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A Geometric Based Connection between Fractional Calculus and Fractal Functions

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Abstract Establishing the accurate relationship between fractional calculus and fractals is an important research content of fractional calculus theory. In the present paper, we investigate the relationship between fractional calculus and fractal functions, based only on fractal dimension considerations. Fractal dimension of the Riemann–Liouville fractional integral of continuous functions seems no more than fractal dimension of functions themselves. Meanwhile fractal dimension of the Riemann–Liouville fractional differential of continuous functions seems no less than fractal dimension of functions themselves when they exist. After further discussion, fractal dimension of the Riemann–Liouville fractional integral is at least linearly decreasing and fractal dimension of the Riemann–Liouville fractional differential is at most linearly increasing for the Hölder continuous functions. Investigation about other fractional calculus, such as the Weyl–Marchaud fractional derivative and the Weyl fractional integral has also been given elementary. This work is helpful to reveal the mechanism of fractional calculus on continuous functions. At the same time, it provides some theoretical basis for the rationality of the definition of fractional calculus. This is also helpful to reveal and explain the internal relationship between fractional calculus and fractals from the perspective of geometry.

Keywords Fractional calculus, fractal functions, fractal dimension, fractional calculus equation, relationship

MR(2010) Subject Classification 26A33, 28A80

1 Introduction

In [33], a topic about "Fractional Calculus: Quo Vadimus? (Where are we going?)" had been discussed by some most famous scientists who work about fractional calculus and its applications. The intention was to pose open problems, challenging hypotheses and questions "Where to go", to discuss them and try to find ways to resolve. From 2.9 (Kiryakova), 2.13 (Mathai)

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and 2.16 (Nigmatullin) in the paper, scholars want to establish the accurate relationship between fractional calculus and fractals. Thus in the present paper, we will make research on the relationship between fractional calculus, such as the Riemann–Liouville fractional calculus and the Weyl–Marchaud fractional derivative, and fractal functions. In fact, similar to the study of classical calculus on continuous functions, many scholars make research on the mechanism of fractional calculus on fractal functions, and pay attention to its consistency with the classical case. Another aspect is to test the rationality and applicability of many definitions of fractional calculus. Of course, fractional calculus has a lot of effective applications in fractional operators, fractional control, especially in fractional calculus equations, but its main research direction is the mechanism of continuous functions such as fractal functions.

Revealing the relationship between fractional calculus and fractals or fractal functions has become the core content of fractional calculus theory [1, 8]. The description of physical meaning and geometric structure, as well as other aspects of exploration such as geometric interpolation, are the important research contents of fractional calculus and fractals. However, the exact effect of fractional calculus on fractal structure is still unknown. In [3], authors showed certain relationship between fractional calculus and fractals, based only on physical and geometrical considerations. The link had been found in the physical origins of the power-laws, ruling the evolution of many natural phenomena, whose long memory and hereditary properties were mathematically modelled by differential operators of non integer order from certain examples elementary. By [40, 41], relationships between the averaging procedure of a smooth function over $1D$-fractal sets and fractional integral of the Riemann–Liouville-type had been established. The numerical verifications were realized for confirmation of the analytical results and the physical meaning of these obtained formulas had been discussed. Most of these works are explored from the perspective of physical meaning, and the internal relationship between them is not studied from the perspective of mathematical theory. We think it is necessary to discuss and explore this relationship from the perspective of mathematical theory.

On the basis of previous work, we will explore the relationship between order of fractional calculus and fractal dimension of fractal functions, and reveal the relationship between fractional calculus and fractals from the perspective of fractal dimension from point of view of geometric meaning.

1.1 Problem

The purpose of extending order of calculus from integer to non integer is to characterize the local variation structure of non differentiable functions. While the local structure of non differentiable functions is often described by the fractal dimension [6]. Therefore, scholars put forward an important scientific problem of fractional calculus theory. How fractional calculus changes the fractal dimension of continuous functions. From this, people begin to study the objective relationship between the fractal dimension of continuous functions and the fractal dimension of their fractional calculus. We express this problem in the following mathematical language.

Let $f(x)$ be a continuous function defined on a closed interval $[a, b]$ $(a < b)$. Write $\dim f$ as the fractal dimension of $f(x)$ on $[a, b]$. $D^{-\nu} f(x)$ and $D^{\mu} f(x)$ mean fractional integral of $f(x)$ of positive order ν and fractional differential of $f(x)$ of positive order μ respectively when they exist on $[a, b]$. So the problem that we study belongs to the correct estimation of

$$\dim D^{-\nu} f \quad \text{and} \quad \dim D^{\mu} f.$$

On this basis, the relationship among $\dim f$, $\dim D^{-\nu} f$ and $\dim D^{\mu} f$ could be further studied. In Ref. [54], Tatom discussed the fractal dimension of fractional calculus of some special functions and curves, and gave certain numerical results and simulation curves. He thought that graph lines of these functions or curves became smoother under the action of fractional integral, while the numerical results showed that the fractal dimension decreased. At the same time, their fractional differential became more rough, while the numerical results showed that the fractal dimension increased. Thus for the reasonable definition of the fractional calculus, we think there are the following basic conclusions.

Problem one *Let $f(x)$ be a continuous function and* $\dim f$ *be the fractal dimension of $f(x)$ on $[a, b]$ $(a < b)$. If fractional integral of $f(x)$ of positive order ν as $D^{-\nu} f(x)$ exists on $[a, b]$,*

$$\dim D^{-\nu} f \leq \dim f.$$

If fractional differential of $f(x)$ of positive order μ as $D^{\mu} f(x)$ exists on $[a, b]$,

$$\dim f \leq \dim D^{\mu} f.$$

Problem one given above reflects the consistency of the action mechanism between the fractional calculus and classical calculus to a certain extent. That is, from the perspective of the fractal dimension, a fractional integral function is smoother than the original function and a fractional differential function is rougher than the original one. In fact, this conclusion holds for the Riemann–Liouville fractional calculus which has been given in the following several sections. On the basis of Refs. [75, 76], Zähle considered that the fractal dimension of continuous functions may keep a linear relationship with the fractal dimension of their fractional calculus. It has been written as follows.

Problem two *Let $f(x)$ be a continuous function and* $\dim f$ *be the fractal dimension of $f(x)$ on $[a, b]$ $(a < b)$. If fractional integral of $f(x)$ of positive order ν as $D^{-\nu} f(x)$ exists on $[a, b]$,*

$$\dim D^{-\nu} f = 1$$

when $\dim f = 1$. *Furthermore, if $f(x)$ is a regular fractal function,*

$$\dim D^{-\nu} f = \dim f - \nu \quad (\nu \leq \dim f - 1)$$

when $\dim f > 1$. *Meanwhile if fractional differential of $f(x)$ of positive order μ as $D^{\mu} f(x)$ exists on $[a, b]$,*

$$\dim D^{\mu} f = \dim f + \mu \quad (\mu \leq 2 - \dim f)$$

when $\dim f < 2$.

Problem two shows that fractional integral of one-dimensional continuous functions must still be one-dimensional continuous functions. Fractional integral of non one-dimensional continuous functions which are regular decreases linearly with respect to order, while fractional differential of non two-dimensional continuous functions which are regular increases linearly with respect to order if they exist.

For problems given above, scholars mainly considered special continuous functions with real expressions such as the Weierstrass function, the Besicovitch function and the linear fractal interpolation functions before 2010. After 2010, people began to make research on the fractional calculus of ordinary continuous functions without real expressions. Through the unremitting efforts of many scholars, the conclusion of problems has been partially solved and important research progress has been made. Therefore, progress of the work over the past two decades and the problems still to be solved are summarized above for further study by scholars. Thus the accurate relationship between fractional calculus and fractal functions could be established. It is helpful to further explore the geometric interpretation and physical significance of this kind of relationship.

1.2 Previous Work

For the problems raised in the last subsection, scholars have made a lot of efforts and made rich progress in the past 30 years. Before 2000, many people had been trying to use fractional calculus to study the local structure and the fractal dimension of special fractal functions and curves. Works on the fractal dimension estimation for the fractional calculus of special fractal curves and functions were probably first given by Tatom in 1995 [54]. The fractal dimension of certain functions and curves is shown to be a linear function of order of fractional integro-differentiation with the numerical simulation while he did not provide a proof of theory. In the conclusion Tatom insisted that there appears to be a very interesting relationship between fractional calculus and fractals.

Patzschke and Zähle [43] studied the fractional derivatives of self-affine functions in 1993. In the following work, Zähle [76] described the Weyl–Marchaud fractional derivative of the Weierstrass function and the Weierstrass–Mandelbrot function, and mainly gave numerical results in 1996. At the same time, in [75], he used the Riemann–Liouville fractional calculus to describe fractal functions, curves and surfaces with actual background, and revealed certain relationship between the fractal dimension of function images and fractional order of the Riemann–Liouville fractional calculus by means of computer numerical fitting and software simulation in 1997. Kolwankar and Gangal [17, 18] mainly discussed the local derivative of a class of continuous functions represented by the Weierstrass function, and gave the upper and lower bounds of the fractal dimension of the local derivative in 1997. These important research results indicate that there is a certain connection between order of fractional calculus and the fractal dimension of fractal functions and fractal curves. These numerical simulations give a preliminary result and provide a theoretical basis for the corresponding mathematical proof.

From 2000 to 2010, scholars used different methods to accurately characterize the fractal dimension of fractional calculus of some special fractal functions, and preliminarily revealed the relationship between the fractal dimension of functions and their fractional calculus fractal dimension. The earliest discussion may be seen from Sun's 2002 work of Ref. [53] on the basic estimation of the fractal dimension of a class of continuous functions fractional calculus. They gave the corresponding example of the Weierstrass function fractional calculus. In 2004, Yao proved that there is a linear relationship between the Box dimension of the Weierstrass function and the Box dimension of its Riemann–Liouville fractional calculus in [74]. We [19, 20] had discussed fractal dimension of fractional calculus of the Besicovitch function in 2008 elementary.

In 2009, Ruan [48] gave the Box dimension and the Riemann–Liouville fractional integral of the linear fractal interpolation functions.

From 2010, based on the investigation of special continuous functions, such as the Weierstrass function and the linear fractal interpolation function, scholars began to further study the fractal dimension of general continuous functions fractional calculus. In [21], we have proved that the Riemann–Liouville fractional integral of a continuous function with bounded variation on a closed interval is still a continuous function of bounded variation in 2010. Therefore, the Riemann–Liouville fractional integral can maintain the invariance of the fractal dimension of the bounded variation function. Results given in [22] show that the fractal dimension of fractional integral of any continuous functions is no more than that of functions themselves uniformly in 2018. Thus if fractional differential of any continuous functions exists, its fractal dimension must be no less than that of functions themselves uniformly. Verma and Viswanathan [58] studied the bounded variation function with the Katugampola fractional integral, and obtained that the fractal dimension of the corresponding fractional integral is still one in 2018. At the same time, a class of one-dimensional unbounded variation functions was constructed, and a nontrivial upper bound of the Katugampola fractional integral was given. On this basis, they [59] discussed the case of multivariate functions and obtained the result consistent with one-dimensional functions in 2020. Other works about the fractal dimension estimation of fractional calculus of functions can be found in Ref. [77]. In 2020, the Box dimension for the Weyl fractional integral of the Hölder continuous functions had been estimated and certain relationship had been got by Tian in [57]. In the same year, Wu had discussed the Riemann–Liouville fractional integral of certain continuous functions in Refs. [65, 66]. We made research on the fractal dimension of the Riemann–Liouville fractional integral of fractal functions elementary in [30].

Many other attempts have been proposed in the last decade. Relevant researches were carried out in discrete spaces by Su in [52], especially in the estimation of the fractal dimension of functions fractional calculus in p-adic domain, and “fractal calculus” was constructed. Similar discussion can also be found in Refs. [64, 65]. By using the ability of fractional calculus to control the fractal dimension of fractal functions, Navascués applied it to the approximation by the fractal interpolation functions and their fractional calculus [38, 39]. We have also done a simple work on the fractal interpolation functions and their approximation by fractional calculus. The relationship between the fractal dimension of the Von Koch curve and fractional differential of its complex-valued expression had been investigated in [28]. Other works about the fractal dimension estimation of fractal functions and their fractional calculus can be found in Ref. [29].

2 Preliminary

As is known to all, fractional calculus can act on various objects, such as fractal curves or fractal functions whether they are certain or uncertain. In the present paper, we mainly make research on continuous functions defined on closed intervals whether they are of bounded variation or not. Thus in the following first subsection, we will introduce certain continuous functions which will be discussed below. Then, we give certain definitions of fractal dimension such as the Box dimension and the Hausdroff dimension. Definitions of the Riemann–Liouville fractional calculus and the Weyl–Marchaud fractional derivative have also been shown in the third subsection. Other definitions and notations which will be used have been given in the last part of the present section.

2.1 Functions

Functions discussed in this paper are all continuous on closed intervals. We write all continuous functions defined on the unit closed interval $I = [0,1]$ as the set

$$C_I = \{f(x) : f(x) \text{ is continuous on } I,\, x \in I\}.$$

If $f(x)$ belongs to C_I, it is either differentiable or not. We write C'_I as the set of all differentiable functions and $C_{0,I}$ as the set of all non differentiable functions. Thus

$$C_I = C'_I \cup C_{0,I}.$$

In the following discussion, we mainly investigate functions in $C_{0,I}$ by the fractional calculus. For convenience of discussion, let BC_I mean all continuous functions on I with bounded variation and UC_I mean all continuous functions on I with unbounded variation. That is,

$$C_I = BC_I \cup UC_I.$$

It is obvious that

$$UC_I \subset C_{0,I} \quad \text{and} \quad C'_I \subset BC_I.$$

Here we give some examples of UC_I.

Example 2.1 ([6, 34]) The Weierstrass function:

Let $0 < \alpha < 1, \lambda > 4$. The Weierstrass function is defined as

$$W(x) = \sum_{j=1}^{\infty} \lambda^{-\alpha j} \sin(\lambda^j x).$$

From definition of the Weierstrass function given in Example 2.1 above, one can get

$$W(x) \in UC_I \subset C_{0,I}.$$

In fact, $W(x)$ is a differentiable nowhere on I and its fractal dimension is greater than its topological dimension.

Example 2.2 ([2]) The Bush function:

Let b be a positive integer no less than 3. If $x \in I$, its b-adic fraction is

$$x = 0.x_1x_2x_3\cdots x_n\cdots = \sum_{n=1}^{\infty} \frac{x_n}{b^n}, \quad x_n \in \{0,1,2,\ldots,b-1\}.$$

Write the Bush function $f(x)$ as

$$f(x) = \sum_{n=1}^{\infty} \frac{\mu_n(x)}{\lambda_n}$$

where $\lambda_n \geq 2$ is a determined constant and $\mu_1(x) = 1$. Define $\mu_n\,(n > 1)$ as u_{n-1} when $x_n = x_{n-1}$ and $\mu_n = (1-\lambda)\mu_{n-1}$ when $x_n \neq x_{n-1}$. For suitably chosen λ_n, $f(x)$ is continuous but differentiable nowhere on I.

Example 2.2 shows the Bush function could be a nowhere differentiable function. With different parameters, the fractal dimension of the Bush function can be taken to certain value larger than its topological dimension.

Example 2.3 ([32]) The linear fractal interpolation function:

Let $N(\in \mathbb{N})$ be no less than one and $0 = x_0 < x_1 < \cdots < x_N = 1$ be real numbers. For $i \in D = \{1, 2, \ldots, N\}, L_i(x)$ is the linear map satisfying

$$L_i(0) = x_{i-1}, \quad L_i(1) = x_i.$$

Let $K = I \times \mathbb{R}$ and $\{y_i\}_{i=0}^N$ be a certain data set. A continuous map $F_i : K \to \mathbb{R}$ is defined as

$$F_i(0, y_0) = y_{i-1}, \quad F_i(1, y_N) = y_i.$$

For $\alpha \in (0, 1)$, F_i satisfies

$$|F_i(x, t) - F_i(x, u)| \leq \alpha|t - u|, \quad \forall x \in I, \quad \forall t, u \in \mathbb{R}.$$

Let functions $\psi_i : K \to K$ be defined as $\psi_i(x, y) = (L_i(x), F_i(x, y))$. Then $\{K, \psi_i : i \in D\}$ is an iterated function system. It has a unique attractor

$$G = \bigcup_{i=1}^{N} \psi_i(G).$$

$\Gamma(G, I)$ can be looked as graph of certain continuous function $g : I \to \mathbb{R}$ which satisfies $g(x_i) = y_i$. It is called as the fractal interpolation function. Let $|d_i| < 1$ and q_i be continuous with $F_i(x, y) = d_i y + q_i(x)$. Then, a fractal interpolation function $g(x)$ defined as above is called as a linear fractal interpolation function.

From definition of a linear fractal interpolation function given above, it must belong to UC_I. Of course, it is differentiable nowhere on I.

For continuous functions with fractal structure on closed intervals and definition of fractal functions, please refer to Ref. [23].

2.2 Fractal Dimension

In the present paper, continuous functions are typically characterized using the Box dimension and the Hausdorff dimension defined below. The reason is that the Box dimension has the largest value and the Hausdorff dimension has the smallest value generally.

Definition 2.4 ([6, 63]) *Let $F(\neq \emptyset)$ be any bounded subset of $\mathbb{R}^2$ and $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower Box dimension and the upper Box dimension of F respectively are defined as*

$$\underline{\dim}_B(F) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{2.1}$$

and

$$\overline{\dim}_B(F) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{2.2}$$

If (2.1) *and* (2.2) *are equal we refer to the common value as the Box dimension of F*

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{2.3}$$

Diameter of U is defined as

$$|U| = \sup\{|x-y| : x, y \in U\},$$

i.e., the greatest distance apart of any pair of points in U. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover F, i.e., $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of F. Suppose that $F \subset \mathbb{R}^n$ and $s \ge 0$. For any $\delta > 0$ define

$$H_\delta^s(F) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}.$$

Write

$$H^s(F) = \lim_{\delta \to 0} H_\delta^s(F).$$

$H_\delta^s(F)$ is called as the s-dimensional Hausdorff measure of F. The Hausdorff dimension of F is given as follows.

Definition 2.5 ([6]) *Let $F \subset \mathbb{R}^n$ and $s \ge 0$. The Hausdorff dimension of F is*

$$\dim_H(F) = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

For the Weierstrass function $W(x)$, we know that its Box dimension is $2-\alpha$ by Example 11.3 of [6] and its Hausdorff dimension is $2-\alpha$ too by [50]. The lower Box dimension of the Bush function is bigger than one and the upper Box dimension of the Bush function is less than two. In other words, its Hausdorff dimension is between one and its lower Box dimension. All univariate real valued continuous functions have topological dimension one and exist in two-dimensional plane. Therefore, the minimum fractal dimension is one and the maximum is two. x and $\sin x$ have the Box and Hausdroff dimension one while functions constructed in [69, 70] have the Box and Hausdroff dimension two.

In addition to these two fractal dimensions given above, people also use the Packing dimension [6] and the K-dimension [10, 11] in specific research.

2.3 Fractional Calculus

The Riemann–Liouville fractional calculus is the most widely used fractional calculus in both practical and theoretical calculations, especially in terms of the fractional calculus equations.

Definition 2.6 ([35, 47]) *Let $f(x) \in C_I, \nu > 0$. Let $D^{-\nu} f(0) = 0$, and for $x \in (0,1]$, we call*

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt$$

the Riemann–Liouville fractional integral of $f(x)$ of order ν. For $\mu > 0$, let $D^\mu f(0) = 0$. For $x \in (0,1]$, we call

$$D^\mu f(x) = D(D^{\mu-1} f(x)) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x (x-t)^{-\mu} f(t) dt$$

the Riemann–Liouville fractional differential of $f(x)$ of order μ when it exists.

Its physical and geometric significance can be seen in Refs. [44, 45]. Definitions of the Weyl–Marchaud fractional derivative and the Weyl fractional integral have been given as follows.

Definition 2.7 ([35, 42]) *Let $f(x)$ be a continuous function defined on $\mathbb{R}$ and $0 < \mu < 1$. Write*

$$D^\mu f(x) = \frac{\mu}{\Gamma(1-\mu)} \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\mu}} dt$$

as the Weyl–Marchaud fractional derivative of $f(x)$ of order μ when it exists.

Definition 2.8 ([42]) *Let $f(x)$ be a continuous function defined on $\mathbb{R}$ and $0 < \nu < 1$. Then if*

$$W_\nu f(x) = \Gamma(\nu) W^\nu f(x) = \int_x^\infty (t-x)^{\nu-1} f(t) dt$$

is well defined, it is called as the Weyl fractional integral of $f(x)$ of order ν.

Other definitions of fractional integral and differential can be found in Refs. [14, 16, 49, 55].

2.4 Other Notations

In addition to the fact that the fractal dimension can explain some structural characteristics of continuous functions, we sometimes need variation to describe functions. Definition of total variation of continuous functions on closed intervals is given as follows.

Definition 2.9 ([80]) *Let $f(x) \in C_I$ and $\{x_i\}_{i=0}^n$ be arbitrary points which satisfy*

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1.$$

Write

$$V_f(x_0, x_1, \ldots, x_n) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|.$$

It is called as variation of $f(x)$ about $\{x_i\}_{i=0}^n$ on I. Let

$$V_0^1 f = \sup\{V_f(x_0, x_1, \ldots, x_n) : 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}.$$

$V_0^1 f$ is called as total variation of $f(x)$ on I. If $V_0^1 f < +\infty$, we say $f(x)$ is of bounded variation on I. If $V_0^1 f = +\infty$, we say $f(x)$ is of unbounded variation on I.

The Weierstrass function, the Bush function and the linear fractal interpolation functions given in Subsection 2.2 are all of unbounded variation on I. It is obvious that functions such as x and $\sin x$ are all of bounded variation on I.

For a point on the closed interval of a continuous function, the following definition is given to determine the local structure near the point which are called as bounded variation points and unbounded variation points of the functions.

Definition 2.10 *Let $f(x) \in C_{[a,b]}$ and $s \in [a, b]$.*

(1) *If $s = a$ and there exists a closed subinterval $J = [a, a']$ $(a < a' \leq b)$ of $[a, b]$ such that variation of $f(x)$ on J being of bounded variation, we say $s = a$ is a bounded variation point of $f(x)$, or we say $s = a$ is an unbounded variation point of $f(x)$.*

(2) *If $s = b$ and there exists a closed subinterval $J = [b', b]$ $(a \leq b' < b)$ of $[a, b]$ such that variation of $f(x)$ on J being of bounded variation, we say $s = b$ is a bounded variation point of $f(x)$, or we say $s = b$ is an unbounded variation point of $f(x)$.*

(3) *If $a < s < b$ and there exists a closed subinterval $J = [a', b']$ $(a \leq a' < s < b' \leq b)$ of $[a, b]$ such that variation of $f(x)$ on J being of bounded variation, we say s is a bounded variation point of $f(x)$, or we say s is an unbounded variation point of $f(x)$.*

Points of the Weierstrass function, the Bush function and the linear fractal interpolation functions are all of unbounded variation. It is obvious that all points of functions, such as x, $\sin x$, are of bounded variation.

Let $\Gamma(f, I)$ be graph of $f(x)$ on I as

$$\Gamma(f, I) = \{(x, f(x)), x \in I\}.$$

Thus the lower Box dimension, the upper Box dimension, the Box dimension and the Hausdorff dimension of graph of $f(x)$ on I can be written as

$$\overline{\dim}_B \Gamma(f, I), \quad \underline{\dim}_B \Gamma(f, I), \quad \dim_B \Gamma(f, I) \quad \text{and} \quad \dim_H \Gamma(f, I),$$

respectively. Write R_f for the maximum range of $f(x)$ over $[a, b]$ as

$$R_f[a, b] = \sup_{a \le x < y \le b} |f(x) - f(y)|.$$

3 Works about the Riemann–Liouville Fractional Calculus

The Riemann–Liouville fractional calculus is the most common one among the numerous fractional calculus used to study general continuous functions and fractal functions. Therefore, here we first discuss the relationship between the fractal dimension of continuous functions and the fractal dimension of their Riemann–Liouville fractional calculus, that is, the mechanism of the Riemann–Liouville fractional calculus on the local structure of functions which have been discussed. Check whether it is consistent with the problems mentioned above.

3.1 Special Functions and Curves

Works about the fractal dimension estimation of the Riemann–Liouville fractional calculus of fractal functions and fractal curves is the earliest available in Refs. [17, 54]. Tatom made research on the relationship between the Riemann–Liouville fractional calculus and fractals such as the Von Koch curve by numerical results and computer simulation in 1995 [54]. Kolwankar discussed the connection between critical order of the Riemann–Liouville fractional differential and the Box dimension of certain fractal curve such as graph of the Weierstrass function in 1996 [17]. These work laid the foundation for the later theoretical proof.

In 2002, Sun and Su investigated the relationship between order of the Riemann–Liouville fractional integral and the fractal dimension of certain continuous functions [53]. A numerical result about the Weierstrass function had been given basically. Subsequently, Yao gave the linear connection between order of the Riemann–Liouville fractional calculus and the Box dimension of the Weierstrass function [74] as follows in 2004.

Theorem 3.1 ([74]) *Let $W(x)$ be given as Example 2.1 and $0 < \mu < \alpha$, $0 < \nu < 1 - \alpha$. It holds*

$$\dim_B \Gamma(D^{-\nu} W, I) = \dim_B \Gamma(W, I) - \nu$$

and

$$\dim_B \Gamma(D^{\mu} W, I) = \dim_B \Gamma(W, I) + \mu.$$

The discussion of the above theorem shows that the fractal dimension of the Riemann–Liouville fractional calculus of the Weierstrass function changes linearly compared with that of the Weierstrass function itself. This proves Zähle's conjecture about the fractal dimension of fractional calculus of fractal functions and curves changes linearly [76] to certain extent.

Discussion about the relationship between the fractal dimension of the Besicovitch function and order of the Riemann–Liouville fractional calculus can be found in Ref. [19]. Here we give certain part of the theorem.

Theorem 3.2 ([19]) *Let $B(x)$ be defined as*

$$B(x) = \sum_{j=1}^{\infty} \lambda_j^{-\alpha} \sin(\lambda_j x), \quad 0 < \alpha < 1, \lambda_j \nearrow \infty.$$

Whether the Box dimension of $B(x)$ exists or not, one can lead the following conclusions

$$\overline{\dim}_B \Gamma(D^{-\nu} B, I) = \overline{\dim}_B \Gamma(B, I) - \nu, \quad \overline{\dim}_B \Gamma(D^{\mu} B, I) = \overline{\dim}_B \Gamma(B, I) + \mu$$

for $0 < \nu < 1 - \alpha$ and $0 < \mu < \alpha$.

Theorem 3.2 tells us that the upper Box dimension of the Riemann–Liouville fractional calculus of the Besicovitch function is linear with order. We also give corresponding conclusion about linear fractal interpolation functions.

Theorem 3.3 ([32, 48]) *Let $g(x)$ be given as Example 2.3 and $0 < \mu < \alpha$, $0 < \nu < 1 - \alpha$. For*

$$\dim_B \Gamma(g, I) = s \quad (1 < s < 2),$$

it holds

$$\dim_B \Gamma(D^{-\nu} g, I) = \dim_B \Gamma(g, I) - \nu$$

and

$$\dim_B \Gamma(D^{\mu} g, I) = \dim_B \Gamma(g, I) + \mu.$$

Theorem 3.3 means the linear relationship between order of the Riemann–Liouville fractional calculus and the Box dimension of linear fractal interpolation functions. Ref. [28] majored on estimation of the fractal dimension of the Riemann–Liouville fractional calculus of the Von koch curve elementary. The related research on the connection between order of the Riemann–Liouville fractional calculus and the fractal dimension of Weierstrass-type function can be seen in Refs. [28, 62].

We find that for some special fractal functions, fractal dimension of functions themselves has a corresponding linear relationship with the fractal dimension of their fractional calculus, which is consistent with part of Problem two. Investigation about the fractal dimension of the Riemann–Liouville fractional calculus of other fractal functions, such as the Bush function and the Takagi function, could be further made.

3.2 One-dimensional Continuous Functions

Before considering general continuous functions, we first discuss a class of simple continuous functions, that is, continuous functions with the Box dimension one. In order to be consistent with Problem one and two, a basic expected conclusion is that the Riemann–Liouville fractional integral of any order of one-dimensional continuous functions still has Box dimension one. If their Riemann–Liouville fractional derivative exists, the corresponding upper Box dimension increases linearly at most.

For a continuous function with Box dimension one, it is generally considered to be bounded. Even if there is no bounded variation, the fractal dimension estimation of the Riemann–Liouville fractional calculus should be very simple. But that's not the case, and as the research continues, more interesting results emerge. For example, it is widely believed that the fractal dimension of the Riemann–Liouville fractional integral of a continuous function of bounded variation must still be one. But this seemingly simple result was not proved to be true until 2010 in [21].

Theorem 3.4 ([21]) *If $f(x) \in BC_I$,*

$$\dim_H \Gamma(D^{-\nu} f, I) = \dim_B \Gamma(D^{-\nu} f, I) = 1$$

for any positive order ν and $D^{-\nu} f(x) \in BC_I$.

As we all know, the Box dimension of a continuous function with bounded variation must be one, but a continuous function having the Box dimension one may not be of bounded variation. In Ref. [77], Zhang constructed a continuous function with only one unbounded variable point, and proved that the Box dimension of any order of the Riemann–Liouville fractional integral of this function is still one. In fact, there exist one-dimensional continuous functions with infinite but countable unbounded variation points [61] and one-dimensional continuous functions with uncountable but zero measure unbounded variation points [31]. To our surprise, there even exist one-dimensional full measure functions which have no bounded variation points [24].

It has been proved that the Box dimension of Riemann–Liouville fractional integral is still one-dimensional for all kinds of one-dimensional continuous functions with different local structures. In 2018, we reached a consensus on the calculation of fractal dimension of the Riemann–Liouville fractional integral of any one-dimensional continuous functions which is in accordance with first part of Problem two as follows.

Theorem 3.5 ([22]) *Let $f(x) \in C_I$ and*

$$\dim_B \Gamma(f, I) = 1.$$

Then

$$\dim_H \Gamma(D^{-\nu} f, I) = \dim_B \Gamma(D^{-\nu} f, I) = 1$$

for any positive order ν.

Theorem 3.5 finally solved the problem that the Box dimension of any order of the Riemann–Liouville fractional integral of a one-dimensional continuous function is still one. But the work is far from over, and new problems are emerging. For example, whether variation of the Riemann–Liouville fractional integral of a one-dimensional continuous function is finite or not. Under what condition they are differentiable. Here we give an example of the Riemann–Liouville fractional calculus of the Lipschitz continuous functions.

Theorem 3.6 *Let $f(x)$ be a Lipschitz continuous function on I and $f(0) = 0$, $0 < \nu < 1$. Then $D^{-\nu} f(x) \in C'_I$ and*

$$\overline{\dim}_B \Gamma\left(\frac{d}{dx} D^{-\nu} f, I\right) \leq 2 - \nu.$$

$D^{\mu} f(x)$ must exist on I for $0 < \mu < 1$ and

$$\overline{\dim}_B \Gamma(D^{\mu} f, I) \leq 1 + \mu.$$

Proof Let $0 \le x < x + 2h \le 1$. Since $f(x)$ is a Lipschitz continuous function on I,

$$|f(x+h) - f(x)| \le Mh$$

for certain positive number M unanimously. By $0 < \nu < 1$,

$$\begin{aligned}
&|D^{-\nu} f(x+2h) - 2D^{-\nu} f(x+h) + D^{-\nu} f(x)| \\
&= \frac{1}{\Gamma(\nu)} \left| \int_0^{x+2h} (x+2h-t)^{\nu-1} f(t)dt - 2\int_0^{x+h} (x+h-t)^{\nu-1} f(t)dt + \int_0^x (x-t)^{\nu-1} f(t)dt \right| \\
&= \frac{1}{\Gamma(\nu)} \left| \int_h^{x+h} (x+2h-t)^{\nu-1} f(t)dt - \int_0^x (x+h-t)^{\nu-1} f(t)dt - \int_h^{x+h} (x+h-t)^{\nu-1} f(t)dt \right. \\
&\quad + \int_0^x (x-t)^{\nu-1} f(t)dt + \int_0^h (x+2h-t)^{\nu-1} f(t)dt + \int_{x+h}^{x+2h} (x+2h-t)^{\nu-1} f(t)dt \\
&\quad \left. - \int_0^h (x+h-t)^{\nu-1} f(t)dt - \int_x^{x+h} (x+h-t)^{\nu-1} f(t)dt \right|.
\end{aligned}$$

Through variable translation,

$$\begin{aligned}
&|D^{-\nu} f(x+2h) - 2D^{-\nu} f(x+h) + D^{-\nu} f(x)| \\
&= \frac{1}{\Gamma(\nu)} \left| \int_0^x (x+h-t)^{\nu-1} f(t+h)dt - \int_0^x (x+h-t)^{\nu-1} f(t)dt - \int_0^x (x-t)^{\nu-1} f(t+h)dt \right. \\
&\quad + \int_0^x (x-t)^{\nu-1} f(t)dt + \int_0^h (x+2h-t)^{\nu-1} f(t)dt + \int_x^{x+h} (x+h-t)^{\nu-1} f(t+h)dt \\
&\quad \left. - \int_0^h (x+h-t)^{\nu-1} f(t)dt - \int_x^{x+h} (x+h-t)^{\nu-1} f(t)dt \right| \\
&= \frac{1}{\Gamma(\nu)} \left| \int_0^x [(x+h-t)^{\nu-1} - (x-t)^{\nu-1}][f(t+h) - f(t)]dt \right. \\
&\quad + \int_0^h [(x+2h-t)^{\nu-1} - (x+h-t)^{\nu-1}] f(t)dt \\
&\quad \left. + \int_x^{x+h} (x+h-t)^{\nu-1} [f(t+h) - f(t)]dt \right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left| \int_0^x [(x+h-t)^{\nu-1} - (x-t)^{\nu-1}][f(t+h) - f(t)]dt \right| \\
&\le \int_0^x |(x+h-t)^{\nu-1} - (x-t)^{\nu-1}| \cdot |f(t+h) - f(t)|dt \\
&\le Mh \int_0^x [(x+h-t)^{\nu-1} - (x-t)^{\nu-1}]dt \\
&\le \frac{M}{\nu} h \cdot h^{\nu}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_x^{x+h} (x+h-t)^{\nu-1} [f(t+h) - f(t)]dt \right| \\
&\le Mh \int_x^{x+h} (x+h-t)^{\nu-1} dt
\end{aligned}$$

$$\leq \frac{M}{\nu} h \cdot h^{\nu}.$$

Since $f(0) = 0$, it holds

$$\begin{aligned}
&\left| \int_0^h [(x+2h-t)^{\nu-1} - (x+h-t)^{\nu-1}] f(t) dt \right| \\
&\leq \int_0^h |(x+2h-t)^{\nu-1} - (x+h-t)^{\nu-1}| \cdot |f(t) - f(0)| dt \\
&\leq Mh \int_0^x [(x+h-t)^{\nu-1} - (x+2h-t)^{\nu-1}] dt \\
&\leq \frac{M}{\nu} h \cdot h^{\nu}.
\end{aligned}$$

So one can get

$$\begin{aligned}
&|D^{-\nu} f(x+2h) - 2D^{-\nu} f(x+h) + D^{-\nu} f(x)| \\
&\leq \frac{1}{\Gamma(\nu)} \left(\frac{M}{\nu} h \cdot h^{\nu} + \frac{M}{\nu} h \cdot h^{\nu} + \frac{M}{\nu} h \cdot h^{\nu} \right) \\
&\leq \frac{3M}{\Gamma(\nu+1)} h^{\nu+1}.
\end{aligned}$$

By Propositions 8 and 9 of [51], $D^{-\nu} f(x)$ is a differentiable function on I while

$$\left| \frac{d}{dx} D^{-\nu} f(x+h) - \frac{d}{dx} D^{-\nu} f(x) \right| \leq N h^{\nu}$$

for certain positive number N unanimously. This means $\frac{d}{dx} D^{-\nu} f(x)$ satisfies the Hölder condition of order ν. Since $0 < \nu < 1$, using Corollary 11.2 (a) of [6],

$$\overline{\dim}_B \Gamma\left(\frac{d}{dx} D^{-\nu} f(x), I \right) \leq 2 - \nu.$$

For $0 < \mu < 1$, by [42]

$$\begin{aligned}
D^{\mu} f(x) &= D[D^{\mu-1} f(x)] \\
&= \frac{d}{dx} \{ D^{\mu-1} f(x) \} \\
&= \frac{d}{dx} \{ D^{-(1-\mu)} f(x) \}.
\end{aligned}$$

Let $1 - \mu = \nu'$. Then,

$$D^{-(1-\mu)} f(x) = D^{-\nu'} f(x)$$

while $0 < \nu' < 1$. Since $f(x)$ is a Lipschitz continuous function on I, $D^{-\nu'} f(x)$ is a differentiable function which means

$$\frac{d}{dx} \{ D^{-(1-\mu)} f(x) \}$$

exists on I. Furthermore,

$$\begin{aligned}
\overline{\dim}_B \Gamma(D^{\mu} f, I) &= \overline{\dim}_B \Gamma\left(\frac{d}{dx} \{ D^{-(1-\mu)} f(x) \}, I \right) \\
&= \overline{\dim}_B \Gamma\left(\frac{d}{dx} \{ D^{-\nu'} f(x) \}, I \right) \\
&\leq 2 - \nu = 1 + \mu.
\end{aligned}$$

□

Theorem 3.6 means that derivative of the Riemann–Liouville fractional integral of certain order ν of a Lipschitz continuous function exists, and the upper Box dimension of its differential does not exceed $2-\nu$ when $0<\nu<1$. Meanwhile the upper Box dimension of the Riemann–Liouville fractional differntial of order μ does not exceed $1+\mu$ when $0<\mu<1$. This is in consistent with part of Problems one and two.

The Riemann–Liouville fractional differential of a differentiable function on I has been given in the following example.

Example 3.7 *If $f(x)\in C_I'$, $D^{\mu}f(x)$ exists on $(0,1]$ for $\mu\in(0,1)$. Furthermore, if $f(0)=0$, $D^{\mu}f(x)$ exists on I.*

The latest progress in fractal dimension estimation and other related work of the Riemann–Liouville fractional calculus for one-dimensional continuous functions can be seen in Refs. [24, 26]. At the end of this subsection, we present certain open problems about one-dimensional continuous functions and their Riemann–Liouville fractional calculus to be solved for further study.

Remark 3.8 Let $f(x)\in C_I$ and $\dim_B\Gamma(f,I)=1$. Problem one: Under what conditions, the Riemann–Liouville fractional integral of $f(x)$ is of bounded variation on I?
Problem two: Under what conditions, the Riemann–Liouville fractional integral of $f(x)$ is differentiable on I? (This means the Riemann–Liouville fractional derivative of $f(x)$ exists on I.)
Problem three: Under what conditions, fractal dimension of the Riemann–Liouville fractional differential of $f(x)$ increases linearly?

3.3 The Hölder Continuous Functions

With the development of fractal dimension estimation and other research work of one-dimensional continuous functions Riemann–Liouville fractional calculus, scholars have gradually begun to explore the change law of fractal dimension of general non expression continuous functions Riemann–Liouville fractional calculus and other related problems. By simply calculation, the upper Box dimension of the Riemann–Liouville fractional integral of $f(x)\in C_I$ of order $\nu\,(0<\nu<1)$ is no more than $2-\nu$. While in Ref. [27], by using the method of auxiliary function and the basis of previous studies, the following results are obtained for the Hölder continuous functions in 2016.

Theorem 3.9 ([27]) *Let $f(x)\in C_I$ and*

$$|f(x)-f(y)|\le M|x-y|^{\alpha}\quad (x,y\in I, 0<\alpha<1)$$

for certain positive M unanimously. Then

$$\overline{\dim}_B\Gamma(D^{-\nu}f,I)\le 2-\frac{\nu}{1-\alpha}$$

for $0<\nu<1-\alpha$.

Results given in Theorem 3.9 show that the upper Box dimension of graph of $D^{-\nu}f(x)$ is no more than $2-\frac{\nu}{1-\alpha}$ which is strictly less than $2-\nu$ when $f(x)$ is a Hölder continuous function on I.

Theorem 3.10 *Let $f(x)\in C_I$ satisfy the Hölder condition of order $\alpha\in(0,1)$ and $f(0)=0$. Then $\overline{\dim}_B\Gamma(D^{-\nu}f,I)$ for order $\nu\in(0,1-\alpha)$ is no more than $2-\alpha-\nu$ and $\overline{\dim}_B\Gamma(D^{\mu}f,I)$ for order $\mu\in(0,\alpha)$ is no more than $2-\alpha+\mu$.*

Proof We consider the Riemann–Liouville fractional differential of $f(x)$ first. Because $f(x)\in C_I$ satisfies the Hölder condition of order $\alpha\in(0,1)$,

$$|f(x)-f(y)|\le M|x-y|^{\alpha}$$

for a positive constant M unanimously for arbitrary $x,y\in I$. From [42], if the Riemann–Liouville fractional derivative of $f(x)$ of order $\mu\in(0,\alpha)$ exists,

$$D^{\mu}f(x)=D[D^{\mu-1}f(x)]=\frac{d}{dx}\{D^{\mu-1}f(x)\}=\frac{d}{dx}\{D^{-(1-\mu)}f(x)\}.$$

Because $0<1-\mu<1$, $D^{-(1-\mu)}f(x)$ can also be written as

$$D^{-(1-\mu)}f(x)=D^{-(\alpha-\mu)}[D^{-(1-\alpha)}f(x)].$$

Let $0\le x<x+2h\le 1$. We have

$$\begin{aligned}&|D^{-(1-\alpha)}f(x+2h)-2D^{-(1-\alpha)}f(x+h)+D^{-(1-\alpha)}f(x)|\\ &\quad=\frac{1}{\Gamma(1-\alpha)}\bigg|\int_h^{x+h}\frac{f(t)dt}{(x+2h-t)^{\alpha}}-\int_0^{x}\frac{f(t)dt}{(x+h-t)^{\alpha}}-\int_h^{x+h}\frac{f(t)dt}{(x+h-t)^{\alpha}}\\ &\qquad+\int_0^{x}\frac{f(t)dt}{(x-t)^{\alpha}}+\int_0^{h}\frac{f(t)dt}{(x+2h-t)^{\alpha}}+\int_{x+h}^{x+2h}\frac{f(t)dt}{(x+2h-t)^{\alpha}}\\ &\qquad-\int_0^{h}\frac{f(t)dt}{(x+h-t)^{\alpha}}-\int_x^{x+h}\frac{f(t)dt}{(x+h-t)^{\alpha}}\bigg|=:I.\end{aligned}$$

Through variable substitution,

$$\begin{aligned}I&=\frac{1}{\Gamma(1-\alpha)}\bigg|\bigg(\int_0^{x}\frac{f(t+h)dt}{(x+h-t)^{\alpha}}-\int_0^{x}\frac{f(t)dt}{(x+h-t)^{\alpha}}-\int_0^{x}\frac{f(t+h)dt}{(x-t)^{\alpha}}+\int_0^{x}\frac{f(t)dt}{(x-t)^{\alpha}}\bigg)\\ &\quad+\bigg(\int_0^{h}\frac{f(t)dt}{(x+2h-t)^{\alpha}}-\int_0^{h}\frac{f(t)dt}{(x+h-t)^{\alpha}}\bigg)+\bigg(\int_x^{x+h}\frac{f(t+h)dt}{(x+h-t)^{\alpha}}-\int_x^{x+h}\frac{f(t)dt}{(x+h-t)^{\alpha}}\bigg)\bigg|\\ &\le\frac{1}{\Gamma(1-\alpha))}\bigg|\int_0^{x}[(x+h-t)^{-\alpha}-(x-t)^{-\alpha}][f(t+h)-a(t)]dt\bigg|\\ &\quad+\frac{1}{\Gamma(1-\alpha)}\bigg|\int_x^{x+h}\frac{[f(t+h)-f(t)]dt}{(x+h-t)^{\alpha}}\bigg|\\ &\quad+\frac{1}{\Gamma(1-\alpha)}\bigg|\int_0^{h}\frac{1}{(x+2h-t)^{\alpha}}-\frac{1}{(x+h-t)^{\alpha}}\bigg|f(t)dt.\end{aligned}$$

Thus,

$$\begin{aligned}&\bigg|\int_0^{x}[(x+h-t)^{-\alpha}-(x-t)^{-\alpha}][f(t+h)-f(t)]dt\bigg|\\ &\quad\le Mh^{\alpha}\int_0^{x}|(x+h-t)^{-\alpha}-(x-t)^{-\alpha}|dt\\ &\quad\le\frac{Mh}{1-\alpha}\end{aligned}$$

and

$$\left|\int_x^{x+h} \frac{[f(t+h)-f(t)]dt}{(x+h-t)^\alpha}\right| \le Mh^\alpha \int_x^{x+h} (x+h-t)^{-\alpha}dt \le \frac{Mh}{1-\alpha}.$$

Meanwhile for $h(0) = 0$,

$$\begin{aligned}
&\left|\int_0^h [(x+2h-t)^{-\alpha} - (x+h-t)^{-\alpha}]f(t)dt\right| \\
&\quad = \left|\int_0^h [(x+2h-t)^{-\alpha} - (x+h-t)^{-\alpha}][f(t)-f(0)]dt\right| \\
&\quad \le Mh^\alpha \int_0^h |(x+2h-t)^{-\alpha} - (x+h-t)^{-\alpha}|dt \\
&\quad \le \frac{Mh}{1-\alpha}.
\end{aligned}$$

Thus

$$I \le \frac{1}{\Gamma(1-\alpha)}\left(\frac{Mh}{1-\alpha} + \frac{Mh}{1-\alpha} + \frac{Mh}{1-\alpha}\right) = \frac{3Mh}{\Gamma(2-\alpha)},$$

which means

$$|D^{-(1-\alpha)}f(x+2h) - 2D^{-(1-\alpha)}f(x+h) + D^{-(1-\alpha)}f(x)| \le \frac{3Mh}{\Gamma(2-\alpha)}.$$

From Proposition 8 of [51],

$$|D^{-(1-\alpha)}f(x+h) - D^{-(1-\alpha)}f(x)| \le M' \cdot h,$$

which shows

$$|D^{-(1-\alpha)}f(x) - D^{-(1-\alpha)}f(y)| \le M'|x-y| \quad (\forall x, y \in I)$$

holds for $M' = \frac{3M}{\Gamma(2-\alpha)}$. Thus $D^{-(1-\alpha)}f(x)$ is a Lipschitz continuous function.

Write $g(x) = D^{-(1-\alpha)}f(x)$. Then, $D^{-(1-\mu)}f(x) = D^{-(\alpha-\mu)}g(x)$. From Theorem 3.5, $D^{-(1-\mu)}a(x)$ is a differentiable function. Furthermore, derivative of $D^{-(1-\mu)}f(x)$ is a Hölder continuous function of order $\alpha - \mu$. This means for any $0 \le x < x+h \le 1$,

$$\left|\frac{d}{dx}D^{-(1-\mu)}f(x+h) - \frac{d}{dx}D^{-(1-\mu)}f(x)\right| \le Nl^{\alpha-\mu},$$

while N is a positive constant. By 11.2 (a) of [6],

$$\overline{\dim}_B\Gamma(D^\mu f, I) = \overline{\dim}_B\Gamma\left(\frac{d}{dx}D^{-(1-\mu)}f, I\right) \le 2 - \alpha + \mu. \qquad \square$$

Corollary 3.11 *Let $f(x) \in C_I$ satisfy the Hölder condition of order $\alpha \in (0,1)$ and $f(0) = 0$. Then $\dim_B \Gamma(D^{\alpha-1}f, I)$ exists and equals to one.*

Proof Since $f(x) \in C_I$ and $f(x)$ satisfies the Hölder condition of order $\alpha \in (0,1)$ on I, by Theorem 3.10 the upper Box dimension of the Riemann–Liouville fractional integral of $f(x)$ of order $1-\alpha$ is no more than $2 - \alpha - (1-\alpha) = 1$ as

$$\overline{\dim}_B\Gamma(D^{\alpha-1}f, I) \le 1.$$

By Definition 2.4,

$$\underline{\dim}_B\Gamma(D^{\alpha-1}f, I) \ge 1.$$

Thus

$$\dim_B \Gamma(D^{\alpha-1}f, I) = 1. \qquad \square$$

From Corollary 3.11, if a continuous function on I satisfies the Hölder condition of order $\alpha \in (0,1)$ having the Box dimension $2-\alpha$, its Riemann–Liouville fractional integral decreases linearly with order $1-\alpha$.

Corollary 3.12 *Let $f(x) \in C_I$ satisfy the Hölder condition of order $\alpha \in (0,1)$ and $f(0) = 0$. Then the Box dimension of the Riemann–Liouville fractional integral of $f(x)$ of order $\nu \geq 1-\alpha$ exists and equals to one.*

Proof From Theorem 3.5 and Corollary 3.11, we can immediately get the conclusion of the present corollary.

Other works about the fractal dimension estimation of the Riemann–Liouville fractional calculus of the Hölder continuous functions can be found in Refs. [30, 65, 66]. The fractal dimension of the Riemann–Liouville fractional integral of a bivariate continuous function of bounded variation in the sense of Arzelá has been investigated in [59]. $\square$

Remark 3.13 Let $f(x) \in C_I$ satisfy the Hölder and the count Hölder condition with the same order $\alpha \in (0,1)$ and $f(0) = 0$.

Problem one: Is the lower Box dimension of the Riemann–Liouville fractional integral of $f(x)$ of order $\nu \in (0, 1-\alpha)$ no less than $2-\alpha-\nu$? (This means the Box dimension of the Riemann–Liouville fractional integral of $f(x)$ of order $\nu \in (0, 1-\alpha)$ is $2-\alpha-\nu$.)

Problem two: Is the lower Box dimension of the Riemann–Liouville fractional derivative of $f(x)$ of order $\mu \in (0,\alpha)$ no less than $2-\alpha+\mu$? (This means the Box dimension of the Riemann–Liouville fractional derivative of $f(x)$ of order $\mu \in (0,\alpha)$ is $2-\alpha+\mu$.)

If conclusions of Remark 3.13 hold, the linear relationship between the Box dimension of continuous functions satisfying the Hölder and certain order of the Riemann–Liouville fractional calculus could be set up. A similar conclusion about the count Hölder condition can also be proved. This is consistent with the conclusion of Problem two.

3.4 Ordinary Continuous Functions

Now we make research on the estimation of fractal dimension of the Riemann–Liouville fractional calculus for any continuous functions defined on I. Using definition of the upper Box dimension given in Definition 2.4, we get the following conclusion by simple calculation.

Theorem 3.14 *Let $f(x) \in C_I$ and $\nu > 0$.*

(1) *If $0 < \nu < 1$,*

$$1 \leq \dim_H \Gamma(f, I) \leq \overline{\dim}_B \Gamma(f, I) \leq 2-\nu < 2.$$

(2) *If $\nu \geq 1$,*

$$1 = \dim_H \Gamma(f, I) = \dim_B \Gamma(f, I) = 1.$$

In Ref. [22], by using definition of the upper Box dimension, one can get the following results.

Theorem 3.15 ([22]) *Let $f(x) \in C_I$ and $\nu > 0$.*

$$\overline{\dim}_B \Gamma(D^{-\nu} f, I) \leq \overline{\dim}_B \Gamma(f, I).$$

If $D^{\mu} f(x)$ exists for certain positive μ,

$$\overline{\dim}_B \Gamma(D^{\mu} f, I) \geq \overline{\dim}_B \Gamma(f, I).$$

Theorem 3.15 settles the conclusion of Problem one about the Riemann–Liouville fractional calculus, but it is still far from the conclusion of Problem two.

Remark 3.16 Let $f(x) \in C_I$ and

$$\dim_B \Gamma(f, I) = s \quad (1 < s < 2).$$

Problem one: Is $\overline{\dim}_B \Gamma(D^{-\nu} f, I)$ for order $\nu \in (0, s-1)$ no more than $s - \nu$?
Problem two: Is $\underline{\dim}_B \Gamma(D^{-\nu} f, I)$ for order $\nu \in (0, s-1)$ no less than $2 - \nu$? (This means $\dim_B \Gamma(D^{-\nu} f, I)$ for order $\nu \in (0, s-1)$ is $s - \nu$ which is the conclusion of Problem two.)
Problem three: Is $\overline{\dim}_B \Gamma(D^{-\mu} f, I)$ for order $\mu \in (0, 2-s)$ no more than $s + \mu$ if it exists on I?
Problem four: Is $\underline{\dim}_B \Gamma(D^{-\mu} f, I)$ for order $\mu \in (0, 2-s)$ no less than $s + \mu$ if it exists on I? (This means $\dim_B \Gamma(D^{-\nu} f, I)$ for order $\mu \in (0, 2-s)$ is $s + \mu$ when it exists on I which is the conclusion of Problem two.)

At present we are exploring the proposition that fractal dimension of the Riemann–Liouville fractional integral of any continuous functions is at least linearly decreasing and we have made some preliminary progress.

3.5 Two-dimensional Continuous Functions

By [69, 70], we know there exist two-dimensional continuous functions on closed intervals. By simply calculation, one can get the following conclusion.

Theorem 3.17 *Let $f(x) \in C_I$ and*

$$\dim_B \Gamma(f, I) = 2.$$

Then for $0 < \nu < 1$, it holds

$$\dim_H \Gamma(D^{-\nu} f, I) \leq \overline{\dim}_B \Gamma(D^{-\nu} f, I) \leq 2 - \nu.$$

Theorem 3.17 means that the upper Box dimension of the Riemann–Liouville fractional integral of two-dimensional continuous functions of certain positive order decreases at least linearly corresponding to that of the original functions.

Cui [5] investigated fractal dimension of the Riemann–Liouville fractional integral of a two-dimensional continuous function. She got the linear connection between the Box dimension of this two-dimensional continuous function and the Box dimension of its Riemann–Liouville fractional integral under certain condition.

3.6 Other Works

Verma had investigated the fractal dimension estimation of the bivariate Riemann–Liouville fractional integral of certain bivariate continuous functions with bounded variation elementary in [59]. Relevant researches were carried out in discrete spaces by Su in [52], especially in the estimation of fractal dimension of discrete functions Riemann–Liouville fractional calculus in

p-adic domain, and "fractal calculus" was constructed. In [3], Butera discussed the relationship between a class of fractal curves and fractional calculus, and gave numerical results. Nigmatullin believed in that the study of the accurate relationship between fractal and fractional integral had become a "hot" point in the theoretical work of fractional calculus at present, and the in-depth discussion of this problem was of great significance to reveal the solutions of fractional calculus equations describing physical phenomena.

4 Works about the Weyl–Marchaud Fractional Derivative

Here we discuss the fractal dimension of the Weyl–Marchaud fractional derivative which is given by Weyl and Marchaud [7] of certain continuous functions. More details about definition and applications of the Weyl–Marchaud fractional derivative can be found in Ref. [47].

Zähle's work on the calculation of the fractal dimension of the Weyl–Marchaud fractional derivative of continuous functions such as the Weierstrass function may be the earliest discussion [76]. He gave certain numerical computation about fractal dimension of the Weyl–Marchaud fractional derivative of the Weierstrass and Weierstrass–Mandelbrot functions elementary in 1996. Yao first studied the Box dimension of the Weyl–Marchaud fractional derivative of the Weierstrass function in 2008 [72]. In 2012, the relationship between the Box dimension of certain self-affine functions such as the Besicovitch function and order of the Weyl–Marchaud fractional derivative has been investigated by theoretical proof in [78]. The linear connection between the fractal dimension of certain fractal functions and order of the Weyl–Marchaud fractional derivative has been set up. As an example, we give the following theorem that there is a linear relationship between the Box dimension of self-affine functions and order of the Weyl–Marchaud fractional derivative.

Theorem 4.1 ([78]) *Let $M(x)$ be a self-affine function given as Example* 11.4 *in* [6] *with the Box dimension $s\,(1 < s < 2)$. If $0 < \mu < 2 - s$, it holds*

$$\dim_B \Gamma(D^\mu M, I) = \dim_B \Gamma(M, I) + \mu = s + \mu.$$

Discussion about the fractal dimension estimation of the Weyl–Marchaud fractional derivative of special fractal functions, such as the Weierstrass-type function and the linear fractal interpolation functions with real expression can be found in Refs. [72]. Other works about fractal curves such as self-affine functions can be found in Refs. [36, 73].

Because the Weyl–Marchaud fractional derivative is global, it is more difficult to estimate the fractal dimension of the corresponding functions derivative than the Riemann–Liouville fractional calculus. But for certain Lipschitz continuous functions, we still have the following conclusion that the corresponding fractal dimension increases linearly at most in accordance with that of Problem two.

Theorem 4.2 ([25]) *If a continuous function $f(x)$ defined on $\mathbb{R}$ satisfies the Lipschitz condition, for $0 < \mu < 1$,*

$$1 \le \underline{\dim}_B \Gamma(D^\mu f, I) \le \overline{\dim}_B \Gamma(D^\mu f, I) \le 1 + \mu.$$

Conclusions of Theorem 4.2 mean the Weyl–Marchaud fractional derivative of a Lipschitz continuous function exists and its fractal dimension increases at most linearly with order. Based on the discussion of Theorem 4.2, one can lead the following conclusion about the Hölder continuous functions.

Theorem 4.3 ([27, 71]) *Let a continuous function $f(x)$ satisfy the Hölder condition of order α on $\mathbb{R}$. If $0 < \mu < \alpha$,*

$$\overline{\dim}_B \Gamma(D^\mu f, I) \leq 2 - \alpha + \mu.$$

Theorem 4.3 shows that the fractal dimension of the Weyl–Marchaud fractional derivative of continuous functions satisfying the Hölder condition increases at most linearly with order too. So it is natural to ask whether the fractal dimension is no less than the corresponding fractal dimension if continuous functions satisfy the count Hölder condition with the same order. Thus, we give a remark about the fractal dimension estimation of the Weyl–Marchaud fractional derivative of certain continuous functions as follows.

Remark 4.4 Let $f(x)$ be a continuous function on $\mathbb{R}$.
Problem one: If $f(x)$ satisfies the Hölder and the count Hölder condition with the same order $\alpha \in (0,1)$, is the lower Box dimension of the Weyl–Marchaud fractional derivative of $f(x)$ of order $\mu \in (0,\alpha)$ no less than $2-\alpha+\mu$? (This means the Box dimension of the Weyl–Marchaud fractional derivative of $f(x)$ of order $\mu \in (0,\alpha)$ is $2-\alpha+\mu$ which means the fractal dimension of the corresponding functions increases linearly about the order.)
Problem two: Is the lower Box dimension of the Weyl–Marchaud fractional derivative of $f(x)$ of order $\mu \in (0, 2-s)$ no less than $s+\mu$?

5 Works about Other Fractional Calculus

In the last two sections, the relationship between order of the Riemann–Liouville fractional calculus and the Weyl–Marchaud fractional derivative and the fractal dimension of continuous functions has been discussed. Here we deal with the connection between other fractional calculus and the fractal dimension of continuous functions elementary.

5.1 Works about the Weyl Fractional Integral

The Weyl fractional integral [47] is close to the Riemann–Liouville fractional integral. However, because of the infinity of the integral limit, there is a big difficulty in describing the fractal dimension of functions with Weyl fractional integral.

For the special fractal functions, such as the Weierstrass function and the Besicovitch function, people have seldom explored the fractal dimension variation law of their fractional calculus. Similar argument with that of [72], one can lead the following conclusion elementary.

Theorem 5.1 *Let $f(x)$ be the Weierstrass function which is given as Example 2.1 and $0 < \nu < 1-\alpha$. Thus*

$$\dim_B \Gamma(W^\nu f, I) = \dim_B \Gamma(f, I) - \nu.$$

Fractal dimension of the Weyl fractional integral of the Besicovitch function and the linear fractal interpolation functions can also be investigated similarly.

If a continuous function has bounded variation on a closed interval, we can establish the following basic result.

Theorem 5.2 *If $f(x)$ is continuous and of bounded variation on $[0,+\infty)$,*

$$\dim_B \Gamma(W^\nu f, I) = \dim_B \Gamma(f, I) = 1$$

for order $\nu \in (0,1)$ and $W^\nu f(x)$ also is continuous and of bounded variation on $[0,+\infty)$.

Proof Since $f(x)$ is continuous and of bounded variation on $[0,+\infty)$, by Theorem 6.6 of [80] $f(x)$ can be represented by difference of two monotone increasing and continuous functions $g_1(x)$ and $g_2(x)$. That is,

$$f(x) = g_1(x) - g_2(x).$$

(1) If $f(0) \geq 0$, we can choose $g_1(0) \geq 0$ and $g_2(0) = 0$. Thus

$$G_1(x) = W^\nu g_1(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} g_1(t)dt, \quad 0 < \nu < 1.$$

We know $G_1(x)$ is still continuous when $g_1(x)$ is continuous.

Let $0 \leq x_1 < x_2 \leq 1$ and $0 < \nu < 1$. It holds

$$\begin{aligned}
G_1(x_2) - G_1(x_1) &= W^\nu g_1(x_2) - W^\nu g_1(x_1) \\
&= \frac{1}{\Gamma(\nu)} \int_{x_2}^\infty (t-x_2)^{\nu-1} g_1(t)dt - \frac{1}{\Gamma(\nu)} \int_{x_1}^\infty (t-x_1)^{\nu-1} g_1(t)dt \\
&= \frac{1}{\Gamma(\nu)} \int_{x_2}^\infty (t-x_2)^{\nu-1} g_1(t)dt \\
&\quad - \frac{1}{\Gamma(\nu)} \int_{x_2}^\infty (t-x_2)^{\nu-1} g_1(t-x_2+x_1)dt \\
&= \frac{1}{\Gamma(\nu)} \int_{x_2}^\infty (t-x_2)^{\nu-1} [g_1(t) - g_1(t-x_2+x_1)]dt \\
&\geq 0.
\end{aligned}$$

Thus function $G_1(x)$ is still a monotone increasing and continuous function. If

$$G_2(x) = W^\nu g_2(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} g_2(t)dt, \quad 0 < \nu < 1,$$

$G_2(x)$ is also a monotone increasing and continuous function.

(2) If $f(0) < 0$, we can choose $g_1(0) = 0$ and $g_2(0) > 0$. Write $f(x)$ as $g_1(x) - g_2(x)$. $g_1(x)$ and $g_2(x)$ both are monotone increasing and continuous functions. For $0 < \nu < 1$, similarly argument with (1), we can get both $W^\nu g_1(x)$ and $W^\nu g_2(x)$ are monotone increasing and continuous.

From Theorem 6.6 of [80] again, $W^\nu f(x)$ is still a continuous function with bounded variation. Thus conclusions of Theorem 5.2 hold. □

Theorem 5.2 shows that the Weyl fractional integral of any positive order of a continuous function with bounded variation is still a continuous function with bounded variation which is similar with results of the Riemann–Liouville fractional integral. More details can be found in Ref. [37].

For the Hölder continuous functions, we can get the following conclusion.

Theorem 5.3 ([23]) *Let a continuous function $f(x)$ satisfy α-order Hölder condition on $\mathbb{R}$. If $0 < \alpha, \nu, \alpha + \nu < 1$, it holds*

$$\overline{\dim}_B \Gamma(W_\nu f, I) \leq 2 - \frac{\nu}{1-\alpha}.$$

Here we give another conclusion about fractal dimension estimation of the Weyl fractional integral of certain continuous functions satisfying the Hölder condition as follows.

Theorem 5.4 *Let $f(x)$ be a continuous and nonnegative function on $\mathbb{R}$. If $f(x)$ satisfies α-order Hölder condition on $\mathbb{R}$, then it holds*

$$\overline{\dim}_B\Gamma(W_\nu f, I) \le 2 - \frac{\alpha+\nu}{1+\nu}, \quad 0<\alpha,\nu<1.$$

Proof Let $0<\nu<1$ and $0\le x<x+h\le 1$. For $x<x_0<1$,

$$\begin{aligned}|W_\nu f(x+h)-W_\nu f(x)| &= \left|\int_{x+h}^{\infty}(t-x-h)^{\nu-1}f(t)dt - \int_x^{\infty}(t-x)^{\nu-1}f(t)dt\right| \\ &= \left|\int_x^{\infty}(t-x)^{\nu-1}[f(t+h)-f(t)]dt\right| \\ &= \left|\left(\int_x^{x_0}+\int_{x_0}^{\infty}\right)(t-x)^{\nu-1}[f(t+h)-f(t)]dt\right| \\ &\le Ch^\alpha(x_0-x)^\nu + \left|\int_{x_0}^{\infty}(t-x)^{\nu-1}[f(t+h)-f(t)]dt\right|.\end{aligned}$$

Define

$$I_2 := \left|\int_{x_0}^{\infty}(t-x)^{\nu-1}[f(t+h)-f(t)]dt\right|$$

and it holds

$$\begin{aligned}I_2 &= \left|\int_{x_0+h}^{\infty}(t-x-h)^{\nu-1}f(t)dt - \int_{x_0}^{\infty}(t-x)^{\nu-1}f(t)dt\right| \\ &\le \left|\left(1-\frac{h}{x_0-x}\right)^{\nu-1}\int_{x_0+h}^{\infty}(t-x)^{\nu-1}f(t)dt - \int_{x_0}^{\infty}(t-x)^{\nu-1}f(t)dt\right| \\ &\le \left(\left[\frac{x_0-x}{x_0-x-h}\right]^{1-\nu}-1\right)\int_{x_0}^{\infty}(t-x)^{\nu-1}f(t)dt + \int_{x_0}^{x_0+h}(t-x)^{\nu-1}f(t)dt \\ &\le C\frac{h}{x_0-x} + C(t-x)^\nu\Big|_{x_0}^{x_0+h} \\ &\le C\frac{h}{x_0-x} + Ch(x_0-x)^{\nu-1}.\end{aligned}$$

Let $x_0 = x+h^{-l}$. We have

$$\begin{aligned}|W_\nu f(x+h)-W_\nu f(x)| &\le Ch^\alpha(x_0-x)^\nu + I_2 \\ &\le Ch^\alpha(x_0-x)^\nu + C\frac{h}{x_0-x} + Ch(x_0-x)^{\nu-1} \\ &\le C(h^{\alpha-l\nu}+h^{l+1}+h^{1+l-l\nu}).\end{aligned}$$

Choose $l=\frac{\alpha-1}{1+\nu}$. Since $0<\alpha,\nu<1$,

$$\begin{aligned}|W_\nu f(x+h)-W_\nu f(x)| &\le C(h^{\frac{\alpha+\nu}{1+\nu}}+h^{\frac{\alpha+\nu}{1+\nu}}+h^{\frac{\alpha+\nu-\alpha\nu}{1+\nu}}) \\ &\le C(2h^{\frac{\alpha+\nu}{1+\nu}}+h^{\frac{\alpha+\nu-\alpha\nu}{1+\nu}}) \\ &\le C(2h^{\frac{\alpha+\nu}{1+\nu}}+h^{\frac{\alpha+\nu}{1+\nu}}) \\ &\le 3Ch^{\frac{\alpha+\nu}{1+\nu}}\end{aligned}$$

$$\leq Ch^{\frac{\alpha+\nu}{1+\nu}}.$$

By Corollary 11.2 (a) in [6],

$$\overline{\dim}_B \Gamma(W_\nu f, I) \leq 2 - \frac{\alpha+\nu}{1+\nu}$$

which means conclusion of the present theorem holds. □

More details about fractal dimension estimation of the Weyl fractional integral of certain positive order can be found in Ref. [57].

5.2 The Katugampola Fractional Calculus

In addition to the Riemann–Liouville fractional calculus, the Weyl–Marchaud fractional derivative and the Weyl fractional integral, there are many different kinds of fractional integrals and fractional derivatives [55]. They are based on their own physical background or mathematical meaning, have a certain range of use.

Katugampola [12] produced a fractional integral that generalizes both the Riemann–Liouville and the Hadamard fractional integrals, but is a particular case of the modified Erdélyi–Kober fractional integrals (see for example in [16]).

In [58], authors dealt with the Katugampola fractional integral of a continuous function of bounded variation. They deduced that fractal dimension of the Katugampola fractional integral of a continuous function of bounded variation is one. Definition of the Katugampola fractional integral has been given as follows.

Definition 5.5 ([12, 13]) *Let $f(x) \in C_I$. The Katugampola fractional integral of $f(x)$ is defined as*

$$^{\rho}_{0}D^{\nu} f(x) = \frac{(\rho+1)^{1-\nu}}{\Gamma(\nu)} \int_0^x (x^{\rho+1} - t^{\rho+1})^{\nu-1} t^{\rho} f(t) dt$$

where $\nu > 0$ and $\rho \neq -1$ are real numbers.

They also got at an upper bound for the upper Box dimension of the Katugampola fractional derivative of a differentiable function. We write it as the following theorem.

Theorem 5.6 ([12]) *Let $f(x) \in BC_I$ and $\rho > -1$, $\nu > 0$. Then the Box dimension of the Katugampola fractional integral of $f(x)$ of positive order ν is still one on I. Furthermore, $^{\rho}_{0}D^{\nu} f(x)$ is a continuous function with bounded variation on I.*

Zhang made research on connection between order of the Katugampola fractional integral and the Box dimension of the Weierstrass function in [79]. She got the corresponding relationship as the following theorem.

Theorem 5.7 ([9]) *Let $W(x)$ be given as Example 2.1 and $0 < \nu < 1 - \alpha$. Thus*

$$\dim_B \Gamma(^{\rho}_{0}D^{\nu} W, I) = \dim_B \Gamma(W, I) - \nu$$

where $\rho \neq -1$.

5.3 The Hadamard Fractional Integral

The Hadamard fractional integral differs from the other ones in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. The background material of the Hadamard fractional derivative and integral can be found in Refs. [9, 46]. Definition of the Hadamard fractional integral has been given as follows.

Definition 5.8 ([9, 46]) *Let $f(x) \in C_I$. Then the left Hadamard fractional integral $I_\nu f(x)$ and the right Hadamard fractional integral $\tilde{I}_\nu f(x)$ are*

$$I_\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x \left(\ln \frac{x}{t}\right)^{\nu-1} \frac{f(t)}{t} dt$$

and

$$\tilde{I}_\nu f(x) = \frac{1}{\Gamma(\nu)} \int_x^1 \left(\ln \frac{t}{x}\right)^{\nu-1} \frac{f(t)}{t} dt.$$

Recently, people began to study the properties of the Hadamard fractional integral such as Ref. [4]. Applications in differential equations can be found in [15].

Some linear relationship between order of the Hadamard fractional integral and fractal dimension of the Weierstrass function has been proved in [68].

Theorem 5.9 ([68]) *Let $W(x)$ be given as Example 2.1 and $0 < \nu < 1 - \alpha$. Thus*

$$\dim_B \Gamma(I_\nu W, I) = \dim_B \Gamma(W, I) - \nu.$$

In [67], it had been proved that the Hadamard fractional integral of a continuous function with bounded variation is still a continuous function with bounded variation. Furthermore, fractal dimension of the Hadamard fractional integral of certain continuous functions with unbounded variation had also been proved to be still with the Box dimension one.

Theorem 5.10 ([56]) *If $f(x) \in BC_I$, $I_\nu f(x) \in BC_I$ for any $\nu > 0$.*

Based on discussion of works about the Weyl fractional integral of certain continuous functions, Tian had made research on the fractal dimension estimation of the Hadamard fractional integral of certain continuous functions satisfying the Hölder condition and got several results [56].

5.4 The Caputo Fractional Derivative

Another kind of fractional derivative with practical application background has been given as follows.

Definition 5.11 ([44, 45]) *Let $f(x) \in C_I$ and*

$$D^\nu f(x) = \frac{1}{\Gamma(n-\nu)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\nu+1-n}} dt, \quad (n-1) < \nu < n$$

for a positive integer n and a real number ν. $D^\nu f(x)$ given above is called as the Caputo fractional derivative of $f(x)$ of order ν.

6 Applications

In the last three sections, we mainly make research on the fractal dimension estimation of fractional calculus of different kinds of continuous functions. Here we give a simple application of fractional calculus in the study of fractional calculus equations, and explain the concrete significance of this kind of work at one time [60]. Then we show a method to judge the rationality of definition of fractional calculus.

6.1 Fractional Calculus Equations

By using the change law of fractal dimension of continuous functions fractional calculus, we can discuss fractional calculus equations as follows elementary.

Example 6.1 Let $f(x)$ and $g(x)$ be two continuous functions on I. Let $0 < \mu < 1$ and

$$\dim_B \Gamma(g, I) = s$$

while $1 < s < 2$ and

$$D^{\mu} f(x) = g(x).$$

Here $D^{\mu} f(x)$ means the Riemann–Liouville fractional derivative of $f(x)$ of positive order μ. If the Box dimension of $f(x)$ exists on I, it must be

$$\dim_B \Gamma(f, I) \leq s.$$

Furthermore, if $g(x)$ satisfies the Hölder condition of order α and $0 < \mu < \alpha$, by Theorem 3.10

$$s \leq 2 - \alpha + \mu.$$

The above example shows that the fractal dimension of continuous functions can be judged by the conclusion that the fractal dimensions of both ends of fractional order integral and differential equations are equal. Here, we give two examples to illustrate conclusions of the present paper.

Example 6.2 Let $f(x)$ be a Lipschitz continuous function and $g(x)$ be a continuous function on I. Let $\nu > 0$ and $D^{-\nu} f(x)$ be the Riemann–Liouville fractional integral of $f(x)$ of order ν on I. If

$$D^{-\nu} f(x) = g(x),$$

$g(x)$ must be a differentiable function on I by Theorem 3.6.

Example 6.3 Let $f(x)$ and $g(x)$ be two continuous functions on I. Let $\nu > 0$ and $D^{-\nu} f(x)$ be the Riemann–Liouville fractional integral of $f(x)$ of order ν on I. If

$$D^{-\nu} f(x) = g(x)$$

and

$$\dim_B \Gamma(g, I) > 2 - \nu,$$

the fractional integral equation given above has no solution.

Proof In fact, for any continuous function $f(x)$ on I,

$$\overline{\dim}_B \Gamma(D^{-\nu} f, I) \leq 2 - \nu$$

by Theorem 3.17. From the fractional integral equation given above, we know

$$\overline{\dim}_B \Gamma(D^{-\nu} f, I) = \overline{\dim}_B \Gamma(g, I).$$

But the upper Box dimension of the left side of the equation is no more than $2 - \nu$ while the upper Box dimension of the right side of the equation is strictly large than $2 - \nu$. This leads to contradictions. Therefore, the above fractional integral equation has no solution. $\square$

6.2 Judgment on the Rationality of Definitions of Fractional Calculus

We believe that the fractal dimension estimation of fractional calculus of continuous functions in this paper of the above three sections is of great theoretical significance to discuss the rationality of definition of fractional calculus. In fact, classical integral of a continuous function seems smoother than the function itself, while classical derivative of a continuous function seems coarser than the function itself when it exists. Therefore, the corresponding properties should be maintained for fractional calculus. It just uses the fractal dimension to replace the general smoothness.

In fact, people could find the discussion about the fractal dimension of the Riemann–Liouville fractional integral and differential of certain fractal functions such as the Weierstrass function, the Besicovitch function and so on in Refs. [24, 53, 74]. The results of computer simulation and numerical verification have also been given. Under the action of the Riemann–Liouville fractional integral, fractal dimension of these fractal functions decreases. Under the action of the Riemann–Liouville fractional differential, fractal dimension of the obtained function increases. In a sense, it has the same mechanism as the classical calculus.

The above discussion also shows that definition of certain fractional calculus such as the Riemann–Liouville fractional calculus is reasonable from this point of view by discussion of Section 3. The rationality of other definitions of fractional calculus needs to be further verified by a similar discussion process.

7 Conclusions

Exploring the relationship between fractional calculus and fractal geometry has become the core content of fractional calculus theory. Many scholars have studied this issue from different angles. For example, Nigmatullin made research on the accurate relationship between fractional integral and fractals of the given symmetry in space and try to give the geometrical/physical interpretation of this relationship [40, 41]. Butera showed a relationship between fractional calculus and fractals, based on certain physical and geometrical considerations by the physical origins of the power-laws [3]. We expect to establish the relationship between fractional calculus and fractal functions from fractal dimension. On one hand, from the geometric point of view, one can explore the internal relationship between them. On the other hand, it also provides a certain basis for judging the rationality of definitions of fractional calculus.

It is of great theoretical significance to study the mechanism of fractional calculus on continuous functions, especially the variation of fractal dimension. As we all know, for the classical calculus, the integral function is smoother than the original one, and the differential function is more rough than the original function. Therefore, a similar relationship should be established for fractional calculus. In other words, the fractal dimension of fractional integral function is not increased compared with the original one, and fractal dimension of fractional differential function is not reduced correspondingly. Especially for the special class of fractal functions which meet certain conditions, the fractal dimension of their fractional calculus is linear compared with that of the original functions which is the same as described in [54]. We think that definition of fractional calculus, which satisfies this conclusion at least discussed above, is reasonable.

In the specific research process, we found an interesting result. That is to say, fractal dimension of the Riemann–Liouville fractional integral of regular fractal functions decreases at least linearly compared with the original functions. We think that for regular fractal functions, fractal dimension of their Riemann–Liouville fractional integral should change linearly with the corresponding one. This conclusion is consistent with what is described in Ref. [3] to certain extent. In other words, it seems that fractional calculus can control or change the fractal dimension of a fractal function according to certain rules from the geometric view.

The results of this paper have an important internal relationship with the description of the essential characteristics of fractional calculus in Ref. [3], and can be used as an important evidence of the physical interpretation and geometric significance of fractional calculus. We think it could be helpful of establish the accurate relationship between fractional calculus and fractals from geometric view. We also believe that the essence of geometric of power-laws is revealed from the perspective of mathematical theory.

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