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Multiplicity of Periodic Bouncing Solutions for Sublinear Damped Variation Systems via Nonsmooth Variational Methods

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Abstract Two results about the multiplicity of nontrivial periodic bouncing solutions for sublinear damped vibration systems $-\ddot{x} = g(t)\dot{x} + f(t,x)$ are obtained via the Generalized Nonsmooth Saddle Point Theorem and a technique established by Wu Xian and Wang Shaomin. Both of them imply the condition "$f \geq 0$" required in some previous papers can be weakened, furthermore, one of them also implies the condition about $\frac{\partial F(t,x)}{\partial t}$ required in some previous papers, such as "$|\frac{\partial F(t,x)}{\partial t}| \leq \sigma_0 F(t,x)$" and "$|\frac{\partial F(t,x)}{\partial t}| \leq C(1+F(t,x))$", is unnecessary, where $F(t,x) := \int_0^x f(t,s)\,ds$, and σ_0, C are positive constants.

Keywords Damped vibration systems, generalized Nonsmooth Saddle Point Theorem, sublinear conditions, periodic bouncing solutions, multiplicity

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1 Introduction

This paper is devoted to the solutions of

$$-\ddot{x} = g(t)\dot{x} + f(t,x), \quad \text{if } t \in \mathbb{R} \setminus W, \tag{1.1}$$

satisfying

$$\begin{cases} \dot{x}(t^-) = -\dot{x}(t^+), & \text{if } t \in W, \\ x(t) \geq a_0, & \forall t \in \mathbb{R}, \\ x(t) = x(t+T), & \forall t \in \mathbb{R}, \end{cases} \tag{1.2}$$

where a_0 is an arbitrary given constant, $T>0$, $W := \{t \in \mathbb{R} \mid x(t) = a_0\}$, $f \in C(\mathbb{R} \times [a_0, +\infty), \mathbb{R})$ is T-periodic in t for every $x \geq a_0$, $g \in C(\mathbb{R}, \mathbb{R})$ is also T-periodic with $G(T) = 0$, where $G(t) := \int_0^t g(s)\,ds$, $\dot{x}({t_0}^-)$ and $\dot{x}({t_0}^+)$ are left-derivative and right-derivative of x at t_0 respectively.

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Definition 1.1 ([16, Definition 1.1]) *Continuous map $x : \mathbb{R} \to \mathbb{R}$ is a nontrivial periodic bouncing solution of system* (1.1) *with collision axis at $x = a_0$, if it satisfies* (1.1), (1.2) *and*

(i) *the set W is discrete and nonempty,*

(ii) *there exists at least one $t_0 \in W$ such that $\dot{x}(t_0^-) \neq 0$.*

As $g \equiv 0$, systems having solutions satisfying Definition 1.1 are called the *impact Hamiltonian systems.* As $g \not\equiv 0$, the system (1.1) is called the *damped vibration systems* widely applied in physics and engineering. In mechanics, (1.1) and (1.2) mean that the particle only moves in $x \geq a_0$ and bounces in a perfectly elastic way when it hits the obstacle at equilibrium axis $x = a_0$.

Impact systems have been considered by numerous authors using topological methods, such as papers [1, 2, 10, 14–16]. It's well known that variational methods are powerful tools for finding solutions of differential equations. But the main difficulties of using variational methods in studying the solutions for system (1.1) are that action functional is non-differential, and the set W of a periodic bouncing solution is difficult to be specified. In 1981, Chang ([3]) firstly considered variational methods for non-differentiable functionals and their applications to partial differential equations, so it is possible to apply the variational methods to study the solutions of system (1.1). To the authors' knowledge, there are two methods to solve the second difficulty. One needs the following condition (F) or (F2) (see papers [4, 5, 7, 8, 12]), another is established in [18, Theorem 3.2].

As $g \equiv 0$ and $a_0 = 0$: in 2006, Jiang ([8]) overcame some technical difficulties and took the lead in applying variational methods to study the multiplicity of bouncing solutions for system (1.1) with the classical superquadratic condition and condition (F), that is,

(F)

$$\left|\frac{\partial F_1(t,x)}{\partial t}\right| \leq C(1 + F_1(t,x)), \quad x \in [0,+\infty),\ C > 0,$$

where $F_1(t,x) := \int_0^x f(t,s)\,ds$. In 2011, Ding ([4]) considered the existence of subharmonic bouncing solutions for system (1.1) with sublinear conditions and

$$\frac{F_1(t,x)}{|x|^{2\alpha}} \to +\infty, \quad |x| \to +\infty, \tag{1.3}$$

where $\alpha \in [0,1)$. In 2016, Ding et al. ([5]) generalized [4, Theorem 1.3] by weakening condition (1.3). In 2017, Nie and Guo ([12]) firstly proved a Generalized Nonsmooth Saddle Point Theorem; as its applications, the multiplicity of periodic bouncing solutions for system (1.1) was obtained with new sublinear conditions, which generalized [4, Theorem 1.3]. Generally, the condition "$f \geq 0$" is required (see [4, 5, 8, 12, 18] and etc.).

As $g \equiv 0$ and $a_0 \neq 0$: in 2019, Huang and Guo ([7]) considered the multiplicity of periodic bouncing solutions for system (1.1) with collision axis at $x = a_0$ based on paper [12]. The result is as follows.

Define set $\tilde{\Gamma} = \{h \in C([0,+\infty),[0,+\infty)) \,|\, h \text{ satisfies (h1)} - \text{(h4)}\}$, where

(h1) $h(s) \leq h(t) + C_1$ for a certain constant $C_1 > 0$ and arbitrary $s, t \in [0,+\infty)$ with $s \leq t$,

(h2) $h(s+t) \leq C_2\big(h(s) + h(t)\big)$ for a certain constant $C_2 > 0$ and arbitrary $s, t \in [0,+\infty)$,

(h3) $th(t) - 2H(t) \to -\infty$ as $t \to +\infty$, (h4) $\frac{H(t)}{t^2} \to 0$ as $t \to +\infty$,

and $H(t) := \int_0^t h(s)\,ds$. For example, $h(t) = t^\alpha\ (\alpha \in [0,1))$ and $h(t) = \ln(1+t)$ are in $\tilde{\Gamma}$, so $\tilde{\Gamma} \neq \emptyset$.

Theorem 1.2 ([7, Theorem 1.1]) *Suppose function f satisfies the following conditions*:

(f) *there exist two T-periodic functions $\gamma,\ l \in L^1\big([0,T],(0,+\infty)\big)$ and a function $h \in \tilde{\Gamma}$ such that*

$$|f(t,|x|+a_0)| \le \gamma(t)h(|x|) + l(t), \quad \forall x \in \mathbb{R} \text{ and } t \in [0,T],$$

(F1) *the function h coming from condition* (f) *satisfies*

$$\lim_{|x|\to+\infty} \frac{1}{H(|x|)} \int_0^T F(t,|x|+a_0)dt = +\infty \quad \text{or} \quad \liminf_{|x|\to+\infty} \frac{1}{H(|x|)} \int_0^T F(t,|x|+a_0)dt > 0,$$

where $F(t,x) := \int_{a_0}^x f(t,s)\,ds$,

(F2) *function $f(t,x)$ is differentiable in a.e. $t \in [0,T]$ for every $x \in \mathbb{R}$ and there is a constant $\sigma_0 > 0$ such that*

$$\left|\frac{\partial F(t,x)}{\partial t}\right| \le \sigma_0 F(t,x), \qquad \text{a.e.} \qquad t \in [0,T] \text{ and } x \in [a_0,+\infty),$$

(B) *$f(t,x) \ge 0$ holds for all $t \in [0,T]$ and $x \ge a_0$, furthermore,*

$$\lim_{x\to+\infty} f(t,x) = +\infty \quad \text{or} \quad \liminf_{x\to+\infty} f(t,x) > 0, \quad \forall t \in [0,T].$$

Then system (1.1) (*with $g \equiv 0$*) *possesses non-trivial kT-periodic bouncing solutions x_k for any sufficiently large integer k. Furthermore, $\|x_k\|_{L^\infty} \to +\infty$ holds as $k \to +\infty$.*

As $g \not\equiv 0$ and $a_0 = 0$: in 2010, Wu and Wang ([18]) obtained the existence and multiplicity of periodic bouncing solutions for superlinear and asymptotically linear system (1.1) respectively. In addition, they also proved that sublinear system (1.1) has at least two distinct solutions. Importantly, during the proof, a new method on how to prove the points in W are isolated was obtained (see [18, Theorem 3.2]), which leads to the unnecessity of condition (F) or (F2).

In 2015, Wang and Xiao ([17]) established a class of sublinear conditions different from those in Theorem 1.2 when they studied the existence of periodic solutions for second-order Hamiltonian systems.

Motivated by the idea in paper [18] and sublinear conditions in paper [17], the multiplicity of periodic bouncing solutions for system (1.1) will be proved (see Theorem 1.3) under sublinear conditions similar to those in paper [17] via Generalized Nonsmooth Saddle Point Theorem proved in paper [12] and a technique in paper [18]. And motivated by the idea in paper [18] and [7, Theorem 1.1], the multiplicity of periodic bouncing solutions for system (1.1) will be obtained under conditions (f′), (F1) and (B″) (see Theorem 1.4), which generalizes those in papers [4, 7] and [12], and also implies the condition "$f \ge 0$" can be weakened and the condition (F2) is unnecessary. But the condition (F) or (F2) was generally required (see [4, 5, 7, 8, 12]).

Now, we list the main results as follows.

Define set $\mathcal{H} = \{\theta \in C([0,+\infty),[0,+\infty)) \,|\, \theta \text{ satisfies } (\theta 1) - (\theta 2)\}$, where

(θ1) $\theta(s) > 0,\ \forall s \ge 0$,

(θ2) there exists a constant $M_0 > 0$ such that $\int_{M_0}^t \frac{1}{s\theta(s)}\,ds \to +\infty$ holds as $t \to +\infty$.

Theorem 1.3 *Suppose f satisfies the following conditions*

(A) *there are two functions $a_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in C([0,T], \mathbb{R}^+)$ with T-periodic such that*

$$|f(t, |x| + a_0)| \le a_1(|x|)b(t), \quad \forall x \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

(H1) *there exist a constant $M_0 > 0$ and a function $\theta \in \mathcal{H}$ with $0 < \frac{1}{\theta(s)} < 2$ such that*

$$f(t, x + a_0) \cdot x \le \left(2 - \frac{1}{\theta(x)}\right) F(t, x + a_0), \quad \forall x \ge M_0 \text{ and a.e. } t \in [0,T],$$

where $F(t,x) := \int_{a_0}^{x} f(t,s)\, ds$,

(H2) $\lim_{x \to +\infty} \frac{F(t, x+a_0)}{\theta(x)} = +\infty$ *holds uniformly for a.e. $t \in [0,T]$,*

(B′) $\lim_{x \to +\infty} f(t,x) = +\infty$ *or* $\liminf_{x \to +\infty} f(t,x) > 0$ *holds for all $t \in [0,T]$.*

Then system (1.1) *possesses a non-trivial kT-periodic bouncing solution x_k for any sufficiently large integer k with $\|x_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$.*

Theorem 1.4 *Suppose f satisfies condition $(F1)$ and the following conditions*

(f′) *there exist two T-periodic functions $\gamma \in L^1([0,T], [0, +\infty))$, $l \in L^1([0,T], (0, +\infty))$ and a function $h \in \tilde{\Gamma}$ such that*

$$|f(t, |x| + a_0)| \le \gamma(t)h(|x|) + l(t), \quad \forall x \in \mathbb{R} \text{ and } t \in [0,T],$$

(B″) $\liminf_{x \to +\infty} F(t,x) \ge 0$ *holds uniformly for a.e. $t \in [0,T]$, furthermore,* $\lim_{x \to +\infty} f(t, x) = +\infty$ *or* $\liminf_{x \to +\infty} f(t,x) > 0$ *holds for all $t \in [0,T]$.*

Then system (1.1) *possesses non-trivial kT-periodic bouncing solutions x_k for every sufficiently large integer k with $\|x_k\|_{L^\infty} \to +\infty$ as $k \to +\infty$.*

Remark 1.5 (1) In view of condition (A) and the definition of F, one has

$$|f(t, |x| + a_0)| \le a(|x|)b(t), \quad |F(t, |x| + a_0)| \le a(|x|)b(t), \quad \forall x \in \mathbb{R} \text{ and a.e. } t \in [0,T],$$

where $a(s) := \max\{a_1(s), a_1(s)s\}$.

(2) The conditions (B′) and (B″) imply that the condition "$f \ge 0$" required in papers [4, 5, 7, 8, 12, 18] can be weakened.

(3) In Section 4, there are examples to show that functions satisfying Theorem 1.3 and Theorem 1.4 do really exist.

2 Preliminaries

Firstly, we recall some notions for locally Lipschitzian functionals. The details can be found in book [6] and paper [3].

Let E be a real Banach space with the norm $\|\cdot\|$ and E^* be the dual space of E. A functional $\varphi : E \to \mathbb{R}$ is called locally Lipschitzian, if for each $u \in E$ there exist a neighborhood U of u and a constant $\mathcal{L} > 0$ such that

$$|\varphi(w) - \varphi(v)| \le \mathcal{L} \cdot \|w - v\|, \quad \forall w, v \in U.$$

The generalized directional derivative $\varphi^0(x_0; v)$ of functional φ at $x_0 \in E$ in the direction of $v \in E$ is defined as

$$\varphi^0(x_0; v) = \limsup_{h \to 0, t \to 0^+} \frac{\varphi(x_0 + h + tv) - \varphi(x_0 + h)}{t},$$

and the functional $v \to \varphi^0(x_0; v)$ is subadditive, convex, positively homogeneous and continuous. The generalized gradient $\partial\varphi(x_0)$ of φ at x_0 is the set defined as

$$\partial\varphi(x_0) = \{w \in E^* \big| \langle w, v\rangle \le \varphi^0(x_0; v),\ \forall v \in E\},$$

which is a nonempty, convex, and weak*-compact subset of E^*. A point $x_0 \in E$ is said to be a critical point of φ, if $\mathbf{0} \in \partial\varphi(x_0)$.

Definition 2.1 ([6, Definition 2.1.1]) *We say that φ satisfies nonsmooth* (PS) *condition, if every sequence $\{x_n\} \subset E$, such that $\{\varphi(x_n)\}$ is bounded and $\lambda(x_n) \to 0$ holds as $n \to \infty$, has a strongly convergent subsequence. We say that φ satisfies nonsmooth* (C) *condition, if every sequence $\{x_n\} \subset E$, such that $\{\varphi(x_n)\}$ is bounded and $(1 + \|x_n\|)\lambda(x_n) \to 0$ holds as $n \to \infty$, has a strongly convergent subsequence, where $\lambda(x) := \inf_{x^* \in \partial\varphi(x)} \|x^*\|_{E^*}$.*

Remark 2.2 [3, p. 105, (7)] implies that the function $\lambda(x) = \min_{x^* \in \partial\varphi(x)} \|x^*\|_{E^*}$ exists for every $x \in E$.

Lemma 2.3 (Generalized Nonsmooth Saddle Point Theorem, [12, Theorem 2.1]) *Let E be a real Banach space with $E = V \oplus X$, where $V \ne \{\mathbf{0}\}$ and $\dim V < +\infty$. Assume that functional φ satisfies nonsmooth* (PS) *condition, and there exists a constant $r > 0$ for each $x_0 \in X$ such that*

$$\max_{v \in V \cap \partial B_r} \varphi(v + x_0) < \inf_{x \in X} \varphi(x).$$

If c can be characterized as

$$c = \inf_{\chi \in \Gamma} \max_{v \in V \cap \bar{B}_r} \varphi(\chi(v + x_0)),$$

then c is a critical value of φ, where $\Gamma := \{\chi \in C(V \cap \bar{B}_r + x_0, E) \,|\, \chi(v + x_0) = v + x_0, \text{if } v \in V \cap \partial B_r\}$ and $B_r := \{x \in E \,|\, \|x\| < r\}$. Furthermore, one has $c \ge \inf_{x \in X} \varphi(x)$.

Remark 2.4 If nonsmooth (PS) condition is replaced by nonsmooth (C) condition, the conclusion in Lemma 2.3 is still valid via the proof of [12, Theorem 2.1].

By the same analysis in papers [18] and [7], $|x|$ is a kT-periodic bouncing solution of system (1.1), if $x : \mathbb{R} \to \mathbb{R}$ is a kT-periodic solution with isolated zeros of

$$-\ddot{x} = g(t)\dot{x} + f(t, |x| + a_0)\mathrm{sgn}(x), \tag{2.1}$$

where $\mathrm{sgn}(x)$ is the sign function.

$H^1_{kT} := \big\{x : [0, kT] \to \mathbb{R} \,|\, x(t) \text{ is absolutely continuous},\ x(0) = x(kT),\ \dot{x} \in L^2([0, kT], \mathbb{R})\big\}$, in which $k \in \mathbb{N}^*$, then H^1_{kT} is a Hilbert space with the norm

$$\|x\| = \left[\int_0^{kT} (|\dot{x}(t)|^2 + |x(t)|^2)\, dt\right]^{\frac{1}{2}}.$$

For $x \in H^1_{kT}$, let $\bar{x} = \frac{1}{kT}\int_0^{kT} x(t)\, dt$ and $\tilde{x}(t) = x(t) - \bar{x}$, then book [11] tells us the following Wirtinger's inequality

$$\int_0^{kT} |\tilde{x}(t)|^2\, dt \le \frac{k^2T^2}{4\pi^2} \int_0^{kT} |\dot{x}(t)|^2\, dt,$$

and Sobolev's inequality

$$\|\tilde{x}\|^2_{L^\infty} \le \frac{kT}{12} \int_0^{kT} |\dot{x}(t)|^2\, dt.$$

Let $\|x\|_1 = (|\bar{x}|^2 + \int_0^{kT} |\dot{x}(t)|^2\,dt)^{\frac{1}{2}}$, then book [11] implies that $\|\cdot\|_1$ is equivalent to $\|\cdot\|$. Moreover, H^1_{kT} can be decomposed into $\mathbb{R} \oplus \tilde{H}^1_{kT}$, where $\tilde{H}^1_{kT} := \{x \in H^1_{kT} \mid \bar{x} = 0\}$.

Lemma 2.5 (Sobolev Embedding Theorem, [6, Theorem 1.1.5]) *H^1_{kT} can be compactly embedded into $L^r([0,kT],\mathbb{R})$ for any $r \in [1,+\infty]$ and $C([0,kT],\mathbb{R})$, so there exists a constant $\tau_r > 0$ such that $\|x\|_{L^r} \le \tau_r \|x\|$ holds for all $x \in H^1_{kT}$ and all $r \in [1,+\infty]$.*

Let $J_k(x) = \int_0^{kT} \mathrm{e}^{G(t)} F(t,|x(t)|+a_0)\,dt$, $\forall x \in H^1_{kT}$. Define the corresponding functional of system (2.1) given by

$$\varphi_k(x) = \frac{1}{2}\int_0^{kT} \mathrm{e}^{G(t)} |\dot{x}(t)|^2\,dt - J_k(x), \quad \forall x \in H^1_{kT}.$$

Lemma 2.6 ([18, Theorem 3.2]) *Let f be a continuous function, $f(t+T,x) = f(t,x)$ hold for all $x \in \mathbb{R}$, and u_k be a critical point of the functional φ_k on H^1_{kT} for every $k \in \mathbb{N}^*$.*

(1) *If all zero points of u_k (i.e., the points in set W) are isolated, then u_k is a kT-periodic solution of (2.1) with periodic boundary condition for every $k \in \mathbb{N}^*$.*

(2) *If the following conditions hold:*

(a) $\liminf_{|x|\to+\infty} F(t,|x|) \ge 0$, *uniformly for a.e. $t \in [0,T]$,*

(b) *there exists a $t_0 \in [0,kT]$ such that $u_k(t_0) = \dot{u}_k(t_0) = 0$,*

then $u_k \equiv 0$ on $[0,kT]$. Particularly, if $u_k \not\equiv 0$ and (a) *holds, then the zeros of u_k in $[0,kT]$ are isolated.*

Lemma 2.7 *If f satisfies* (f′) (*or* (A)), *then functionals J_k and φ_k are locally Lipschitzian on H^1_{kT} and*

$$\partial J_k(x) \subseteq \mathrm{e}^{G(t)}[f^-(t,|x(t)|+a_0), f^+(t,|x(t)|+a_0)], \quad a.e.\ t \in [0,kT], \tag{2.2}$$

where

$$f^-(t,|s|+a_0) := \min\Big\{\lim_{v\to s^-} f(t,|s|+a_0)\mathrm{sgn}(v), \lim_{v\to s^+} f(t,|s|+a_0)\mathrm{sgn}(v)\Big\},$$

$$f^+(t,|s|+a_0) := \max\Big\{\lim_{v\to s^-} f(t,|s|+a_0)\mathrm{sgn}(v), \lim_{v\to s^+} f(t,|s|+a_0)\mathrm{sgn}(v)\Big\}.$$

Proof The main idea comes from papers [3] and [7].

Let $d_1 = \min_{t\in[0,T]} \mathrm{e}^{G(t)}$, $d_2 = \max_{t\in[0,T]} \mathrm{e}^{G(t)}$, then $0 < d_1 \le d_2 < +\infty$. By the definition of $J_k(x)$, (f′)(or (A)), (h1), the Hölder's inequality and Lemma 2.5, one can prove that $J_k(x)$ is locally Lipschitzian continuous, then $\varphi_k(x)$ is also locally Lipschitzian continuous.

Similarly, for fixed t, the function $F(t,x+a_0) = \int_{a_0}^{x+a_0} f(t,s)\,ds$ is also locally Lipschitzian continuous in variable x, then generalized directional derivative $F^0(t,x+a_0;v)$ does exist, namely, $F^0(t,x+a_0;v) = \limsup_{h\to0,\ \mu\to0^+} \frac{1}{\mu}\int_{x+h}^{x+h+\mu v} f(t,s+a_0)\,ds$, one has

$$F^0(t,x+a_0;v) \le \begin{cases} v\lim_{\sigma\to0^+}\min_{s\in[x-\sigma,x+\sigma]} f(t,s+a_0) = f^-(t,x+a_0)v, & v \le 0, \\ v\lim_{\sigma\to0^+}\max_{s\in[x-\sigma,x+\sigma]} f(t,s+a_0) = f^+(t,x+a_0)v, & v \ge 0. \end{cases} \tag{2.3}$$

In view of the Fatou's Lemma and (2.3), there exists $\{h_i\} \subset H^1_{kT}$ with $h_i \to 0$ $(i \to +\infty)$ in H^1_{kT} such that

$$J_k^0(x;v) \le \int_0^{kT} \mathrm{e}^{G(t)} F^0(t,|x(t)|+a_0;v(t))\,dt$$

$$\leq \int_{v(t)>0} \mathrm{e}^{G(t)} v(t) f^{+}(t, |x(t)| + a_0)\, dt$$
$$+ \int_{v(t)<0} \mathrm{e}^{G(t)} v(t) f^{-}(t, |x(t)| + a_0)\, dt, \quad \forall v \in H^1_{kT}. \tag{2.4}$$

If $\omega_0 \in \partial J_k(x)$, we claim that

$$\mathrm{e}^{G(t)} f^{-}(t, |x(t)| + a_0) \leq \omega_0 \leq \mathrm{e}^{G(t)} f^{+}(t, |x(t)| + a_0), \quad \text{a.e. } t \in \mathbb{R}.$$

Otherwise, there would be a set $E_0 \subset \mathbb{R}$ with $\mathrm{meas}(E_0) > 0$ such that

$$\mathrm{e}^{G(t)} f^{-}(t, |x(t)| + a_0) > \omega_0(t), \quad \forall t \in E_0. \tag{2.5}$$

Let $v_0(t) = -\chi_{E_0}(t)$, the characteristic function of E_0. From the definition of $\partial J_k(x)$, one has

$$J_k^0(x; v_0) \geq \langle \omega_0(t), v_0(t) \rangle = -\int_{E_0} \omega_0(t)\, dt. \tag{2.6}$$

From the definition of $v_0(t)$, (2.4) and (2.6), one has

$$-\int_{E_0} \mathrm{e}^{G(t)} f^{-}(t, |x(t)| + a_0)\, dt \geq J_k^0(x; v_0) \geq -\int_{E_0} \omega_0(t)\, dt,$$

which contradicts (2.5). Similarly, $\omega_0 \leq \mathrm{e}^{G(t)} f^{+}(t, |x(t)| + a_0)$ holds for a.e. $t \in \mathbb{R}$. □

Meaning of (2.2) is understood as: for $\omega \in \partial J_k(x)$, there is a function ϖ with $\varpi(t) \in [f^{-}(t, |x(t)| + a_0), f^{+}(t, |x(t)| + a_0)]$ for a.e. $t \in [0, kT]$ such that

$$\langle \omega, v \rangle = \int_0^{kT} \mathrm{e}^{G(t)} \varpi(t) v(t)\, dt, \quad \forall v \in H^1_{kT}.$$

Lemma 2.8 ([12, Lemma 2.3]) *Suppose that h satisfies* (h1)–(h4)*, then for any $\varepsilon > 0$, there exists a positive constant C_ε depending on ε such that*

(1) $0 \leq h(t) < \varepsilon t + C_\varepsilon,\ \forall t \in [0, +\infty)$,
(2) $\frac{h^2(t)}{H(t)} \to 0,\ t \to +\infty$,
(3) $H(t) \to +\infty,\ t \to +\infty$,
(4) $\frac{h(t)}{H(t)} \to 0,\ t \to +\infty$.

Lemma 2.9 ([17, Proposition 2.1]) *Suppose that $F(t, x)$ satisfies conditions* (A) *and* (H1)*, then*

$$F(t, |x| + a_0) \leq \frac{h_1(t)}{M_0^2} |x|^2 W(|x|) + h_1(t), \quad \forall x \in \mathbb{R} \text{ and a.e. } t \in [0, T],$$

where $h_1(t) := \max_{|x| \leq M_0} a(|x|) b(t)$, $W(t) := \mathrm{e}^{-\int_{M_0}^t \frac{1}{s\theta(s)}\, ds}$.

Remark 2.10 ([17, Remark 2.1]) (1) Making use of the property ($\theta 2$) of $\mathcal{H}$, we know that $W(t) \to 0$ holds as $t \to +\infty$, which means that there exists a constant $\tilde{M} > 0$ such that $W(t) < \tilde{M}$ holds for all $t \in \mathbb{R}$.

(2) Assume (H1) holds, then function $t \to t^2 W(t)$ is increasing as $(t^2 W(t))' = tW(t)(2 - \frac{1}{\theta(t)}) > 0$.

3 Bouncing Solutions for Damped Vibration Systems via Nonsmooth Variational Methods

Proposition 3.1 *Assume f satisfies* (A), (H1) *and* (H2)*. Then functional φ_k satisfies the nonsmooth* (C) *condition for every given $k \in \mathbb{N}^*$.*

Proof Let $\{x_n\} \subset H^1_{kT}$ be a nonsmooth (C) sequence, that is, $\{\varphi_k(x_n)\}$ is bounded and $(1+\|x_n\|)\lambda(x_n) \to 0$ holds as $n \to \infty$. Then there exists a constant $K > 0$ such that

$$|\varphi_k(x_n)| \le K, \quad (1+\|x_n\|)\lambda(x_n) \le K, \quad \forall n \in \mathbf{N}. \tag{3.1}$$

Step 1 We claim that $\{\|x_n\|\}$ is bounded. The main idea comes from paper [17].

Otherwise, there exists a subsequence of $\{x_n\}$ still written as $\{x_n\}$, one can assume that $\|x_n\| \to +\infty$ holds as $n \to +\infty$. Let $z_n = \frac{x_n}{\|x_n\|}$. Then $\|z_n\| = 1$. Hence, there exists a subsequence of $\{z_n\}$, also denoted by $\{z_n\}$ for convenience, such that

$$z_n \rightharpoonup z_0 \text{ in } H^1_{kT} \quad \text{and} \quad z_n \to z_0 \text{ in } C([0,kT],\mathbb{R}). \tag{3.2}$$

According to Remark 2.2, Lemma 2.7 and the definition of $\lambda(x_n)$, for each $n \in \mathbb{N}$, there are a function $\varpi_n(t) \in [f^-(t,|x_n(t)|+a_0), f^+(t,|x_n(t)|+a_0)]$ for a.e. $t \in [0,kT]$ and $x_n^* \in \partial\varphi_k(x_n)$ with $\lambda(x_n) = \|x_n^*\| \to 0$ as $n \to \infty$ such that

$$\langle x_n^*, v\rangle = \int_0^{kT} \mathrm{e}^{G(t)}\dot{x}_n(t)\dot{v}(t)\,dt - \int_0^{kT} \mathrm{e}^{G(t)}\varpi_n(t)v(t)\,dt, \quad \forall v \in H^1_{kT}. \tag{3.3}$$

It is easy to check that (A) and (H1) yield

$$f(t,|x|+a_0)|x| \le \left(2 - \frac{1}{\theta(|x|)}\right)F(t,|x|+a_0) + K_1, \quad \forall x \in \mathbb{R} \text{ and a.e. } t \in [0,T], \tag{3.4}$$

where $K_1 := (2+M_0)\max_{0\le s\le M_0} a(s)\max_{t\in[0,T]} b(t) > 0$.

(1) If $z_0(t) \not\equiv 0$. Remark 2.2, (3.1), (3.3), Lemma 2.7 and (3.4) imply there exist a function $\varpi_n(t) \in [f^-(t,|x_n(t)|+a_0), f^+(t,|x_n(t)|+a_0)]$ for a.e. $t \in [0,kT]$ and an $x_n^* \in \partial\varphi_k(x_n)$ with $\lambda(x_n) = \|x_n^*\|$ such that

$$\begin{aligned}
3K &\ge (1+\|x_n\|)\lambda(x_n) - 2\varphi_k(x_n) \ge \langle x_n^*, x_n\rangle - 2\varphi_k(x_n)\\
&= \int_0^{kT} \mathrm{e}^{G(t)}(2F(t,|x_n(t)|+a_0) - \varpi_n(t)x_n(t))\,dt\\
&= \int_0^{kT} \mathrm{e}^{G(t)}(2F(t,|x_n(t)|+a_0) - f(t,|x_n(t)|+a_0)|x_n(t)|)\,dt\\
&\ge \int_0^{kT} \mathrm{e}^{G(t)}\left[2F(t,|x_n(t)|+a_0) - \left(2 - \frac{1}{\theta(|x_n(t)|)}\right)F(t,|x_n(t)|+a_0) - K_1\right]dt\\
&\ge \int_0^{kT} \mathrm{e}^{G(t)}\frac{F(t,|x_n(t)|+a_0)}{\theta(|x_n(t)|)}\,dt - d_2kTK_1, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Hence, there is a constant $K_{2,k} > 0$ such that

$$\int_0^{kT} \mathrm{e}^{G(t)}\frac{F(t,|x_n(t)|+a_0)}{\theta(|x_n(t)|)}\,dt \le K_{2,k}, \quad \forall n \in \mathbb{N}, \tag{3.5}$$

where $K_{2,k} := 3K + d_2kTK_1 > 0$.

In view of (3.1), Lemma 2.9, (2) of Remark 2.10 and Lemma 2.5, one has

$$\begin{aligned}
K \ge \varphi_k(x_n) &= \frac{1}{2}\int_0^{kT} \mathrm{e}^{G(t)}|\dot{x}_n(t)|^2\,dt - \int_0^{kT} \mathrm{e}^{G(t)}F(t,|x_n(t)|+a_0)\,dt\\
&\ge \frac{1}{2}d_1\|\dot{x}_n\|^2_{L^2} - d_2\int_0^{kT}\left[\frac{h_1(t)}{M_0^2}|x_n(t)|^2W(|x_n(t)|) + h_1(t)\right]dt
\end{aligned}$$

$$\geq \frac{1}{2}d_1\|\dot{x}_n\|_{L^2}^2 - d_2\|x_n\|_{L^\infty}^2 W(\|x_n\|_{L^\infty})\int_0^{kT}\frac{h_1(t)}{M_0^2}\,dt - K_{4,k}$$
$$\geq \frac{1}{2}d_1\|\dot{x}_n\|_{L^2}^2 - K_{3,k}\tau_\infty^2\|x_n\|^2 W(\tau_\infty\|x_n\|) - K_{4,k}, \quad \forall n\in\mathbb{N}, \tag{3.6}$$

where $K_{3,k} := d_2\int_0^{kT}\frac{h_1(t)}{M_0^2}\,dt > 0,\ K_{4,k} := d_2\int_0^{kT} h_1(t)\,dt > 0$.

Let $z_n(t) = \tilde{z}_n(t)+\bar{z}_n$, $z_0(t) = \tilde{z}_0(t)+\bar{z}_0$, $\forall t\in[0,kT]$, where $\tilde{z}_n(t),\ \tilde{z}_0(t)\in\tilde{H}_{kT}^1$, $\bar{z}_n,\ \bar{z}_0\in\mathbb{R}$. Divided by $\|x_n\|^2$ on both sides of (3.6), (1) of Remark 2.10 yields $\|\dot{z}_n\|_{L^2}\to 0$ as $n\to+\infty$, which together with the Sobolev's inequality yields $\|\tilde{z}_n\|_{L^\infty}^2\to 0$ as $n\to+\infty$, that is, $\tilde{z}_n(t)\to 0$ holds uniformly for all $t\in[0,kT]$ as $n\to+\infty$, which together with the property of the sequence of functions implies that

$$\bar{z}_0 = \frac{1}{kT}\int_0^{kT}\lim_{n\to+\infty} z_n(t)\,dt = \lim_{n\to+\infty}\frac{1}{kT}\int_0^{kT} z_n(t)\,dt = \lim_{n\to+\infty}\bar{z}_n = \lim_{n\to+\infty} z_n.$$

Then one has $\bar{z}_0 = z_0$ holds for all $t\in[0,kT]$ via (3.2). Note that $kT|\bar{z}_0|^2 = \|\bar{z}_0\|^2 > 0$, then $|x_n(t)| = |z_n(t)|\cdot\|x_n\|\to+\infty$ holds uniformly for all $t\in[0,kT]$ as $n\to+\infty$. The Fatou's Lemma and (H2) yield

$$\liminf_{n\to+\infty}\int_0^{kT}\mathrm{e}^{G(t)}\frac{F(t,|x_n(t)|+a_0)}{\theta(|x_n(t)|)}\,dt \geq d_1\int_0^{kT}\liminf_{n\to+\infty}\frac{F(t,|x_n(t)|+a_0)}{\theta(|x_n(t)|)}\,dt = +\infty,$$

which contradicts (3.5).

(2) If $z_0(t)\equiv 0$. Remark 2.2, (3.3), (3.4), Lemma 2.9, (1) of Remark 2.10 and the Fatou's Lemma imply that there are a function $\varpi_n(t)\in[f^-(t,|x_n(t)|+a_0), f^+(t,|x_n(t)|+a_0)]$ for a.e. $t\in[0,kT]$ and an $x_n^*\in\partial\varphi_k(x_n)$ with $\lambda(x_n) = \|x_n^*\|\to 0$ as $n\to\infty$ such that

$$\begin{aligned}
d_1 &\leq \limsup_{n\to+\infty}\frac{1}{\|x_n\|^2}\left(\int_0^{kT}\mathrm{e}^{G(t)}|\dot{x}_n(t)|^2\,dt + \int_0^{kT}\mathrm{e}^{G(t)}|x_n(t)|^2\,dt\right)\\
&= \limsup_{n\to+\infty}\frac{1}{\|x_n\|^2}\left(\langle x_n^*, x_n\rangle + \int_0^{kT}\mathrm{e}^{G(t)}\varpi_n(t)x_n(t)dt + \int_0^{kT}\mathrm{e}^{G(t)}|x_n(t)|^2dt\right)\\
&\leq \limsup_{n\to+\infty}\frac{1}{\|x_n\|^2}\left(\int_0^{kT}\mathrm{e}^{G(t)}f(t,|x_n(t)|+a_0)|x_n(t)|dt + d_2\int_0^{kT}|x_n(t)|^2dt\right)\\
&\leq \limsup_{n\to+\infty}\frac{d_2}{\|x_n\|^2}\int_0^{kT}\left[\left(2-\frac{1}{\theta(|x_n(t)|)}\right)F(t,|x_n(t)|+a_0) + K_1 + |x_n(t)|^2\right]dt\\
&\leq \limsup_{n\to+\infty}\frac{d_2}{\|x_n\|^2}\int_0^{kT}\left[2\left(\frac{h_1(t)}{M_0^2}|x_n(t)|^2 W(|x_n(t)|) + h_1(t)\right) + K_1 + |x_n(t)|^2\right]dt\\
&\leq \limsup_{n\to+\infty} d_2\int_0^{kT}\left(\frac{2h_1(t)}{M_0^2}|z_n(t)|^2\tilde{M} + \frac{2h_1(t)}{\|x_n\|^2} + \frac{K_1}{\|x_n\|^2}\right)dt = 0,
\end{aligned}$$

which is a contradiction. Thus, $\{x_n\}$ is bounded in H_{kT}^1.

Step 2 We claim that $\{x_n\}$ has a convergent subsequence in H_{kT}^1.

The idea comes from paper [13]. Because $\{\|x_n\|\}$ is bounded, there exists a subsequence of $\{x_n\}$, also denoted by $\{x_n\}$ for convenience, such that $x_n \rightharpoonup x$ holds as $n\to\infty$ in H_{kT}^1, then $x_n\to x$ holds as $n\to\infty$ in $C([0,kT],\mathbb{R})$ and $L^2([0,kT],\mathbb{R})$ via Lemma 2.5. Since H_{kT}^1 is reflexive and $\partial\varphi_k(x_n)$ is a non-empty convex weak*-compact subset of H_{kT}^1, and set-valued mapping $x\to\partial\varphi_k(x)$ is upper semicontinuous (see [3, Propositions]), for $x_n^*\in\partial\varphi_k(x_n)$, there is an $x^* \in \partial\varphi_k(x)$ such that $\langle x_n^* - x^*, x_n - x\rangle \to 0$ holds as $n \to \infty$. Then

$$\int_0^{kT} \mathrm{e}^{G(t)}|\dot{x}_n(t) - \dot{x}(t)|^2\, dt = \int_0^{kT} \mathrm{e}^{G(t)}(\varpi_n(t) - \varpi(t), x_n(t) - x(t))\, dt \\ + \langle x_n^* - x^*, x_n - x\rangle \to 0,\ n \to \infty, \tag{3.7}$$

where $\varpi_n \in [f^-(t, |x_n(t)|+a_0), f^+(t, |x_n(t)|+a_0)]$ and $\varpi \in [f^-(t, |x(t)|+a_0), f^+(t, |x(t)|+a_0)]$. (3.7) implies that $\int_0^{kT} |\dot{x}_n(t) - \dot{x}(t)|^2\, dt \to 0$ holds as $n \to \infty$, together with $\|x_n - x\|_{L^2} \to 0$, one has $\|x_n - x\| \to 0$, so φ_k satisfies nonsmooth (C) condition. □

Lemma 3.2 *Assume f satisfies* (A) *and* (H1)*. Then $\varphi_k(x) \to +\infty$ holds as $\|x\| \to +\infty$ with $x \in \tilde{H}^1_{kT}$ for every given $k \in \mathbb{N}^*$, so there is a certain constant $M_{1,k}$ such that $\varphi_k(x) \geq M_{1,k}$ holds for all $x \in \tilde{H}^1_{kT}$.*

Proof In view of Lemma 2.9, (2) of Remark 2.10 and the Sobolev's inequality, for every $\tilde{x} \in \tilde{H}^1_{kT}$, one has

$$\begin{aligned}\varphi_k(\tilde{x}) &\geq \frac{1}{2}d_1 \int_0^{kT} |\dot{x}(t)|^2\, dt - d_2 \int_0^{kT} \Big(\frac{h_1(t)}{M_0^2}|\tilde{x}(t)|^2 W(|\tilde{x}(t)|) + h_1(t)\Big) dt \\ &\geq \frac{1}{2}d_1 \int_0^{kT} |\dot{x}(t)|^2\, dt - K_{5,k}\|\tilde{x}\|^2_{L^\infty} W(\|\tilde{x}\|_{L^\infty}) - K_{6,k} \\ &\geq \Big[\frac{1}{2}d_1 - K_{7,k} W\Big(\Big(\frac{kT}{12}\int_0^{kT} |\dot{x}(t)|^2 dt\Big)^{\frac{1}{2}}\Big)\Big] \int_0^{kT} |\dot{x}(t)|^2 dt - K_{6,k},\end{aligned} \tag{3.8}$$

where $K_{5,k} := d_2 \int_0^{kT} \frac{h_1(t)}{M_0^2}\, dt > 0$, $K_{6,k} := d_2 \int_0^{kT} h_1(t)\, dt > 0$, $K_{7,k} := \frac{K_{5,k}kT}{12} > 0$. Due to $\|\tilde{x}\| \to +\infty \Leftrightarrow (\int_0^{kT} |\dot{x}(t)|^2\, dt)^{\frac{1}{2}} \to +\infty$ via $\tilde{x} \in \tilde{H}^1_{kT}$, (3.8) and (1) of Remark 2.10 imply that $\varphi_k(\tilde{x}) \to +\infty$ holds as $\|\tilde{x}\| \to +\infty$, which means that for all $M_2 > 0$, there exists a constant $N_{1,k} > 0$ such that $\varphi_k(\tilde{x}) > M_2$ holds for $\|\tilde{x}\| \geq N_{1,k}$. If $\|\tilde{x}\| \leq N_{1,k}$, then $\|\tilde{x}\|_{L^\infty} \leq \tau_\infty\|\tilde{x}\| \leq \tau_\infty N_{1,k}$. One has $\varphi_k(\tilde{x}) \geq -d_2 kTK_8 := M_{3,k}$ via condition (A), where $K_8 := \max_{0\leq s\leq \tau_\infty N_{1,k}} a(s) \max_{t\in[0,T]} b(t) > 0$. Let $M_{1,k} = \min\{M_2, M_{3,k}\}$. One has $\varphi_k(x) \geq M_{1,k}, \forall x \in \tilde{H}^1_{kT}$. □

Lemma 3.3 *Assume f satisfies* (H1) *and* (H2)*. Then $\varphi_k(x + e_k) \to -\infty$ holds as $|x| \to +\infty$ in $\mathbb{R} \subseteq H^1_{kT}$ for every given $k \in \mathbb{N}^*$, so one has $\sup_{\bar{x}\in\mathbb{R}} \varphi_k(\bar{x} + e_k) < +\infty$, where $e_k(t) := k\cos(\frac{2\pi t}{kT}) \in \tilde{H}^1_{kT}$.*

Proof In view of condition (H2), for every constant $\beta \geq \frac{4d_2\pi^2}{d_1T^2} > 0$, there exists a constant $m_1 > 0$ independent of t such that

$$\frac{F(t, |x| + a_0)}{\theta(|x|)} \geq \beta, \quad \forall |x| \geq m_1 \text{ and a.e. } t \in [0, T], \tag{3.9}$$

which yields

$$F(t, |x| + a_0) \geq 0, \quad \forall |x| \geq m_1 \text{ and a.e. } t \in [0, T]. \tag{3.10}$$

Condition (H1), (3.10) and (3.9) yield

$$\begin{aligned}\varphi_k(x + e_k) &\leq \frac{d_2 k\pi^2}{T} - \frac{d_1}{2}\int_0^{kT} \frac{F(t, |x + e_k(t)| + a_0)}{\theta(|x + e_k(t)|)}\, dt \\ &\leq \frac{d_2 k\pi^2}{T} - \frac{d_1 kT}{2}\beta\end{aligned}$$

$$\leq -\frac{d_1 kT}{4}\beta, \quad \forall x \in \mathbb{R} \text{ with } |x| \geq m_1 + k,$$

which implies that $\varphi_k(x+e_k) \to -\infty$ holds as $|x| \to +\infty$ in $\mathbb{R} \subseteq H^1_{kT}$ by the arbitrariness of β. □

Proposition 3.4 *Assume f satisfies* (A), (H1) *and* (H2). *Then for every fixed $k \in \mathbb{N}^*$, there exists a constant $\tilde{r}_{1,k} > 0$ large enough such that functional φ_k possesses at least a critical value c_k characterized as*

$$c_k = \inf_{\chi \in \Gamma_1} \max_{x \in [-\tilde{r}_{1,k}, \tilde{r}_{1,k}]} \varphi_k(\chi(x+e_k)),$$

where $\Gamma_1 := \{\chi \in C([-\tilde{r}_{1,k}, \tilde{r}_{1,k}] + e_k, H^1_{kT}) | \chi(e_k \pm \tilde{r}_{1,k}) = e_k \pm \tilde{r}_{1,k}\}$. In addition, one has

$$-\infty < \inf_{\tilde{H}^1_{kT}} \varphi_k \leq c_k \leq \sup_{x \in \mathbb{R}} \varphi_k(x+e_k) < +\infty. \tag{3.11}$$

Proof Let $V = \mathbb{R}$ and $X = \tilde{H}^1_{kT}$, then for every $k \in \mathbb{N}^*$, Lemma 3.2 and Lemma 3.3 imply that there exists a constant $\tilde{r}_{1,k} > 0$ large enough such that

$$\max_{x \in V \cap \partial B_{\tilde{r}_{1,k}}} \varphi_k(x+e_k) < \inf_{x \in X} \varphi_k(x). \tag{3.12}$$

Together with Proposition 3.1 and (3.12), c_k is the critical value of φ_k and $c_k \geq \inf_{\tilde{H}^1_{kT}} \varphi_k$ via Lemma 2.3 and Remark 2.4. The definition of c_k and Lemma 3.3 imply that $c_k \leq \sup_{x\in\mathbb{R}} \varphi_k(x+e_k) < +\infty$. Furthermore, Lemma 3.2 implies that $\inf_{\tilde{H}^1_{kT}} \varphi_k > -\infty$ holds. Thus, (3.11) holds. □

Proposition 3.5 *Assume f satisfies* (A), (H1) *and* (H2). *Let x_k be a critical point of functional φ_k for every $k \in \mathbb{N}^*$, then $\|x_k\|_{L^\infty} \to +\infty$ holds as $k \to +\infty$.*

Proof Otherwise, there is a constant $K_9 > 0$ such that $\|x_k\|_{L^\infty} \leq K_9$ holds for all $k \in \mathbb{N}^*$. In view of condition (A), one has

$$\begin{aligned} \frac{\varphi_k(x_k)}{k} &\geq -\frac{1}{k}\int_0^{kT} \mathrm{e}^{G(t)} \max_{0\leq s\leq K_9} a(s) \max_{t\in[0,T]} b(t)\,dt \\ &\geq -d_2 T \max_{0\leq s\leq K_9} a(s) \max_{t\in[0,T]} b(t) := \tilde{L}, \end{aligned} \tag{3.13}$$

where $\tilde{L}$ is a constant independent of k. Together with (3.11) and (3.13), one has

$$\limsup_{k\to+\infty} \sup_{x\in\mathbb{R}} \frac{\varphi_k(x+e_k)}{k} \geq \liminf_{k\to+\infty} \frac{c_k}{k} \geq \tilde{L} > -\infty. \tag{3.14}$$

On the other hand, for the constant m_1 in (3.9), for fixed $x \in \mathbb{R} \subset H^1_{kT}$, set $\tilde{B}_k = \{t \in [0,kT] \big| |x+e_k(t)| \leq m_1\}$, motivated by the result of [9, page 387], we can claim that

$$\mathrm{meas}(\tilde{B}_k) \leq \frac{kT}{4}, \quad k \text{ large enough}. \tag{3.15}$$

In fact, if $\mathrm{meas}(\tilde{B}_k) > \frac{kT}{4}$, there exists a $t_1 \in \tilde{B}_k$ such that (see Figure 1)

$$\frac{kT}{16} \leq t_1 \leq \frac{7}{16}kT \quad \text{or} \quad \frac{9}{16}kT \leq t_1 \leq \frac{15}{16}kT. \tag{3.16}$$

Moreover, there exists a $t_2 \in \tilde{B}_k$ such that (see Figure 1)

$$|t_2 - t_1| \geq \frac{kT}{16}, \tag{3.17}$$

and

$$|t_2 - (kT - t_1)| \geq \frac{kT}{16}. \tag{3.18}$$

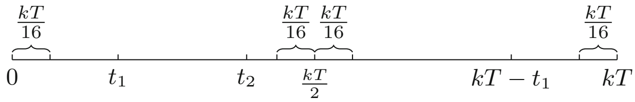

Figure 1 The locations of t_1 and t_2

It follows from (3.16) and (3.18) that

$$\frac{\pi}{16} \leq \frac{t_1 + t_2}{kT}\pi \leq \frac{15}{16}\pi \quad \text{or} \quad \frac{17}{16}\pi \leq \frac{t_1 + t_2}{kT}\pi \leq \frac{31}{16}\pi. \tag{3.19}$$

(3.16) and (3.17) yield

$$-\frac{15}{16}\pi \leq \frac{t_1 - t_2}{kT}\pi \leq -\frac{\pi}{16} \quad \text{or} \quad \frac{\pi}{16} \leq \frac{t_1 - t_2}{kT}\pi \leq \frac{15}{16}\pi. \tag{3.20}$$

(3.19) and (3.20) yield (see Figure 2)

$$\left|\sin\left(\frac{t_1 + t_2}{kT}\pi\right)\right| \geq \sin\frac{\pi}{16} \quad \text{and} \quad \left|\sin\left(\frac{t_1 - t_2}{kT}\pi\right)\right| \geq \sin\frac{\pi}{16},$$

then one has

$$\left|\cos\frac{2\pi t_1}{kT} - \cos\frac{2\pi t_2}{kT}\right| = 2\left|\sin\left(\frac{t_1 + t_2}{kT}\pi\right)\right|\left|\sin\left(\frac{t_1 - t_2}{kT}\pi\right)\right| \geq 2\sin^2\frac{\pi}{16}. \tag{3.21}$$

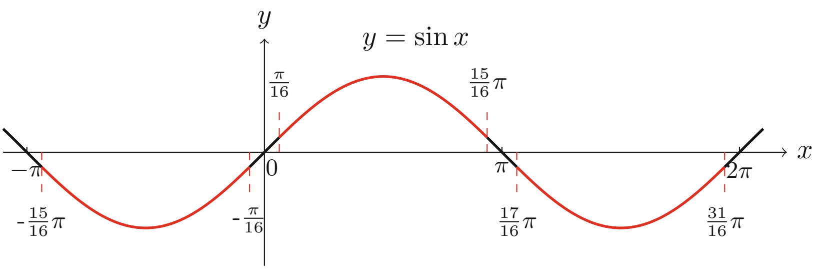

Figure 2 The ranges of $\sin(\frac{t_1+t_2}{kT}\pi)$ and $\sin(\frac{t_1-t_2}{kT}\pi)$

From the definition of $\tilde{B}_k$, one has

$$\left|\cos\frac{2\pi t_1}{kT} - \cos\frac{2\pi t_2}{kT}\right| = \frac{1}{k}|(x + e_k(t_1)) - (x + e_k(t_2))| \leq \frac{2m_1}{k},$$

which contradicts (3.21) for k large enough. Hence (3.15) holds. Then it follows (3.15). One has

$$\text{meas}([0, kT]\backslash\tilde{B}_k) \geq \frac{3}{4}kT > \frac{1}{2}kT, \quad k \text{ large enough}. \tag{3.22}$$

In view of (H1), (3.10), (A), (3.9) and (3.22), one has

$$\begin{aligned}\frac{\varphi_k(x + e_k)}{k} &\leq d_2\frac{\pi^2}{T} - \frac{1}{k}\int_{\tilde{B}_k} \mathrm{e}^{G(t)}F(t, |x + e_k(t)| + a_0)\,dt \\ &\quad - \frac{d_1}{2}\frac{1}{k}\int_{[0,kT]\backslash\tilde{B}_k} \frac{F(t, |x + e_k(t)| + a_0)}{\theta(|x + e_k(t)|)}\,dt\end{aligned}$$

$$\leq d_2\frac{\pi^2}{T}+d_2T\max_{0\leq s\leq m_1}a(s)\max_{t\in[0,T]}b(t)-\frac{d_1T}{4}\beta$$
$$\leq -\frac{d_1T}{8}\beta,\quad \beta\geq\frac{8K_{10}}{d_1T}\text{ and } k \text{ large enough},$$

where $K_{10}:=\frac{d_2\pi^2}{T}+d_2T\max_{0\leq s\leq m_1}a(s)\max_{t\in[0,T]}b(t)$. The arbitrariness of β yields

$$\limsup_{k\to+\infty}\sup_{x\in\mathbb{R}}\frac{\varphi_k(x+e_k)}{k}=-\infty,$$

which contradicts (3.14). Thus, $\|x_k\|_{L^\infty}\to+\infty$ holds as $k\to+\infty$. □

Proof of Theorem 1.3 For every $k\in\mathbb{N}^*$, φ_k has a critical point x_k and $\|x_k\|_{L^\infty}\to+\infty$ holds as $k\to+\infty$ via Proposition 3.4 and Proposition 3.5. (3.10) implies that φ_k satisfies condition (a) of (2) in Lemma 2.6 and Proposition 3.5 implies that $x_k(t)\not\equiv 0$ holds for k large enough. Then (2) in Lemma 2.6 implies that the zero set of x_k is isolated, so $x_k(t)$ is a periodic solution of (2.1) via (1) in Lemma 2.6. Thus $|x_k(t)|$ is a periodic bouncing solution of (1.1) for any sufficiently large integer k.

Next, we claim that x_k is nontrivial if $k\in\mathbb{N}^*$ is large enough. In fact, if $x_k(t)\equiv b_k(\neq 0)$ holds for all $t\in[0,kT]$, then Proposition 3.5 implies that $b_k\to+\infty$ holds as $k\to+\infty$ and $f(t,|b_k|+a_0)=0$ via (2.1), which contradicts condition (B′). □

From now on, we focus on the proof of Theorem 1.4.

Proposition 3.6 *Assume f satisfies* (f′) *and* (F1). *Then functional φ_k satisfies the nonsmooth* (PS) *condition for every given $k\in\mathbb{N}^*$.*

Proof Let $\{x_n\}\subset H^1_{kT}$ be a nonsmooth (PS) sequence, that is, $\{\varphi_k(x_n)\}$ is bounded and $\lambda(x_n)\to 0$ holds as $n\to\infty$.

Step 1 We claim that $\{\|x_n\|\}$ is bounded. The main idea comes from papers [7] and [12].

For every $x_n\in H^1_{kT}$, x_n can be written as $x_n(t)=\bar{x}_n+\tilde{x}_n(t)$ for all $t\in[0,kT]$, where $\bar{x}_n\in\mathbb{R}$, $\tilde{x}_n\in\tilde{H}^1_{kT}$. It follows (f′), (h1) and (h2) that

$$\begin{aligned}|\varpi_n(t)|&\leq\big|f(t,|x_n(t)|+a_0)\big|\\&\leq\gamma(t)h(|x_n(t)|)+l(t)\\&\leq\gamma(t)(h(|\bar{x}_n|+|\tilde{x}_n(t)|)+C_1)+l(t)\\&\leq C_2\gamma(t)h(|\bar{x}_n|)+C_2\gamma(t)h(|\tilde{x}_n(t)|)+C_1\gamma(t)+l(t).\end{aligned}\tag{3.23}$$

Due to (3.23), (h1), the Young's inequality, (1) in Lemma 2.8 and the Sobolev's inequality, one has

$$\begin{aligned}&\Big|\int_0^{kT}\mathrm{e}^{G(t)}\varpi_n(t)\tilde{x}_n(t)\,dt\Big|\\&\leq d_2\int_0^{kT}\big[C_2\gamma(t)h(|\bar{x}_n|)+C_2\gamma(t)h(|\tilde{x}_n(t)|)+C_1\gamma(t)+l(t)\big]\cdot|\tilde{x}_n(t)|\,dt\\&\leq d_2C_2\|\tilde{x}_n\|_{L^\infty}h(|\bar{x}_n|)\int_0^{kT}\gamma(t)\,dt+d_2C_2(h(\|\tilde{x}_n\|_{L^\infty})+C_1)\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}\gamma(t)\,dt\\&\quad+d_2C_1\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}\gamma(t)\,dt+d_2\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}l(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&\le C_2\left[\frac{d_1}{2C_2kT}\|\tilde{x}_n\|^2_{L^\infty}+\frac{C_2kT}{2d_1}d_2^2h^2(|\bar{x}_n|)\left(\int_0^{kT}\gamma(t)\,dt\right)^2\right]\\
&\quad+d_2C_2\big(\varepsilon\|\tilde{x}_n\|_{L^\infty}+C_\varepsilon+C_1\big)\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}\gamma(t)\,dt\\
&\quad+d_2C_1\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}\gamma(t)\,dt+d_2\|\tilde{x}_n\|_{L^\infty}\int_0^{kT}l(t)\,dt\\
&\le\frac{d_1}{24}\int_0^{kT}|\dot{x}_n(t)|^2\,dt+\frac{\varepsilon d_2C_2kT}{12}\int_0^{kT}|\dot{x}_n(t)|^2\,dt\int_0^{kT}\gamma(t)\,dt\\
&\quad+\frac{C_2^2kTd_2^2}{2d_1}h^2(|\bar{x}_n|)\left(\int_0^{kT}\gamma(t)\,dt\right)^2+d_2\left(\frac{kT}{12}\int_0^{kT}|\dot{x}_n(t)|^2\,dt\right)^{\frac12}\int_0^{kT}l(t)\,dt\\
&\quad+d_2(C_2C_\varepsilon+C_2C_1+C_1)\left(\frac{kT}{12}\int_0^{kT}|\dot{x}_n(t)|^2\,dt\right)^{\frac12}\int_0^{kT}\gamma(t)\,dt\\
&=\left(\frac{d_1}{24}+\varepsilon C_{3,k}\right)\int_0^{kT}|\dot{x}_n(t)|^2dt+C_{4,k}\left(\int_0^{kT}|\dot{x}_n(t)|^2dt\right)^{\frac12}+C_{5,k}h^2(|\bar{x}_n|),
\end{aligned}
\tag{3.24}
$$

where $C_{3,k}:=\frac{d_2C_2kT}{12}\int_0^{kT}\gamma(t)dt\ge 0$, $C_{4,k}:=d_2\sqrt{\frac{kT}{12}}(C_2C_\varepsilon+C_2C_1+C_1)\int_0^{kT}\gamma(t)dt+d_2\sqrt{\frac{kT}{12}}\cdot\int_0^{kT}l(t)dt>0$ and $C_{5,k}:=\frac{C_2^2kTd_2^2}{2d_1}(\int_0^{kT}\gamma(t)\,dt)^2\ge 0$. Remark 2.2, (3.3) and (3.24) imply that there are a function $\varpi_n(t)\in[f^-(t,|x_n(t)|+a_0),f^+(t,|x_n(t)|+a_0)]$ for a.e. $t\in[0,kT]$ and an $x_n^*\in\partial\varphi_k(x_n)$ with $\lambda(x_n)=\|x_n^*\|\to 0$ as $n\to\infty$. One has

$$
\begin{aligned}
\|\tilde{x}_n\|\ge\langle x_n^*,\tilde{x}_n\rangle&=\int_0^{kT}\mathrm{e}^{G(t)}|\dot{x}_n(t)|^2\,dt-\int_0^{kT}\mathrm{e}^{G(t)}\varpi_n(t)\tilde{x}_n(t)\,dt\\
&\ge d_1\int_0^{kT}|\dot{x}_n(t)|^2\,dt-\left|\int_0^{kT}\mathrm{e}^{G(t)}\varpi_n(t)\tilde{x}_n(t)\,dt\right|\\
&\ge\left(\frac{23}{24}d_1-\varepsilon C_{3,k}\right)\int_0^{kT}|\dot{x}_n(t)|^2\,dt-C_{4,k}\left(\int_0^{kT}|\dot{x}_n(t)|^2\,dt\right)^{\frac12}\\
&\quad-C_{5,k}h^2(|\bar{x}_n|),\qquad n\text{ large enough.}
\end{aligned}
\tag{3.25}
$$

On the other hand, the Wirtinger's inequality yields

$$
\|\tilde{x}_n\|\le\left(1+\frac{k^2T^2}{4\pi^2}\right)^{\frac12}\left(\int_0^{kT}|\dot{x}_n(t)|^2\,dt\right)^{\frac12},\quad\forall n\in\mathbb{N}.
\tag{3.26}
$$

$\varepsilon>0$ in (3.24) small enough can be chosen such that $\frac{23}{24}d_1-\varepsilon C_{3,k}>0$, using the property of parabola, (3.25) and (3.26) imply that there exist constants $C_{6,k}>0,C_{7,k}\ge 0$ such that

$$
\left(\int_0^{kT}|\dot{x}_n(t)|^2\,dt\right)^{\frac12}\le C_{6,k}+C_{7,k}h(|\bar{x}_n|),\quad n\text{ large enough.}
\tag{3.27}
$$

The Mean Value Theorem of locally Lipschitzian functional (see [6, Proposition 1.3.14]) implies that there exist a $z_n\in\{(1-s)x_n+s\bar{x}_n\,|\,0\le s\le 1\}$ and a $z_n^*\in\partial J_k(z_n)$ such that

$$
\int_0^{kT}\mathrm{e}^{G(t)}F(t,|x_n(t)|+a_0)\,dt-\int_0^{kT}\mathrm{e}^{G(t)}F(t,|\bar{x}_n|+a_0)\,dt=\langle z_n^*,x_n-\bar{x}_n\rangle.
\tag{3.28}
$$

Due to Lemma 2.7, there exists a function $\varpi_{z_n}(t)\in[f^-(t,|z_n(t)|+a_0),f^+(t,|z_n(t)|+a_0)]$ for a.e. $t \in [0, kT]$ such that

$$\langle z_n^*, x_n - \bar{x}_n \rangle = \int_0^{kT} \mathrm{e}^{G(t)} (\varpi_{z_n}(t), x_n(t) - \bar{x}_n)\, dt. \tag{3.29}$$

Similarly to (3.24), together with (3.28) and (3.29), one has

$$\begin{aligned}
&\left| \int_0^{kT} \mathrm{e}^{G(t)} F(t, |x_n(t)| + a_0)\, dt - \int_0^{kT} \mathrm{e}^{G(t)} F(t, |\bar{x}_n| + a_0)\, dt \right| \\
&\quad = \left| \int_0^{kT} \mathrm{e}^{G(t)} \big(\varpi_{z_n}(t), x_n(t) - \bar{x}_n\big)\, dt \right| \\
&\quad \le d_2 \int_0^{kT} |\, f(t, |z_n(t)| + a_0)| \cdot |x_n(t) - \bar{x}_n|\, dt \\
&\quad = d_2 \int_0^{kT} \big| f(t, |\bar{x}_n + (1-s)\tilde{x}_n(t)| + a_0) \big| \cdot |\tilde{x}_n(t)|\, dt \\
&\quad \le d_2 \int_0^{kT} \big[\gamma(t) h(|\bar{x}_n + (1-s)\tilde{x}_n(t)|) + l(t) \big] \cdot |\tilde{x}_n(t)|\, dt \\
&\quad \le d_2 \int_0^{kT} \big[C_2 \gamma(t) h(|\bar{x}_n|) + C_2 \gamma(t) h(|\tilde{x}_n(t)|) + C_1 \gamma(t) + l(t) \big] \cdot |\tilde{x}_n(t)|\, dt \\
&\quad \le \left(\frac{d_2}{24} + \varepsilon C_{3,k} \right) \int_0^{kT} |\dot{x}_n(t)|^2 dt + C_{4,k} \left(\int_0^{kT} |\dot{x}_n(t)|^2 dt \right)^{\frac{1}{2}} + C_{8,k} h^2(|\bar{x}_n|),
\end{aligned} \tag{3.30}$$

where $C_{8,k} := \frac{C_2^2 kT d_2}{2} (\int_0^{kT} \gamma(t)\, dt)^2 \ge 0$.

Suppose $|\bar{x}_n| \to +\infty$ holds as $n \to \infty$, (3.30), (B'') and (3.27) yield

$$\begin{aligned}
\varphi_k(x_n) &\le \frac{1}{2} d_2 \int_0^{kT} |\dot{x}_n(t)|^2\, dt - \int_0^{kT} \mathrm{e}^{G(t)} F(t, |\bar{x}_n| + a_0)\, dt \\
&\quad + \left| \int_0^{kT} \mathrm{e}^{G(t)} [F(t, |x_n(t)| + a_0) - F(t, |\bar{x}_n| + a_0)]\, dt \right| \\
&\le \left(\frac{13}{24} d_2 + \varepsilon C_{3,k} \right) \int_0^{kT} |\dot{x}_n(t)|^2\, dt + C_{4,k} \left(\int_0^{kT} |\dot{x}_n(t)|^2\, dt \right)^{\frac{1}{2}} \\
&\quad + C_{8,k} h^2(|\bar{x}_n|) - d_1 \int_0^{kT} F(t, |\bar{x}_n| + a_0)\, dt \\
&\le \left(\frac{13}{24} d_2 + \varepsilon C_{3,k} \right) (C_{7,k} h(|\bar{x}_n|) + C_{6,k})^2 + C_{8,k} h^2(|\bar{x}_n|) \\
&\quad + C_{4,k} (C_{7,k} h(|\bar{x}_n|) + C_{6,k}) - d_1 \int_0^{kT} F(t, |\bar{x}_n| + a_0) dt \\
&\le C_{9,k} h^2(|\bar{x}_n|) + C_{10,k} h(|\bar{x}_n|) + C_{11,k} \\
&\quad - d_1 \int_0^{kT} F(t, |\bar{x}_n| + a_0) dt, \quad n \text{ large enough},
\end{aligned} \tag{3.31}$$

where constants $C_{9,k}, C_{10,k} \ge 0$, $C_{11,k} > 0$. (3.31), Lemma 2.8 and (F1) yield

$$\begin{aligned}
\varphi_k(x_n) &\le H(|\bar{x}_n|) \left[C_{9,k} \frac{h^2(|\bar{x}_n|)}{H(|\bar{x}_n|)} + C_{10,k} \frac{h(|\bar{x}_n|)}{H(|\bar{x}_n|)} - \frac{d_1 \int_0^{kT} F(t, |\bar{x}_n| + a_0) dt}{H(|\bar{x}_n|)} \right] + C_{11,k} \\
&\to -\infty, \quad n \to \infty,
\end{aligned}$$

which contradicts the boundedness of $\{\varphi_k(x_n)\}$, thus $\{|\bar{x}_n|\}$ is bounded, so $\{\|x_n\|_1\}$ is bounded via (3.27). Then $\{\|x_n\|\}$ is bounded via the equivalence of norm $\|\cdot\|_1$ and $\|\cdot\|$.

Step 2 We claim that $\{x_n\}$ has a convergent subsequence in H^1_{kT}.

Using similar argument in the proof of Proposition 3.1, one can also show that φ_k satisfies the nonsmooth (PS) condition. □

Lemma 3.7 *Assume f satisfies* (f′). *Then $\varphi_k(x) \to +\infty$ holds as $\|x\| \to +\infty$ with $x \in \tilde{H}^1_{kT}$ for every given $k \in \mathbb{N}^*$, so there is a certain constant $M_{4,k}$ such that $\varphi_k(x) \geq M_{4,k}$ holds for all $x \in \tilde{H}^1_{kT}$.*

Proof The proof is similar to that in [12, Lemma 3.2]. □

Lemma 3.8 *Assume f satisfies* (f′) *and* (F1). *Then $\varphi_k(\bar{x}+e_k) \to -\infty$ holds as $|\bar{x}| \to +\infty$ in $\mathbb{R} \subseteq H^1_{kT}$ for every given $k \in \mathbb{N}^*$, so one has $\sup_{\bar{x}\in\mathbb{R}} \varphi_k(\bar{x}+e_k) < +\infty$, where $e_k(t) := k\cos(\frac{2\pi t}{kT}) \in \tilde{H}^1_{kT}$.*

Proof Using (3.30), (F1) and Lemma 2.8, one has

$$\begin{aligned}
\varphi_k(\bar{x}+e_k) &= \frac{1}{2}\int_0^{kT} \mathrm{e}^{G(t)}|\dot{e}_k(t)|^2\,dt - \int_0^{kT} \mathrm{e}^{G(t)}F(t,|\bar{x}|+a_0)\,dt \\
&\quad - \int_0^{kT} \mathrm{e}^{G(t)}[F(t,|\bar{x}+e_k(t)|+a_0) - F(t,|\bar{x}|+a_0)]\,dt \\
&\leq d_2\frac{k\pi^2}{T} + \left(\frac{d_2}{24} + \varepsilon C_{3,k}\right)\frac{2k\pi^2}{T} + \frac{C_{4,k}\pi\sqrt{2k}}{\sqrt{T}} \\
&\quad + H(|\bar{x}|)\left(C_{8,k}\frac{h^2(|\bar{x}|)}{H(|\bar{x}|)} - \frac{d_1\int_0^{kT} F(t,|\bar{x}|+a_0)\,dt}{H(|\bar{x}|)}\right) \\
&\to -\infty, \quad |\bar{x}| \to +\infty,\ \bar{x} \in \mathbb{R}.
\end{aligned}$$
□

Proposition 3.9 *Assume f satisfies* (f′) *and* (F1). *Then for every given $k \in \mathbb{N}^*$, there exists a constant $r_{2,k} > 0$ large enough such that functional φ_k possesses at least a critical value c_k characterized as*

$$c_k = \inf_{\chi\in\Gamma_2}\max_{x\in[-r_{2,k},r_{2,k}]} \varphi_k(\chi(x+e_k)),$$

where $\Gamma_2 := \{\chi \in C([-r_{2,k}, r_{2,k}]+e_k, H^1_{kT}) \big| \chi(e_k \pm r_{2,k}) = e_k \pm r_{2,k}\}$. In addition, one has

$$-\infty < \inf_{\tilde{H}^1_{kT}} \varphi_k \leq c_k \leq \sup_{x\in\mathbb{R}} \varphi_k(x+e_k) < +\infty.$$

Proof The proof is similar to that in Proposition 3.4. □

Proposition 3.10 *Assume f satisfies* (f′). *Let x_k be a critical point of functional φ_k, then $\|x_k\|_{L^\infty} \to +\infty$ holds as $k \to +\infty$ for every given $k \in \mathbb{N}^*$.*

Proof The proof is similar to that in [12, Lemma 3.6] via the proof in Proposition 3.5. □

Proof of Theorem 1.4 For every $k \in \mathbb{N}^*$, φ_k has a critical point x_k with $\|x_k\|_{L^\infty} \to +\infty$ $(k \to +\infty)$ via Proposition 3.9 and Proposition 3.10. Condition (B″) implies that φ_k satisfies condition (a) of (2) in Lemma 2.6 and Proposition 3.10 implies that $x_k(t) \not\equiv 0$ holds for k large enough. Then (2) in Lemma 2.6 implies that the zero set of x_k is isolated, so $x_k(t)$ is a periodic solution of (2.1) via (1) in Lemma 2.6. Thus $|x_k(t)|$ is a periodic bouncing solution of (1.1) for any sufficiently large integer k.

Next, we claim that x_k is nontrivial if $k \in \mathbb{N}^*$ is large enough. In fact, if $x_k(t) \equiv b_k(\neq 0)$ holds for all $t \in [0, kT]$, then Proposition 3.10 implies that $b_k \to +\infty$ holds as $k \to +\infty$, and $f(t, |b_k| + a_0) = 0$ holds via (2.1), which contradicts condition (B$''$). □

4 Example

Example 4.1 Function $f \in C(\mathbb{R} \times [a_0, +\infty), \mathbb{R})$ is defined as

$$f(t,x) = \eta(t)\frac{2(x-a_0)\ln(2+(x-a_0)^2) - 2(x-a_0)}{\ln^2(2+(x-a_0)^2)},$$

where $\eta \in C(\mathbb{R}, [0, +\infty))$ is a 2π-periodic function with

$$\eta(t) = \begin{cases} \sin t + 1, & t \in [0, \pi), \\ 1, & t \in [\pi, 2\pi]. \end{cases}$$

Then $f(t,x)$ is 2π-periodic, $F(t,x) := \int_{a_0}^{x} f(t,s)\, ds = \eta(t)(\frac{2+(x-a_0)^2}{\ln(2+(x-a_0)^2)} - \frac{2}{\ln 2})$. Set $g(t) = \sin t$, then $g \in C(\mathbb{R}, \mathbb{R})$, $g(t+2\pi) = g(t)$ and $G(t) = \int_0^t g(s)\, ds = 1 - \cos t$ with $G(2\pi) = 0$.

Let $\theta(s) = \ln(2+s^2) > \frac{1}{2}$. Then for the given constant $\tau > 0$, one has

$$\int_{\tau}^{t} \frac{1}{s\ln(2+s^2)} ds > \int_{\tau}^{t} \frac{s}{\ln(2+s^2)\cdot(2+s^2)} ds = \frac{1}{2}\ln\ln(2+s^2)|_{\tau}^{t} \to +\infty, \quad t \to +\infty.$$

Thus, θ satisfies ($\theta 1$) and ($\theta 2$).

Obviously, conditions (A), (H2) and (B$'$) hold. At the same time, one has

$$\begin{aligned}
&\lim_{|x|\to+\infty} \left[f(t,|x|+a_0)\cdot|x| - \left(2 - \frac{1}{\theta(|x|)}\right) F(t,|x|+a_0) \right] \\
&= \lim_{|x|\to+\infty} \frac{\eta(t)}{\ln^2(2+|x|^2)}[2 - |x|^2 - 4\ln(2+|x|^2)] - \lim_{|x|\to+\infty} \frac{2\eta(t)}{\ln 2 \cdot \ln(2+|x|)} + \frac{4\eta(t)}{\ln 2} \\
&= -\infty, \quad \forall t \in [0, 2\pi],
\end{aligned}$$

which implies that (H1) holds.

From the definition of f, one has $f(t,x) < 0$ holds for all $x \in [a_0, \sqrt{\mathrm{e}-2} + a_0)$ and $t \in \mathbb{R}$. Thus f dissatisfies the condition (B).

Example 4.2 Define functions $\lambda(t) = |\sin t|$ and $f \in C(\mathbb{R} \times [a_0, +\infty), \mathbb{R})$ with

$$f(t,x) = \lambda(t)\frac{2(x-a_0)\ln(2+(x-a_0)^2) - 2(x-a_0)}{\ln^2(2+(x-a_0)^2)} + \frac{1}{100},$$

then $f(t,x)$ is 2π-periodic, $F(t,x) := \int_{a_0}^{x} f(t,s) ds = \lambda(t)(\frac{2+(x-a_0)^2}{\ln(2+(x-a_0)^2)} - \frac{2}{\ln 2}) + \frac{1}{100}(x-a_0)$. Set $g(t) = \sin t$, then $g \in C(\mathbb{R}, \mathbb{R})$, $g(t+2\pi) = g(t)$ and $G(t) = \int_0^t g(s)\, ds = 1 - \cos t$ with $G(2\pi) = 0$.

Obviously, condition (B$''$) holds. Let $h(t) = \frac{t}{\ln(2+t^2)}$, $\gamma(t) = 6|\lambda(t)|$ and $l(t) \equiv \frac{1}{100}$. The calculation similar to [12, Example 4.1] implies that h satisfies (h1) – (h4), then f satisfies (f$'$).

$$\lim_{|x|\to+\infty} \frac{\int_0^{2\pi} F(t,|x|+a_0) dt}{H(|x|)}$$

$$
\begin{aligned}
&= \lim_{|x|\to+\infty}\left(\frac{\frac{2+x^2}{\ln(2+x^2)}-\frac{2}{\ln 2}}{\int_0^{|x|}\frac{s}{\ln(2+s^2)}ds}\int_0^{2\pi}\lambda(t)dt+\frac{2\pi|x|}{100\int_0^{|x|}\frac{s}{\ln(2+s^2)}ds}\right)\\
&= \lim_{|x|\to+\infty}\frac{\frac{2|x|\ln(2+x^2)-\frac{2|x|^3+4|x|}{2+x^2}}{\ln^2(2+x^2)}}{\frac{|x|}{\ln(2+x^2)}}\int_0^{2\pi}\lambda(t)dt+\lim_{|x|\to+\infty}\frac{\pi\ln(2+x^2)}{50|x|}\\
&= 2\int_0^{2\pi}\lambda(t)dt>0,
\end{aligned}
$$

thus (F1) holds.

But, we claim that f dissatisfies (F2). Otherwise, for a.e. $t\in[0,2\pi]$ and every $x\in\mathbb{R}$, there is a constant $\sigma_0>0$ such that

$$
\left|\lambda'(t)\left(\frac{2+x^2}{\ln(2+x^2)}-\frac{2}{\ln 2}\right)\right|\le\sigma_0\left[\lambda(t)\left(\frac{2+x^2}{\ln(2+x^2)}-\frac{2}{\ln 2}\right)+\frac{|x|}{100}\right]. \tag{4.1}
$$

Dividing both sides of (4.1) by $\frac{2+x^2}{\ln(2+x^2)}-\frac{2}{\ln 2}$ ($|x|\ge 2$), one has

$$
|\lambda'(t)|\le\sigma_0\left(\lambda(t)+\frac{\ln 2\cdot\ln(2+x^2)\cdot|x|}{200\ln 2+100x^2\ln 2-200\ln(2+x^2)}\right),\quad \text{a.e. } t\in[0,2\pi],|x|\ge 2. \tag{4.2}
$$

Let $|x|\to+\infty$ on both sides of (4.2). Then

$$
|\cos t|\le\sigma_0|\sin t|,\quad \text{a.e. } t\in[0,2\pi],
$$

that is, $|\cot t|\le\sigma_0$ holds for a.e. $t\in[0,2\pi]$, which is impossible.

From the definition of f, one has $f(t,x)<0$ holds for all $x\in[\frac{\sqrt{e-2}}{5}+a_0,\frac{\sqrt{e-2}}{3}+a_0)$ and $t\in[\frac{\pi}{6},\frac{5\pi}{6}]\cup[\frac{7\pi}{6},\frac{11\pi}{6}]$. Thus f dissatisfies the condition (B).

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