fx, x - w_{N_k}\rangle \geq \langle Fx - F(x + \epsilon_k q_{N_k}), x + \epsilon_k q_{N_k} - w_{N_k}\rangle - \epsilon_k\langle Fx, q_{N_k}\rangle. \tag{3.27}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k q_{N_k} = 0$. Indeed, we have $w_{n_k} \rightharpoonup z$ as $k \to \infty$ and F satisfies the condition (3.1), which imply that

$$0 < \|Fz\| \leq \liminf_{k\to\infty} \|F w_{n_k}\| \ \ \text{(note that } Fz \neq 0 \text{ otherwise, } z \text{ is a solution).}$$

Since $\{w_{N_k}\} \subset \{w_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k q_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\epsilon_k}{\|F w_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty}\|F w_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k q_{N_k} = 0$.

Now, letting $k \to \infty$, then the right hand side of (3.27) tends to zero by F is uniformly continuous, $\{w_{N_k}\}, \{q_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k q_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty}\langle Fx, x - w_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Fx, x - z\rangle = \lim_{k\to\infty}\langle Fx, x - w_{N_k}\rangle = \liminf_{k\to\infty}\langle Fx, x - w_{N_k}\rangle \geq 0.$$

By Lemma 2.4 we obtain $z \in \mathrm{Sol}(C, F)$ and the proof is complete. □

Theorem 3.7 *Assume that Conditions* 1–3 *hold. If the factor* γ *of Algorithm* 3.1 *is chosen such that* $\gamma \in (0, \frac{2(1-\theta)^2}{1-\theta+2\theta^2})$ *then the sequence* $\{s_n\}$ *generated by Algorithm* 3.1 *converges weakly to an element* $z \in \mathrm{Sol}(C, F)$.

Proof From Lemma 3.4, we get

$$\|s_{n+1} - x^*\| \leq \|w_n - x^*\| - \frac{2-\gamma}{\gamma}\|s_{n+1} - w_n\|^2. \tag{3.28}$$

On the other hand, from the definition of w_n, we get

$$\begin{aligned}\|w_n - x^*\|^2 &= \|s_n + \theta(s_n - s_{n-1}) - x^*\|^2\\ &= \|(1+\theta)(s_n - x^*) - \theta(s_{n-1} - x^*)\|^2\\ &= (1+\theta)\|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 + (1+\theta)\theta\|s_n - s_{n-1}\|^2\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\|s_{n+1} - w_n\|^2 &= \|s_{n+1} - s_n - \theta(s_n - s_{n-1})\|^2\\ &= \|s_{n+1} - s_n\|^2 + \theta^2\|s_n - s_{n-1}\|^2 - 2\theta\langle s_{n+1} - s_n, s_n - s_{n-1}\rangle\\ &\geq \|s_{n+1} - s_n\|^2 + \theta^2\|s_n - s_{n-1}\|^2 - 2\theta\|s_{n+1} - s_n\|\|s_n - s_{n-1}\|\\ &\geq \|s_{n+1} - s_n\|^2 + \theta^2\|s_n - s_{n-1}\|^2 - \theta\|s_{n+1} - s_n\|^2 - \theta\|s_n - s_{n-1}\|^2\\ &\geq (1-\theta)\|s_{n+1} - s_n\|^2 - \theta(1-\theta)\|s_n - s_{n-1}\|^2.\end{aligned} \tag{3.30}$$

Substituting (3.29) and (3.30) into (3.28), we have

$$\begin{aligned}\|s_{n+1} - x^*\|^2 \leq{} & (1+\theta)\|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 + (1+\theta)\theta\|s_n - s_{n-1}\|^2\\ & - (1-\theta)\frac{2-\gamma}{\gamma}\|s_{n+1} - s_n\|^2 + \theta(1-\theta)\frac{2-\gamma}{\gamma}\|s_n - s_{n-1}\|^2.\end{aligned} \tag{3.31}$$

It follows from (3.31) that

$$\begin{aligned}&\|s_{n+1} - x^*\|^2 - \theta\|s_n - x^*\|^2 + (1-\theta)\frac{2-\gamma}{\gamma}\|s_{n+1} - s_n\|^2\\ &\leq \|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 + (1-\theta)\frac{2-\gamma}{\gamma}\|s_n - s_{n-1}\|^2\\ &\quad - \left((1-\theta)\frac{2-\gamma}{\gamma} - \theta(1-\theta)\frac{2-\gamma}{\gamma} - (1+\theta)\theta\right)\|s_n - s_{n-1}\|^2.\end{aligned} \tag{3.32}$$

Let

$$\Lambda_n := \|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 + (1-\theta)\frac{2-\gamma}{\gamma}\|s_n - s_{n-1}\|^2.$$

Using (3.32), we get

$$\Lambda_{n+1} - \Lambda_n \leq -\left((1-\theta)\frac{2-\gamma}{\gamma} - \theta(1-\theta)\frac{2-\gamma}{\gamma} - (1+\theta)\theta\right)\|s_n - s_{n-1}\|^2.$$

Now, let $\epsilon := (1-\theta)\frac{2-\gamma}{\gamma} - \theta(1-\theta)\frac{2-\gamma}{\gamma} - (1+\theta)\theta$. Using the assumption $\gamma \in (0, \frac{2(1-\theta)^2}{1-\theta+2\theta^2})$ we deduce

$$\epsilon := (1-\theta)\frac{2-\gamma}{\gamma} - \theta(1-\theta)\frac{2-\gamma}{\gamma} - (1+\theta)\theta > 0. \tag{3.33}$$

Using (3.33), we get

$$\Lambda_{n+1} - \Lambda_n < -\epsilon\|s_{n+1} - s_n\|^2. \tag{3.34}$$

Therefore, we have

$$\begin{aligned}\Lambda_n &= \|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 + (1-\theta)\frac{2-\gamma}{\gamma}\|s_n - s_{n-1}\|^2 \\ &\geq \|s_n - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2.\end{aligned}$$

This follows that

$$\begin{aligned}\|s_n - x^*\|^2 &\leq \theta\|s_{n-1} - x^*\|^2 + \Lambda_n \\ &\leq \theta\|s_{n-1} - x^*\|^2 + \Lambda_1 \\ &\leq \cdots\cdots \\ &\leq \theta^{n-1}\|s_1 - x^*\|^2 + \Lambda_1(\theta^{n-1} + \cdots + 1) \\ &\leq \theta^{n-1}\|s_1 - x^*\|^2 + \frac{\Lambda_1}{1-\theta}. \end{aligned} \tag{3.35}$$

We also have

$$\begin{aligned}\Lambda_{n+1} &\geq \|s_{n+1} - x^*\|^2 - \theta\|s_n - x^*\|^2 \\ &\geq -\theta\|s_n - x^*\|^2. \end{aligned} \tag{3.36}$$

From (3.35) and (3.36), we obtain

$$-\Lambda_{n+1} \leq \theta\|s_n - x^*\|^2 \leq \theta^n\|s_1 - x^*\|^2 + \frac{\theta\Lambda_1}{1-\theta}.$$

It follows from (3.34) that

$$\begin{aligned}\epsilon\sum_{n=1}^{k}\|s_{n+1} - s_n\|^2 &\leq \Lambda_1 - \Lambda_{k+1} \\ &\leq \theta^k\|s_1 - x^*\|^2 + \frac{\Lambda_1}{1-\theta} \\ &\leq \|s_1 - x^*\|^2 + \frac{\Lambda_1}{1-\theta}, \quad \forall k > 0.\end{aligned}$$

This implies

$$\sum_{n=1}^{\infty}\|s_{n+1} - s_n\|^2 < +\infty. \tag{3.37}$$

This follows that

$$\lim_{n\to\infty}\|s_{n+1} - s_n\| = 0. \tag{3.38}$$

Moreover, from (3.37) and Lemma 2.6, we have

$$\lim_{n\to\infty}\|s_n - x^*\|^2 = l.$$

On the other hand, by (3.29), we get

$$\lim_{n\to\infty}\|w_n - x^*\|^2 = l.$$

Since (3.28) we get

$$\lim_{n\to\infty}\|s_{n+1} - w_n\| = 0. \tag{3.39}$$

Combining (3.38) and (3.39) we deduce

$$\lim_{n\to\infty}\|s_n - w_n\| = 0. \tag{3.40}$$

It follows from Lemma 3.5 and (3.39) that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{3.41}$$

The sequence $\{s_n\}$ converges weakly to an element of $\mathrm{Sol}(C,F)$. Indeed, since $\lim_{n\to\infty} \|s_n - x^*\|$ exists, it follows that the sequence $\{s_n\}$ is bounded. Now, we choose a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \rightharpoonup z^*$. By (3.40), we have $w_{n_k} \rightharpoonup z^*$. Since Lemma 3.6 and (3.41) we get $z^* \in \mathrm{Sol}(C,F)$. Therefore, we proved that, for all $x^* \in \mathrm{Sol}(C,F)$, $\lim_{n\to\infty} \|s_n - x^*\|$ exists and each sequential weak cluster point of the sequence $\{s_n\}$ is in $\mathrm{Sol}(C,F)$. By Lemma 2.7, the sequence $\{s_n\}$ converges weakly to an element of $\mathrm{Sol}(C,F)$. This completes the proof. □

Remark 3.8 Our result improves the related results in the literature and hence might be applied to a wider class of mappings. For example, we present the advantage of our method compared with the recent result [10, Theorem 3.1]. Our Theorem 3.7, $F : H \to H$ is assumed to be pseudomonotone on H instead of monotone on H in [10]. In particular, unlike [10, Algorithm 3.1] the weak convergence is proved with the variational inequality mapping is uniformly continuous on bounded subsets of H instead of F is Lipschitz continuous.

4 Convergence Rate

In this section, we provide a result on the convergence rate of the iterative sequence generated by Algorithm 3.1.

Theorem 4.1 *Assume that F is L-Lipschitz continuous on H and κ-strongly pseudomonotone on C. Let $\delta \in (0,1)$ be arbitrary and θ be such that*

$$0 \le \theta \le \min\left\{\frac{\xi}{2+\xi}, \frac{\sqrt{(1+\delta\xi)^2 + 4\delta\xi} - (1+\delta\xi)}{2}, (1-\delta)\left(1 - \frac{(1-\xi)(1-\mu)}{2(1+\mu)}\right)\right\}, \tag{4.1}$$

where $\xi := \theta\frac{2-\gamma}{\gamma}$. Then the sequence $\{s_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution x^* of (1.1) with an R-linear rate.

Proof Under assumptions made, it was proved that (1.1) has a unique solution [18]. From the κ-strong pseudomonotonicity of F, we have

$$\langle Fy_n, y_n - x^*\rangle \ge \kappa\|y_n - x^*\|^2. \tag{4.2}$$

Adding (3.9) and (4.2), we obtain

$$\langle y_n - x^*, w_n - y_n - \tau_n(Fw_n - Fy_n)\rangle \ge \tau_n\kappa\|y_n - x^*\|^2. \tag{4.3}$$

Adding (3.13) and (4.3), we obtain

$$\langle w_n - x^*, d_n\rangle \ge (1-\mu)\|w_n - y_n\|^2 + \tau_n\kappa\|y_n - x^*\|^2. \tag{4.4}$$

Combining (3.14) and (4.4), we deduce

$$\begin{aligned}\|s_{n+1} - x^*\|^2 &\le \|w_n - x^*\|^2 - 2\gamma\eta_n(1-\mu)\|w_n - y_n\|^2 - 2\gamma\eta_n\tau_n\kappa\|y_n - x^*\|^2 + \gamma^2\eta_n^2\|d_n\|^2 \\ &\le \|w_n - x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n - s_{n+1}\|^2 - 2\gamma\eta_n\tau_n\kappa\|y_n - x^*\|^2. \end{aligned} \tag{4.5}$$

It is easy to see that

$$\eta_n = \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} \ge \frac{1-\mu}{(1+\mu)^2}. \tag{4.6}$$

Again, substituting (4.6) into (4.5), we have

$$\|s_{n+1}-x^*\|^2 \le \|w_n-x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - 2\gamma\frac{(1-\mu)}{(1+\mu)^2}\tau_n\kappa\|y_n-x^*\|^2. \tag{4.7}$$

Now we show that $\tau_n > \frac{\mu l}{L}$ for all n. Indeed, by the search rule (3.2), we know that $\frac{\tau_n}{l}$ must violate inequality (3.2), i.e.,

$$\|Fy_n - Fw_n\| > \frac{\mu}{\frac{\tau_n}{l}}\|y_n-w_n\|.$$

This follows that

$$L\|y_n-w_n\| > \frac{\mu}{\frac{\tau_n}{l}}\|y_n-w_n\|.$$

Thus

$$L > \frac{\mu}{\frac{\tau_n}{l}},$$

that is

$$\tau_n > \frac{\mu l}{L}, \quad \forall n. \tag{4.8}$$

Combining (4.7) and (4.8) we get

$$\begin{aligned}\|s_{n+1}-x^*\|^2 &\le \|w_n-x^*\|^2 - \frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - 2\gamma\frac{(1-\mu)}{(1+\mu)^2}\frac{\mu l}{L}\kappa\|y_n-x^*\|^2\\ &= \|w_n-x^*\|^2 - \theta\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - (1-\theta)\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2\\ &\quad - 2\gamma\frac{(1-\mu)}{(1+\mu)^2}\frac{\mu l}{L}\kappa\|y_n-x^*\|^2. \end{aligned}\tag{4.9}$$

Substituting (3.8) into (4.9), we get

$$\begin{aligned}\|s_{n+1}-x^*\|^2 &\le \|w_n-x^*\|^2 - \theta\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - (1-\theta)(2-\gamma)\gamma\frac{(1-\mu)^2}{(1+\mu)^2}\|w_n-y_n\|^2\\ &\quad - 2\gamma\frac{(1-\mu)}{(1+\mu)^2}\frac{\mu l}{L}\kappa\|y_n-x^*\|^2\\ &= \|w_n-x^*\|^2 - \theta\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - 2\beta(\|w_n-y_n\|^2+\|y_n-x^*\|^2)\\ &\le \|w_n-x^*\|^2 - \theta\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2 - \beta\|w_n-x^*\|^2\\ &= (1-\beta)\|w_n-x^*\|^2 - \theta\frac{2-\gamma}{\gamma}\|w_n-s_{n+1}\|^2, \end{aligned}\tag{4.10}$$

where $\beta := \frac{1}{2}\min\{(1-\xi)(2-\gamma)\gamma\frac{(1-\mu)^2}{(1+\mu)^2}; 2\gamma\frac{(1-\mu)}{(1+\mu)^2}\frac{\mu l}{L}\kappa\}$. Let $\varphi := 1-\beta$ and $\xi := \theta\frac{2-\gamma}{\gamma}$. From (4.10) we get

$$\|s_{n+1}-x^*\|^2 \le \varphi\|w_n-x^*\|^2 - \xi\|w_n-s_{n+1}\|^2. \tag{4.11}$$

We have

$$\begin{aligned}\|w_n-x^*\|^2 &= \|(1+\theta)(s_n-x^*) - \theta(s_{n-1}-x^*)\|^2\\ &= (1+\theta)\|s_n-x^*\|^2 - \theta\|s_{n-1}-x^*\|^2 + \theta(1+\theta)\|s_n-s_{n-1}\|^2 \end{aligned}\tag{4.12}$$

and

$$\begin{aligned}\|s_{n+1}-w_n\|^2 &= \|s_{n+1}-s_n-\theta(s_n-s_{n-1})\|^2\\ &= \|s_{n+1}-s_n\|^2+\theta^2\|s_n-s_{n-1}\|^2-2\theta\left\langle s_{n+1}-s_n, s_n-s_{n-1}\right\rangle\\ &\geq \|s_{n+1}-s_n\|^2+\theta^2\|s_n-s_{n-1}\|^2-2\theta\|s_{n+1}-s_n\|\|s_n-s_{n-1}\|\\ &\geq \|s_{n+1}-s_n\|^2+\theta^2\|s_n-s_{n-1}\|^2-\theta\|s_{n+1}-s_n\|^2-\theta\|s_n-s_{n-1}\|^2\\ &\geq (1-\theta)\|s_{n+1}-s_n\|^2-\theta(1-\theta)\|s_n-s_{n-1}\|^2. \end{aligned} \tag{4.13}$$

Substituting the inequalities (4.12) and (4.13) into (4.11), we obtain

$$\begin{aligned}\|s_{n+1}-x^*\|^2 \leq \varphi(1+\theta)\|s_n-x^*\|^2-\varphi\theta\|s_{n-1}-x^*\|^2+\varphi\theta(1+\theta)\|s_n-s_{n-1}\|^2\\ -\xi(1-\theta)\|s_{n+1}-s_n\|^2+\xi\theta(1-\theta)\|s_n-s_{n-1}\|^2,\end{aligned}$$

or equivalently

$$\begin{aligned}&\|s_{n+1}-x^*\|^2-\varphi\theta\|s_n-x^*\|^2+\xi(1-\theta)\|s_{n+1}-s_n\|^2\\ &\quad\leq \varphi[\|s_n-x^*\|^2-\theta\|s_{n-1}-x^*\|^2+\xi(1-\theta)\|s_n-s_{n-1}\|^2]\\ &\qquad-(\varphi\xi(1-\theta)-\varphi\theta(1+\theta)-\xi\theta(1-\theta))\|s_n-s_{n-1}\|^2.\end{aligned}$$

Setting

$$a_n := \|s_n-x^*\|^2-\theta\|s_{n-1}-x^*\|^2+\xi(1-\theta)\|s_n-s_{n-1}\|^2,$$

since $\varphi\in(0,1)$ we can write

$$\begin{aligned}a_{n+1} &\leq \|s_{n+1}-x^*\|^2-\varphi\theta\|s_n-x^*\|^2+\xi(1-\theta)\|s_{n+1}-s_n\|^2\\ &\leq \varphi a_n-\left(\varphi\xi(1-\theta)-\varphi\theta(1+\theta)-\xi\theta(1-\theta)\right)\|s_n-s_{n-1}\|^2.\end{aligned}$$

Note that from (4.1) we have

$$\begin{aligned}\theta &\leq (1-\delta)\left(1-\frac{(1-\xi)(1-\mu)}{2(1+\mu)}\right)\\ &\leq (1-\delta)(1-\beta)=(1-\delta)\varphi,\end{aligned}$$

which implies

$$\xi\theta(1-\theta)\leq(1-\delta)\varphi\xi(1-\theta). \tag{4.14}$$

Since

$$\theta\leq\frac{\sqrt{(1+\delta\xi)^2+4\delta\xi}-(1+\delta\xi)}{2}$$

it holds

$$\theta^2+(1+\delta\xi)\theta-\delta\xi\leq 0,$$

or equivalently

$$\theta(1+\theta)\leq\delta\xi(1-\theta).$$

Hence

$$\varphi\theta(1+\theta)\leq\delta\varphi\xi(1-\theta). \tag{4.15}$$

From (4.14) and (4.15) we deduce

$$\varphi\xi(1-\theta)-\varphi\theta(1+\theta)-\xi\theta(1-\theta)\geq 0.$$

Moreover, since $\theta \leq \frac{\xi}{2+\xi}$, we have $\theta \leq \frac{\xi(1-\theta)}{2}$, which implies

$$\begin{aligned} a_n &= (1-\xi(1-\theta))\|s_n - x^*\|^2 + \xi(1-\theta)\left(\|s_n - x^*\|^2 + \|s_n - s_{n-1}\|^2\right) - \theta\|s_{n-1} - x^*\|^2 \\ &\geq (1-\xi(1-\theta))\|s_n - x^*\|^2 + \frac{\xi(1-\theta)}{2}\|s_{n-1} - x^*\|^2 - \theta\|s_{n-1} - x^*\|^2 \\ &\geq (1-\xi(1-\theta))\|s_n - x^*\|^2 \geq 0. \end{aligned}$$

Therefore, we deduce

$$a_{n+1} \leq \varphi a_n \leq \cdots \leq \varphi^n a_1.$$

This follows that

$$\|s_n - x^*\|^2 \leq \frac{a_1}{\varphi(1-\xi(1-\theta))}\varphi^n,$$

which implies that $\{s_n\}$ converges R-linearly to x^*. □

5 Numerical Illustrations

In this section, we illustrate the convergence of Algorithm 3.1 (Alg. 1 in Figs. 1–4) and compare it with other algorithms. All the numerical experiments are performed on an HP laptop with Intel(R) Core(TM) i5-6200U CPU 2.3GHz with 4 GB RAM. All the programs are written in Matlab2015a.

Example 4.1 Let $F(x) := Mx + q$ where

$$M = BB^T + C + D,$$

and B is an $m \times m$ matrix, C is an $m \times m$ skew-symmetric matrix, D is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M is positive semidefinite), q is a vector in $\mathbb{R}^m$. The feasible set $C \subset \mathbb{R}^m$ is a box constraints in $\mathbb{R}^m$ defined by

$$C := \{x \in \mathbb{R}^m : 0 \leq x \leq 2\}.$$

It is clear that F is monotone and Lipschitz-continuous with constant $L = \|M\|$. Let $q = 0$. Then, we obtain the solution set $\Gamma = \{0\}$. The parameters are chosen as follows:

Algorithm 3.1: $\rho = 0.01$, $\mu = 0.5$, $l = 0.5$, $\theta = 0.4$, $\gamma = 1.8$.

Algorithm 2 in [28]: $\gamma = 0.01$, $\mu = 0.5$, $l = 0.5$, $\beta_n = \frac{1}{n+2}$.

Algorithm 3.1 in [3]: $f(x) = 0.8x$, $\gamma = 0.01$, $\mu = 0.5$, $l = 0.5$, $\beta_n = \frac{1}{n+2}$.

Algorithm 3.2 in [14]: $f(x) = 0.8x$, $\lambda = 0.01$, $\gamma = 1.8$, $\mu = 0.5$, $l = 0.5$, $\beta_n = \frac{1}{n+2}$.

Algorithm 3.1 in [27]: $\tau_0 = 0.001$, $\mu = 0.5$, $\alpha_n = \frac{1}{n+2}$, $\beta_n = 0.99(1-\alpha_n)$. For experiment, all entries of B, C and D are generated randomly from a normal distribution with mean zero and unit variance. The process is started with the initial $x_0 = (1, \ldots, 1)^T \in \mathbb{R}^m$ and $x_1 = 0.9x_0$. We use stopping rule $\|x_n\| < 10^{-7}$ or Iter ≥ 1000 for all algorithms. The numerical results are described in Table 1 and Figures 1–2.

Methods	$m = 100$			$m = 200$		
	Sec.	Iter.	Error	Sec.	Iter.	Error
Algorithm 3.1	0.0330	49	7.9682e-08	0.1060	49	8.4458e-08
Algorithm 2 in [28]	0.036	82	9.4183e-08	0.1260	75	8.1403e-08
Algorithm 3.1 in [3]	0.8390	1000	0.0028	4.1920	1000	0.0063
Algorithm 3.2 in [14]	0.1430	422	9.8859e-08	0.4460	254	8.9049e-08
Algorithm 3.1 in [27]	0.1470	1000	4.5955e-04	0.5680	1000	5.6284e-04

Table 1 Numerical results obtained by other algorithms

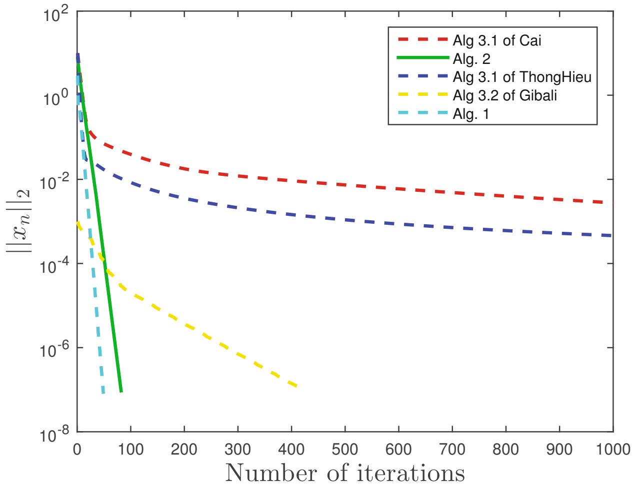

Figure 1 Comparison of all algorithms with $m = 100$

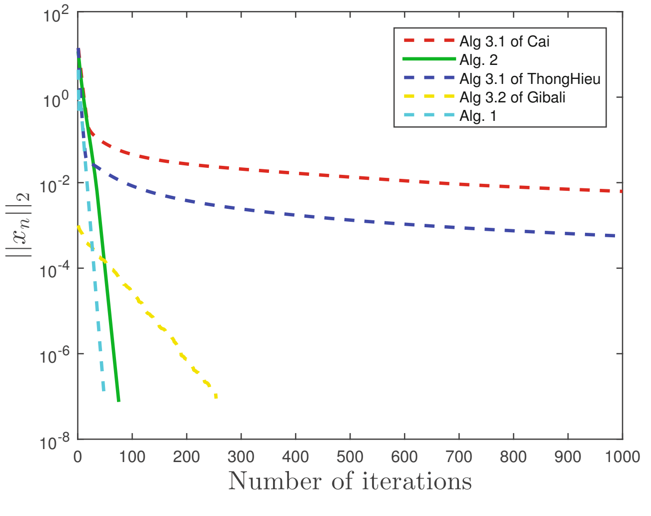

Figure 2 Comparison of all algorithms with $m = 200$

Figures 1–2 and Table 1 demonstrate that Algorithm 3.1 performs better than Algorithm 2 in [28], Algorithm 3.1 in [3], Algorithm 3.2 in [14] and Algorithm 3.1 in [27].

Example 4.2 Consider the following fractional programming problem:

$$\min f(x) = \frac{x^TQx + a^Tx + a_0}{b^Tx + b_0}$$
$$\text{subject to} x \in X := \{x \in \mathbb{R}^m : b^Tx + b_0 > 0\},$$

where Q is an $m \times m$ symmetric matrix, $a, b \in \mathbb{R}^m$, and $a_0, b_0 \in \mathbb{R}$. It is well known that f is pseudoconvex on X when Q is positive-semidefinite.

For experiment, let matrix $A : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, vectors $c, d, y_0 \in \mathbb{R}^m$ and c_0, d_0 are generated from a normal distribution with mean zero and unit variance. We put $e = (1, 1, ..., 1)^T \in \mathbb{R}^m$, $Q = A^TA + I$, $a := e + c$, $b := e + d$, $a_0 = 1 + c_0$, $b_0 = 1 + d_0$. We minimize f over $C := \{x \in \mathbb{R}^m : 0 \le x_i \le 2, i = 1, \dots, m\} \subset X$. Matrix Q is symmetric and positive definite in $\mathbb{R}^m$ and consequently f is pseudo-convex on X.

The process is started with the initial $x_0 := (1, 1, \dots, 1)^T$ and $x_1 = 0.9 * x_0$ and stopping conditions is Residual $:= \|s_{n+1} - s_n\| \le 10^{-7}$ or the number of iterations ≥ 1000 for all algorithms. We choose $\theta = 10^{-3}$ for Algorithm 3.1 and other parameters as Example 4.1. The numerical results are described in Figures 3–4 and Table 2.

Methods	$m = 30$			$m = 50$		
	Sec.	Iter.	Error	Sec.	Iter.	Error
Algorithm 3.1	0.044	286	7.0118e-08	0.15	451	9.8304e-08
Algorithm 3.1 in [3]	0.3050	1000	1.6016e-05	0.57	1000	4.5064e-06
Algorithm 3.2 in [14]	0.1890	1000	8.5671e-07	0.3460	1000	1.0000e-05
Algorithm 3.1 in [27]	0.09	1000	7.3282e-07	0.17	1000	1.1839e-05

Table 2 Numerical results obtained by other algorithms

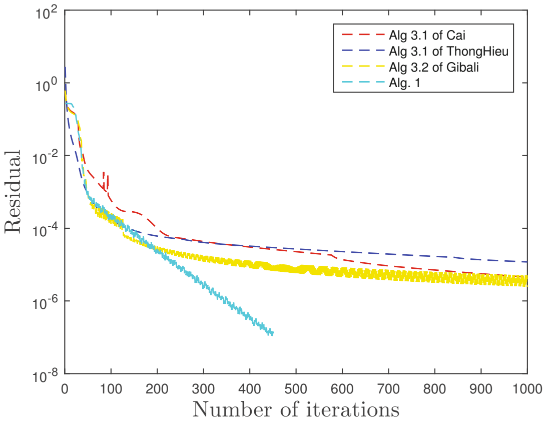

Figure 3 Comparison of all algorithms with $m = 30$

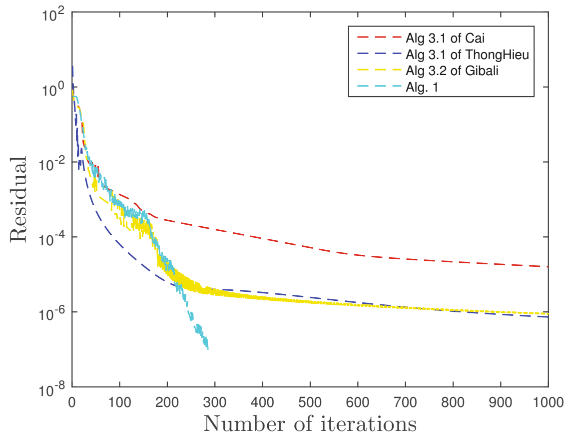

Figure 4 Comparison of all algorithms with $m = 50$

6 Conclusions

In this paper, based on the inertial method and projection and contraction method we introduce a class of inertial projection methods for solving a variational inequality problem involving pseudomonotone and non-Lipschitz continuous mappings in Hilbert spaces. We improve their line search conditions and some parameters to obtain the convergence results. We show that the infinite sequences generated by the algorithm globally weakly converge to some solution of the variational inequality problem and establish the linear convergence of the algorithm, respectively. The proposed algorithm can be considered as continuous versions of the existing ones for variational inequalities. Numerical experiments are presented to illustrate the performance of the proposed method.

Conflict of Interest

The authors declare no conflict of interest.

References

[1] Alvarez, F., Attouch, H.: An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Anal.*, **9**, 3–11 (2001)

[2] Antipin, A. S.: On a method for convex programs using a symmetrical modification of the Lagrange function. *Ekon. Mat. Metody*, **12**, 1164–1173 (1976)

[3] Cai, G., Dong, Q. L., Peng, Y.: Strong convergence theorems for solving variational inequality problems with pseudo-monotone and non-Lipschitz operators. *J. Optim. Theory Appl.*, **188**, 447-472 (2021)

[4] Cai, X., Gu, G., He, B.: On the $O(1/t)$ convergence rate of the projection and contraction methods for variational inequalities with Lipschitz continuous monotone operators. *Comput. Optim. Appl.*, **57**, 339–363 (2014)

[5] Cegielski, A.: Iterative Methods for Fixed Point Problems in Hilbert Spaces, Lecture Notes in Mathematics, vol. 2057, Springer, Berlin, 2012

[6] Censor, Y., Gibali A, Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.*, **148**, 318–335 (2011)

[7] Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. *Optimization*, **61**, 1119–1132 (2011)

[8] Cottle, R. W., Yao, J. C.: Pseudo-monotone complementarity problems in Hilbert space. *J. Optim. Theory Appl.*, **75**, 281–295 (1992)

[9] Denisov, S. V., Semenov, V. V., Chabak, L. M.: Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. *Cybern. Syst. Anal.*, **51**, 757–765 (2015)

[10] Dong, L. Q., Cho, J. Y., Zhong, L. L., et al.: Inertial projection and contraction algorithms for variational inequalities. *J. Glob. Optim.*, **70**, 687–704 (2018)

[11] Dong, Q. L., Gibali, A., Jiang, D.: A modified subgradient extragradient method for solving the variational inequality problem. *Numer. Algorithms*, **79**, 927–940 (2018)

[12] Facchinei, F., Pang, J. S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer Series in Operations Research, vols. I and II. Springer, New York, 2003

[13] Fichera, G.: Sul problema elastostatico di Signorini con ambigue condizioni al contorno. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.*, **34**, 138–142 (1963)

[14] Gibali, A., Thong, D. V., Tuan, P. A.: Two simple projection-type methods for solving variational inequalities. *Anal. Math. Phys.*, **9**, 220–2225 (2019)

[15] Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984

[16] He, B. S.: A class of projection and contraction methods for monotone variational inequalities. *Appl. Math. Optim.*, **35**, 69–76 (1997)

[17] Karamardian, S., Schaible, S.: Seven kinds of monotone maps. *J. Optim. Theory Appl.*, **66**, 37–46 (1990)

[18] Kim, D. S., Vuong, P. T., Khanh P. D.: Qualitative properties of strongly pseudomonotone variational inequalities. *Optim. Lett.*, **10**, 1669–1679 (2016)

[19] Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980

[20] Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. *Ekon. Mat. Metody*, **12**, 747–756 (1976)

[21] Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bull. Amer. Math. Soc.*, **73**, 591–597 (1967)

[22] Ortega, J. M., Rheinboldt, W. C.: Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970

[23] Polyak, B. T.: Some methods of speeding up the convergence of iteration methods. *U.S.S.R. Comput. Math. Math. Phys.*, **4**, 1–17 (1964)

[24] Stampacchia, G.: Formes bilineaires coercitives sur les ensembles convexes. *C. R. Acad. Sci.*, **258**, 4413–4416 (1964)

[25] Sun, D. F.: A class of iterative methods for solving nonlinear projection equations. *J. Optim. Theory Appl.*, **91**, 123–140 (1996)

[26] Thong, D. V., Hieu, D. V.: Modified subgradient extragradient method for variational inequality problems. *Numer. Algorithms*, **79**, 597–610 (2018)

[27] Thong, D. V., Hieu, D. V., Rassias, T. M.: Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems. *Optimization Lett.*, **14**, 115–144 (2020)

[28] Thong, D. V.: Extragradient method with a new adaptive step size for solving non-Lipschitzian pseudo-monotone variational inequalities. *Carpathian J. Math.*, **38**, 503–516 (2022)

[29] Vuong, P. T.: On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities. *J. Optim. Theory Appl.*, **176**, 399–409 (2018)

[30] Vuong P. T., Shehu, Y.: Convergence of an extragradient-type method for variational inequality with applications to optimal control problems. *Numer. Algorithms*, **81**, 269–291 (2019)