Acta Mathematica Sinica, English Series
Mar., 2023, Vol. 39, No. 3, pp. 481–496
Published online: March 15, 2023
https://doi.org/10.1007/s10114-023-1376-7
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Jacobson's Lemma for Generalized Drazin–Riesz Inverses

Soufiane HADJI Hassane ZGUITTI[1)]
Department of Mathematics, Dhar El Mahraz Faculty of Science, Sidi Mohamed Ben Abdellah University, P.O.Box 1796 *Fes-Atlas,* 30003 *Fez Morocco*
E-mail: *soufiane.hadji@usmba.ac.ma* *hassane.zguitti@usmba.ac.ma*

Abstract For bounded linear operators A, B, C and D on a Banach space X, we show that if $BAC = BDB$ and $CDB = CAC$ then $I - AC$ is generalized Drazin–Riesz invertible if and only if $I - BD$ is generalized Drazin–Riesz invertible, which gives a positive answer to Question 4.9 in Yan, Zeng and Zhu [*Complex Anal. Oper. Theory* 14, Paper No. 12 (2020)]. In particular, we show that Jacobson's lemma holds for generalized Drazin–Riesz inverses.

Keywords Jacobson's lemma, Drazin inverse, generalized Drazin–Riesz inverse

MR(2010) Subject Classification 15A09, 47A10, 47A53, 47A55

1 Introduction

Let $\mathcal{L}(X)$ denote the set of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in \mathcal{L}(X)$, we denote by $\sigma(T)$, $\rho(T)$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the spectrum, the resolvent set, the null space and the range of T, respectively. $T \in \mathcal{L}(X)$ is said to be a *Fredholm* operator if $\dim \mathcal{N}(T) < \infty$ and $\operatorname{codim} \mathcal{R}(T) < \infty$. T is *Riesz* if $T - \lambda I$ is Fredholm for every nonzero λ in $\mathbb{C}$. Let M and N be two closed T-invariant subspaces of X. If $X = M \oplus N$, then we say that T is *completely reduced* by the pair (M, N) and we denote it by $(M, N) \in \operatorname{Red}(T)$.

For a subset Δ of $\mathbb{C}$, we denote by $\operatorname{acc}\Delta$ the accumulation points of Δ. We denote also by $D(\lambda, r)$ (resp., $\overline{D}(\lambda, r)$) the open (resp., closed) disc centered at $\lambda \in \mathbb{C}$ and with radius $r > 0$.

Following Drazin [9], an operator $T \in \mathcal{L}(X)$ is said to be *Drazin invertible* if there exists an $S \in \mathcal{L}(X)$ satisfying

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is nilpotent.} \tag{1.1}$$

If such S exists then it is unique and it is called the *Drazin inverse* of T. Notice that T is Drazin invertible if and only if 0 is a pole of T.

A generalization of the Drazin inverse was given by Koliha [14]: $T \in \mathcal{L}(X)$ is said to be *generalized Drazin invertible* if there exists an $S \in \mathcal{L}(X)$ satisfying

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is quasinilpotent.} \tag{1.2}$$

If such S exists then it is unique and it is called the *generalized Drazin inverse* of T. Recall that T is generalized Drazin invertible if and only if $0 \notin \operatorname{acc}\sigma(T)$.

Received August 5, 2021, revised February 2, 2022, accepted April 13, 2022
1) Corresponding author

The (generalized) Drazin inverse has important applications in matrix theory and computations and fields such as linear systems theory, differential equations, Markov chains, and so on, see [1, 3–5, 16] and the references therein.

Recently, Živkovic–Zlatanović and Cvetković [23] generalized the concept of generalized Drazin invertible operators:

Definition 1.1 ([23]) *An operator $T \in \mathcal{L}(X)$ is generalized Drazin–Riesz invertible if there exists $S \in \mathcal{L}(X)$ satisfying*

$$TS = ST,\ STS = S \text{ and } T - T^2S \text{ is Riesz};$$

and such S is called a generalized Drazin–Riesz inverse of T.

The generalized Drazin–Riesz spectrum of T is defined by

$$\sigma_{\mathrm{gDR}}(T) = \{\lambda \in \mathbf{C}, T - \lambda I \text{ is not generalized Drazin–Riesz invertible}\}.$$

T is generalized Drazin–Riesz invertible if and only if there exists $(M, N) \in \mathrm{Red}(T)$ such that $T_M = T|M$ is invertible and $T_N = T|N$ is Riesz [23]. Moreover, if we assume that $0 \in \operatorname{acc} \sigma(T)$ then T_N is Riesz with infinite spectrum. Hence $\sigma(T_N) = \{\lambda_n\} \cup \{0\}$ where λ_n converges to zero. Since T_M is invertible, there is some $r > 0$ such that $\overline{D}(0, r) \cap \sigma(T_M) = \emptyset$ and there is some positive integer n_0 such that $\{\lambda_n\}_{n \geq n_0} \subset D(0, r) \cap \sigma(T_N)$ and $\{\lambda_n\}_{n \geq n_0}$ are non-zero Riesz points of T (see the proof of [23, Proposition 2.7]).

Given two operators A and $B \in \mathcal{L}(X)$, Jacobson's lemma asserts that $I - AB$ is invertible if and only if $I - BA$ is invertible, see for instance [2]. Independently, [6] and [7] extended Jaconbson's lemma to Drazin inverse by showing that $I - AB$ is Drazin invertible if and only if $I - BA$ is Drazin invertible. Lam and Nielsen [15] gave an explicit expressions for the Drazin inverses of $I - AB$ and $I - BA$ in terms of each other. In [22], the authors extended Jacobson's lemma to generalized Drazin inverse. [20] extended Jacobson's lemma to AC and BA with the assumption $ABA = ACA$. Under other conditions, [18] and [21] extended Jacobson's lemma to Drazin and generalized inverses for AC and BD.

Recently, Yan et al. [19] introduced conditions

$$BAC = BDB \text{ and } CDB = CAC, \tag{1.3}$$

and investigated Drazin (resp., generalized Drazin) inverses for operators $I - AC$ and $I - BD$. For generalized Drazin–Riesz inverse, the authors proved that AC is gneralized Drazin–Riesz invertible if and only if BD is generalized Drazin–Riesz invertible. The following question was raised in [19, Question 4.9]:

Question *If $BAC = BDB$ and $CDB = CAC$ then $I - AC$ is generalized Drazin–Riesz invertible if and only if $I - BD$ is generalized Drazin–Riesz invertible?*

The aim of this paper is to answer this question. In Section 2, we present some preliminary results that we need in the sequel. In particular, we investigate the generalized Drazin–Riesz invertibility for upper triangular operator matrices. In Section 3, under conditions (1.3) we show that $I - AC$ is generalized Drazin–Riesz invertible if and only if $I - BD$ is generalized Drazin–Riesz invertible, and we also give expressions for generalized Drazin–Riesz inverse of $I - AC$ and $I - BD$. Therefore, we answer positively [19, Question 4.9]. Consequently, we show that Jacobson's lemma holds for generalized Drazin–Riesz inverse, i.e., $I - AB$ is generalized Drazin–Riesz invertible if and only if $I - BA$ is generalized Drazin–Riesz invertible.

2 Preliminary Results

Lemma 2.1 *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$ be similar operators. Then A is generalized Drazin–Riesz invertible if, and only if, B is generalized Drazin–Riesz invertible.*

Proof Assume that B is generalized Drazin–Riesz invertible and let B_1 be a generalized Drazin–Riesz inverse, i.e.,

$$BB_1 = B_1B, \; B_1BB_1 = B_1 \; \text{ and } \; B - BB_1B \text{ is a Riesz operator.}$$

Since A is similar to B, there exists an invertible operator $J \in \mathcal{L}(X)$ such that $A = JBJ^{-1}$. Set $A_1 = JB_1J^{-1}$.

Then

$$A_1A = JB_1J^{-1}JBJ^{-1} = JB_1BJ^{-1} = JBB_1J^{-1} = JBJ^{-1}A_1 = AA_1.$$

Next

$$A_1AA_1 = JB_1BB_1J^{-1} = JB_1J^{-1} = A_1.$$

Hence

$$A' = A(I - AA_1) = JBJ^{-1}(I - JBJ^{-1}JB_1J^{-1}) = JB(I - BB_1)J^{-1} = JB'J^{-1}.$$

Then A', B' are similar. Thus, A' is a Riesz operator. Therefore A is generalized Drazin–Riesz invertible with generalized Drazin–Riesz inverse A_1. The converse goes similarly. □

Theorem 2.2 *Let $A \in \mathcal{L}(X)$ be generalized Drazin–Riesz invertible operator with $0 \in \operatorname{acc}\sigma(A)$ and a generalized Drazin–Riesz inverse A_1. Then there exists a nonzero λ small enough such that the following holds*:

$$(\lambda I - A)^{-1} = (I - AA_1)\sum_{n=1}^{\infty} A^{n-1}\lambda^{-n} - A_1\sum_{n=0}^{\infty}(A_1)^n\lambda^n.$$

Proof It is easy to see that A_1 is Drazin invertible with Drazin inverse A^2A_1. Then according to [23, Theorem 2.3] it follows that the expansion $\sum_{n=0}^{\infty}(A_1)^n\lambda^n$ exists for every nonzero λ small enough, since it represents the function $\lambda \mapsto (I - \lambda A_1)^{-1}$.

For $P_A = I - AA_1$, we have AP_A is a Riesz and then $0 \in \operatorname{acc}\rho(AP_A)$. The expansion $\sum_{n=1}^{\infty} A^{n-1}\lambda^{-n}(I - AA_1)$ exists for some nonzero λ small enough, since it represents the function $\lambda \mapsto (\lambda I - AP_A)^{-1}$. Thus, we obtain $(\lambda I - A)^{-1}$ exists for some $0 < |\lambda| < \epsilon$. Therefore,

$$\begin{aligned}
&(\lambda I - A)\Big((I - AA_1)\sum_{n=1}^{\infty} A^{n-1}\lambda^{-n} - A_1\sum_{n=0}^{\infty}(A_1)^n\lambda^n\Big)\\
&= (I - AA_1)\sum_{n=1}^{\infty} A^{n-1}\lambda^{-n+1} - \sum_{n=0}^{\infty}(A_1)^{n+1}\lambda^{n+1}\\
&\quad - (I - AA_1)\sum_{n=1}^{\infty} A^{n}\lambda^{-n} + AA_1\sum_{n=0}^{\infty}(A_1)^n\lambda^n\\
&= (I - AA_1)\Big(\sum_{n=0}^{\infty} A^{n}\lambda^{-n} - \sum_{n=1}^{\infty} A^{n}\lambda^{-n}\Big)
\end{aligned}$$

$$+ AA_1 + \sum_{n=1}^{\infty} A(A_1)^{n+1}\lambda^n - \sum_{n=1}^{\infty}(A_1)^n\lambda^n$$
$$= I.$$

This completes the proof. □

For A, B and $C \in \mathcal{L}(X)$, we consider the upper triangular operator matrix

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{L}(X \oplus X).$$

Now, we present an additive result concerning some sufficient conditions for M_C to be generalized Drazin–Riesz invertible for every $C \in \mathcal{L}(X)$.

Theorem 2.3 *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$ be generalized Drazin–Riesz invertible operators such that $0 \in \operatorname{acc}\sigma(A) \cap \operatorname{acc}\sigma(B)$ and with generalized Drazin–Riesz inverses A_1 and B_1, respectively. Then for every $C \in \mathcal{L}(X)$, M_C is generalized Drazin–Riesz invertible and*

$$M'_C = \begin{bmatrix} A_1 & S \\ 0 & B_1 \end{bmatrix}$$

is a generalized Drazin–Riesz inverse of M_C, where

$$S = (A_1)^2\Big(\sum_{n=0}^{\infty}(A_1)^n CB^n\Big)(I - BB_1) + (I - AA_1)\Big(\sum_{n=0}^{\infty} A^n C(B_1)^n\Big)(B_1)^2 - A_1CB_1.$$

Proof Assume that A and B are generalized Drazin–Riesz invertible operators, then by [23, Proposition 2.7] there exist $(\mu_n)_n$ and $(\nu_n)_n$ sequences of nonzero Riesz points of A and B respectively, both converging to zero. We can assume that $(|\mu_n|)_n$ and $(|\nu_n|)_n$ are decreasing sequences. Then there exists $r > 0$ small enough such that $|\nu_{n+1}| < r < |\nu_n|$ and $|\mu_{m+1}| < r < |\mu_m|$ for some positive integers n and m. Then for every $\lambda \in \mathbb{C}$ such that $|\lambda| = r$ we have $A - \lambda I$ and $B - \lambda I$ are invertible. [12, Lemma 1] implies that $\lambda I - M_C$ is also invertible and so

$$(\lambda - M_C)^{-1} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}C(\lambda - B)^{-1} \\ 0 & (\lambda - B)^{-1} \end{bmatrix}.$$

By applying Theorem 2.2 and comparing the coefficients at $\lambda^0 = 1$, we obtain the operator

$$M'_C = \begin{bmatrix} A_1 & S \\ 0 & B_1 \end{bmatrix}$$

as a generalized Drazin–Riesz inverse of M_C, where

$$S = (A_1)^2\Big(\sum_{n=0}^{\infty}(A_1)^n CB^n\Big)(I - BB_1) + (I - AA_1)\Big(\sum_{n=0}^{\infty} A^n C(B_1)^n\Big)(B_1)^2 - A_1CB_1.$$

Indeed,

Claim 1 $M_C M'_C = M'_C M_C$:
We have

$$M_C M'_C = \begin{bmatrix} AA_1 & AS + CB_1 \\ 0 & BB_1 \end{bmatrix} \quad \text{and} \quad M'_C M_C = \begin{bmatrix} A_1A & A_1C + SB \\ 0 & B_1B \end{bmatrix}.$$

But

$$SB = (A_1)^2\bigg(\sum_{n=0}^{\infty}(A_1)^nCB^n\bigg)(I-BB_1)B + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^n\bigg)(B_1)^2B - A_1CB_1B$$
$$= \bigg(\sum_{n=0}^{\infty}(A_1)^{n+1}CB^n\bigg)(I-BB_1) + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg) - A_1C.$$

Then

$$SB + A_1C = \bigg(\sum_{n=0}^{\infty}(A_1)^{n+1}CB^n\bigg)(I-BB_1) + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg).$$

Also

$$AS = A(A_1)^2\bigg(\sum_{n=0}^{\infty}(A_1)^nCB^n\bigg)(I-BB_1) + A(I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^n\bigg)(B_1)^2 - A_1CB_1B$$
$$= \bigg(\sum_{n=0}^{\infty}(A_1)^{n+1}CB^n\bigg)(I-BB_1) + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg) - CB_1,$$

which implies

$$AS + CB_1 = \bigg(\sum_{n=0}^{\infty}(A_1)^{n+1}CB^n\bigg)(I-BB_1) + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg).$$

Thus

$$M_CM'_C = M'_CM_C.$$

Claim 2 $M'_CM_CM'_C = M'_C$:

We have

$$M'_CM_CM'_C = \begin{bmatrix} A_1AA_1 & A_1AS + (A_1C+SB)B_1 \\ 0 & B_1BB_1 \end{bmatrix}.$$

Since

$$AS + CB_1 = \bigg(\sum_{n=0}^{\infty}(A_1)^{n+1}CB^n\bigg)(I-BB_1) + (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg),$$

then

$$(AS+CB_1)B_1 = (I-AA_1)\bigg(\sum_{n=0}^{\infty}A^nC(B_1)^{n+1}\bigg)B_1.$$

Hence

$$A_1AS = (A_1)^2\bigg(\sum_{n=0}^{\infty}(A_1)^nCB^n\bigg)(I-BB_1) - A_1CB_1.$$

Consequently,

$$A_1AS + (A_1C+SB)B_1 = S.$$

Therefore,

$$M_CM'_CM_C = M'_C.$$

Claim 3 $M = M_C - M_CM'_CM_C$ is a Riesz operator:

We have

$$M = \begin{bmatrix} A' & R \\ 0 & B' \end{bmatrix},$$

where

$$R = C - AA_1C - ASB - CB_1B, \ A' = A - AA_1 \ \text{ and } \ B' = B - BB_1B.$$

Since A' and B' are Riesz operators, we have $A' - \lambda I$ and $B' - \lambda I$ are Fredholm for every $\lambda \in \mathbb{C} \setminus \{0\}$. Then according to [8, Proposition 3.1] it implies that $M - \lambda I$ is Fredholm for every $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, M is a Riesz operator.

Finally, M_C is generalized Drazin–Riesz invertible and M'_C is a generalized Drazin–Riesz inverse of M_C. □

It follows from Lemma 2.1 and Theorem 2.3 that the result is also valid for block 2×2 lower triangular matrices.

Corollary 2.4 *For $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$ are generalized Drazin–Riesz invertible operators such that $0 \in \operatorname{acc}\sigma(A) \cap \operatorname{acc}\sigma(B)$ and with generalized Drazin–Riesz inverses A_1 and B_1, respectively. Then*

$$T = \begin{bmatrix} B & 0 \\ C & A \end{bmatrix}$$

is generalized Drazin–Riesz invertible for every $C \in \mathcal{L}(X)$ and

$$T' = \begin{bmatrix} B_1 & 0 \\ R & A_1 \end{bmatrix}$$

is a generalized Drazin–Riesz inverse of T, where

$$R = (A_1)^2\left(\sum_{n=0}^{\infty}(A_1)^nCB^n\right)(I - BB_1) + (I - AA_1)\left(\sum_{n=0}^{\infty}A^nC(B_1)^n\right)(B_1)^2 - A_1CB_1.$$

According to Theorem 2.3 we get the following result.

Theorem 2.5 *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$. Then*

$$\sigma_{\mathrm{gDR}}(M_C) \subset \sigma_{\mathrm{gDR}}(A) \cup \sigma_{\mathrm{gDR}}(B), \quad \textit{for every } C \in \mathcal{L}(X).$$

Proposition 2.6 *If any two of operators A, B and M_C are generalized Drazin invertible, then so is the third.*

Proof It is enough to show that A and M_C are generalized Drazin–Riesz invertible then B is generalized Drazin–Riesz invertible. When A and M_C are generalized Drazin–Riesz invertible, likewise in proof of Theorem 2.3 there exists λ such that $A - \lambda I$ and $M_C - \lambda I$ are invertible. It follows from [12] that $B - \lambda I$ are invertible. Hence

$$(\lambda - M_C)^{-1} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}C(\lambda - B)^{-1} \\ 0 & (\lambda - B)^{-1} \end{bmatrix}.$$

By applying Theorem 2.2 and comparing the coefficients at $\lambda^0 = 1$, we obtain the operator

$$M'_C = \begin{bmatrix} A_1 & S \\ 0 & B_1 \end{bmatrix}$$

as a generalized Drazin–Riesz inverse of M_C, where M'_C and A_1 are respectively generalized Drazin–Riesz inverses of M_C and A. It is easy to see that B_1 is a generalized Drazin–Riesz inverse of B, then B is generalized Drazin–Riesz invertible. □

Example 2.7 For $S, C \in \mathcal{L}(X)$, let M_C be the operator matrix defined on $X \oplus X$ by

$$M_C = \begin{bmatrix} S^* & C \\ 0 & S \end{bmatrix}.$$

Then for every $C \in \mathcal{L}(X)$, $I - M_C$ is generalized Drazin–Riesz invertible provided that $I - S$ is generalized Drazin–Riesz invertible. By Theorem 2.5 we have

$$\sigma_{\mathrm{gDR}}(M_C) \subset \sigma_{\mathrm{gDR}}(S).$$

Now if S is the backward shift operator on $X = l^2(\mathbb{N})$ and $C = I - S^*S$, then we have M_C is unitary. Hence $\sigma_{\mathrm{gDR}}(M_C) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ while $\sigma_{\mathrm{gDR}}(S) = \overline{D}(0,1)$; which proves that the inclusion in Theorem 2.5 maybe strict.

3 Extended Jacobson's Lemma for Generalized Drazin–Riesz Inverse

We are now ready to answer Question 4.9 in [19] affirmatively.

Theorem 3.1 *Let A, B, C and $D \in \mathcal{L}(X)$ satisfy $BAC = BDB$ and $CDB = CAC$. Then $\Gamma_{AC} = I - AC$ is generalized Drazin–Riesz invertible if, and only if, $\Gamma_{BD} = I - BD$ is generalized Drazin–Riesz invertible.*

Proof Assume that Γ_{BD} is generalized Drazin–Riesz invertible with a generalized Drazin–Riesz inverse Γ'_{BD}. Set $\Gamma^\pi_{BD} = I - \Gamma_{BD}\Gamma'_{BD}$. Since $\Gamma_{BD}\Gamma^\pi_{BD}$ is a Riesz and commutes with $(I + BD)$, then according to Theorem 2.2 the expansion $\Gamma^\pi_{BD}\sum_{k=0}^{\infty}\left(I - (BD)^2\right)^k \lambda^{-k-1}$ exists since it represents the function $\lambda \mapsto \left(\lambda - \Gamma_{BD}\Gamma^\pi_{BD}\,(I + BD)\right)^{-1}$. Hence

$$\Gamma^\pi_{BD}\sum_{k=0}^{\infty}\left(\lambda I - (BD)^2\right)^k \lambda^{-k-1} = \Gamma^\pi_{BD}\sum_{k=0}^{\infty}\sum_{j=0}^{k} C^j_k\,(\lambda - 1)^j\left(I - (BD)^2\right)^{k-j}\lambda^{-k-1}.$$

Then the expansion $\Gamma^\pi_{BD}\sum_{k=0}^{\infty}\left(\lambda I - (BD)^2\right)^k \lambda^{-k-1}$ exists. Let

$$\Gamma'_{AC} = \left(I - ACD\Gamma^\pi_{BD}\sum_{k=0}^{\infty}(\lambda I - (BD)^2)^k\lambda^{-k-1}B\right)(I + AC) + ACD\Gamma'_{BD}B.$$

It is easy to check that Γ'_{AC} commutes with Γ_{AC}. Then

$$\begin{aligned}\Gamma'_{AC}\Gamma_{AC} &= I - (AC)^2 - ACD\Gamma^\pi_{BD}\sum_{k=0}^{\infty}(\lambda I - (BD)^2)^k\lambda^{-k-1}B(I + AC)\Gamma_{AC} \\ &\quad + ACD\Gamma'_{BD}B\Gamma_{AC} \\ &= I - ACDB - ACD\Gamma^\pi_{BD}\sum_{k=0}^{\infty}(\lambda I - (BD)^2)^k\lambda^{-k-1}(I + BD)B\Gamma_{AC} \\ &\quad + ACD\Gamma'_{BD}\Gamma_{BD}B \\ &= I - ACD\Gamma^\pi_{BD}B - ACD\Gamma^\pi_{BD}\sum_{k=0}^{\infty}(\lambda I - (BD)^2)^k\lambda^{-k-1}\Gamma^\pi_{BD}(I - (BD)^2)B\end{aligned}$$

$$= I - ACD\Gamma^{\pi}_{BD}B - ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}\Gamma^{\pi}_{BD}B$$
$$+ ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}\Gamma^{\pi}_{BD}(BD)^2B$$
$$= I - ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B.$$

We have

$$\begin{aligned}
&ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(I-(BD)^2)^k\lambda^{-k-1}BACD{\Gamma'}_{BD}B\\
&= ACD\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}\Gamma^{\pi}_{BD}BDBD{\Gamma'}_{BD}B\\
&= ACD\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}\Gamma^{\pi}_{BD}{\Gamma'}_{BD}BDBDB\\
&= ACD\Gamma^{\pi}_{BD}{\Gamma'}_{BD}\\
&= 0.
\end{aligned}$$

And

$$\begin{aligned}
&ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}BACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B\\
&= ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}BDBD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B\\
&= ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B.
\end{aligned}$$

We get

$$\begin{aligned}
&{\Gamma'}_{AC}\Gamma_{AC}{\Gamma'}_{AC}\\
&= {\Gamma'}_{AC} - ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B{\Gamma'}_{AC}\\
&= {\Gamma'}_{AC} - ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}B(I+AC)\\
&\quad + ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}BACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}\left(\lambda I-(BD)^2\right)^k\lambda^{-k-1}B(I+AC)\\
&\quad - ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}(\lambda I-(BD)^2)^k\lambda^{-k-1}BACD{\Gamma'}_{BD}B\\
&= {\Gamma'}_{AC}.
\end{aligned}$$

Finally

$$\Gamma_{AC}(I-{\Gamma'}_{AC}\Gamma_{AC}) = \Gamma_{AC}ACD\Gamma^{\pi}_{BD}\sum_{k=0}^{\infty}\left(I-(BD)^2\right)^k\lambda^{-k-1}B$$

$$= ACD\Gamma_{BD}\Gamma_{BD}^{\pi}\sum_{k=0}^{\infty}\left(\lambda I-(BD)^2\right)^k\lambda^{-k-1}B.$$

We have

$$BACD\Gamma_{BD}\Gamma_{BD}^{\pi}\sum_{k=0}^{\infty}\left(\lambda I-(BD)^2\right)^k\lambda^{-k-1}=(BD)^2\Gamma_{BD}\Gamma_{BD}^{\pi}\sum_{k=0}^{\infty}\left(\lambda I-(BD)^2\right)^k\lambda^{-k-1}$$
$$=\Gamma_{BD}\Gamma_{BD}^{\pi}.$$

Since $\Gamma_{BD}\Gamma_{BD}^{\pi}$ is a Riesz operator, we have $\Gamma_{AC}(I-\Gamma'_{AC}\Gamma_{AC})$ is a Riesz.

Therefore Γ_{AC} is generalized Drazin–Riesz invertible. Similarly, we get the converse. □

The following result follows from Theorem 3.1 and [19, Theorem 3.9]

Theorem 3.2 *Let $A,B,C,D\in\mathcal{L}(X)$ satisfy $BAC=BDB$ and $CDB=CAC$. Then*

$$\sigma_{\mathrm{gDR}}(AC)=\sigma_{\mathrm{gDR}}(BD).$$

Remark 3.3 If $BAC=BDB$ and $CDB=CAC$, then we have $BA_nC=BD_nB$ and $CD_nB=CA_nC$. Note that $I-A_nC=(I-AC)^n$ and $I-BD_n=(I-BD)^n$.

$$D_n=\sum_{k=1}^{n}\binom{n}{k}(-1)^kD(BD)^{k-1}$$

and

$$A_n=\sum_{k=1}^{n}\binom{n}{k}(-1)^k(AC)^{k-1}A.$$

Then $(I-AC)^n$ is generalized Drazin–Riesz invertible if and only if $(I-BD)^n$ is generalized Drazin–Riesz invertible.

Example 3.4 For Banach spaces X and Y, let $S_1\in\mathcal{L}(Y),S_2,T_1\in\mathcal{L}(Y,X)$ and $T_2\in\mathcal{L}(X,Y)$ be arbitrary nonzero operators satisfying $S_1=T_2S_2$. We consider $A,B,C\in\mathcal{L}(X\oplus Y)$ as follows:

$$B=\begin{bmatrix}0&0\\0&S_1\end{bmatrix},\quad C=\begin{bmatrix}0&S_2\\0&0\end{bmatrix}\quad\text{and}\quad A=D=\begin{bmatrix}I&T_1\\T_2&I\end{bmatrix},$$

respectively. Then we have $BAC=BDB$ and $CDB=CAC$. Hence Corollary 2.5 implies that the operator

$$\lambda I-AC=\begin{bmatrix}\lambda I&-S_2\\0&\lambda I-S_1\end{bmatrix}$$

is generalized Drazin–Riesz invertible if, and only if,

$$\lambda I-BD=\begin{bmatrix}\lambda I&0\\-S_1T_2&\lambda I-S_1\end{bmatrix}$$

is generalized Drazin–Riesz invertible. On the other hand, according to Theorem 2.3 and Theorem 3.2, we have

$$\sigma_{\mathrm{gDR}}(AC)=\sigma_{\mathrm{gDR}}(BD)=\sigma_{\mathrm{gDR}}(S_1).$$

On the other hand, we have

$$ACD = \begin{bmatrix} S_1 & S_2 \\ T_2S_1 & T_2S_2 \end{bmatrix} \quad \text{and} \quad DBD = \begin{bmatrix} T_1S_1T_2 & T_1S_1 \\ T_2S_1T_2 & S_1 \end{bmatrix}.$$

Example 3.5 Let A, B, C and D be the operators defined on the separable Hilbert space $l_2(\mathbb{N})$, respectively, by

$$\begin{aligned} A(x_1, x_2, x_3, x_4, \ldots) &= (0, x_2, 0, x_4, 0, x_6, \ldots), \\ B(x_1, x_2, x_3, x_4, \ldots) &= (0, x_1, x_2, x_4, x_5, \ldots), \\ C(x_1, x_2, x_3, x_4, \ldots) &= (0, 0, x_1, x_4, x_5, \ldots), \\ D(x_1, x_2, x_3, x_4, \ldots) &= (x_1, 0, x_3, x_4, 0, x_6, \ldots). \end{aligned}$$

Then we have $BAC = BDB$ and $CDB = CAC$, but $DBA \neq ACA$. Hence applying Theorem 3.2 we have

$$\sigma_{\mathrm{gDR}}(AC) = \sigma_{\mathrm{gDR}}(BD).$$

Theorem 3.1 shows that Jabobson's Lemma holds for generalized Drazin–Riesz inverses:

Corollary 3.6 *Let A and $B \in \mathcal{L}(X)$. Then $\Gamma_{AB} = I - AB$ is generalized Drazin–Riesz invertible if, and only if, $\Gamma_{BA} = I - BA$ is generalized Drazin–Riesz invertible.*

Combining Corollary 3.6 and [13, Theorem 2.3] we get the following result. We point out that in [13], the result was presented but without proof.

Corollary 3.7 *Let A and $B \in \mathcal{L}(X)$. Then*

$$\sigma_{\mathrm{gDR}}(AB) = \sigma_{\mathrm{gDR}}(BA).$$

Example 3.8 Let H be complex Hilbert space. Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(H)$ where $|T| = (T^*T)^{\frac{1}{2}}$. The Aluthge transform of T is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Let $A = U|T|^{\frac{1}{2}}$ and $B = |T|^{\frac{1}{2}}$. Then $AB = T$ and $BA = \widetilde{T}$. Thus

$$\sigma_{\mathrm{gDR}}(T) = \sigma_{\mathrm{gDR}}(\tilde{T}).$$

Corollary 3.9 *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy $BAC = BDB$ and $CDB = CAC$. Then*

$$\sigma_{\mathrm{gDR}}((AC)^n) = \sigma_{\mathrm{gDR}}((BD)^n) \quad \text{for all } n \geq 1.$$

Proof It is easy to see that $(AC)^n = (AC)^{n-1}(DB)$ and $(BD)^n = B(AC)^{n-1}D$. Then for each $n \geq 1$, we have

$$\begin{aligned} \sigma_{\mathrm{gDR}}((BD)^n) &= \sigma_{\mathrm{gDR}}(B(AC)^{n-1}D) \\ &= \sigma_{\mathrm{gDR}}((AC)^{n-1}DB) \qquad \text{(by Corollary 3.7)} \\ &= \sigma_{\mathrm{gDR}}((AC)^n). \end{aligned}$$

Proposition 3.10 *Let $S, T, A,$ and $B \in \mathcal{L}(X)$. Then*

(i) *If ST is generalized Drazin–Riesz invertible and T is invertible, then S is also generalized Drazin–Riesz invertible.*

(ii) *If A and B are invertible and $T = ASB$, then T is generalized Drazin–Riesz invertible if and only if S is generalized Drazin–Riesz invertible.*

Proof (i) Since $U = ST$ is generalized Drazin–Riesz invertible, there exists $(M, N) \in \mathrm{Red}(U)$ such that $U = U_M \oplus U_N$ where U_M is invertible and U_N is a Riesz. Since

$$T^{-1}(X) = X = M \oplus N$$

then

$$S(X) = UT^{-1}(X) = (U_1 \oplus U_2)(M \oplus N)$$

where U_1 is invertible on M and U_2 is a Riesz on N. Therefore S is generalized Drazin–Riesz invertible.

(ii) Suppose that A and B are invertible, and $T = ASB$. If T is generalized Drazin–Riesz invertible, it follows from i) that AS is generalized Drazin–Riesz invertible. By [13, Theorem 2.3], SA is also generalized Drazin–Riesz invertible. Since A is invertible, we have again by i) S is generalized Drazin–Riesz invertible. The converse goes by the same way. □

Remark 3.11 In general, we observe that even though ST is generalized Drazin–Riesz invertible and S is invertible, T may not be generalized Drazin–Riesz invertible. For example, let R be not generalized Drazin–Riesz invertible on X. If S and T have the following operator matrix forms

$$S = \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} -R & 0 \\ I & 0 \end{bmatrix},$$

then

$$ST = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

is generalized Drazin–Riesz invertible and S is invertible. However, $S^{-1}(ST) = T$ is not generalized Drazin–Riesz invertible by virtue of Proposition 2.6.

Example 3.12 For Banach spaces X and Y, let $S \in \mathcal{L}(Y)$ and $T \in \mathcal{L}(X, Y)$ be arbitrary nonzero operators. Let A and $B \in \mathcal{L}(X \oplus Y)$ be defined by

$$A = \begin{bmatrix} S^* & 0 \\ S & I \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & 0 \\ T & I \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} S^* & 0 \\ S + T & S \end{bmatrix}$$

is generalized Drazin–Riesz invertible if, and only if,

$$BA = \begin{bmatrix} S^* & 0 \\ TS^* + S & S \end{bmatrix}$$

is generalized Drazin–Riesz invertible. According to Theorem 2.3 and Proposition 2.6, $I - S$ is generalized Drazin–Riesz invertible if and only if $I - AB$ (resp., $I - BA$) is generalized Drazin–Riesz invertible.

Since B is invertible, then if $\lambda I - AB$ is generalized Drazin–Riesz invertible and using Proposition 3.10 we get

$$\lambda B^{-1} - A = \begin{bmatrix} \lambda I - S^* & 0 \\ -\lambda T - S & (\lambda - 1)I \end{bmatrix}$$

is generalized Drazin–Riesz invertible. Hence $\lambda I - S^*$ is generalized Drazin–Riesz invertible. Thus

$$\sigma_{\mathrm{gDR}}(AB) = \sigma_{\mathrm{gDR}}(BA) = \sigma_{\mathrm{gDR}}(S).$$

4 Concluding Remarks

4.1 The Class $\mathcal{P}_k$

For $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the set of bounded linear operators A and $B \in \mathcal{L}(X)$ satisfying equalities

$$A^k B^k A^k = A^{k+1} \quad \text{and} \quad B^k A^k B^k = B^{k+1}.$$

In [10, 11], Jacobson's Lemma was studied for various kind of inverses.

Lemma 4.1 *Let $(A, B) \in \mathcal{P}_k$, $k \in \mathbb{N}$, and N and M be subsets in X such $X = M \oplus N$. Then we have*

(1) $(M, N) \in \mathrm{Red}(I - B^k A^k)$ *if and only if* $(M, N) \in \mathrm{Red}(I - A)$.

(2) $(M, N) \in \mathrm{Red}(I - A^k B^k)$ *if and only if* $(M, N) \in \mathrm{Red}(I - B)$.

Proof (1) Suppose that $(M, N) \in \mathrm{Red}(I - A)$. Then $(M, N) \in \mathrm{Red}(A)$. Let $x \in N$, then there exists $y \in N$ such that $A^{k+1}x = y$. Using the fact $A^k B^k A^k = A^{k+1}$, we have $A^k B^k A^k x = y$. Since N is A^k-invariant, implies that N is $(B^k A^k)$-invariant. Therefore N is $(I - B^k A^k)$-invariant. With the same argument we show that M is $(I - B^k A^k)$-invariant. Conversely, suppose that (M, N) is $(I - B^k A^k)$-invariant. Then $(M, N) \in \mathrm{Red}(B^k A^k)^2$. Let $z \in N$, then there exists $t \in N$ such that $(B^k A^k)^2 z = t$. Using the fact $A^k B^k A^k = A^{k+1}$, we have $B^k A^{k+1} z = (B^k A^k)^2 z = t$. Since (M, N) is $(B^k A^k)$-invariant, it implies that N is A-invariant. Therefore $(M, N) \in \mathrm{Red}(I - A)$.

(2) is similar to (1). □

Theorem 4.2 *Let $(A, B) \in \mathcal{P}_k$, $k \in \mathbb{N}$. Then the following statements are equivalent*:

(i) $I - B$ *is generalized Drazin–Riesz invertible.*

(ii) $I - A^k B^k$ *is generalized Drazin–Riesz invertible.*

(iii) $I - B^k A^k$ *is generalized Drazin–Riesz invertible.*

(vi) $I - A$ *is generalized Drazin–Riesz invertible.*

In particular,

$$\sigma_{\mathrm{gDR}}(A) = \sigma_{\mathrm{gDR}}(A^k B^k) = \sigma_{\mathrm{gDR}}(B^k A^k) = \sigma_{\mathrm{gDR}}(B).$$

Proof (i) ⇔ (ii): Suppose that $I - B$ is generalized Drazin–Riesz invertible, then there exists $(M, N) \in \mathrm{Red}(I - B)$ such that $I - B = (I - B)_M \oplus (I - B)_N$, $(I - B)_M$ is invertible and $(I - B)_N$ is a Riesz. By [10, Theorem 2.10], $(I - A^k B^k)_M$ is invertible and $(I - A^k B^k)_N$ is a Riesz. According to Lemma 4.1, $I - A^k B^k = (I - A^k B^k)_M \oplus (I - A^k B^k)_N$. Hence $I - A^k B^k$ is generalized Drazin–Riesz invertible. By the same reasoning we get the converse.

(ii) ⇔ (iii): By Corollary 3.6.

(iii) ⇔ (vi): It is similar to (i) ⇔ (ii). □

Example 4.3 Let A and B be the bounded operators defined on $l^2(\mathbb{N})$ by:

$$A(x_1, x_2, x_3, x_4, x_5, \ldots) = (0, x_1, 0, 0, 0, x_3, 0, 0, 0, x_5, \ldots),$$
$$B(x_1, x_2, x_3, x_4, x_5, \ldots) = (0, x_1, 0, x_3, 0, x_5, \ldots).$$

It is easy to see $B^2 = A^2 = BA = AB = 0$. This implies that $(A, B) \in \mathcal{P}_k$ for all $k \in \mathbb{N}$. Applying Theorem 4.2 we have

$$\sigma_{\mathrm{gDR}}(A) = \sigma_{\mathrm{gDR}}(B).$$

4.2 A Second Proof of Jacobson's Lemma

For A and $B \in \mathcal{L}(X)$, let us consider $\mathbb{A}, \mathbb{B} \in \mathcal{L}(X \oplus X)$, defined by

$$\mathbb{A} = \begin{bmatrix} I & A \\ B & I \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} I & B \\ A & I \end{bmatrix}.$$

Then $\mathbb{A}$ and $\mathbb{B}$ are similar operators. Indeed, $\mathbb{A} = \mathbb{J}\mathbb{B}\mathbb{J}^{-1}$, where the operators $\mathbb{J}$ and $\mathbb{J}^{-1}$ are given by

$$\mathbb{J} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \mathbb{J}^{-1}.$$

Let $\mathbb{A}_1$ and $\mathbb{B}_1 \in \mathcal{L}(X \oplus X)$ be the operator matrices defined by

$$\mathbb{A}_1 = \begin{bmatrix} I - AB & 0 \\ 0 & I \end{bmatrix}, \quad \mathbb{B}_1 = \begin{bmatrix} I - BA & 0 \\ 0 & I \end{bmatrix}.$$

Then the following equalities hold:

$$\mathbb{A} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \mathbb{A}_1 \begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \mathbb{U}\mathbb{A}_1\mathbb{V},$$
$$\mathbb{B} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \mathbb{B}_1 \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \mathbb{U}_1\mathbb{B}_1\mathbb{V}_1;$$

where the matrices $\mathbb{U}$, $\mathbb{U}_1$, $\mathbb{V}$ and $\mathbb{V}_1$ have the inverses

$$\mathbb{U}^{-1} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}, \quad \mathbb{U}_1^{-1} = \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix},$$

$$\mathbb{V}^{-1} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}, \quad \mathbb{V}_1^{-1} = \begin{bmatrix} I & 0 \\ -A & I \end{bmatrix}.$$

Now assume that $\Gamma_{AB} = I - AB$ is generalized Drazin–Riesz invertible with a generalized Drazin–Riesz inverse Γ'_{AB}. It follows from Theorem 2.3 that

$$\mathbb{U}\mathbb{A}_1 = \begin{bmatrix} \Gamma_{AB} & A \\ 0 & I \end{bmatrix}$$

is generalized Drazin–Riesz invertible with an inverse

$$S=\begin{bmatrix} \Gamma'_{AB} & S_{\Gamma_{AB}} \\ 0 & I \end{bmatrix},$$

where

$$S_{\Gamma_{AB}}=(I-\Gamma'_{AB})\Big(\sum_{n=0}^{\infty}\Gamma_{AB}^{n}\Big)A.$$

Since $\mathbb{A}=\mathbb{U}\mathbb{A}_1\mathbb{V}$, we have $\mathbb{A}\mathbb{V}^{-1}$ is generalized Drazin–Riesz invertible. By Proposition 3.10 (i), it follows that $\mathbb{A}$ is generalized Drazin–Riesz invertible. Since we have $\mathbb{A}$ and $\mathbb{B}$ are similar, then by Lemma 2.1, $\mathbb{B}$ is also generalized Drazin–Riesz invertible. Since $\mathbb{B}=\mathbb{U}_1\mathbb{B}_1\mathbb{V}_1$ and $\mathbb{U}_1$ and $\mathbb{V}_1$ are invertible, it follows from Proposition 3.10 (ii), that $\mathbb{B}_1$ is generalized Drazin–Riesz invertible. As a result, $\Gamma_{BA}=I-BA$ is generalized Drazin–Riesz invertible. By the same argument as above we get the converse.

Now assume that Γ'_{BA} is a generalized Drazin–Riesz inverse of Γ_{BA}. We will show that

$$\Gamma'_{AB}=I-A\Big(\sum_{n=0}^{\infty}\Gamma_{BA}^{n}\Big)(I-(\Gamma'_{BA})^2)B-A(\Gamma'_{BA})^2B$$

is a generalized Drazin–Riesz inverse of Γ_{AB}.

Since

$$\mathbb{A}\mathbb{V}^{-1}=\begin{bmatrix} \Gamma_{AB} & A \\ 0 & I \end{bmatrix} \quad\text{and}\quad \mathbb{V}^{-1}\mathbb{A}=\begin{bmatrix} I & A \\ 0 & \Gamma_{BA} \end{bmatrix},$$

then using Cline's formula [13, Theorem 2.2], we obtain a generalized Drazin–Riesz inverse of $\mathbb{A}\mathbb{V}^{-1}$ of the form

$$\begin{bmatrix} I-S_{\Gamma_{BA}}(I-\Gamma'_{BA})B-A(\Gamma'_{BA})^2B & S_{\Gamma_{BA}}(I-\Gamma'_{BA})+A(\Gamma'_{BA})^2 \\ B-BS_{\Gamma_{BA}}(I-\Gamma'_{BA})B-(\Gamma'_{BA})^2B & BS_{\Gamma_{BA}}(I-\Gamma'_{BA})B-(\Gamma'_{BA})^2 \end{bmatrix},$$

where

$$S_{\Gamma_{BA}}=A\Big(\sum_{n=0}^{\infty}\Gamma_{BA}^{n}\Big)(I-\Gamma'_{BA}).$$

Set

$$\Gamma'_{AB}=I-A\Big(\sum_{n=0}^{\infty}\Gamma_{BA}^{n}\Big)(I-(\Gamma'_{BA})^2)B-A(\Gamma'_{BA})^2B.$$

First

$$\begin{aligned}\Gamma_{AB}\Gamma'_{AB}&=\Gamma_{AB}-\Gamma_{AB}A\Big(\sum_{n=0}^{\infty}\Gamma_{BA}^{n}\Big)(I-(\Gamma'_{BA})^2)B-\Gamma_{AB}A(\Gamma'_{BA})^2B\\&=\Gamma_{AB}-A\Big(\sum_{n=1}^{\infty}\Gamma_{BA}^{n}\Big)(I-(\Gamma'_{BA})^2)B-A\Gamma'_{BA}B,\end{aligned}$$

and

$$\Gamma'_{AB}\Gamma_{AB}=\Gamma_{AB}-A\Big(\sum_{n=0}^{\infty}\Gamma_{BA}^{n}\Big)(I-(\Gamma'_{BA})^2)B\Gamma_{AB}-A(\Gamma'_{BA})^2B\Gamma_{AB}$$

$$= \Gamma_{AB} - A\bigg(\sum_{n=1}^{\infty} \Gamma_{BA}^n\bigg)(I - (\Gamma'_{BA})^2)B - A\Gamma'_{BA}B.$$

Then

$$\Gamma'_{AB}\Gamma_{AB} = \Gamma_{AB}\Gamma'_{AB}.$$

Next

$$\Gamma'_{AB}\Gamma_{AB} = I - A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})B.$$

Since

$$\begin{aligned}\Gamma'_{AB}\bigg(- A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})B\bigg) &= A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})B \\ &\quad + A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})(I - \Gamma_{BA}\Gamma'_{BA})B \\ &\quad + A(\Gamma'_{BA})^2(I - \Gamma_{BA}\Gamma'_{BA})B \\ &= 0,\end{aligned}$$

then

$$\Gamma'_{AB}\Gamma_{AB}\Gamma'_{AB} = \Gamma'_{AB}.$$

Also

$$\Gamma_{AB}\Gamma'_{AB}\Gamma_{AB} = \Gamma_{AB} - A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})\Gamma_{BA}B.$$

Since

$$BA\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})\Gamma_{BA} = (I - \Gamma_{BA}\Gamma'_{BA})\Gamma_{BA}$$

and $(I - \Gamma_{BA}\Gamma'_{BA})\Gamma_{BA}$ is a Riesz operator, we have by [2, Theorem 6]

$$A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - \Gamma_{BA}\Gamma'_{BA})\Gamma_{BA}B$$

is a Riesz operator. Thus $\Gamma_{AB}\Gamma'_{AB}\Gamma_{AB} - \Gamma_{AB}$ is a Riesz operator.

Therefore

$$\Gamma'_{AB} = I - A\bigg(\sum_{n=0}^{\infty} \Gamma_{BA}^n\bigg)(I - (\Gamma'_{BA})^2)B - A(\Gamma'_{BA})^2B$$

is a generalized Drazin–Riesz inverse of Γ_{AB}.

Acknowledgements The authors are grateful to the referee for several helpful remarks and suggestions concerning this paper.

References

[1] Abdeljabbar, A., Tran, T. D.: Singular differential equations and g-Drazin invertible operators. *Int. J. Anal.*, **2016**, 7975107 (2016)

[2] Barnes, B. A.: Common operator properties of the linear operators *RS* and *SR*. *Proc. Amer. Math. Soc.*, **126**, 1055–1061 (1998)

[3] Ben-Israel, A., Greville, T. N. E.: Generalized Inverses: Theory and Applications, Second Ed., Wiley-Interscience New York, 1974, Springer, New York, 2002

[4] Campbell, S. L., Meyer, C. D.: Generalized Inverses of Linear Transformations, Pitman, London, 1979

[5] Campbell, S. L., Meyer, C. D., Rose, N. J.: Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. *SIAM J. Appl. Math.*, **31**, 411–425 (1976)

[6] Castro-González, N., Mendes-Araújo, C., Patrício, P.: Generalized inverses of a sum in rings. *Bull. Austral. Math. Soc.*, **82**, 156–164 (2010)

[7] Cvetković-Ilić, D., Harte, R.: On Jacobson's lemma and Drazin invertibility. *Appl. Math. Lett.*, **23**, 417–420 (2010)

[8] Djordjević, D. S.: Perturbations of spectra of operator matrices. *J. Operator Theory*, **24**, 467–486 (2002)

[9] Drazin, M. P.: Pseudo-inverse in associative rings and semigroups. *Amer. Math. Monthly*, **65**, 506–514 (1958)

[10] Gupta, A., Kumar, A.: Common spectral properties of linear operators A and B satisfying $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$. *Asian-Eur. J. Math.*, **12**, 1950084 (2019)

[11] Hadji, S., Zguitti, H.: Back to the common spectral properties of operators satisfying $A^kB^kA^k = A^{k+1}$ and $B^kA^kB^k = B^{k+1}$. *Asian-Eur. J. Math.*, **14**, 2150177 (2021)

[12] Han, J. K., Lee, H. Y., Lee, W. Y.: Invertible completions of 2×2 upper triangular operator matrices. *Proc. Amer. Math. Soc.*, **128**, 119–123 (2000)

[13] Karmouni, M., Tajmouati, A.: A Cline's formula for the generalized Drazin–Riesz inverses. *Funct. Anal. Approx. Comput.*, **10**, 35–39 (2018)

[14] Koliha, J. J.: A generalized Drazin inverse. *Glasg. Math. J.*, **38**, 367–381 (1996)

[15] Lam, T. Y., Nielsen, P. P.: Jacobson's lemma for Drazin inverses. *Contemporary Math.*, **609**, 185–195 (2014)

[16] Nashed, M. Z., Zhao, Y.: The Drazin inverse for singular evolution equations and partial differential operators. *World Sci. Ser. Appl. Anal.*, **1**, 441–456 (1992)

[17] Patrício, P., Da Costa, A. V.: On the Drazin index of regular elements. *Cent. Eur. J. Math.*, **7**, 200–205 (2009)

[18] Yan, K., Fang, X.: Common properties of the operator products in spectral theory. *Ann. Funct. Anal.*, **6**, 60–69 (2015)

[19] Yan, K., Zeng, Q., Zhu, Y.: On Drazin spectral equation for the operator products. *Complex Anal. Oper. Theory*, **14**, Paper No. 12 (2020)

[20] Zeng, Q. P., Zhong, H. J.: Common properties of bounded linear operators AC and BA: spectral theory. *Math. Nachr.*, **287**, 717–725 (2014)

[21] Zguitti, H.: A note on the common spectral properties for bounded linear operators. *Filomat*, **33**, 4575–4584 (2019)

[22] Zhuang, G., Chen, J., Cui, J.: Jacobson's lemma for the generalized Drazin inverse. *Linear Algebra Appl.*, **436**, 742–746 (2012)

[23] Živković-Zlatanović, S. Č., Cvetković, M. D.: Generalized Kato–Riesz decomposition and generalized Drazin–Riesz invertible operators. *Linear Multilinear Algebra*, **65**, 1171–1193 (2017)