The Sufficient and Necessary Conditions of the Strong Law of Large Numbers under Sub-linear Expectations

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Abstract   In this paper, by establishing a Borel–Cantelli lemma for a capacity which is not necessarily continuous, and a link between a sequence of independent random variables under the sub-linear expectation and a sequence of independent random variables on $\mathbb{R}^\infty$ under a probability, we give the sufficient and necessary conditions of the strong law of large numbers for independent and identically distributed random variables under the sub-linear expectation, and the sufficient and necessary conditions for the convergence of an infinite series of independent random variables, without the assumption on the continuity of the capacities. A purely probabilistic proof of a weak law of large numbers is also given.

Keywords   Sub-linear expectation, capacity, strong convergence, law of large numbers

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1   Introduction and Notations

Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.) on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Denote $S_n = \sum_{i=1}^n X_i$. One of the most famous results of probability theory is Kolmogorov [3]'s strong law of large numbers (see Theorem 3.2.2 of Stout [7]), which states that

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = b\right) = 1 \tag{1.1}$$

if and only if

$$E_\mathsf{P}[|X_1|] < \infty \quad \text{and} \quad E_\mathsf{P}[X_1] = b, \tag{1.2}$$

where E_P is the expectation with respect to the probability measure P. When the probability measure P is uncertain, one may consider a family $\mathscr{P}$ of probability measures and define $\widehat{\mathbb{E}}[X] = \sup_{P \in \mathscr{P}} E_P[X]$. Then $\widehat{\mathbb{E}}$ is no longer a linear expectation. It is sub-linear in sense that $\widehat{\mathbb{E}}[aX + bY] \leq a\widehat{\mathbb{E}}[X] + b\widehat{\mathbb{E}}[Y]$ if $a, b \geq 0$. Peng [4, 5] introduced the concepts of independence, identical distribution and G-normal random variables under the sub-linear expectation,

and established the weak law of large numbers and central limit theorem for independent and identically distributed random variables. Fang et al. [2] obtained the rate of convergence of the weak law of large numbers and central limit theorem.

As for the strong law of large numbers, Chen [1] established a Kolmogorov type result. Let $\{X_n; n \geq 1\}$ be a sequence of random variables in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ with a related upper capacity $\widehat{\mathbb{V}}$. Chen [1] showed that, if $\{X_n; n \geq 1\}$ is a sequence of i.i.d. random variables, the capacity $\mathbb{V}$ is continuous, and the following moment condition is satisfied

$$\widehat{\mathbb{E}}[|X_1|^{1+\alpha}] < \infty \quad \text{for some } \alpha > 0, \tag{1.3}$$

then

$$\widehat{\mathbb{V}}\left(\liminf_{n\to\infty} \frac{S_n}{n} < -\widehat{\mathbb{E}}[-X_1] \text{ and } \limsup_{n\to\infty} \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1]\right) = 0 \tag{1.4}$$

and

$$\widehat{\mathbb{V}}\left(\liminf_{n\to\infty} \frac{S_n}{n} = -\widehat{\mathbb{E}}[-X_1]\right) = 1 \quad \text{and} \quad \widehat{\mathbb{V}}\left(\limsup_{n\to\infty} \frac{S_n}{n} = \widehat{\mathbb{E}}[X_1]\right) = 1. \tag{1.5}$$

By establishing the moment inequalities of the maximum partial sums, Zhang [9] weakened the condition (1.3) to

$$C_{\widehat{\mathbb{V}}}(|X_1|) := \int_0^\infty \widehat{\mathbb{V}}(|X_1| > x) dx < \infty \tag{1.6}$$

and

$$\widehat{\mathbb{E}}[(|X_1| - c)^+] \to 0 \quad \text{as } c \to \infty. \tag{1.7}$$

The conditions (1.6) and (1.7) are very close to Kolmogorov's condition (1.2). Zhang [9] showed that (1.6) is also a necessary condition. Nevertheless, whether (1.7) is necessary or not is unknown. On the other hand, to make both the direct part and converse part of the Borel–Cantelli lemma are valid for a capacity, it is usually needed to assume that the capacity is continuous when the strong convergence is considered as in Chen [1] and Zhang [9] etc. However, Zhang [12] showed that the assumption of the continuity of a capacity is very stringent. It is showed that a sub-linear expectation with a continuous capacity is nearly linear.

The purpose of this paper is to obtain the sufficient and necessary conditions for the strong law of large numbers of independent random variables under the sub-linear expectation without the assumption of the continuity of the capacities. In particular it will be shown that, if $\{X_n; n \geq 1\}$ is sequence of i.i.d. random variables in sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ with a regular sub-linear expectation $\widehat{\mathbb{E}}$ and a related upper capacity $\widehat{\mathbb{V}}$ is countably sub-additive (otherwise, $\widehat{\mathbb{V}}$ can be replaced by a countably sub-additive extension), then

$$\widehat{\mathcal{V}}\left(\lim_{n\to\infty} \frac{S_n}{n} = b\right) = 1 \quad \text{and } b \text{ is finite}$$

if and only if

$$(1.6) \text{ holds and } b = \breve{\mathbb{E}}[X_1] = \breve{\mathcal{E}}[X_1],$$

where $\widehat{\mathcal{V}}(A) = 1 - \widehat{\mathbb{V}}(A^c)$, $\breve{\mathbb{E}}[X] = \lim_{c\to\infty} \widehat{\mathbb{E}}[(-c) \vee X_1 \wedge c]$ and $\breve{\mathcal{E}}[X] = -\lim_{c\to\infty} \widehat{\mathbb{E}}[(-c) \vee (-X_1) \wedge c]$. As in the classical probability space, the almost sure convergence of a random infinite series together with Kronecker's lemma is a powerful tool for studying the strong law of large numbers (see Pages 208–212 of Petrov [6]). The paper also gives the sufficient and

necessary conditions for an infinite series of independent random variables under the sub-linear expectation to be convergent.

Our main tools are a Borel–Cantelli lemma for a capacity which is not necessarily continuous, and a comparison theorem for the random variables defined on the product space $\mathbb{R}^\infty$ which gives a link between a sequence of independent random variables on $\mathbb{R}^\infty$ under the sub-linear expectation and a sequence of independent random variables under a probability. By the comparison theorem, a Kolmogorov's maximal inequality is obtained and a weak law of large numbers is given with a purely probabilistic proof.

To state the results, we shall first recall the framework of sub-linear expectation in this section. We use the framework and notations of Peng [4, 5]. If one is familiar with these notations, he or she can skip the following paragraphs. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathscr{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$, where $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$

$$\text{for some } C > 0, m \in \mathbb{N} \text{ depending on } \varphi.$$

We also denote $C_{b,\mathrm{Lip}}(\mathbb{R}^n)$ the space of bounded Lipschitz functions.

Definition 1.1 *A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathscr{H}$ is a function $\widehat{\mathbb{E}} : \mathscr{H} \to \overline{\mathbb{R}}$ satisfying the following properties*: *for all $X, Y \in \mathscr{H}$, we have*

(a) *Monotonicity*: *If $X \geq Y$ then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;*

(b) *Constant preserving*: *$\widehat{\mathbb{E}}[c] = c$;*

(c) *Sub-additivity*: *$\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;*

(d) *Positive homogeneity*: *$\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda > 0$.*

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathscr{H}.$$

By Theorem 1.2.1 of Peng [5], there exists a family of finite additive linear expectations $E_\theta : \mathscr{H} \to \overline{R}$ indexed by $\theta \in \Theta$, such that

$$\widehat{\mathbb{E}}[X] = \max_{\theta \in \Theta} E_\theta[X] \quad \text{for } X \in \mathscr{H} \text{ with } \widehat{\mathbb{E}}[X] \text{ being finite.} \tag{1.8}$$

Moreover, for each $X \in \mathscr{H}$, there exists $\theta_X \in \Theta$ such that $\widehat{\mathbb{E}}[X] = E_{\theta_X}[X]$ if $\widehat{\mathbb{E}}[X]$ is finite.

Definition 1.2 (Peng [4, 5]) (i) (Identical distribution) *Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be two n-dimensional random vectors, respectively, defined in sub-linear expectation spaces $(\Omega_1, \mathscr{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathscr{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$ if*

$$\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)], \quad \forall \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^n).$$

A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) (Independence) *In a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, a random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$ is said to be independent to another random vector $\boldsymbol{X} = (X_1, \ldots, X_m)$,*

$X_i \in \mathscr{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]|_{\boldsymbol{x}=\boldsymbol{X}}]$, whenever $\overline{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|] < \infty$ for all $\boldsymbol{x}$ and $\widehat{\mathbb{E}}[|\overline{\varphi}(\boldsymbol{X})|] < \infty$.

A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent, if X_{i+1} is independent to $(X_1, \ldots, X_i)$ for each $i \geq 1$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0,1]$ is called a capacity if

$$V(\emptyset) = 0, \ V(\Omega) = 1 \ \text{ and } V(A) \leq V(B), \quad \forall A \subset B, \ A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$.

Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space. We denote $(\widehat{\mathbb{V}}, \widehat{\mathcal{V}})$ to be a pair of capacities by

$$\widehat{\mathbb{V}}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathscr{H}\}, \quad \widehat{\mathcal{V}}(A) = 1 - \widehat{\mathbb{V}}(A^c), \quad \forall A \in \mathcal{F}, \tag{1.9}$$

where A^c is the complement set of A. Then $\widehat{\mathbb{V}}$ is a sub-additive capacity with the property that

$$\widehat{\mathbb{E}}[f] \leq \widehat{\mathbb{V}}(A) \leq \widehat{\mathbb{E}}[g] \quad \text{if } 0 \leq f \leq I_A \leq g, \ f, g \in \mathscr{H} \text{ and } A \in \mathcal{F}. \tag{1.10}$$

We call $\widehat{\mathbb{V}}$ and $\widehat{\mathcal{V}}$ the upper and the lower capacity, respectively.

Also, we define the Choquet integrals/expecations $(C_{\widehat{\mathbb{V}}}, C_{\widehat{\mathcal{V}}})$ by

$$C_V[X] = \int_0^\infty V(X \geq t) dt + \int_{-\infty}^0 [V(X \geq t) - 1] dt$$

with V being replaced by $\widehat{\mathbb{V}}$ and $\widehat{\mathcal{V}}$ respectively. If $\mathbb{V}$ on the sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ and $\widetilde{\mathbb{V}}$ on the sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ are two capacities having the property (1.10), then for any random variables $X \in \mathscr{H}$ and $\tilde{X} \in \widetilde{\mathscr{H}}$ with $X \overset{d}{=} \tilde{X}$, we have

$$\mathbb{V}(X \geq x + \epsilon) \leq \widetilde{\mathbb{V}}(\tilde{X} \geq x) \leq \mathbb{V}(X \geq x - \epsilon) \quad \text{for all } \epsilon > 0 \text{ and } x. \tag{1.11}$$

In fact, let $f \in C_{b,\text{Lip}}(\mathbb{R})$ such that $I\{y \geq x + \epsilon\} \leq f(y) \leq I\{y \geq x\}$. Then

$$\mathbb{V}(X \geq x + \epsilon) \leq \widehat{\mathbb{E}}[f(X)] = \widetilde{\mathbb{E}}[f(\tilde{X})] \leq \widetilde{\mathbb{V}}(\tilde{X} \geq x),$$

and similar $\widetilde{\mathbb{V}}(\tilde{X} \geq x + \epsilon) \leq \mathbb{V}(X \geq x)$. From (1.11), it follows that $\mathbb{V}(X \geq x) = \widetilde{\mathbb{V}}(\tilde{X} \geq x)$ if x is a continuous point of both functions $\mathbb{V}(X \geq y)$ and $\widetilde{\mathbb{V}}(\tilde{X} \geq y)$. Since, a monotone function has at most countable number of discontinuous points. So

$$\mathbb{V}(X \geq x) = \widetilde{\mathbb{V}}(\tilde{X} \geq x) \quad \text{for all but except countable many } x,$$

and then

$$C_{\mathbb{V}}(X) = C_{\widetilde{\mathbb{V}}}(\tilde{X}). \tag{1.12}$$

Because a capacity $\widehat{\mathbb{V}}$ may be not countably sub-additive so that the Borel–Cantelli lemma is not valid, we consider its countably sub-additive extension $\widehat{\mathbb{V}}^*$ which defined by

$$\widehat{\mathbb{V}}^*(A) = \inf\left\{\sum_{n=1}^\infty \widehat{\mathbb{V}}(A_n) : A \subset \bigcup_{n=1}^\infty A_n\right\}, \quad \widehat{\mathcal{V}}^*(A) = 1 - \widehat{\mathbb{V}}^*(A^c), \quad A \in \mathcal{F}. \tag{1.13}$$

As shown in Zhang [9], $\widehat{\mathbb{V}}^*$ is countably sub-additive, and $\widehat{\mathbb{V}}^*(A) \leq \widehat{\mathbb{V}}(A)$. Furthermore, $\widehat{\mathbb{V}}$ (resp. $\widehat{\mathbb{V}}^*$) is the largest sub-additive (resp. countably sub-additive) set function in sense that if V is

also a sub-additive (resp. countably sub-additive) set function satisfying $V(A) \leq \widehat{\mathbb{E}}[g]$ whenever $I_A \leq g \in \mathscr{H}$, then $V(A) \leq \mathbb{V}(A)$ (resp. $V(A) \leq \widehat{\mathbb{V}}^*(A)$).

Besides $\widehat{\mathbb{V}}^*$, another countably sub-additive capacity generated by $\widehat{\mathbb{E}}$ can be defined as follows:

$$\mathbb{C}^*(A) = \inf \left\{ \lim_{n \to \infty} \widehat{\mathbb{E}}\left[\sum_{i=1}^{n} g_i\right] : I_A \leq \sum_{n=1}^{\infty} g_n, 0 \leq g_n \in \mathscr{H} \right\}, \quad A \in \mathcal{F}. \tag{1.14}$$

Then $\mathbb{C}^* \leq \widehat{\mathbb{V}}^*$. It can be shown that the out capacity c' defined in Example 6.5.1 of Peng [5] coincides with $\mathbb{C}^*$ if $\mathscr{H}$ is chosen as the family of (bounded) continuous functions on a metric space Ω.

For real numbers x and y, denote $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$. For a random variable X, because $XI\{|X| \leq c\}$ may be not in $\mathscr{H}$, we will truncate it in the form $(-c) \vee X \wedge c$ denoted by $X^{(c)}$, and define $\breve{\mathbb{E}}[X] = \lim_{c \to \infty} \widehat{\mathbb{E}}[X^{(c)}]$ if the limit exists, and $\breve{\mathcal{E}}[X] = -\breve{\mathbb{E}}[-X]$.

Proposition 1.3 *Consider a subspace of $\mathscr{H}$ as*

$$\mathscr{H}_1 = \{X \in \mathscr{H} : \lim_{c,d \to \infty} \widehat{\mathbb{E}}[(|X| \wedge d - c)^+] = 0\}. \tag{1.15}$$

Then for any $X \in \mathscr{H}_1$, $\breve{\mathbb{E}}[X]$ is well defined, and $(\Omega, \mathscr{H}_1, \breve{\mathbb{E}})$ is a sub-linear expectation space.

Proof For any $X \in \mathscr{H}_1$ and $0 < c_1, c_2 \leq d$ we have

$$\widehat{\mathbb{E}}[|(-c_1) \vee X \wedge c_2 - X^{(d)}|] \leq \widehat{\mathbb{E}}[(|X| \wedge d - (c_1 \wedge c_2))^+].$$

Hence

$$|\widehat{\mathbb{E}}[X^{(c)}] - \widehat{\mathbb{E}}[X^{(d)}]| \to 0 \quad \text{as } c, d \to \infty,$$

which implies that $\breve{\mathbb{E}}[X] = \lim_{c \to \infty} \widehat{\mathbb{E}}[X^{(c)}]$ exists and is finite. Furthermore,

$$\lim_{c_1,c_2 \to \infty} \widehat{\mathbb{E}}[|(-c_1) \vee X \wedge c_2|] = \breve{\mathbb{E}}[X]. \tag{1.16}$$

Notice $(\lambda X)^{(c)} = \lambda X^{(c/\lambda)}$ for $\lambda > 0$. It is obvious that $\breve{\mathbb{E}}[\lambda X] = \lambda \breve{\mathbb{E}}[X]$ for $\lambda > 0$. Finally, for any $X, Y \in \mathscr{H}_1$ and $c > 0$, we have $X + Y \in \mathscr{H}_1$ and

$$(X + Y)^{(c)} \leq (-c/2) \vee X \wedge (3c/2) + (-c/2) \vee Y \wedge (3c/2).$$

By (1.16), $\breve{\mathbb{E}}[X + Y] \leq \breve{\mathbb{E}}[X] + \breve{\mathbb{E}}[Y]$. The monotonicity and constant preserving for $\breve{\mathbb{E}}$ are obvious. The proof is completed. $\square$

Let

$$\mathscr{E} = \{E : \mathscr{H}_1 \to \mathbb{R} \text{ is a finite additive linear expectation with } E \leq \breve{\mathbb{E}}\}. \tag{1.17}$$

By Theorem 1.2.1 of Peng [5],

$$\breve{\mathbb{E}}[X] = \max_{E \in \mathscr{E}} E[X] \quad \text{for } X \in \mathscr{H}_1, \tag{1.18}$$

and moreover, for each $X \in \mathscr{H}_1$, there exists $E \in \mathscr{E}$ such that $\breve{\mathbb{E}}[X] = E[X]$. For the vector $\boldsymbol{X} = (X_1, \ldots, X_d)$, we denote $\breve{\mathbb{E}}[\boldsymbol{X}] = (\breve{\mathbb{E}}[X_1], \ldots, \breve{\mathbb{E}}[X_d])$, $\widehat{\mathbb{E}}[\boldsymbol{X}] = (\widehat{\mathbb{E}}[X_1], \ldots, \widehat{\mathbb{E}}[X_d])$ and $E[\boldsymbol{X}] = (E[X_1], \ldots, E[X_d])$ for $E \in \mathscr{E}$.

Finally, a random variable X is called tight (under a capacity $\mathbb{V}$ satisfying (1.10)) if $\mathbb{V}(|X| \geq c) \to 0$ as $c \to \infty$. It is obvious that if $\widehat{\mathbb{E}}[|X|] < \infty$, or $\breve{\mathbb{E}}[|X|] < \infty$ or $C_{\mathbb{V}}(|X|) < \infty$, then X is tight.

2 Basic Tools

In this section, we give some results which are basic tools for establishing the law of large numbers as well as other limit theorems. The first one gives a link between the capacity and a probability measure.

Proposition 2.1   *Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear expectation space with a capacity $\mathbb{V}$ satisfying (1.10), and $\{X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. We can find a new sub-linear space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ defined on a metric space $\widetilde{\Omega} = \mathbb{R}^\infty$, with a sequence $\{\tilde{X}_n; n \geq 1\}$ of random variables and a set function $\widetilde{V}: \widetilde{\mathcal{F}} \to [0,1]$ on it satisfying the following properties, where $\widetilde{\mathcal{F}} = \sigma(\widetilde{\mathscr{H}})$.*

(a) $(X_1, X_2, \ldots, X_n) \overset{d}{=} (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n)$, $n = 1, 2, \ldots$, *i.e.*,

$$\widetilde{\mathbb{E}}[\varphi(\tilde{X}_1, \ldots, \tilde{X}_n)] = \widehat{\mathbb{E}}[\varphi(X_1, \ldots, X_n)], \quad \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n), n \geq 1,$$

whenever the sub-linear expectation in the right hand is finite. In particular, if $\{X_n; \geq 1\}$ are independent under $\widehat{\mathbb{E}}$, then $\{\tilde{X}_n; n \geq 1\}$ are independent under $\widetilde{\mathbb{E}}$.

(b) *Define*

$$\widetilde{V}(A) = \widetilde{\mathbb{V}}^{\widetilde{\mathscr{P}}}(A) = \sup_{P \in \widetilde{\mathscr{P}}} P(A), \quad A \in \widetilde{\mathcal{F}},$$

where $\widetilde{\mathscr{P}}$ is the family of all probability measures P on $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ with the property

$$P[\varphi] \leq \widetilde{\mathbb{E}}[\varphi] \ \text{ for bounded } \varphi \in \widetilde{\mathscr{H}},$$

and $\widetilde{V} \equiv 0$ if $\widetilde{\mathscr{P}}$ is empty. Then $\widetilde{V}: \widetilde{\mathcal{F}} \to [0,1]$ is a countably sub-additive and nondecreasing function, and $\widetilde{V} \leq \widetilde{\mathbb{C}}^* \leq \widetilde{\mathbb{V}}^* \leq \widetilde{\mathbb{V}}$, where $\widetilde{\mathbb{V}}$, $\widetilde{\mathbb{V}}^*$ and $\widetilde{\mathbb{C}}^*$ are defined on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ in the same way as $\widehat{\mathbb{V}}$, $\widehat{\mathbb{V}}^*$ and $\mathbb{C}^*$ on $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, respectively.

Here and in the sequel, for a probability measure P and a measurable function X, $P[X]$ is defined to be the expectation $\int X dP$.

(c) *If each X_n is tight, then $\widetilde{\mathscr{P}}$ is a weakly compact family of probability measures on the metric space $\widetilde{\Omega}$,*

$$\widetilde{\mathbb{E}}[\varphi] = \sup_{P \in \widetilde{\mathscr{P}}} P[\varphi] \ \text{ for bounded } \varphi \in \widetilde{\mathscr{H}}, \tag{2.1}$$

and $\widetilde{V}$ is a countably sub-additive capacity with the property (1.10), i.e.,

$$\widetilde{\mathbb{E}}[f] \leq \widetilde{V}(A) \leq \widetilde{\mathbb{E}}[g] \quad \text{if } 0 \leq f \leq I_A \leq g, \ f, g \in \widetilde{\mathscr{H}} \text{ and } A \in \widetilde{\mathcal{F}}. \tag{2.2}$$

(d) *If $\{X_n; \geq 1\}$ are independent under $\widehat{\mathbb{E}}$ and each X_n is tight, then for any sequence of vectors $\{\boldsymbol{\xi}_k = (X_{n_{k-1}+1}, \ldots, X_{n_k}); k \geq 1\}$ and a sequence $\{E_k; k \geq 1\}$ of finite additive linear expectations on $\mathscr{H}_b = \{f \in \mathscr{H}; f \text{ is bounded}\}$ with $E_k \leq \widehat{\mathbb{E}}$, where $1 = n_0 < n_1 < n_2 < \ldots$, there exists a probability measure Q on $\widetilde{\Omega}$ such that $\{\tilde{\boldsymbol{\xi}}_k = (\tilde{X}_{n_{k-1}+1}, \ldots, \tilde{X}_{n_k}); k \geq 1\}$ is a sequence of independent random vectors under Q,*

$$Q[\varphi(\tilde{\boldsymbol{\xi}}_k)] = E_k[\varphi(\boldsymbol{\xi}_k)] \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}), \tag{2.3}$$

$$Q[\varphi(\tilde{X}_1, \ldots, \tilde{X}_n)] \leq \widehat{\mathbb{E}}[\varphi(X_1, \ldots, X_d)] \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^n) \tag{2.4}$$

and

$$\widetilde{v}((\tilde{X}_1, \tilde{X}_2, \ldots) \in B) \leq Q((\tilde{X}_1, \tilde{X}_2, \ldots) \in B) \leq \widetilde{V}((\tilde{X}_1, \tilde{X}_2, \ldots) \in B) \tag{2.5}$$

$$\text{for all } B \in \mathscr{B}(\mathbb{R}^\infty),$$

where $\widetilde{v}(A) = 1 - \widetilde{V}(A^c)$.

Remark 2.2   When $X_1, X_2, \ldots$ are bounded random variables, then (2.3) and (2.4) hold for all $\varphi \in C_{l,\mathrm{Lip}}$. When $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are multi-dimensional random vectors, Proposition 2.1 remains true.

Proof   A special case of this lemma can be found in the proofs in Zhang [12]. We summarize the results and the proof here for the convenience of reading and the completeness of this paper. We use the key idea in Lemma 1.3.5 of Peng [5] to construct the new sub-linear expectation in the real space. Let $\widetilde{\Omega} = \mathbb{R}^\infty$, $\widetilde{\mathcal{F}} = \mathscr{B}(\mathbb{R}^\infty)$ and

$$\widetilde{\mathscr{H}} = \{\varphi(x_1, \ldots, x_n) : \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n), n \geq 1, \text{for } \boldsymbol{x} = (x_1, x_2, \ldots) \in \widetilde{\Omega}\}.$$

Define

$$\widetilde{\mathbb{E}}[\varphi] = \widehat{\mathbb{E}}[\varphi(X_1, \ldots, X_n)], \quad \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n).$$

Then $\widetilde{\mathbb{E}}$ is a sub-linear expectation on $(\widetilde{\Omega}, \widetilde{\mathscr{H}})$. Define the random variable $\tilde{X}_i$ by $\tilde{X}_i(\tilde{\omega}) = x_i$ for $\tilde{\omega} = (x_1, x_2, \ldots) \in \widetilde{\Omega}$. Then

$$\widetilde{\mathbb{E}}[\varphi(\tilde{X}_1, \ldots, \tilde{X}_n)] = \widetilde{\mathbb{E}}[\varphi] = \widehat{\mathbb{E}}[\varphi(X_1, \ldots, X_n)], \quad \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n).$$

It follows that $(\tilde{X}_1, \ldots, \tilde{X}_n) \overset{d}{=} (X_1, \ldots, X_n)$ for $n = 1, 2 \ldots$. (a) is proved, and (b) is obvious.

For (c), suppose that each X_n is tight. For the new sub-linear expectation, we also have the expression (1.8):

$$\widetilde{\mathbb{E}}[\tilde{X}] = \max_{\widetilde{\theta} \in \widetilde{\Theta}} E_{\widetilde{\theta}}[\tilde{X}] \quad \text{for } \tilde{X} \in \widetilde{\mathscr{H}} \text{ with } \widetilde{\mathbb{E}}[\tilde{X}] \text{ being finite,}$$

for a family of finite additive linear expectations $E_{\widetilde{\theta}} : \widetilde{\mathscr{H}} \to \overline{\mathbb{R}}$ indexed by $\widetilde{\theta} \in \widetilde{\Theta}$. Furthermore, for each $\tilde{X} \in \widetilde{\mathscr{H}}$, there exists $\widetilde{\theta}_{\tilde{X}} \in \widetilde{\Theta}$ such that $\widetilde{\mathbb{E}}[\tilde{X}] = E_{\widetilde{\theta}_{\tilde{X}}}[\tilde{X}]$ if $\widetilde{\mathbb{E}}[\tilde{X}]$ is finite. For each $E_{\widetilde{\theta}}$, consider the finite additive linear expectation $E_{\widetilde{\theta}}$ on $C_{b,\mathrm{Lip}}(\mathbb{R}^p)$. For any sequence $C_{b,\mathrm{Lip}}(\mathbb{R}^p) \ni \varphi_n \searrow 0$, we have $\sup_{|\boldsymbol{x}| \leq c} |\varphi_n(\boldsymbol{x})| \to 0$, and so

$$E_{\widetilde{\theta}}[\varphi_n] \leq \widehat{\mathbb{E}}[\varphi_n(X_1, \ldots, X_p)] \leq \sup_{|\boldsymbol{x}| \leq c} |\varphi_n(\boldsymbol{x})| + \sum_{j=1}^{p} \|\varphi_1\| \mathbb{V}(|X_j| > c) \to 0$$

as $n \to \infty$ and then $c \to \infty$, by the tightness of X_j, where $\|\varphi\| = \sup_{\boldsymbol{x}} |\varphi(\boldsymbol{x})|$. Then, as shown in Lemma 1.3.5 of Peng [5], by Daniell–Stone's theorem, there exists a family of probability measures $P_{\widetilde{\theta},p}$ on $(\mathbb{R}^p, \mathscr{B}(\mathbb{R}^p))$ such that

$$E_{\widetilde{\theta}}[\varphi] = P_{\widetilde{\theta},p}[\varphi] = \int \varphi(x_1, \ldots, x_p) P_{\widetilde{\theta},p}(dx_1, \ldots, dx_p), \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^p).$$

It is obvious that $\{P_{\widetilde{\theta},p}; p \geq 1\}$ is a Kolmogorov's consistency system. By Kolmogorov's existence theorem, there is a unique probability measure $P_{\widetilde{\theta}}$ on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ such that $P_{\widetilde{\theta}}|_{\mathscr{B}(\mathbb{R}^p)} = P_{\widetilde{\theta},p}$. Hence

$$P_{\widetilde{\theta}}[\varphi] = E_{\widetilde{\theta}}[\varphi] \leq \widetilde{\mathbb{E}}[\varphi], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^p).$$

Recall that $\widetilde{\mathscr{P}}$ is the family of all probability measures P on $(\mathbb{R}^\infty, \mathscr{B}(\mathbb{R}^\infty))$ with the property

$$P[\varphi] \leq \widetilde{\mathbb{E}}[\varphi], \quad \text{for bounded } \varphi \in \widetilde{\mathscr{H}}.$$

Then for any bounded $\varphi \in \widetilde{\mathscr{H}}$,

$$\widetilde{\mathbb{E}}[\varphi] = \sup_{\tilde{\theta} \in \widetilde{\Theta}} E_{\tilde{\theta}}[\varphi] = \sup_{\tilde{\theta} \in \widetilde{\Theta}} P_{\tilde{\theta}}[\varphi] \leq \sup_{P \in \widetilde{\mathscr{P}}} P[\varphi] \leq \widetilde{\mathbb{E}}[\varphi].$$

It follows that (2.1) holds and for each bounded $\varphi \in \widetilde{\mathscr{H}}$ there exists a $P \in \widetilde{\mathscr{P}}$ such that $P[\varphi] = \widetilde{\mathbb{E}}[\varphi]$.

Suppose $0 \leq f \leq I_A \leq g$, $f(\boldsymbol{x}) = f(x_1, \ldots, x_p)$, $g(\boldsymbol{x}) = g(x_1, \ldots, x_p) \in \widetilde{\mathscr{H}}$ and $A \in \widetilde{\mathcal{F}}$. Then

$$P[f] \leq P(A) \leq P[g \wedge 1].$$

By (2.1) and taking the supremum over $P \in \widetilde{\mathscr{P}}$, it follows that

$$\widehat{\mathbb{E}}[f(X_1, \ldots, X_p)] = \widetilde{\mathbb{E}}[f] \leq \widetilde{V}(A) \leq \widetilde{\mathbb{E}}[g \wedge 1] \leq \widetilde{\mathbb{E}}[g] = \widehat{\mathbb{E}}[g(X_1, \ldots, X_p)].$$

(2.2) is proved. At last, we show that $\widetilde{\mathscr{P}}$ is weakly compact. For any $\epsilon > 0$, by the tightness of X_i, there exists a constant C_i such that $\mathbb{V}(|X_i| \geq C_i) < \epsilon/2^i$. Then $\widetilde{V}(\boldsymbol{x} : |x_i| \geq 2C_i) \leq \mathbb{V}(|X_i| \geq C_i) < \epsilon/2^i$ by (1.11). Let $K = \bigotimes_{i=1}^{\infty}[-2C_i, 2C_i]$. Then K is a compact subset in the space $\mathbb{R}^\infty$ with a metric defined by $d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{\infty}(|x_i - y_i| \wedge 1)/2^i$. Notice

$$\widetilde{V}(\boldsymbol{x} \notin K) \leq \sum_{i=1}^{\infty} \widetilde{V}(\boldsymbol{x} : |x_i| \geq 2C_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i < \epsilon.$$

It follows that $\widetilde{\mathscr{P}}$ is tight and so is relatively weakly compact. Assume $\widetilde{\mathscr{P}} \ni P_n \implies P$. It is obvious that

$$P[f] = \lim_{n \to \infty} P_n[f] \leq \widehat{\mathbb{E}}[f] \quad \text{for bounded } f \in \widetilde{\mathscr{H}}.$$

Hence $P \in \widetilde{\mathscr{P}}$. It follows that $\widetilde{\mathscr{P}}$ is closed and so is weakly compact. (c) is proved.

Now, we show (d). Consider the linear operator $\widetilde{E}_k$ on $C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}})$ defined by

$$\widetilde{E}_k[\varphi] = E_k[\varphi(\boldsymbol{\xi}_k)], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}).$$

Then

$$\widetilde{E}_k[\varphi] \leq \widehat{\mathbb{E}}[\varphi(\boldsymbol{\xi}_k)], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}).$$

If $C_{l,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}) \ni \varphi_n \searrow 0$, then $\sup_{|\boldsymbol{x}| \leq c} |\varphi_n(\boldsymbol{x})| \to 0$ and

$$\widetilde{E}_k[\varphi_n] \leq \widehat{\mathbb{E}}[\varphi_n(\boldsymbol{\xi}_k)] \leq \sup_{|\boldsymbol{x}| \leq c} |\varphi_n(\boldsymbol{x})| + \|\varphi_1\| \mathbb{V}(|\boldsymbol{\xi}_k| > c) \to 0$$

as $n \to \infty$ and then $c \to \infty$, where $\|\varphi\| = \sup_{\boldsymbol{x}} |\varphi(\boldsymbol{x})|$. By Daniell–Stone's theorem again, there exists a probability measure Q_k on $\mathbb{R}^{n_k - n_{k-1}}$ such that

$$Q_k[\varphi] = \widetilde{E}_k[\varphi] \leq \widehat{\mathbb{E}}[\varphi(\boldsymbol{\xi}_k)], \quad \forall \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}).$$

Now, we introduce a product probability measure on $\mathbb{R}^\infty$ defined by

$$Q = Q_1|_{\mathbb{R}^{n_1}} \times Q_2|_{\mathbb{R}^{n_2 - n_1}} \times \cdots.$$

Then, under the probability measure Q, for any $A_i \in \mathscr{B}(\mathbb{R}^{n_i - n_{i-1}})$, $i = 1, \ldots, d$, $d \geq 1$,

$$Q(\{\boldsymbol{x}; \boldsymbol{z}_1 \in A_1, \ldots, \boldsymbol{z}_d \in A_d\}) = Q_1(A_1) \cdots Q_d(A_d) = Q(\{\boldsymbol{x}; \boldsymbol{z}_1 \in A_1\}) \cdots Q(\{\boldsymbol{x}; \boldsymbol{z}_d \in A_d\}),$$

where $\boldsymbol{z}_i = (x_{n_{i-1}+1}, \ldots, x_{n_i})$. That is

$$Q(\tilde{\boldsymbol{\xi}}_1 \in A_1, \ldots, \tilde{\boldsymbol{\xi}}_d \in A_d) = Q(\tilde{\boldsymbol{\xi}}_1 \in A_1) \cdots Q(\tilde{\boldsymbol{\xi}}_d \in A_d).$$

So, $\tilde{\boldsymbol{\xi}}_1, \tilde{\boldsymbol{\xi}}_2, \ldots$ is a sequence of independent random variables under Q. Furthermore,

$$Q[\varphi(\tilde{\boldsymbol{\xi}}_k)] = Q_k[\varphi] = \widetilde{E}_k[\varphi] = E_k[\varphi(\boldsymbol{\xi}_k)] \le \widehat{\mathbb{E}}[\varphi(\boldsymbol{\xi}_k)], \quad \forall \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}}). \tag{2.6}$$

(2.3) is proved.

Notice that for every $\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_d) \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_d})$, where $\boldsymbol{z}_i = (x_{n_{i-1}+1}, \ldots, x_{n_i})$,

$$Q[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1}, \tilde{\boldsymbol{\xi}}_d)] = Q_d[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1}, \cdot)] \le \widehat{\mathbb{E}}[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1}, \boldsymbol{\xi}_d)]$$

by (2.6). Write the functions of $(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1})$ in the left hand and right hand by $\varphi_1(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1})$ and $\varphi_2(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-1})$, respectively. Notice that $\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_d$ are independent under both Q and $\widetilde{\mathbb{E}}$, and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_d$ are independent under $\widehat{\mathbb{E}}$. We have that

$$\begin{aligned} Q[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \tilde{\boldsymbol{\xi}}_{d-1}, \tilde{\boldsymbol{\xi}}_d)] &= Q[\varphi_1(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \tilde{\boldsymbol{\xi}}_{d-1})] \\ &\le Q[\varphi_2(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \tilde{\boldsymbol{\xi}}_{d-1})] \le \widehat{\mathbb{E}}[\varphi_2(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \boldsymbol{\xi}_{d-1})] \\ &= \widehat{\mathbb{E}}[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \boldsymbol{\xi}_{d-1}, \boldsymbol{\xi}_d)] = \widehat{\mathbb{E}}[\varphi(\boldsymbol{z}_1, \ldots, \boldsymbol{z}_{d-2}, \tilde{\boldsymbol{\xi}}_{d-1}, \tilde{\boldsymbol{\xi}}_d)], \end{aligned}$$

by (2.6) again. By induction, we conclude that

$$Q[\varphi(\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_d)] \le \widetilde{\mathbb{E}}[\varphi(\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_d)], \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n_d}), \ d \ge 1.$$

Now, for each $\varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^n)$, $\varphi \circ \pi_{n_d \to n}$ is also a function in $C_{b,\mathrm{Lip}}(\mathbb{R}^{n_d})$ when $n \le n_d$, where $\pi_{n_d \to n} : \mathbb{R}^{n_d} \to \mathbb{R}^n$ is the projection map. It follows that

$$\begin{aligned} Q[\varphi] = Q[\varphi(\tilde{X}_1, \ldots, \tilde{X}_n)] &= Q[\varphi \circ \pi_{n_d \to n}(\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_d)] \\ &\le \widetilde{\mathbb{E}}[\varphi \circ \pi_{n_d \to n}(\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_d)] = \widetilde{\mathbb{E}}[\varphi(\tilde{X}_1, \ldots, \tilde{X}_n)] \\ &= \widehat{\mathbb{E}}[\varphi(X_1, \ldots, X_n)] = \widetilde{\mathbb{E}}[\varphi]. \end{aligned}$$

That is, $Q[\varphi] \le \widetilde{\mathbb{E}}[\varphi]$ for all bounded $\varphi \in \widetilde{\mathscr{H}}$. Hence, $Q \in \widetilde{\mathscr{P}}$ and (2.4) holds. So, for each $B \in \mathscr{B}(\mathbb{R}^\infty)$,

$$Q((\tilde{X}_1, \tilde{X}_2, \ldots) \in B) = Q(B) \le \widetilde{V}(B) = \widetilde{V}((\tilde{X}_1, \tilde{X}_2, \ldots) \in B),$$

by the definition of $\widetilde{V}$. The right hand of (2.5) is proved. The left hand is obvious by noting $Q(B) = 1 - Q(B^c)$ and $\widetilde{v}(B) = 1 - \widetilde{V}(B^c)$. The proof is completed. $\qquad\square$

The next lemma is the Borel–Cantelli lemma for a countably sub-additive capacity.

Lemma 2.3 *Let V be a countably sub-additive capacity and $\sum_{n=1}^\infty V(A_n) < \infty$. Then*

$$V(A_n \ i.o.) = 0, \quad where \ \{A_n \ i.o.\} = \bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i.$$

Proof Easy and omitted. $\qquad\square$

The following lemma is the converse part of Borel–Cantelli lemma under $\widetilde{\mathbb{V}}^*$ or $\widetilde{V}$.

Lemma 2.4 *Let $\{X_n; n \ge 1\}$ be a sequence of independent random variables in the sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ for which each X_n is tight, $\{\tilde{X}_n; n \ge 1\}$ be its copy on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ as defined in Proposition 2.1. Suppose $\boldsymbol{\xi}_k = (X_{n_{k-1}+1}, \ldots, X_{n_k})$, $1 = n_0 < n_1 < \ldots$, $f_{k,j} \in C_{l,\mathrm{Lip}}(\mathbb{R}^{n_k - n_{k-1}})$ and $\sum_{k=1}^\infty \mathbb{V}(f_{k,j}(\boldsymbol{\xi}_k) \ge 1 + \epsilon_{k,j}) = \infty$, $j = 1, 2, \ldots$, where $\epsilon_{k,j} > 0$, $k, j = 1, 2, \ldots$. Then on the space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$,*

$$\widetilde{\mathbb{V}}(A) = \widetilde{\mathbb{V}}^*(A) = \widetilde{V}(A) = 1, \quad A = \bigcap_{j=1}^\infty \{f_{k,j}(\tilde{\boldsymbol{\xi}}_k) \ge 1 \ i.o.\},$$

where $\tilde{\boldsymbol{\xi}}_k = (\tilde{X}_{n_{k-1}+1}, \ldots, \tilde{X}_{n_k})$.

Proof   Let $g_{k,j} \in C_{b,\mathrm{Lip}}(\mathbb{R})$ such that $I\{x \geq 1\} \geq g_{k,j}(x) \geq I\{x \geq 1 + \epsilon_{k,j}\}$. Then

$$\sum_{k=1}^{\infty} \widehat{\mathbb{E}}[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))] = \infty, \quad j = 1, 2, \ldots.$$

By the expression (1.8), for each pair of k and j there exists $\theta_{k,j} \in \Theta$ such that

$$E_{\theta_{k,j}}[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))] = \widehat{\mathbb{E}}[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))].$$

Define the linear operator E_k by

$$E_k = \sum_{j=1}^{\infty} 2^{-j} E_{\theta_{k,j}}.$$

Then $E_k \leq \widehat{\mathbb{E}}$. By Proportion 2.1 (d), there exists a probability measure Q on $\widetilde{\Omega}$ such that $\{\boldsymbol{\xi}_k; k \geq 1\}$ is a sequence of independent random variables under Q, and (2.3)–(2.5) hold. By (2.3),

$$\sum_{k=1}^{\infty} Q(f_{k,j}(\tilde{\boldsymbol{\xi}}_k) \geq 1) \geq \sum_{k=1}^{\infty} Q[g_{k,j}(f_{k,j}(\tilde{\boldsymbol{\xi}}_k))] = \sum_{k=1}^{\infty} E_k[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))]$$

$$\geq \frac{1}{2^j} \sum_{k=1}^{\infty} E_{\theta_{k,j}}[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))] = \frac{1}{2^j} \sum_{k=1}^{\infty} \widehat{\mathbb{E}}[g_{k,j}(f_{k,j}(\boldsymbol{\xi}_k))] = \infty.$$

So, by the Borel–Cantelli lemma for a probability measure,

$$Q(f_{k,j}(\boldsymbol{\xi}_k) \geq 1 \ i.o.) = 1.$$

It follows that

$$Q\bigg(\bigcap_{j=1}^{\infty} \{f_{k,j}(\tilde{\boldsymbol{\xi}}_k) \geq 1 \ i.o.\} \bigg) = 1.$$

By (2.5), it follows that

$$\widetilde{\mathbb{V}}(A) \geq \widetilde{\mathbb{V}}^*(A) \geq \widetilde{V}(A) = 1.$$

The proof is now completed. $\square$

The next lemma tells us that the converse part of the Borel–Cantelli lemma remains valid in the original sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ under certain conditions.

Lemma 2.5   *Let* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ *be a sub-linear expectation space with a capacity* V *having the property (1.10), and* $v(A) = 1 - V(A^c)$. *Suppose that one of the following conditions is satisfied.*

(a) *The sub-linear expectation* $\widehat{\mathbb{E}}$ *satisfies*

$$\widehat{\mathbb{E}}[X] = \max_{P \in \mathscr{P}} P[X], \quad X \in \mathscr{H}_b,$$

where $\mathscr{H}_b = \{f \in \mathscr{H}; f \text{ is bounded}\}$, $\mathscr{P}$ *is a countable-dimensionally weakly compact family of probability measures on* $(\Omega, \sigma(\mathscr{H}))$ *in sense that, for any* $Y_1, Y_2, \ldots \in \mathscr{H}_b$ *and any sequence* $\{P_n\} \subset \mathscr{P}$ *there is a subsequence* $\{n_k\}$ *and a probability measure* $P \in \mathscr{P}$ *for which*

$$\lim_{k \to \infty} P_{n_k}[\varphi(Y_1, \ldots, Y_d)] = P[\varphi(Y_1, \ldots, Y_d)], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), d \geq 1. \tag{2.7}$$

(b) $\widehat{\mathbb{E}}$ on $\mathscr{H}_b$ is regular in sense that $\widehat{\mathbb{E}}[X_n] \downarrow 0$ for any elements $\mathscr{H}_b \ni X_n \downarrow 0$. Let $\mathscr{P}$ be the family of all probability measures on $(\Omega, \sigma(\mathscr{H}))$ for which

$$P[f] \le \widehat{\mathbb{E}}[f], \quad f \in \mathscr{H}_b.$$

(c) Ω is a complete separable metric space, each element $X(\omega)$ in $\mathscr{H}$ is a continuous function on Ω. The capacity V with the property (1.10) is tight in sense that, for any $\epsilon > 0$ there is a compact set $K \subset \Omega$ such that $V(K^c) < \epsilon$. Let $\mathscr{P}$ be defined as in (b).

(d) Ω is a complete separable metric space, each element $X(\omega)$ in $\mathscr{H}$ is a continuous function on Ω. The sub-linear expectation $\widehat{\mathbb{E}}$ is defined by

$$\widehat{\mathbb{E}}[X] = \max_{P \in \mathscr{P}} P[X],$$

where $\mathscr{P}$ is a weakly compact family of probability measures on $(\Omega, \mathscr{B}(\Omega))$.

Denote $\mathbb{V}^{\mathscr{P}}(A) = \max_{P \in \mathscr{P}} P(A)$, $A \in \sigma(\mathscr{H})$. Let $\{X_n; n \ge 1\}$ be a sequence of independent random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$.

(i) If $\sum_{n=1}^{\infty} v(X_n < 1) < \infty$, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathbb{V}\left(\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} \{X_i \ge 1\} \right) = 1, \quad i.e., \ \mathcal{V}(X_i < 1 \ i.o.) = 0. \tag{2.8}$$

(ii) If $\sum_{n=1}^{\infty} V(X_n \ge 1) = \infty$, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathbb{V}(X_n \ge 1 \ i.o.) = 1. \tag{2.9}$$

More generally, suppose that $\{\boldsymbol{X}_n; n \ge 1\}$ is a sequence of independent random vectors in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, where $\boldsymbol{X}_n$ is d_n-dimensional, $f_{n,j} \in C_{l,lip}(\mathbb{R}^{d_n})$ and $\sum_{n=1}^{\infty} V(f_{n,j}(\boldsymbol{X}_n) \ge 1) = \infty$, $j = 1, 2, \ldots$, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathbb{V}\left(\bigcap_{j=1}^{\infty} \{f_{n,j}(\boldsymbol{X}_n) \ge 1 \ i.o.\} \right) = 1. \tag{2.10}$$

(iii) Suppose that $\{\boldsymbol{X}_n; n \ge 1\}$ is a sequence of independent random vectors in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, where $\boldsymbol{X}_n$ is d_n-dimensional. If F_n is a d_n-dimensional closed set with $\sum_{n=1}^{\infty} v(\boldsymbol{X}_n \notin F_n) < \infty$, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathcal{V}(\boldsymbol{X}_n \notin F_n \ i.o.) = 0;$$

If $F_{n,j}s$ are d_n-dimensional closed sets with $\sum_{n=1}^{\infty} V(\boldsymbol{X}_n \in F_{n,j}) = \infty$, $j = 1, 2, \ldots$, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathbb{V}\left(\bigcap_{j=1}^{\infty} \{\boldsymbol{X}_n \in F_{n,j} \ i.o.\} \right) = 1.$$

Proof (i) and (ii) are special cases of (iii). But, to prove the general case (iii), we need to show the two special cases first. Without loss of generality, we can assume $0 \le X_n \le 2$, for otherwise, we can replace it by $0 \vee X_n \wedge 2$. Write $\boldsymbol{X} = (X_1, X_2, \ldots)$. Suppose that (a) is satisfied. Consider the family $\mathscr{P}$ on $\sigma(\boldsymbol{X})$. Notice $|X_n| \le 2$, $n = 1, 2, \ldots$, and the set $K = \bigotimes_{i=1}^{\infty}[-2, 2]$ is a compact set on $\mathbb{R}^{\infty}$. So, $\mathscr{P}\boldsymbol{X}^{-1} =: \{\overline{P} : \overline{P}(A) = P(\boldsymbol{X} \in A), A \in \mathscr{B}(\mathbb{R}^{\infty}), P \in \mathscr{P}\}$ is a tight and so a relatively weakly compact family of probability measures on the metric space $\mathbb{R}^{\infty}$.

Next, we show that $\mathscr{P}\boldsymbol{X}^{-1}$ is closed. Suppose that $P_n\boldsymbol{X}^{-1} \in \mathscr{P}\boldsymbol{X}^{-1}$ is weakly convergent sequence. Then there exists a probability Q on $\mathbb{R}^\infty$ such that $\overline{P}_n \implies Q$, i.e.,

$$Q[f] = \lim_{n \to \infty} P_n[f(\boldsymbol{X})], \quad f \in C_b(\mathbb{R}^\infty). \tag{2.11}$$

It is needed to show that there exists a probability measure $P \in \mathscr{P}$ satisfying $Q(A) = P(\boldsymbol{X} \in A)$ for $A \in \mathscr{B}(\mathbb{R}^\infty)$. By the conditions assumed, for the sequence $\{P_n\}$ there exists a subsequence $\{n_k\}$ and a probability measure $P \in \mathscr{P}$ such that (2.7) holds. Hence

$$Q[f] = P[f(X_1, \ldots, X_d)], \quad \forall f \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), d \geq 1.$$

So, $Q(\{\boldsymbol{x} : (x_1, \ldots, x_d) \in A\}) = P((X_1, \ldots, X_d) \in A)$ for all $A \in \mathscr{B}(\mathbb{R}^d)$, which implies $Q(A) = P(\boldsymbol{X} \in A)$ for all $A \in \mathscr{B}(\mathbb{R}^\infty)$. We conclude that $\mathscr{P}\boldsymbol{X}^{-1}$ is closed and so weakly compact. Denote $\widetilde{V}(A) = \mathbb{V}^{\mathscr{P}}(\boldsymbol{X} \in A)$. By Lemma 6.1.12 of Peng [5], for any sequence of closed sets $F_n \downarrow F$, we have $\widetilde{V}(F_n) \downarrow \widetilde{V}(F)$.

Now, we consider (i). By the independence, we have for any $\delta_i > 0$, and $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ or $\widehat{\mathbb{V}}$,

$$\mathbb{V}\bigg(\bigcap_{i=m}^{n} \{X_i \geq 1 - \delta_i\}\bigg) \geq \prod_{i=m}^{n} V(X_i \geq 1).$$

In fact, we can choose a Lipschitz function f_i such that $I\{x \geq 1 - \delta_i\} \geq f_i(x) \geq I\{x \geq 1\}$. Then

$$\mathbb{V}\bigg(\bigcap_{i=m}^{n} \{X_i \geq 1 - \delta_i\}\bigg) \geq \widehat{\mathbb{E}}\bigg[\prod_{i=m}^{n} f_i(X_i)\bigg] = \prod_{i=m}^{n} \widehat{\mathbb{E}}[f_i(X_i)] \geq \prod_{i=m}^{n} V(X_i \geq 1).$$

Let $\epsilon_i = v(X_i < 1)$ and choose $\delta_i = 1/l$. Then

$$\mathbb{V}^{\mathscr{P}}\bigg(\bigcap_{i=m}^{n} \{X_i \geq 1 - 1/l\}\bigg) \geq \prod_{i=m}^{\infty} V(X_i \geq 1) = \prod_{i=m}^{\infty} (1 - \epsilon_i).$$

Notice that $\{\boldsymbol{x} : \bigcap_{i=m}^{n}\{x_i \geq 1 - \delta_i\}\}$ is a closed set of $\boldsymbol{x}$ on $\mathbb{R}^\infty$. It follows that

$$\mathbb{V}^{\mathscr{P}}\bigg(\bigcap_{i=m}^{n} \{X_i \geq 1 - 1/l\}\bigg) \searrow_l \mathbb{V}^{\mathscr{P}}\bigg(\bigcap_{i=m}^{n} \{X_i \geq 1\}\bigg) \searrow_n \mathbb{V}^{\mathscr{P}}\bigg(\bigcap_{i=m}^{\infty} \{X_i \geq 1\}\bigg).$$

It follows that

$$\mathbb{V}^{\mathscr{P}}\bigg(\bigcap_{i=m}^{\infty} \{X_i \geq 1\}\bigg) \geq \prod_{i=m}^{\infty} V(X_i \geq 1) = \prod_{i=m}^{\infty} (1 - \epsilon_i) \to 1, \quad m \to \infty$$

due the fact that $\sum_{i=1}^{\infty} \epsilon_i < \infty$. Hence (2.8) is proved.

Consider (ii). Write $\epsilon_i = V(X_i \geq 1)$. Now, for $\mathcal{V} = 1 - \mathbb{V}^{\mathscr{P}}$, $1 - \mathbb{C}^*$, $1 - \widehat{\mathbb{V}}^*$ or $1 - \widehat{\mathbb{V}}$, we have

$$\mathcal{V}\bigg(\bigcap_{i=m}^{n} \{X_i < 1 - 1/l\}\bigg) \leq \widehat{\mathcal{E}}\bigg[\prod_{i=m}^{n} (1 - f_i(X_i))\bigg] = \prod_{i=m}^{n} \widehat{\mathcal{E}}[1 - f_i(X_i)] \leq \prod_{i=m}^{n} v(X_i < 1).$$

That is

$$\mathbb{V}\bigg(\bigcup_{i=m}^{n} \{X_i \geq 1 - 1/l\}\bigg) \geq 1 - \prod_{i=m}^{n} (1 - V(X_i \geq 1)) \geq 1 - \exp\bigg\{-\sum_{i=m}^{n} \epsilon_i\bigg\}.$$

Notice that $\bigcup_{i=m}^{n} \{x_i \geq 1 - 1/l\}$ is a closed set of $\boldsymbol{x}$. It follows that

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=m}^{n} \{X_i \geq 1 - 1/l\} \right) \searrow \mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=m}^{n} \{X_i \geq 1\} \right) \quad \text{as } l \to \infty.$$

Hence for each m,

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=m}^{n} \{X_i \geq 1\} \right) \geq 1 - \exp\left\{ -\sum_{i=m}^{n} \epsilon_i \right\} \to 1 \quad \text{as } n \to \infty, \tag{2.12}$$

due to the fact that $\sum_{i=1}^{\infty} \epsilon_i = \infty$. Let $\delta_k = 2^{-k}$. We can choose a sequence $n_k \nearrow \infty$ such that

$$\mathbb{V}^{\mathscr{P}}\left(\max_{n_k+1 \leq i \leq n_{k+1}} X_i \geq 1 \right) = \mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=n_k+1}^{n_{k+1}} \{X_i \geq 1\} \right) \geq 1 - \delta_k.$$

Let $Z_k = \max_{n_k+1 \leq i \leq n_{k+1}} X_i$. Then $\{Z_k; k \geq 1\}$ are independent under $\widehat{\mathbb{E}}$. By (i),

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \{Z_k \geq 1\} \right) = 1.$$

Notice $\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \{Z_k \geq 1\} \subset \{X_n \geq 1 \ i.o.\}$. (2.9) holds.

Now, we consider the general case. Without loss of generality, assume $0 \leq f_{n,j}(\boldsymbol{X}_n) \leq 2$. Similar to (2.12), for each m and j we have

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=m}^{n} \{f_{i,j}(\boldsymbol{X}_i) \geq 1\} \right) \geq 1 - \exp\left\{ -\sum_{i=m}^{n} V(f_{i,j}(\boldsymbol{X}_i) \geq 1) \right\} \to 1, \quad n \to \infty.$$

Let $\delta_k = 2^{-k}$. We choose the sequence $1 = n_{0,0} < n_{1,1} < n_{2,1} < n_{2,2} < \cdots < n_{k,1} < \cdots < n_{k,k} < n_{k+1,1} < \cdots$ such that

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{i=n_{k,j-1}+1}^{n_{k,j}} \{f_{i,j}(\boldsymbol{X}_i) \geq 1\} \right) \geq 1 - \delta_{k+j}, \quad j \leq k, k \geq 1,$$

where $n_{k,0} = n_{k-1,k-1}$. Let $Z_{k,j} = \max_{n_{k,j-1}+1 \leq i \leq n_{k,j}} f_{i,j}(\boldsymbol{X}_i)$. Then the random variables $Z_{1,1}, Z_{2,1}, Z_{2,2}, \ldots, Z_{k,1}, \ldots, Z_{k,k}, Z_{k+1,1}, \ldots$ are independent under $\widehat{\mathbb{E}}$ with

$$\mathcal{V}^{\mathscr{P}}(Z_{k,j} < 1) < \delta_{k+j}.$$

Notice $\sum_{k=1}^{\infty} \sum_{j=1}^{k} \delta_{k+j} < \infty$. By (i), we have

$$\mathbb{V}^{\mathscr{P}}\left(\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \bigcap_{j=1}^{k} \{Z_{k,j} \geq 1\} \right) = 1.$$

On the event $\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \bigcap_{j=1}^{k} \{Z_{k,j} \geq 1\}$, there exists an l_0 such that $Z_{k,j} \geq 1$ for all $k \geq l_0$ and $1 \leq j \leq k$. For each fixed j, when $k \geq j \vee l_0$ we have $Z_{k,j} \geq 1$, and hence $\{f_{n,j}(\boldsymbol{X}_n) \geq 1 \ i.o\}$ occurs. It follows that

$$\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} \bigcap_{j=1}^{k} \{Z_{k,j} \geq 1\} \subset \bigcap_{j=1}^{\infty} \{f_{n,j}(\boldsymbol{X}_n) \geq 1 \ i.o.\}.$$

(2.10) holds.

(iii) Denote $d(\boldsymbol{x}, F) = \inf\{\|\boldsymbol{y} - \boldsymbol{x}\| : \boldsymbol{y} \in F\}$. Then $d(\boldsymbol{x}, F)$ is a Lipschitz function of $\boldsymbol{x}$. If $F_{n,j}$ is a closed set, then

$$\boldsymbol{X}_n \in F_{n,j} \iff d(\boldsymbol{X}_n, F_{n,j}) = 0 \iff f_{n,j}(\boldsymbol{X}_n) := 1 - 1 \wedge d(\boldsymbol{X}_n, F_{n,j}) \geq 1.$$

The results follow from (i) and (ii) immediately.

When the condition (b) is satisfied, it is sufficient to show that the family $\mathscr{P}$ satisfies the assumption in (a). Notice the expression (1.8). Consider the linear expectation E_θ on $\mathscr{H}_b$. If $\mathscr{H}_b \ni f_n \downarrow 0$, then $0 \leq E_\theta[f_n] \leq \widehat{\mathbb{E}}[f_n] \to 0$. Hence, similar to Lemma 1.3.5 and Lemma 6.2.2 of Peng [5], by Daniell–Stone's theorem, there is a unique probability P_θ on $\sigma(\mathscr{H}_b) = \sigma(\mathscr{H})$ such that

$$P_\theta[f] = E_\theta[f] \leq \widehat{\mathbb{E}}[f], \quad f \in \mathscr{H}_b.$$

Hence

$$\widehat{\mathbb{E}}[f] = \sup_{\theta \in \Theta} E_\theta[f] = \sup_{\theta \in \Theta} P_\theta[f], \quad f \in \mathscr{H}_b.$$

Recall that $\mathscr{P}$ is the family of all probability measures P on $\sigma(\mathscr{H})$ which satisfies $P[f] \leq \widehat{\mathbb{E}}[f]$ for all $f \in \mathscr{H}_b$. We have

$$\widehat{\mathbb{E}}[f] = \sup_{\theta \in \Theta} P_\theta[f] \leq \sup_{P \in \mathscr{P}} P[f] \leq \widehat{\mathbb{E}}[f], \quad f \in \mathscr{H}_b.$$

Suppose $Y_1, Y_2, \ldots \in \mathscr{H}_b$ with $|Y_i| \leq C_i$. Write $\boldsymbol{Y} = (Y_1, Y_2, \ldots)$ and $K = \bigotimes_{i=1}^\infty [-C_i, C_i]$. Then $P(\boldsymbol{Y} \in K^c) = 0$ and K is a compact set on the space $\mathbb{R}^\infty$. It follows that $\mathscr{P}\boldsymbol{Y}^{-1}$ is tight and so is relatively weakly compact family of probability measures on the metric space $\mathbb{R}^\infty$. Hence, for any sequence $\{P_n\} \subset \mathscr{P}$, there exists a subsequence $n_k \nearrow \infty$ such that

$$E[f(\boldsymbol{Y})] = \lim_{k \to \infty} P_{n_k}[f(\boldsymbol{Y})], \quad f \in C_b(\mathbb{R}^\infty)$$

is well-defined. It is obvious that E is a linear expectation on $\{\varphi(\boldsymbol{Y}) : \varphi \in C_b(\mathbb{R}^\infty)\}$. Consider E on $\mathscr{L} = \{\varphi(Y_1, \ldots, Y_d) : \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), d \geq 1\}$. It is obvious that

$$E[\varphi(Y_1, \ldots, Y_d)] = \lim_{k \to \infty} P_{n_k}[\varphi(Y_1, \ldots, Y_d)] \leq \widehat{\mathbb{E}}[\varphi(Y_1, \ldots, Y_d)], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d).$$

So, by the Hahn–Banach theorem, there exists a finite additive linear expectation E^e defined on $\mathscr{H}$ such that, $E^e = E$ on $\mathscr{L}$ and, $E^e \leq \widehat{\mathbb{E}}$ on $\mathscr{H}$. For E^e, by the regularity, as shown before there is probability measure P^e on $\sigma(\mathscr{H})$ such that $P^e[f] = E^e[f]$ for all $f \in \mathscr{H}_b \supset \mathscr{L}$. Hence $P^e \in \mathscr{P}$ and

$$\lim_{k \to \infty} P_{n_k}[\varphi(Y_1, \ldots, Y_d)] = E[\varphi(Y_1, \ldots, Y_d)] = P^e[\varphi(Y_1, \ldots, Y_d)], \quad \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), d \geq 1.$$

It follows that $\mathscr{P}$ satisfies the assumption in (a).

For (c), it can be shown that $\widehat{\mathbb{E}}$ is regular on $\mathscr{H}_b$ and so the condition (b) is satisfied. In fact, suppose that $\mathscr{H}_b \ni f_n \downarrow 0$, $f_n \leq M$, and K is a compact set. Then

$$\delta_n =: \sup_{\omega \in K} f_n(\omega) \downarrow 0 \text{ and } 0 \leq \widehat{\mathbb{E}}[f_n] \leq \delta_n + MV(K^c).$$

$\widehat{\mathbb{E}}[f_n] \downarrow 0$ follows from the tightness of V. Finally, (d) is a special case of (a). The proof is completed. $\qquad \square$

Remark 2.6 The condition (d) is popular in the study of sub-linear expectations, c.f. Peng [5]. The condition (a) is an analogue of (d). Since the weak compactness can be only defined for probability measures on a metric space, we assume the condition (a) in the general measurable space.

Actually, the condition (a) implies that $\widehat{\mathbb{E}}$ is regular on $\mathscr{H}_b$. For showing this fact, suppose $\mathscr{H}_b \ni Y_n \searrow 0$. For each n, there exists $P_n \in \mathscr{P}$ such that $\widehat{\mathbb{E}}[Y_n] \geq P_n[Y_n] \geq \widehat{\mathbb{E}}[Y_n] - 1/n^2$. For the sequence $\{P_n\}$ and $Y_1, Y_2, \ldots$, by the condition assumed, there exist a subsequence $\{P_{n_k}\}$ and a probability measure $P \in \mathscr{P}$ such that (2.7) holds. It follows that

$$0 \leq \limsup_{k \to \infty} P_{n_k}[Y_{n_k}] \leq \lim_{k \to \infty} P_{n_k}[Y_m] = P[Y_m], \quad \forall m \geq 1.$$

Notice $P[Y_m] \searrow 0$ as $m \to \infty$ by the continuity of P. Hence, $\lim_{k \to \infty} \widehat{\mathbb{E}}[Y_{n_k}] = \lim_{k \to \infty} P_{n_k}[Y_{n_k}] = 0$, which implies $\widehat{\mathbb{E}}[Y_n] \to 0$ by the monotonicity of $\widehat{\mathbb{E}}[Y_n]$.

Though the conditions (a) and (b) are equivalent, the families $\mathscr{P}$ may be different. The capacities $\mathbb{C}^*$, $\widehat{\mathbb{V}}^*$ and $\widehat{\mathbb{V}}$ do not depend on the choice of $\mathscr{P}$, but $\mathbb{V}^{\mathscr{P}}$ does.

The rest three lemmas give the estimators of the tail capacities of maximum partial sums of independent random variables. Lemma 2.7 below is a kind of Kolmogorov's maximal inequality under $\widehat{\mathcal{V}}$.

Lemma 2.7 *Let $\{\boldsymbol{Z}_{n,k}; k = 1, \ldots, k_n\}$ be an array of independent random vectors taking values in $\mathbb{R}^d$ such that $\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] < \infty$, $k = 1, \ldots, k_n$, here $|\cdot|$ is the Euclidean norm. Then for any $\boldsymbol{\mu}_{n,k} \in \widetilde{\mathbb{M}}[\boldsymbol{Z}_{n,k}] =: \{E[\boldsymbol{Z}_{n,k}] : E \in \mathscr{E}\}$ where $\mathscr{E}$ is defined as (1.17), $k = 1, \ldots, k_n$,*

$$\widehat{\mathcal{V}}\left(\max_{m \leq k_n} \left|\sum_{k=1}^m (\boldsymbol{Z}_{n,k} - \boldsymbol{\mu}_{n,k})\right| \geq x\right) \leq 2x^{-2} \sum_{k=1}^{k_n} (\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] - |\boldsymbol{\mu}_{n,k}|^2), \quad \forall x > 0.$$

Proof For each k there exists $E_k \in \mathscr{E}$ such that $\boldsymbol{\mu}_{n,k} = E_k[\boldsymbol{Z}_{n,k}]$. E_k is a finite additive linear expectation on $\mathscr{H}_b = \{f \in \mathscr{H}; f \text{ is bounded}\}$ with $E \leq \breve{E} = \widehat{\mathbb{E}}$. Notice that each $\boldsymbol{Z}_{n,k}$ is tight by the fact $\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] < \infty$. By Proposition 2.1, $\{\boldsymbol{Z}_{n,k}; k = 1, \ldots, k_n\}$ has a copy $\{\tilde{\boldsymbol{Z}}_{n,k}; k = 1, \ldots, k_n\}$ on a new sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ with a probability measure Q on $\widetilde{\Omega}$ such that $\{\tilde{\boldsymbol{Z}}_{n,1}, \ldots, \tilde{\boldsymbol{Z}}_{n,k_n}\}$ are independent random vectors under Q,

$$Q[\varphi(\tilde{\boldsymbol{Z}}_{n,k})] = E_k[\varphi(\boldsymbol{Z}_{n,k})] \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), \tag{2.13}$$

$$Q[\varphi(\tilde{\boldsymbol{Z}}_{n,1}, \ldots, \tilde{\boldsymbol{Z}}_{n,k_n})] \leq \widehat{\mathbb{E}}[\varphi(\boldsymbol{Z}_{n,1}, \ldots, \boldsymbol{Z}_{n,k_n})] \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{d \times k_n}) \tag{2.14}$$

and

$$\tilde{v}(B) \leq Q(B) \leq \tilde{V}(B) \quad \text{for all } B \in \sigma(\tilde{\boldsymbol{Z}}_{n,1}, \ldots, \tilde{\boldsymbol{Z}}_{n,k_n}). \tag{2.15}$$

Notice $E_k|Z_{n,k,i} - (-c) \vee Z_{n,k,i} \wedge c| \leq \widehat{\mathbb{E}}[(|Z_{n,k,i}| - c)^+] \to 0$ as $c \to \infty$ by $\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] < \infty$. Then

$$Q[\tilde{Z}_{n,k,i}] = \lim_{c \to \infty} Q[(-c) \vee \tilde{Z}_{n,k,i} \wedge c] = \lim_{c \to \infty} E_k[(-c) \vee Z_{n,k,i} \wedge c] = E_k[Z_{n,k,i}] = \mu_{n,k,i}$$

by (2.13), and

$$Q[|\tilde{\boldsymbol{Z}}_{n,k}|^2] = \lim_{c \to \infty} Q[|\tilde{\boldsymbol{Z}}_{n,k}|^2 \wedge c] \leq \lim_{c \to \infty} \widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2 \wedge c] \leq \widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2].$$

by (2.14). Let $Y = \max_{m \leq k_n} |\sum_{k=1}^m (\tilde{\boldsymbol{Z}}_{n,k} - \boldsymbol{\mu}_{n,k})|$. By (2.15) and the Kolmogorov inequality for independent random variables in a probability space, we have

$$\widetilde{\mathcal{V}}(Y \geq x) \leq \tilde{v}(Y \geq x) \leq Q(Y \geq x) \leq 2x^{-2} \sum_{k=1}^{k_n} Q[|\tilde{\boldsymbol{Z}}_{n,k} - Q[\tilde{\boldsymbol{Z}}_{n,k}]|^2]$$

$$= 2x^{-2} \sum_{k=1}^{k_n} (Q[|\tilde{\boldsymbol{Z}}_{n,k}|^2] - |Q[\tilde{\boldsymbol{Z}}_{n,k}]|^2) \leq 2x^{-2} \sum_{k=1}^{k_n} (\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] - |\boldsymbol{\mu}_{n,k}|^2).$$

By (1.11) and noting $\max_{m \le k_n} |\sum_{k=1}^{m} (\tilde{\boldsymbol{Z}}_{n,k} - \boldsymbol{\mu}_{n,k})| \stackrel{d}{=} \max_{m \le k_n} |\sum_{k=1}^{m} (\boldsymbol{Z}_{n,k} - \boldsymbol{\mu}_{n,k})|$, we have

$$\widehat{\mathcal{V}} \left(\max_{m \le k_n} \left| \sum_{k=1}^{m} (\boldsymbol{Z}_{n,k} - \boldsymbol{\mu}_{n,k}) \right| \ge x \right)$$

$$\le \widetilde{\mathcal{V}}(Y \ge y) \le 2y^{-2} \sum_{k=1}^{k_n} (\widehat{\mathbb{E}}[|\boldsymbol{Z}_{n,k}|^2] - |\boldsymbol{\mu}_{n,k}|^2), \quad 0 < y < x.$$

The proof is completed. □

The following lemma is on the exponential inequality under $\widehat{\mathbb{V}}$ whose proof is similar to that of Theorem 4.5 of Zhang [11].

Lemma 2.8 *Let $\{Z_{n,k}; k = 1, \ldots, k_n\}$ be an array of independent random variables under $\widehat{\mathbb{E}}$ such that $\widehat{\mathbb{E}}[Z_{n,k}] \le 0$ and $\widehat{\mathbb{E}}[Z_{n,k}^2] < \infty$, $k = 1, \ldots, k_n$. Then for all $x, y > 0$*

$$\widehat{\mathbb{V}} \left(\max_{m \le k_n} \sum_{k=1}^{m} Z_{n,k} \ge x \right)$$

$$\le \widehat{\mathbb{V}} \left(\max_{k \le k_n} Z_{n,k} \ge y \right) + \exp \left\{ \frac{x}{y} - \frac{x}{y} \left(\frac{B_n^2}{xy} + 1 \right) \ln \left(1 + \frac{xy}{B_n^2} \right) \right\}, \qquad (2.16)$$

where $B_n^2 = \sum_{k=1}^{k_n} \widehat{\mathbb{E}}[Z_{n,k}^2]$. In particular, by letting $y = x$, we have Kolmogorov's maximal inequality under $\widehat{\mathbb{V}}$ as follows:

$$\widehat{\mathbb{V}} \left(\max_{m \le k_n} \sum_{k=1}^{m} Z_{n,k} \ge x \right) \le (\mathrm{e} + 1) \frac{B_n^2}{x^2}, \quad \forall x > 0. \qquad (2.17)$$

The last lemma on the Lévy maximal inequality is Lemma 2.1 of Zhang [10].

Lemma 2.9 *Let $X_1, \ldots, X_n$ be independent random variables in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, $S_k = \sum_{i=1}^{k} X_i$, and $0 < \alpha < 1$ be a real number. If there exist real constants $\beta_{n,k}$ such that*

$$\widehat{\mathbb{V}}(|S_k - S_n| \ge \beta_{n,k} + \epsilon) \le \alpha, \quad \text{for all } \epsilon > 0 \text{ and } k = 1, \ldots, n,$$

then

$$(1 - \alpha) \widehat{\mathbb{V}} \left(\max_{k \le n} (|S_k| - \beta_{n,k}) > x + \epsilon \right) \le \widehat{\mathbb{V}}(|S_n| > x), \quad \text{for all } x > 0, \epsilon > 0. \qquad (2.18)$$

3 The Law of Large Numbers

Our first theorem gives the sufficient and necessary conditions for the strong law of large numbers. Let $\{X_n; n \ge 1\}$ be a sequence of i.i.d. random variables in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. Denote $S_n = \sum_{i=1}^{n} X_i$.

Theorem 3.1 (a) *If*

$$C_{\widehat{\mathbb{V}}}(|X_1|) < \infty, \qquad (3.1)$$

then

$$\widehat{\mathbb{V}}^* \left(\liminf_{n \to \infty} \frac{S_n}{n} < \breve{\mathcal{E}}[X_1] \quad \text{or} \quad \limsup_{n \to \infty} \frac{S_n}{n} > \breve{\mathbb{E}}[X_1] \right) = 0. \qquad (3.2)$$

Furthermore, if the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions (a)–(d) in Lemma 2.5, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$,

$$\mathbb{V} \left(\liminf_{n \to \infty} \frac{S_n}{n} = \breve{\mathcal{E}}[X_1] \quad \text{and} \quad \limsup_{n \to \infty} \frac{S_n}{n} = \breve{\mathbb{E}}[X_1] \right) = 1, \qquad (3.3)$$

$$\mathbb{V}\left(C\left\{\frac{S_n}{n}\right\} = [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]]\right) = 1, \tag{3.4}$$

where $C\{x_n\}$ denotes the cluster set of a sequence of $\{x_n\}$ in $\mathbb{R}$.

(b) *Suppose that the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions* (a)–(d) *in Lemma 2.5. If for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$,*

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|S_n|}{n} = \infty\right) < 1, \tag{3.5}$$

then (3.1) *holds.*

Remark 3.2 Theorem 3.1 tells us that the sufficient and necessary condition for the strong law of large numbers is (3.1). Under (3.1), $\breve{\mathbb{E}}[X_1]$ and $\breve{\mathcal{E}}[X_1]$ are well-defined and finite. In Zhang [9], (3.2) is proved under (3.1) and an extra condition that $\widehat{\mathbb{E}}[(|X_1| - c)^+] \to 0$ as $c \to \infty$. Under this extra condition, we have $\widehat{\mathbb{E}}[X_1] = \breve{\mathbb{E}}[X_1]$ and $\widehat{\mathcal{E}}[X_1] = \breve{\mathcal{E}}[X_1]$. For establishing (3.3) and (3.4) and (b), the continuity of $\widehat{\mathbb{V}}^*$ is also assumed in Zhang [9].

The following corollary gives an analogues of (1.1).

Corollary 3.3 *Suppose that the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions* (a)–(d) *in Lemma 2.5.*

(a) *If* (3.1) *is satisfied, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$,*

$$\mathbb{V}\left(\lim_{n\to\infty}\frac{S_n}{n} = b\right) = \begin{cases} 1, & \text{when } b \in [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]], \\ 0, & \text{when } b \notin [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]]. \end{cases} \tag{3.6}$$

(b) *For $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$, there exists a finite random variable $b(\omega)$ such that*

$$\mathcal{V}\left(\lim_{n\to\infty}\frac{S_n}{n} = b\right) = 1 \tag{3.7}$$

if and only if (3.1), *$\breve{\mathcal{E}}[X_1] = \breve{\mathbb{E}}[X_1]$ and $\mathcal{V}(b = \breve{\mathbb{E}}[X_1]) = 1$.*

The following theorem and corollary are Marcinkiewicz's type laws of large numbers which gives the rate of convergence of Kolmogorov's type law of large numbers.

Theorem 3.4 *Let $1 \le p < 2$. If*

$$C_{\widehat{\mathbb{V}}}(|X_1|^p) < \infty, \tag{3.8}$$

then

$$\widehat{\mathbb{V}}^*\left(\liminf_{n\to\infty}\frac{S_n - n\breve{\mathcal{E}}[X_1]}{n^{1/p}} < 0 \ \text{ or } \ \limsup_{n\to\infty}\frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} > 0\right) = 0. \tag{3.9}$$

Furthermore, if the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions (a)–(d) *in Lemma 2.5, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$,*

$$\mathbb{V}\left(\liminf_{n\to\infty}\frac{S_n - n\breve{\mathcal{E}}[X_1]}{n^{1/p}} = 0 \ \text{ and } \ \limsup_{n\to\infty}\frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} = 0\right) = 1. \tag{3.10}$$

Corollary 3.5 *Suppose that the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions* (a)–(d) *in Lemma 2.5.*

(a) *If* (3.8) *is satisfied, then for $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ or $\widehat{\mathbb{V}}^*$,*

$$\mathbb{V}\left(\lim_{n\to\infty}\frac{S_n - nb}{n^{1/p}} = 0\right) = \begin{cases} 1, & \text{when } b \in [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]], \\ 0, & \text{when } b \notin [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]]. \end{cases} \tag{3.11}$$

(b) *For* $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ *or* $\widehat{\mathbb{V}}^*$, *there exist finite random variables* $b(\omega)$ *and* $c(\omega)$ *such that*

$$\mathcal{V}\left(\lim_{n \to \infty} \frac{S_n - nb}{n^{1/p}} = c\right) = 1 \tag{3.12}$$

if and only if $C_{\widehat{\mathbb{V}}}(|X_1|^p) < \infty$, $\breve{\mathcal{E}}[X_1] = \breve{\mathbb{E}}[X_1]$ *and*

$$\begin{aligned}
&\mathcal{V}(b + c = \breve{\mathbb{E}}[X_1]) = 1, \text{ when } p = 1, \\
&\mathcal{V}(b = \breve{\mathbb{E}}[X_1], c = 0) = 1, \text{ when } 1 < p < 2.
\end{aligned} \tag{3.13}$$

Remark 3.6 When the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ does not satisfy the conditions (a)–(d) in Lemma 2.5. We may consider the copy $\{\tilde{X}_n; n \geq 1\}$. The sub-linear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ satisfies the condition (d) in Lemma 2.5 when each X_n is tight, by Proposition 2.1, and so, Theorems 3.1 and 3.4 and Corollary 3.3 and 3.5 remain true for $\{\tilde{X}_n; n \geq 1\}$.

Terán [8] gave an example as follows which shows that (3.3) and (3.4) do not hold.

Example 3.7 Set $\Omega = \{0, 1, 2, \ldots\}$, take $\mathscr{H}$ to be the set of all bounded (necessarily Borel measurable) functions on Ω, and define $\widehat{\mathbb{E}}[X] = \sup_{\omega \in \Omega} X(\omega)$ for all $X \in \mathscr{H}$. Let X_n be the nth bit in the binary representation of ω. Terán [8] showed that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables under $\widehat{\mathbb{E}}$ with $X_n(\omega) \in \{0, 1\}$, $\widehat{\mathcal{E}}[X_1] = 0$, $\widehat{\mathbb{E}}[X_1] = 1$ and

$$\frac{S_n}{n} \to \widehat{\mathcal{E}}[X_1] \quad \text{for all } \omega \in \Omega.$$

By noting that X_n is bounded, $\breve{\mathcal{E}}[X] = \widehat{\mathcal{E}}[X_1]$ and $\widehat{\mathbb{E}}[X_1] = \breve{\mathbb{E}}[X_1]$. For this sub-linear expectation space, $\widehat{\mathbb{V}}(A) = \widehat{\mathbb{V}}^*(A) = \mathbb{C}^*(A) = \widehat{\mathbb{E}}[I_A] = \sup_{\omega \in \Omega} I_A(\omega) = 1$ if A is not empty and 0 otherwise. Hence, (3.3) and (3.4) do not hold.

In the above example, if let δ_{ω_0} to the unit mass at ω_0, i.e., $\delta_{\omega_0}(A) = I_A(\omega_0)$, and denote $\mathscr{P} = \{\delta_{\omega_0}; \omega_0 \in \Omega\}$, then $\widehat{\mathbb{E}}[X] = \sup_{P \in \mathscr{P}} P[X]$. However, the sub-linear expectation space does not satisfy the conditions in Lemma 2.5. In fact, $Y_n(\omega) = \frac{\omega \wedge n}{n}$ is a bounded sequence in $\mathscr{H}$ with $1 \geq Y_n(\omega) \searrow 0$, but $\widehat{\mathbb{E}}[Y_n] = \sup_{\omega \in \Omega} Y_n(\omega) = 1 \not\to 0$. So, $\widehat{\mathbb{E}}$ is not regular on $\mathscr{H}_b$. As we have shown that, each of the conditions (a)–(d) in Lemma 2.5 implies the regularity of $\widehat{\mathbb{E}}$. Hence, Terán's example shows that the conditions in Lemma 2.5 can not be removed. Whether they can be weakened or not is an open problem.

Zhang [10] studied the convergence of the infinite series $\sum_{n=1}^{\infty} X_n$ of a sequence of independent random variables. But, when the strong convergence is considered, the capacity $\widehat{\mathbb{V}}$ is assumed to be continuous. The next theorem gives the equivalence among various kinds of the convergence of the infinite series $\sum_{n=1}^{\infty} X_n$ without the assumption on the continuity of the capacities.

Theorem 3.8 *Let* $\{X_n; n \geq 1\}$ *be a sequence of independent random variables in a sub-linear expectation space* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ *with a capacity satisfying* (1.10), *and* $\{\tilde{X}_n; n \geq 1\}$ *be its copy on* $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ *as defined in Proposition 2.1. Denote* $S_n = \sum_{i=1}^{n} X_i$, $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i$. *Assume that each* X_n *is tight. Consider the following statements*:

(i) *There exists an* $\mathcal{F}$-*measurable finite random variables* S *such that* $S_n \to S$ *a.s.* $\widehat{\mathbb{V}}^*$, *i.e.*,

$$\widehat{\mathbb{V}}^*(\{\omega : \lim_{n \to \infty} S_n(\omega) \neq S(\omega)\}) = 0; \tag{3.14}$$

(ii) *There exists a $\mathcal{F}$-measurable finite random variables S such that $S_n \to S$ in $\widehat{\mathbb{V}}^*$, i.e.,*

$$\widehat{\mathbb{V}}^*(|S_n - S| \geq \epsilon) \to 0 \quad \text{as } n \to \infty \text{ for all } \epsilon > 0; \tag{3.15}$$

(i′) *For the copy $\{\tilde{X}_n; n \geq 1\}$, there exists a $\sigma(\widetilde{\mathcal{H}})$-measurable finite random variables $\tilde{S}$ such that $\tilde{S}_n \to \tilde{S}$ a.s. $\widetilde{\mathbb{V}}^*$, i.e.,*

$$\widetilde{\mathbb{V}}^*\big(\{\omega : \lim_{n\to\infty} \tilde{S}_n(\omega) \neq \tilde{S}(\omega)\}\big) = 0; \tag{3.16}$$

(ii′) *For the copy $\{\tilde{X}_n; n \geq 1\}$, there exists a $\sigma(\widetilde{\mathcal{H}})$-measurable finite random variables $\tilde{S}$ such that $\tilde{S}_n \to \tilde{S}$ in $\widetilde{\mathbb{V}}^*$, i.e.,*

$$\widetilde{\mathbb{V}}^*(|\tilde{S}_n - \tilde{S}| \geq \epsilon) \to 0 \quad \text{as } n \to \infty \text{ for all } \epsilon > 0; \tag{3.17}$$

(iii) $\{S_n\}$ *is a Cauchy sequence under $\widehat{\mathbb{V}}$, i.e.,*

$$\widehat{\mathbb{V}}(|S_n - S_m| \geq \epsilon) \to 0 \quad \text{as } n, m \to \infty \text{ for all } \epsilon > 0; \tag{3.18}$$

(iv) *For some (equivalently, for any) $c > 0$,*
   (S1) $\sum_{n=1}^{\infty} \widehat{\mathbb{V}}(|X_n| > c) < \infty$,
   (S2) $\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n^{(c)}]$ *and* $\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[-X_n^{(c)}]$ *are both convergent,*
   (S3) $\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[(X_n^{(c)} - \widehat{\mathbb{E}}[X_n^{(c)}])^2] < \infty$ *or/and* $\sum_{n=1}^{\infty} \widehat{\mathbb{E}}[(X_n^{(c)} + \widehat{\mathbb{E}}[-X_n^{(c)}])^2] < \infty$.

(v) S_n *converges in distribution, that is, there is a sub-linear space $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{\mathbb{E}})$ and a random variable $\bar{S}$ on it such that $\bar{S}$ is tight under $\bar{\mathbb{E}}$, i.e., $\bar{\mathbb{V}}(|\bar{S}| > x) \to 0$ as $x \to \infty$, and*

$$\widehat{\mathbb{E}}[\phi(S_n)] \to \bar{\mathbb{E}}[\phi(\bar{S})], \quad \phi \in C_{b,\mathrm{Lip}}(\mathbb{R}). \tag{3.19}$$

Then (i′), (ii′), (iii)–(v) *are equivalent and each of them implies* (i) *and* (ii). *Furthermore, suppose that the space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ satisfies one of the conditions* (a)–(d) *in Lemma* 2.5. *Then* (i′), (ii′), (i)–(v) *are equivalent.*

Remark 3.9 In the theorem, $\widehat{\mathbb{V}}$ can be replaced by any a capacity $\mathbb{V}$ with the property (1.10) by (1.11), $\widetilde{\mathbb{V}}^*$ can be replaced $\widetilde{\mathbb{C}}^*$ or $\widetilde{\mathbb{V}}$, $\widehat{\mathbb{V}}^*$ can be replaced by $\mathbb{C}^*$, and, when one of the conditions (a)–(d) in Lemma 2.5 is satisfied, $\widehat{\mathbb{V}}^*$ can be replaced by $\mathbb{V}^{\mathscr{P}}$.

Remark 3.10 Terán [8]'s example also tells us that (i) may not imply (ii)–(v) and (i′)–(ii′) when the conditions in Lemma 2.5 are not satisfied. In fact, let Ω, $\mathcal{H}$, $\widehat{\mathbb{E}}$ and X_ns are defined as in Example 3. Then (see Terán [8])

$$\sum_{n=1}^{\infty} 2^{n-1} X_n(\omega) = \omega, \quad \text{for all } \omega \in \Omega.$$

If let $\bar{X}_n = 2^{n-1} X_n$, $\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i$, then

$$\mathbb{V}\big(\{\omega : \lim_{n\to\infty} \bar{S}_n(\omega) \neq \omega\}\big) = 0.$$

Hence, (i) holds with $S(\omega) = \omega$. But $\widehat{\mathbb{E}}[\bar{X}_n^{(c)}] = 2^{n-1} \wedge c$. So, (S2) of (iv) does not hold. Then (iii)–(v) and (i′) and (ii′) do not hold by the equivalency. Furthermore, at this case, (ii) implies (iii), since

$$\mathbb{V}(|\bar{S}_n - \bar{S}_m| \geq \epsilon) = \widehat{\mathbb{V}}^*(|\bar{S}_n - \bar{S}_m| \geq \epsilon) \leq \widehat{\mathbb{V}}^*(|\bar{S}_n - S| \geq \epsilon/2) + \widehat{\mathbb{V}}^*(|\bar{S}_m - S| \geq \epsilon/2) \to 0.$$

Hence, none of (ii)–(v) and (i′)–(ii′) holds.

At last, we give an analogues of Theorem 3.1 for random vectors. Now, let $\{\boldsymbol{X}_n; n \geq 1\}$ be a sequence of i.i.d. random vectors in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ who take values in a Euclidean space $\mathbb{R}^d$ with norm $|\boldsymbol{x}| = \sqrt{\sum_{i=1}^{d} x_i^2}$, and $\boldsymbol{X}_i \overset{d}{=} \boldsymbol{X}$. Suppose

$$\lim_{c,d \to \infty} \widehat{\mathbb{E}}[(|\boldsymbol{X}| \wedge d - c)^+] = 0. \tag{3.20}$$

Then by Proposition 1.3, for any $\boldsymbol{p} \in \mathbb{R}^d$, $\breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle]$ is well-defined and finite, and

$$g(\boldsymbol{p}) =: \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle], \quad \boldsymbol{p} \in \mathbb{R}^d$$

is a sub-linear function defined on $\mathbb{R}^d$. The assumption (3.20) is implied by a strong one as

$$C_{\widehat{\mathbb{V}}}(|\boldsymbol{X}|) < \infty \tag{3.21}$$

or $\lim_{c \to \infty} \widehat{\mathbb{E}}[(|\boldsymbol{X}| - c)^+] = 0$. Furthermore, if $\lim_{c \to \infty} \widehat{\mathbb{E}}[(|\boldsymbol{X}| - c)^+] = 0$, then $g(\boldsymbol{p}) = \widehat{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle]$. For the sub-linear function $g(\boldsymbol{p})$, by Theorem 1.2.1 of Peng [5], there exists a (unique) bounded, convex and closed subset $\mathbb{M}$ such that (see Peng ([5, Page 32])

$$g(\boldsymbol{p}) = \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] = \sup_{\boldsymbol{x} \in \mathbb{M}} \langle \boldsymbol{p}, \boldsymbol{x} \rangle, \quad \boldsymbol{p} \in \mathbb{R}^d.$$

We denote this set $\mathbb{M}$ by $\mathbb{M}_{\boldsymbol{X}}$ or $\mathbb{M}[\boldsymbol{X}]$. If $\boldsymbol{X}$ is a one-dimensional random variables, then $\mathbb{M}[\boldsymbol{X}] = [\widehat{\mathcal{E}}[X], \widehat{\mathbb{E}}[X]]$. For the multi-dimension case, recall $\breve{\mathbb{E}}[\boldsymbol{X}] = (\breve{\mathbb{E}}[X_1], \ldots, \breve{\mathbb{E}}[X_d])$ and $E[\boldsymbol{X}] = (E[X_1], \ldots, E[X_d])$ for $\boldsymbol{X} = (X_1, \ldots, X_d)$.

Lemma 3.11 *Under the condition* (3.20) *we have*

$$\mathbb{M}_{\boldsymbol{X}} = \widetilde{\mathbb{M}}_{\boldsymbol{X}} =: \{E[\boldsymbol{X}] : E \in \mathscr{E}\}, \quad \text{where } \mathscr{E} \text{ is defined as } (1.17). \tag{3.22}$$

Proof It is obvious that

$$\sup_{\boldsymbol{x} \in \widetilde{\mathbb{M}}_{\boldsymbol{X}}} \langle \boldsymbol{p}, \boldsymbol{x} \rangle = \sup_{E \in \mathscr{E}} E[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] = \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] = \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}} \langle \boldsymbol{p}, \boldsymbol{x} \rangle \quad \text{for all } \boldsymbol{p} \in \mathbb{R}^d.$$

For (3.22), it is sufficient to show that $\widetilde{\mathbb{M}}_{\boldsymbol{X}}$ is also a bounded, convex and closed subset of $\mathbb{R}^d$. The boundedness and convexity are obvious. Next, we show that it is closed. Suppose $E_i \in \mathscr{E}$, $E_i[\boldsymbol{X}] \to \boldsymbol{b}$. We want to show that $\boldsymbol{b} \in \widetilde{\mathbb{M}}_{\boldsymbol{X}}$. For each E_i, define $\widetilde{E}_i$ by $\widetilde{E}_i[\varphi(\boldsymbol{x})] = E_i[\varphi(\boldsymbol{X})]$, $\varphi \in C_{l,\text{Lip}}(\mathbb{R}^d)$. It is easily checked that, if $C_{b,\text{Lip}}(\mathbb{R}^d) \ni \varphi_n \searrow 0$, then

$$0 \leq \widetilde{E}_i[\varphi_n(\boldsymbol{x})] = E_i[\varphi_n(\boldsymbol{X})] \leq \breve{\mathbb{E}}[\varphi_n(\boldsymbol{X})]$$
$$\leq \sup_{|\boldsymbol{x}| \leq c} |\varphi_n(\boldsymbol{x})| + \|\varphi_1\| c^{-1} \breve{\mathbb{E}}[|\boldsymbol{X}|] \to 0,$$

as $n \to \infty$ and then $c \to \infty$. By Daniell–Stone's theorem, there exists a probability measure P_i on $\mathbb{R}^d$ such that

$$E_i[\varphi(\boldsymbol{X})] = \widetilde{E}_i[\varphi(\boldsymbol{x})] = P_i[\varphi(\boldsymbol{x})], \quad \forall \varphi \in C_{b,\text{Lip}}(\mathbb{R}^d).$$

Notice that $\sup_i P_i(|\boldsymbol{x}| \geq c) \leq c^{-1} \sup_i P_i[|\boldsymbol{x}|] \leq c^{-1} \breve{\mathbb{E}}[|\boldsymbol{X}|] \to 0$ as $c \to \infty$. So, on $\mathbb{R}^d$, the sequence $\{P_i\}$ is tight and so is relatively weakly compact. Then, there exist a subsequence i_j and a probability P on $\mathbb{R}^d$ such that

$$E_{i_j}[\varphi(\boldsymbol{X})] = P_{i_j}[\varphi(\boldsymbol{x})] \to P[\varphi(\boldsymbol{x})] \quad \forall \varphi \in C_{b,\text{Lip}}(\mathbb{R}^d). \tag{3.23}$$

On the space $\mathscr{L} = \{Y = \varphi(\boldsymbol{X}) : \varphi \in C_{l,\text{Lip}}(\mathbb{R}^d), Y \in \mathscr{H}_1\}$ we define an operator E by

$$E[Y] = \lim_{j \to \infty} E_{i_j}[Y], \quad Y \in \mathscr{L}.$$

First, by (3.23), E is well defined for bounded $Y \in \mathscr{L}$. Notice

$$|E_{i_j}[Y] - E_{i_j}[(-c) \vee Y \wedge c]| = |E_{i_j}[Y - (-c) \vee Y \wedge c]| \leq \breve{\mathbb{E}}[(|Y| - c)^+] \to 0 \quad \text{as } c \to \infty$$

for $Y \in \mathscr{L}$. $E[Y]$ is well defined on $\mathscr{L}$ and $E[Y] = \lim_{c \to \infty} E[(-c) \vee Y \wedge c]$. It follows that

$$b = \lim_{j \to \infty} E_{i_j}[\boldsymbol{X}] = E[\boldsymbol{X}].$$

Since each $E_{i_j} \in \mathscr{E}$ is a finite additive linear expectation with $E_{i_j} \leq \breve{\mathbb{E}}$, its limit E is also a finite additive linear expectation on $\mathscr{L}$ with $E \leq \breve{\mathbb{E}}$. By the Hahn–Banach theorem, there exists a finite additive linear expectation E^e defined on $\mathscr{H}_1$ such that, $E^e = E$ on $\mathscr{L}$ and, $E^e \leq \breve{\mathbb{E}}$ on $\mathscr{H}_1$. So $E^e \in \mathscr{E}$. Hence, $b = E[\boldsymbol{X}] = E^e[\boldsymbol{X}] \in \widetilde{\mathbb{M}}_{\boldsymbol{X}}$. It follows that $\widetilde{\mathbb{M}}_{\boldsymbol{X}}$ is a closed set. (3.22) is proved. $\qquad\square$

The following is the strong law of large numbers for i.i.d. random vectors. Let $\{\boldsymbol{X}_n; n \geq 1\}$ be a sequence of i.i.d. random variables in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, $\boldsymbol{X}_i \overset{d}{=} \boldsymbol{X}$. Denote $\boldsymbol{S}_n = \sum_{i=1}^n \boldsymbol{X}_i$.

Theorem 3.12 *If* (3.21) *is satisfied, then*

$$\widehat{\mathcal{V}}^*\left(C\left\{\frac{\boldsymbol{S}_n}{n}\right\} \subset \mathbb{M}_{\boldsymbol{X}}\right) = 1. \tag{3.24}$$

Furthermore, suppose that the space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ *satisfies one of the conditions* (a)–(d) *in Lemma* 2.5. *Then for* $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ *or* $\widehat{\mathbb{V}}^*$,

$$\mathbb{V}\left(C\left\{\frac{\boldsymbol{S}_n}{n}\right\} = \mathbb{M}_{\boldsymbol{X}}\right) = 1 \tag{3.25}$$

and

$$\mathbb{V}\left(\lim_{n \to \infty} \frac{\boldsymbol{S}_n}{n} = \boldsymbol{b}\right) = \begin{cases} 1, & \text{when } \boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}, \\ 0, & \text{when } \boldsymbol{b} \notin \mathbb{M}_{\boldsymbol{X}}. \end{cases} \tag{3.26}$$

(3.25) tells us that, under the upper capacity, the limits of $\frac{\boldsymbol{S}_n}{n}$ fills the set $\mathbb{M}_{\boldsymbol{X}}$. The following corollary tells that, under lower capacity, the limit of $\frac{\boldsymbol{S}_n}{n}$ can only be a point.

Corollary 3.13 *Suppose that the space* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ *satisfies one of the conditions* (a)–(d) *in Lemma* 2.5. *Assume that* (3.21) *is satisfied. If for* $\mathbb{V} = \mathbb{V}^{\mathscr{P}}$, $\mathbb{C}^*$ *or* $\widehat{\mathbb{V}}^*$, *there exists a subset* $\mathbb{O}$ *of* $\mathbb{R}^d$ *such that*

$$\mathcal{V}\left(C\left\{\frac{\boldsymbol{S}_n}{n}\right\} = \mathbb{O}\right) > 0, \tag{3.27}$$

then

$$\breve{\mathbb{E}}[-\boldsymbol{X}] = -\breve{\mathbb{E}}[\boldsymbol{X}] \quad \text{and} \quad \mathbb{O} = \{\breve{\mathbb{E}}[\boldsymbol{X}]\}. \tag{3.28}$$

This is a direct corollary of Theorem 3.12. In fact, combining (3.26) and (3.27) yields

$$\mathbb{V}\left(\lim_{n \to \infty} \frac{\boldsymbol{S}_n}{n} = \boldsymbol{b} \text{ and } C\left\{\frac{\boldsymbol{S}_n}{n}\right\} = \mathbb{O}\right) > 0 \quad \text{for all } \boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}.$$

It follows that $\mathbb{O} = \{\boldsymbol{b}\}$ for all $\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}$. Hence $\mathbb{M}_{\boldsymbol{X}}$ has only one point and then (3.28) holds.

To prove Theorem 3.12, we need a weak law of large number which is of independent interest.

Proposition 3.14 *Let $\{\boldsymbol{X}_n; n \geq 1\}$ be a sequence of i.i.d. random variables in a sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, $\boldsymbol{S}_n = \sum_{i=1}^{n} \boldsymbol{X}_i$. If $\lim_{c,d\to\infty} \widehat{\mathbb{E}}[(|\boldsymbol{X}_1| \wedge d - c)^+] = 0$, then*

$$\widehat{\mathbb{V}}\left(\frac{\boldsymbol{S}_n}{n} \notin \mathbb{M}_{\boldsymbol{X}}^{\epsilon} \right) = \widehat{\mathbb{V}}(\operatorname{dist}(\boldsymbol{S}_n/n, \mathbb{M}_{\boldsymbol{X}}) \geq \epsilon) \to 0 \quad \text{for all } \epsilon > 0 \tag{3.29}$$

and

$$\widehat{\mathbb{V}}\left(\left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| < \epsilon \right) \to 1 \quad \text{for all } \boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}} \text{ and } \epsilon > 0, \tag{3.30}$$

where $\operatorname{dist}(\boldsymbol{y}, \mathbb{M}_{\boldsymbol{X}}) = \inf\{|\boldsymbol{y} - \boldsymbol{x}| : \boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}\}$, $\mathbb{M}_{\boldsymbol{X}}^{\epsilon} = \{\boldsymbol{y} : |\boldsymbol{y} - \boldsymbol{x}| < \epsilon \text{ for some } \boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}\}$ *is the ϵ-neighborhood of $\mathbb{M}_{\boldsymbol{X}}$. In particular,*

$$\lim_{n\to\infty} \widehat{\mathbb{E}}\left[\varphi\left(\frac{\boldsymbol{S}_n}{n} \right) \right] = \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}} \varphi(\boldsymbol{x}), \quad \text{for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d). \tag{3.31}$$

The weak law of large numbers (3.31) is proved by Peng [5] under the condition that $\widehat{\mathbb{E}}[(|\boldsymbol{X}_1| - c)^+] \to 0$ as $c \to \infty$, by considering the solutions of the following parabolic PDEs defined on $[0, \infty) \times \mathbb{R}^d$,

$$\partial_t u - g(Du) = 0, \quad u|_{t=0} = \varphi.$$

For the completeness of this paper, we will give a purely probabilistic proof in which only the probability inequalities are used.

4 Proofs of the Law of Large Numbers

Before the proofs, we need one more lemma.

Lemma 4.1 *Suppose $X \in \mathscr{H}$, $1 \leq p < 2$, $C_{\widehat{\mathbb{V}}}(|X|^p) < \infty$. Then*

$$\sum_{i=1}^{\infty} \widehat{\mathbb{V}}(|X| \geq M i^{1/p}) < \infty, \quad \forall M > 0, \tag{4.1}$$

$$\sum_{i=1}^{\infty} \frac{\widehat{\mathbb{E}}[X^2 \wedge (M i^{2/p})]}{i^{2/p}} < \infty, \quad \forall M > 0 \tag{4.2}$$

and

$$\breve{\mathbb{E}}[(|X| - c)^+] = o(c^{1-p}) \text{ and } \widehat{\mathbb{E}}[X^2 \wedge c^2] = o(c^{2-p}) \quad \text{as } c \to \infty. \tag{4.3}$$

Furthermore,

$$C_{\widehat{\mathbb{V}}}(|X|^p) = \infty \iff \sum_{i=1}^{\infty} \widehat{\mathbb{V}}(|X| \geq M i^{1/p}) = \infty, \quad \forall M > 0. \tag{4.4}$$

Proof   (4.1) and (4.4) are obvious by noting $C_{\widehat{\mathbb{V}}}(|X|^p) = \int_0^{\infty} \widehat{\mathbb{V}}(|X| > x^{1/p}) dx$. (4.2) is similar to Lemma 3.9 (a) of Zhang [9] and is proved in Zhang and Lin [13]. For (4.3), we have

$$\breve{\mathbb{E}}[(|X| - c)^+] \leq C_{\widehat{\mathbb{V}}}((|X| - c)^+) = \int_c^{\infty} \widehat{\mathbb{V}}(|X| > x) dx$$

$$= \frac{1}{p} \int_{c^p}^{\infty} y^{1/p-1} \widehat{\mathbb{V}}(|X|^p > y) dy \leq \frac{1}{p} c^{1-p} \int_{c^p}^{\infty} \widehat{\mathbb{V}}(|X|^p > y) dy = o(c^{1-p})$$

and

$$\breve{\mathbb{E}}[X^2 \wedge c^2] \leq C_{\widehat{\mathbb{V}}}(X^2 \wedge c^2) = \int_0^{c^2} \widehat{\mathbb{V}}(X^2 > x) dx$$

$$= \frac{2}{p} \int_0^{c^p} y^{2/p-1} \widehat{\mathbb{V}}(|X|^p > y) dy = o(c^{2-p}).$$

The proof is completed. □

4.1 One-dimensional Case

Now we turn to the proofs of the main results. We first consider the LLN for one-dimensional random variables.

Proof of Theorems 3.1 *and* 3.4   When (3.1) is satisfied, each X_n is tight. Obviously, (3.4) is implied by (3.3) by noting $\widehat{\mathcal{V}}^*(\frac{S_n}{n} - \frac{S_{n-1}}{n-1} \to 0) = 1$ and the fact that $C(\{x_n\}) = [a, b]$ whenever $\limsup_{n\to\infty} x_n = b$, $\liminf_{n\to\infty} x_n = a$ and $x_n - x_{n-1} \to 0$. (3.2) and (3.3) are special cases of (3.9) and (3.10), respectively. For (3.9), we let $Z_{k,i} = (-2^{k/p}) \vee X_i \wedge 2^{k/p}$, $i = 1, \ldots, 2^k$. Then

$$|\widehat{\mathbb{E}}[Z_{k,i}] - \breve{\mathbb{E}}[X_i]| \le \breve{\mathbb{E}}[(|X_1| - 2^{k/p})^+] = o(2^{k(1/p-1)})$$

by Lemma 4.1. For any $\epsilon > 0$, by Lemma 2.8 and (1.11) we have for k large enough,

$$\widehat{\mathbb{V}}\left(\max_{2^{k-1} \le n \le 2^k} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \epsilon \right)$$

$$\le \widehat{\mathbb{V}}\left(\max_{2^{k-1} \le n \le 2^k} \sum_{i=1}^n (X_i - \breve{\mathbb{E}}[X_i]) \ge \epsilon 2^{(k-1)/p} \right)$$

$$\le \widehat{\mathbb{V}}\left(\max_{n \le 2^k} \sum_{i=1}^n (Z_{k,i} - \breve{\mathbb{E}}[Z_{k,i}]) \ge \epsilon 2^{k/p}/4 \right) + \widehat{\mathbb{V}}\left(\max_{i \le 2^k} |X_i| > 2^{k/p} \right)$$

$$\le C 2^{-2k/p} \sum_{i=1}^{2^k} \widehat{\mathbb{E}}[Z_{k,i}^2] + \sum_{i=1}^{2^k} \widehat{\mathbb{V}}(|X_i| > 2^{k/p})$$

$$\le C 2^{-2k/p} 2^k \widehat{\mathbb{E}}[X_1^2 \wedge 2^{2k/p}] + 2^k \widehat{\mathbb{V}}(|X_1| > 2^{k/p}/2)$$

$$\le 4C \sum_{i=2^k+1}^{2^{k+1}} \frac{\widehat{\mathbb{E}}[X_1^2 \wedge i^{2/p}]}{i^{2/p}} + 2 \sum_{i=2^{k-1}+1}^{2^k} \widehat{\mathbb{V}}(|X_1| > i^{1/p}/2).$$

It follows that

$$\sum_{k=1}^\infty \widehat{\mathbb{V}}^*\left(\max_{2^{k-1} \le n \le 2^k} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \epsilon \right)$$

$$\le \sum_{k=1}^\infty \widehat{\mathbb{V}}\left(\max_{2^{k-1} \le n \le 2^k} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \epsilon \right)$$

$$\le 4C \sum_{i=1}^\infty \frac{\widehat{\mathbb{E}}[X_1^2 \wedge i^{2/p}]}{i^{2/p}} + 2 \sum_{i=1}^\infty \widehat{\mathbb{V}}(|X_1| > i^{1/p}/2) < \infty,$$

by Lemma 4.1. By noting that $\widehat{\mathbb{V}}^*$ is a countably sub-additive capacity and the Borel–Cantelli (Lemma 2.3), we have

$$\widehat{\mathbb{V}}^*\left(\limsup_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \epsilon \right) \le \widehat{\mathbb{V}}^*\left(\max_{2^{k-1} \le n \le 2^k} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \epsilon \ i.o. \right) = 0.$$

By the countable sub-additivity of $\widehat{\mathbb{V}}^*$ again,

$$\widehat{\mathbb{V}}^*\left(\limsup_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} > 0 \right) = \widehat{\mathbb{V}}^*\left(\bigcup_{l=1}^\infty \left\{ \limsup_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \ge \frac{1}{l} \right\} \right) = 0.$$

For $-X_i$s, we have a similar result. (3.9) is proved.

For (3.10), it is sufficient to show that

$$\mathbb{V}^{\mathscr{P}}\left(\liminf_{n\to\infty}\frac{\tilde{S}_n - n\breve{\mathcal{E}}[X_1]}{n^{1/p}} \leq 0 \ \text{ and } \ \limsup_{n\to\infty}\frac{\tilde{S}_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \geq 0\right) = 1. \tag{4.5}$$

Let $Y_{ni} = (-n^{1/p}) \vee X_i \wedge n^{1/p}$, $i = 1, \ldots, n$. Then $\widetilde{\mathbb{M}}[Y_{ni}] = [\widehat{\mathcal{E}}[Y_{ni}], \widehat{\mathbb{E}}[Y_{ni}]]$. By Lemmas 2.7 and 4.1,

$$\widehat{\mathcal{V}}\left(\sum_{i=1}^{n}(-Y_{ni} + \widehat{\mathbb{E}}[Y_{ni}]) \geq \epsilon n^{1/p}\right)$$
$$= \widehat{\mathcal{V}}\left(\sum_{i=1}^{n}(-Y_{ni} - \widehat{\mathcal{E}}[-Y_{ni}]) \geq \epsilon n^{1/p}\right) \leq 2\frac{n\widehat{\mathbb{E}}[X_1^2 \wedge n^{2/p}]}{\epsilon^2 n^{2/p}} \to 0.$$

On the other hand, $n|\widehat{\mathbb{E}}[Y_{ni}] - \breve{\mathbb{E}}[X_1]| \leq n\breve{\mathbb{E}}[(|X_1| - n^{1/p})^+] = o(n^{1/p})$ and

$$\widehat{\mathbb{V}}(Y_{ni} \neq X_i, \exists i = 1, \ldots, n) \leq n\widehat{\mathbb{V}}(|X_1| > n^{1/p}) \to 0,$$

by Lemma 4.1. It follows that

$$\widehat{\mathcal{V}}\left(\sum_{i=1}^{n}(-X_i + \widehat{\mathbb{E}}[X_1]) \geq 2\epsilon n^{1/p}\right) \to 0.$$

That is

$$\widehat{\mathbb{V}}\left(\frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \geq -\epsilon\right) \to 1 \ \text{ for all } \epsilon > 0.$$

By considering $-X_i$s, similarly we have

$$\widehat{\mathbb{V}}\left(\frac{-S_n + n\breve{\mathcal{E}}[X_1]}{n^{1/p}} \geq -\epsilon\right) \to 1 \ \text{ for all } \epsilon > 0.$$

For $\epsilon_k = 1/2^k$, $k = 1, 2, \ldots$, we can choose n_k successively such that $n_k \nearrow \infty$, $n_{k-1}/n_k^{1/p} \to 0$, and

$$\widehat{\mathbb{V}}\left(\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathbb{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k\right) \geq 1 - \epsilon_k,$$
$$\widehat{\mathbb{V}}\left(-\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathcal{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k\right) \geq 1 - \epsilon_k.$$

It follows that

$$\sum_{k=1}^{\infty}\widehat{\mathbb{V}}\left(\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathbb{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k\right) = \infty,$$
$$\sum_{k=1}^{\infty}\widehat{\mathbb{V}}\left(-\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathcal{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k\right) = \infty.$$

Let

$$A = \left\{\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathbb{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k \ i.o.\right\},$$
$$B = \left\{-\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathcal{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq -\epsilon_k \ i.o.\right\}.$$

By the Borel–Cantelli lemma (Lemma 2.5), $\mathbb{V}^{\mathscr{P}}(AB) = 1$. On AB and $C = \{\limsup_{n \to \infty} \frac{|S_n|}{n} < \infty\}$,

$$\limsup_{n \to \infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} \geq \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathbb{E}}[X_1]}{n_k^{1/p}}$$

$$\geq \limsup_{k \to \infty} \frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathbb{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \geq 0,$$

$$\limsup_{n \to \infty} \frac{-S_n + n\breve{\mathcal{E}}[X_1]}{n^{1/p}} \geq \limsup_{k \to \infty} \left(-\frac{S_{n_k} - S_{n_{k-1}} - (n_k - n_{k-1})\breve{\mathcal{E}}[X_1]}{(n_k - n_{k-1})^{1/p}} \right) \geq 0.$$

Notice $\mathbb{V}^{\mathscr{P}}(ABC) \geq \mathbb{V}^{\mathscr{P}}(AB) - \mathbb{V}^{\mathscr{P}}(C^c) = 1 - 0 = 1$ by (3.2). The proof of (4.5) is completed.

For Theorem 3.1 (b), suppose $C_{\widehat{\mathbb{V}}}(|X_1|) = \infty$. Then

$$\sum_{n=1}^{\infty} \widehat{\mathbb{V}}(|X_n| \geq Mn) \geq \sum_{n=1}^{\infty} \widehat{\mathbb{V}}(|X_1| \geq 2Mn) = \infty, \quad \text{for all } M > 0,$$

by (1.11) and Lemma 4.1. So, there exists a sequence $1 < M_n \nearrow \infty$ such that

$$\sum_{n=1}^{\infty} \widehat{\mathbb{V}}(|X_n| \geq M_n n) = \infty.$$

By the Borel–Cantelli Lemma (Lemma 2.5),

$$\mathbb{V}^{\mathscr{P}}(|X_n| \geq M_n n \text{ i.o.}) = 1.$$

On the event $\{|X_n| \geq M_n n \text{ i.o.}\}$, we have

$$\infty = \limsup_{n \to \infty} \frac{|X_n|}{n} \leq 2 \limsup_{n \to \infty} \frac{|S_n|}{n}.$$

It follows that

$$\mathbb{V}^{\mathscr{P}} \left(\limsup_{n \to \infty} \frac{|S_n|}{n} = \infty \right) = 1, \tag{4.6}$$

which contradicts with (3.5). The proof is now completed. $\square$

Proof of Corollaries 3.3 *and* 3.5   It is sufficient to show Corollary 3.5.

(a) When $b \notin [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]]$, the conclusion (3.11) is obvious by (3.9). If $b \in [\breve{\mathcal{E}}[X_1], \breve{\mathbb{E}}[X_1]]$, then there exists an $\alpha \in [0, 1]$ such that $b = \alpha\breve{\mathbb{E}}[X_1] + (1 - \alpha)\breve{\mathcal{E}}[X_1]$. Let $Y_i = (-i^{1/p}) \vee X_i \wedge i^{1/p}$ and $\mu_{\alpha,i} = \alpha\widehat{\mathbb{E}}[Y_i] + (1 - \alpha)\widehat{\mathcal{E}}[Y_i]$. Then $\sum_{i=1}^{n} |\mu_{\alpha,i} - b| \leq \sum_{i=1}^{n} \breve{\mathbb{E}}[(|X_1| - i^{1/p})^+] = o(n^{1/p})$ and $\widehat{\mathbb{V}}^*(X_i \neq Y_i \text{ i.o.}) = 0$. So, it is sufficient to show that for each $\alpha \in [0, 1]$,

$$\mathbb{V}^{\mathscr{P}} \left(\lim_{n \to \infty} \frac{\sum_{i=1}^{n}(Y_i - \mu_{\alpha,i})}{n^{1/p}} = 0 \right) = 1. \tag{4.7}$$

For each i, by the expression (1.8), there exist $\theta_{i,1}, \theta_{i,2} \in \Theta$ such that

$$E_{\theta_{i,1}}[Y_i] = \widehat{\mathbb{E}}[Y_i] \quad \text{and} \quad E_{\theta_{i,2}}[Y_i] = \widehat{\mathcal{E}}[Y_i].$$

Define the linear operator $E_i = \alpha E_{\theta_{i,1}} + (1 - \alpha)E_{\theta_{i,2}}$. Then

$$E_i[Y_i] = \mu_{\alpha,i} \quad \text{and} \quad E_i \leq \widehat{\mathbb{E}}.$$

Notice that each Y_n is tight. By Proposition 2.1, there exist a copy $\{\tilde{Y}_n; n \geq 1\}$ on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ of $\{Y_n; n \geq 1\}$ and a probability measure Q on $\widetilde{\Omega}$ such that such that $\{\tilde{Y}_n; n \geq 1\}$ is a sequence

of independent random variables under Q,

$$Q[\varphi(\tilde{Y}_i)] = E_i[\varphi(Y_i)] \quad \text{for all } \varphi \in C_{b,\text{Lip}}(\mathbb{R}),$$
$$Q[\varphi(\tilde{Y}_1, \ldots, \tilde{Y}_d)] \leq \widehat{\mathbb{E}}[\varphi(Y_1, \ldots, Y_d)] \quad \text{for all } \varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$$

and

$$\widetilde{v}(B) \leq Q(B) \leq \widetilde{V}(B) \quad \text{for all } B \in \sigma(\tilde{Y}_1, \tilde{Y}_2, \ldots). \tag{4.8}$$

Notice $|E_i[Y_i^{(c)}] - E_i[Y_i]| \leq \widehat{\mathbb{E}}[(|Y_i| - c)^+] \to 0$ as $c \to \infty$. We have

$$Q[\tilde{Y}_i] = \lim_{c \to \infty} Q[\tilde{Y}_i^{(c)}] = \lim_{c \to \infty} E_i[Y_i^{(c)}] = E_i[Y_i] = \mu_{\alpha,i}, \tag{4.9}$$

$$Q[\tilde{Y}_i^2] = \lim_{c \to \infty} Q[\tilde{Y}_i^2 \wedge c] = \lim_{c \to \infty} E_i[Y_i^2 \wedge c] \leq \widehat{\mathbb{E}}[Y_i^2]. \tag{4.10}$$

Then

$$\sum_{i=1}^{\infty} \frac{Q[\tilde{Y}_i^2]}{i^{2/p}} \leq \sum_{i=1}^{\infty} \frac{\widehat{\mathbb{E}}[Y_i^2]}{i^{2/p}} = \sum_{i=1}^{\infty} \frac{\widehat{\mathbb{E}}[X_1^2 \wedge i^{2/p}]}{i^{2/p}} < \infty,$$

by Lemma 4.1. Denote $T_n = \sum_{i=1}^n (Y_i - \mu_{\alpha,i})$ and $\tilde{T}_n = \sum_{i=1}^n (\tilde{Y}_i - \mu_{\alpha,i})$, $n_k = 2^k$. Then

$$Q\left(\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |\tilde{T}_n - \tilde{T}_{n_k}|}{n_k^{1/p}} > \epsilon\right)$$

$$\leq 2\epsilon^{-2} n_k^{-2/p} \sum_{i=n_k+1}^{n_{k+1}} Q[\tilde{Y}_i^2] \leq 2\epsilon^{-2} 2^{2/p} \sum_{i=n_k+1}^{n_{k+1}} \frac{Q[\tilde{Y}_i^2]}{i^{2/p}}.$$

It follows that

$$\sum_{k=1}^{\infty} Q\left(\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |\tilde{T}_n - \tilde{T}_{n_k}|}{n_k^{1/p}} > \epsilon\right) < \infty, \quad \text{for all } \epsilon > 0.$$

Then there exists a sequence $\epsilon_k \searrow 0$ such that

$$\sum_{k=1}^{\infty} Q\left(\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |\tilde{T}_n - \tilde{T}_{n_k}|}{n_k^{1/p}} > \epsilon_k\right) < \infty.$$

By (4.8) and (1.11),

$$\sum_{k=1}^{\infty} \mathcal{V}^{\mathscr{P}}\left(\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |T_n - T_{n_k}|}{n_k^{1/p}} > 2\epsilon_k\right)$$

$$\leq \sum_{k=1}^{\infty} \widetilde{v}\left(\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |\tilde{T}_n - \tilde{T}_{n_k}|}{n_k^{1/p}} > \epsilon_k\right) < \infty. \tag{4.11}$$

Notice the independence. By Lemma 2.5,

$$\mathcal{V}^{\mathscr{P}}(A_k \ i.o.) = 0 \quad \text{with} \quad A_k = \left\{\frac{\max_{n_k+1 \leq n \leq n_{k+1}} |T_n - T_{n_k}|}{n_k^{1/p}} > 2\epsilon_k\right\}.$$

Notice that on the event $(A_k \ i.o.)^c$,

$$\lim_{k \to \infty} \frac{\max_{n_k+1 \leq n \leq n_{k+1}} |T_n - T_{n_k}|}{n_k^{1/p}} = 0,$$

which implies $\lim_{n \to \infty} \frac{T_n}{n^{1/p}} = 0$. (4.7) is proved.

(b) First, notice the facts that $\mathbb{V}(AB) = 1$ whenever $\mathbb{V}(A) = \mathcal{V}(B) = 1$, $\mathcal{V}(AB) = 1$ whenever $\mathcal{V}(A) = \mathcal{V}(B) = 1$. If (3.8) and (3.13) hold, and $\breve{\mathcal{E}}[X_1] = \breve{\mathbb{E}}[X_1]$, then (3.12) is

obvious by (3.9). Conversely, suppose that (3.12) holds. Let $A = \{\liminf_{n\to\infty} \frac{S_n - n\breve{\mathcal{E}}[X_1]}{n^{1/p}} = 0\}$, $B = \{\limsup_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} = 0\}$ and $C = \{\lim_{n\to\infty} \frac{S_n - nb}{n^{1/p}} = c\}$.

We first consider the case $p = 1$. If $C_{\widehat{\mathbb{V}}}(|X_1|) = \infty$, then (4.6) holds, which contradicts with (3.12). So, $C_{\widehat{\mathbb{V}}}(|X_1|) < \infty$. By (3.12)) and (3.10) with $p = 1$, $\mathbb{V}(ABC) = 1$. While, on ABC,

$$\breve{\mathcal{E}}[X_1] = \liminf_{n\to\infty} \frac{S_n}{n} = c + b,$$

$$\breve{\mathbb{E}}[X_1] = \limsup_{n\to\infty} \frac{S_n}{n} = c + b.$$

It follows that $\breve{\mathbb{E}}[X_1] = \breve{\mathcal{E}}[X_1]$. Then, by the direct part,

$$\mathcal{V}\left(\lim_{n\to\infty} \frac{S_n}{n} = \breve{\mathbb{E}}[X_1]\right) = 1,$$

which, together with (3.12), implies $\mathcal{V}(b + c = \breve{\mathbb{E}}[X_1]) = 1$.

Now, suppose $1 < p < 2$. Then

$$\mathcal{V}\left(\lim_{n\to\infty} \frac{S_n}{n} = b\right) = 1$$

by (3.12). By the conclusion for the case $p = 1$, we must have $\breve{\mathbb{E}}[X_1] = \breve{\mathcal{E}}[X_1]$ and $\mathcal{V}(b = \breve{\mathcal{E}}[X_1]) = 1$. Suppose $C_{\widehat{\mathbb{V}}}(|X_1|^p) = \infty$. Then $C_{\widehat{\mathbb{V}}}(|X_1 - \breve{\mathbb{E}}[X_1]|^p) = \infty$. Similar to (4.6), we have

$$\mathbb{V}\left(\limsup_{n\to\infty} \frac{|S_n - n\breve{\mathbb{E}}[X_1]|}{n^{1/p}} = \infty\right) \geq \mathbb{V}^{\mathscr{P}}\left(\limsup_{n\to\infty} \frac{|S_n - n\breve{\mathbb{E}}[X_1]|}{n^{1/p}} = \infty\right)$$

$$\geq \mathbb{V}^{\mathscr{P}}\left(\limsup_{n\to\infty} \frac{|X_n - \breve{\mathbb{E}}[X_1]|}{n^{1/p}} = \infty\right)$$

$$= 1,$$

which contradicts with (3.12) by noting $\mathcal{V}(b = \breve{\mathbb{E}}[X_1]) = 1$. So, (3.8) holds. By the direct part, (3.11) holds. Then

$$\mathcal{V}\left(\lim_{n\to\infty} \frac{S_n - nb}{n^{1/p}} = 0\right) = \mathcal{V}\left(\lim_{n\to\infty} \frac{S_n - n\breve{\mathbb{E}}[X_1]}{n^{1/p}} = 0\right) = 1,$$

which, together with (3.12), implies $\mathcal{V}(c = 0) = 1$. The proof is now completed. $\square$

Next, we consider the convergence of infinite series.

Proof of Theorem 3.8   (iii)$\Leftrightarrow$(iv) and (v)$\Rightarrow$(iii) are proved by Zhang [10] (see Theorem 3.2, Theorem 3.3 there).

First, we show that (iii)$\Rightarrow$(v). Let $\bar{\Omega} = \mathbb{R}$, $\bar{\mathscr{H}} = C_{b,\text{Lip}}(\mathbb{R})$. Define $\bar{\mathbb{E}}$ by

$$\bar{\mathbb{E}}[\varphi] = \limsup_{n\to\infty} \widehat{\mathbb{E}}[\varphi(S_n)], \quad \varphi \in \bar{\mathscr{H}},$$

and define the random variable $\bar{S}$ by $\bar{S}(x) = x$. Since each X_i is tight, each S_n is tight. By (3.18), we have $\lim_{c\to\infty} \max_n \widehat{\mathbb{V}}(|S_n| > c) = 0$. Then by choosing a function $\varphi \in C_{b,\text{Lip}}(\mathbb{R})$ with $I\{|x| \geq c\} \leq \varphi(x) \leq I\{|x| \geq c/2\}$ we have

$$\bar{\mathbb{V}}(|\bar{S}| \geq c) \leq \bar{\mathbb{E}}[\varphi(\bar{S})] = \limsup_{n\to\infty} \widehat{\mathbb{E}}[\varphi(S_n)] \leq \limsup_{n\to\infty} \widehat{\mathbb{V}}(|S_n| \geq c/2) \to 0 \quad \text{as } c \to \infty.$$

It follows that $\bar{S}$ is tight. For $\varphi \in \bar{\mathscr{H}}$, let $\varphi_c(x) = \varphi((-c) \vee x \wedge c)$. Then φ_c is a uniformly continuous function. For any $\epsilon > 0$, there is a $\delta > 0$ such that $|\varphi_c(x) - \varphi_c(y)| < \epsilon$ when

$|x - y| < \delta$. Hence

$$|\widehat{\mathbb{E}}[\varphi_c(S_n)] - \widehat{\mathbb{E}}[\varphi_c(S_m)]| < \epsilon + \|\varphi\|\widehat{\mathbb{V}}(|S_n - S_m| > \delta) \to \epsilon$$

as $n, m \to \infty$. Hence $\widehat{\mathbb{E}}[\varphi_c(S_n)]$ converges. It follows that

$$\bar{E}[\varphi_c(\bar{S})] = \lim_{n \to \infty} \widehat{\mathbb{E}}[\varphi_c(S_n)].$$

On the other hand,

$$\max_n |\widehat{\mathbb{E}}[\varphi_c(S_n)] - \widehat{\mathbb{E}}[\varphi(S_n)]| \le \|\varphi\| \max_n \widehat{\mathbb{V}}(|S_n| > c) \to 0$$

and

$$|\bar{\mathbb{E}}[\varphi_c(\bar{S})] - \bar{\mathbb{E}}[\varphi(\bar{S})]| \le \|\varphi\|\bar{\mathbb{V}}(|\bar{S}| > c) \to 0,$$

as $c \to \infty$. It follows that

$$\bar{\mathbb{E}}[\varphi(\bar{S})] = \lim_{n \to \infty} \widehat{\mathbb{E}}[\varphi(S_n)], \quad \forall \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}).$$

(v) holds.

Next, we show (iii)$\Rightarrow$(i) and (ii). By the Lévy inequality (2.18), it follows from (3.18) that

$$\widehat{\mathbb{V}}\big(\max_{m \le i \le n} |S_i - S_m| \ge \epsilon\big) \to 0 \quad \text{as } n, m \to \infty \text{ for all } \epsilon > 0. \tag{4.12}$$

Let $\epsilon_k = 1/2^k$. There exists a sequence $n_k \nearrow \infty$ such that

$$\widehat{\mathbb{V}}^*\big(\max_{n_k \le i \le n_{k+1}} |S_i - S_{n_k}| \ge \epsilon_k\big) \le \widehat{\mathbb{V}}\big(\max_{n_k \le i \le n_{k+1}} |S_i - S_{n_k}| \ge \epsilon_k\big) < \epsilon_k.$$

It follows that

$$\sum_{k=1}^{\infty} \widehat{\mathbb{V}}^*\big(\max_{n_k \le i \le n_{k+1}} |S_i - S_{n_k}| \ge \epsilon_k\big) < \sum_{k=1}^{\infty} \epsilon_k < \infty.$$

Notice the countable sub-additivity of $\widehat{\mathbb{V}}^*$. By the Borel–Cantelli lemma (Lemma 2.3),

$$\widehat{\mathbb{V}}^*(A) = 0 \quad \text{where } A = \big\{\max_{n_k \le i \le n_{k+1}} |S_i - S_{n_k}| \ge \epsilon_k \ i.o.\big\}.$$

on A^c, $S = S_{n_0} + \sum_{k=1}^{\infty}(S_{n_k} - S_{n_{k-1}})$ is finite. Let $S(\omega) = 0$ when $\omega \in A$. On A^c, $S_{n_k} \to S$ and $\max_{n_k \le i \le n_{k+1}} |S_i - S_{n_k}| \to 0$ as $k \to \infty$, and so $S_i \to S$ as $i \to \infty$. Then (i) is proved.

Also, on the event $\bigcap_{m=k}^{\infty} \big\{\max_{n_m \le i \le n_{m+1}} |S_i - S_{n_m}| \le \epsilon_m\big\}$,

$$|S - S_{n_k}| \le \sum_{m=k}^{\infty} |S_{n_{m+1}} - S_{n_m}| \le \sum_{m=k}^{\infty} 2^{-m} = 2^{-k+1}.$$

It follows that

$$\widehat{\mathbb{V}}^*(|S_{n_k} - S| > 2^{-k+1}) \le \sum_{m=k}^{\infty} \widehat{\mathbb{V}}(|S_{n_{m+1}} - S_{n_m}| > \epsilon_m) \le \sum_{m=k}^{\infty} \epsilon_m < 2^{-k+1}.$$

On the other hand, for any $\epsilon > 0$, when k is large enough such that $2^{-k+1} < \epsilon/2$,

$$\widehat{\mathbb{V}}^*(|S_n - S| > \epsilon/2) \le \widehat{\mathbb{V}}^*(|S_{n_k} - S| > 2^{-k+1}) + \widehat{\mathbb{V}}^*(|S_{n_k} - S_n| > \epsilon/2) \to 0,$$

as $n, k \to \infty$. Then (ii) is proved.

Notice (1.11) and $(X_1, \ldots, X_n) \stackrel{d}{=} (\tilde{X}_1, \ldots, \tilde{X}_n)$, $n \ge 1$. (iii) is equivalent to that it holds for $\tilde{S}_n$. So, it implies (i$'$) and (ii$'$).

Notice that the copy space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ satisfies the condition (d) in Lemma 2.5. At last, it is sufficient to show (i)⇒(iii), and (ii)⇒(iii), when one of the conditions (a)–(b) in Lemma 2.5 is satisfied.

Suppose that (iii) does not hold. Then there exist constants $\epsilon_0 > 0$, $\delta_0 > 0$ and sequence $\{m_k\}$ and $\{n_k\}$ with $m_k < n_k \leq m_{k+1}$ such that

$$\widehat{\mathbb{V}}(|S_{n_k} - S_{m_k}| \geq \epsilon_0) \geq \delta_0.$$

Notice the independence of $\{S_{n_k} - S_{m_k}; k \geq 1\}$ and

$$\sum_{k=1}^{\infty} \widehat{\mathbb{V}}(|S_{n_k} - S_{m_k}| \geq \epsilon_0) = \infty.$$

By the Borel–Cantelli lemma (Lemma 2.5),

$$\mathbb{V}^{\mathscr{P}}\left(\limsup_{k \to \infty} |S_{n_k} - S_{m_k}| \geq \epsilon_0/2\right) = 1.$$

However, on the event $\{\lim_{n \to \infty} S_n = S\}$ we have $\limsup_{k \to \infty} |S_{n_k} - S_{m_k}| = 0$. Thus,

$$\mathbb{V}^{\mathscr{P}}\left(\{\omega : \lim_{n \to \infty} S_n(\omega) \neq S(\omega)\}\right) = 1,$$

which contradicts with (3.16). So, (i)⇒(iii) is proved.

Now, suppose $\mathbb{V}^{\mathscr{P}}(|S_n - S| > \epsilon) \to 0$ for all $\epsilon > 0$. Then

$$\mathbb{V}^{\mathscr{P}}(|S_n - S_m| > \epsilon) \to 0 \quad \text{as } n, m \to \infty, \forall \epsilon > 0,$$

which is equivalent to (3.18) by (1.11), since both $\mathbb{V}^{\mathscr{P}}$ and $\widehat{\mathbb{V}}$ have the property (1.10) and $S_n - S_m \in \mathscr{H}$. The proof is completed. $\qquad \square$

4.2 Multi-dimensional Case

Now, we consider the LLN for random vectors.

Proof of Proposition 3.14   Recall $\boldsymbol{X}^{(c)} = (X_1^{(c)}, \ldots, X_d^{(c)})$ and $X_i^{(c)} = (-c) \vee X_i \wedge c$ for $\boldsymbol{X} = (X_1, \ldots, X_d)$. Notice

$$\widehat{\mathbb{V}}\left(\left|\frac{\boldsymbol{S}_n}{n} - \frac{\sum_{i=1}^{n} \boldsymbol{X}_i^{(c)}}{n}\right| \geq \epsilon\right) \leq \epsilon^{-1} \breve{\mathbb{E}}[|\boldsymbol{X}_1 - \boldsymbol{X}_1^{(c)}|] \to 0 \quad \text{as } c \to \infty$$

and

$$\sup_{E \in \mathscr{E}} \left|E[\boldsymbol{X}] - E[\boldsymbol{X}^{(c)}]\right| \leq \breve{\mathbb{E}}[|\boldsymbol{X}_1 - \boldsymbol{X}_1^{(c)}|] \to 0 \quad \text{as } c \to \infty.$$

Hence, without loss of generality we can assume $|\boldsymbol{X}_i| \leq c$ and $|\boldsymbol{X}| \leq c$. Let $\delta = \epsilon^2/(4c)$, and $\mathcal{N}_\delta = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K\} \subset \{\boldsymbol{p} : |\boldsymbol{p}| \leq 2c\}$ be a δ-net of $\{\boldsymbol{p} : |\boldsymbol{p}| \leq 2c\}$. We have the following fact,

$$\boldsymbol{y} \notin \mathbb{M}_{\boldsymbol{X}}^{\epsilon} \text{ and } |\boldsymbol{y}| \leq c \Longrightarrow \langle \boldsymbol{p}_i, \boldsymbol{y} \rangle \geq \breve{\mathbb{E}}[\langle \boldsymbol{p}_i, \boldsymbol{X} \rangle] + \epsilon^2/2 \text{ for some } \boldsymbol{p}_i \in \mathcal{N}_\delta. \qquad (4.13)$$

In fact, for $\boldsymbol{y} \notin \mathbb{M}_{\boldsymbol{X}}^{\epsilon}$, there exists $\boldsymbol{o} = E[\boldsymbol{X}] \in \mathbb{M}_{\boldsymbol{X}}$ such that $\tau =: \inf_{x \in \mathbb{M}_{\boldsymbol{X}}} |\boldsymbol{y} - \boldsymbol{x}| = |\boldsymbol{y} - \boldsymbol{o}| \geq \epsilon$. Let $\boldsymbol{p} = \boldsymbol{y} - \boldsymbol{o}$. Then $|\boldsymbol{p}| \leq 2c$ and $\langle \boldsymbol{p}, \boldsymbol{y} \rangle = \langle \boldsymbol{p}, \boldsymbol{o} \rangle + \tau^2$. For any $\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}$ and $0 \leq \alpha \leq 1$, $\boldsymbol{z} = \alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{o} \in \mathbb{M}_{\boldsymbol{X}}$. Then

$$|\boldsymbol{y} - \boldsymbol{o}|^2 \leq |\boldsymbol{y} - \boldsymbol{z}|^2 = |\boldsymbol{y} - \boldsymbol{o}|^2 + \alpha^2 |\boldsymbol{x} - \boldsymbol{o}|^2 + 2\alpha \langle \boldsymbol{x} - \boldsymbol{o}, \boldsymbol{o} - \boldsymbol{y} \rangle \quad \text{for all } \alpha \in [0, 1].$$

It follows that

$$\langle \boldsymbol{p}, \boldsymbol{o} \rangle - \langle \boldsymbol{p}, \boldsymbol{x} \rangle = \langle \boldsymbol{x} - \boldsymbol{o}, \boldsymbol{o} - \boldsymbol{y} \rangle \geq 0.$$

So $\langle \boldsymbol{p}, \boldsymbol{o} \rangle \geq \langle \boldsymbol{p}, \boldsymbol{x} \rangle$. It follows that $\langle \boldsymbol{p}, \boldsymbol{o} \rangle \geq \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}} \langle \boldsymbol{p}, \boldsymbol{x} \rangle = \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle]$. It follows that $\langle \boldsymbol{p}, \boldsymbol{y} \rangle \geq \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] + \epsilon^2$. Furthermore, for the $\boldsymbol{p}$, there exists a $\boldsymbol{p}_i \in \mathcal{N}_\delta$ such that $|\boldsymbol{p} - \boldsymbol{p}_i| < \delta$. Then

$$\langle \boldsymbol{p}_i, \boldsymbol{y} \rangle - \breve{\mathbb{E}}[\langle \boldsymbol{p}_i, \boldsymbol{X} \rangle] \geq \langle \boldsymbol{p}, \boldsymbol{y} \rangle - \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] - |\boldsymbol{p}_i - \boldsymbol{p}||\boldsymbol{y}| - |\boldsymbol{p}_i - \boldsymbol{p}|\breve{\mathbb{E}}[|\boldsymbol{X}|] \geq \epsilon^2/2.$$

Hence (4.13) follows. Now, it follows from the inequality (2.17) that

$$\widehat{\mathbb{V}}\left(\frac{\boldsymbol{S}_n}{n} \notin \mathbb{M}_{\boldsymbol{X}}^\epsilon \right) \leq \sum_{i \in \mathcal{N}_\delta} \widehat{\mathbb{V}}\left(\langle \boldsymbol{p}_i, \boldsymbol{S}_n/n \rangle \geq \breve{\mathbb{E}}[\langle \boldsymbol{p}_i, \boldsymbol{X} \rangle] + \epsilon^2/2 \right)$$

$$= \sum_{i \in \mathcal{N}_\delta} \widehat{\mathbb{V}}\left(\sum_{k=1}^n (\langle \boldsymbol{p}_i, \boldsymbol{X}_k \rangle - \widehat{\mathbb{E}}[\langle \boldsymbol{p}_i, \boldsymbol{X}_k \rangle]) \geq n\epsilon^2/2 \right)$$

$$\leq 2(e+1) \sum_{i \in \mathcal{N}_\delta} \frac{n\widehat{\mathbb{E}}[\langle \boldsymbol{p}_i, \boldsymbol{X} \rangle^2]}{\epsilon^4 n^2/4} \to 0.$$

The proof of (3.29) is completed.

For (3.30), we suppose $\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}} = \widetilde{\mathbb{M}}_{\boldsymbol{X}} = \{E[\boldsymbol{X}] : E \in \mathscr{E}\}$. Notice $\breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X} \rangle] = \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X}_i \rangle] = g(\boldsymbol{p})$ for all $\boldsymbol{p}$ and i. It follows that

$$\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}} = \mathbb{M}_{\boldsymbol{X}_i} = \widetilde{\mathbb{M}}_{\boldsymbol{X}_i} = \{E[\boldsymbol{X}_i] : E \in \mathscr{E}\}, \quad i = 1, 2, \ldots$$

Hence, by Lemma 2.7,

$$\widehat{\mathcal{V}}\left(\left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| \geq \epsilon \right) \leq 2\epsilon^{-2} n^{-2} n\widehat{\mathbb{E}}[|\boldsymbol{X}|^2] \leq 2c^2 \epsilon^{-2} n^{-1} \to 0.$$

The proof of (3.30) is completed.

Finally, we show that (3.31) is a corollary of (3.29) and (3.30). Without loss of generality, we assume $\varphi(\boldsymbol{x}) \geq 0$, for otherwise we can replace it by $\varphi + \|\varphi\|$, where $\|\varphi\| = \sup_{\boldsymbol{x}} |\varphi(\boldsymbol{x})|$. It follows from (3.29) that

$$\limsup_{n \to \infty} \widehat{\mathbb{E}}\left[\varphi\left(\frac{\boldsymbol{S}_n}{n} \right) \right] \leq \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}^\epsilon} \varphi(\boldsymbol{x}) + \|\varphi\| \limsup_{n \to \infty} \widehat{\mathbb{V}}\left(\frac{\boldsymbol{S}_n}{n} \notin \mathbb{M}_{\boldsymbol{X}}^\epsilon \right)$$

$$= \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}^\epsilon} \varphi(\boldsymbol{x}) \to \sup_{\boldsymbol{x} \in \mathbb{M}_{\boldsymbol{X}}} \varphi(\boldsymbol{x}) \quad \text{as } \epsilon \to 0.$$

Now suppose $b \in \mathbb{M}_{\boldsymbol{X}}$. By (3.30),

$$\liminf_{n \to \infty} \widehat{\mathbb{E}}\left[\varphi\left(\frac{\boldsymbol{S}_n}{n} \right) \right] \geq \liminf_{n \to \infty} \widehat{\mathbb{E}}\left[\varphi\left(\frac{\boldsymbol{S}_n}{n} \right) I\left\{ \left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| < \epsilon \right\} \right]$$

$$\geq \inf_{\boldsymbol{x}: |\boldsymbol{x}-\boldsymbol{b}|<\epsilon} \varphi(\boldsymbol{x}) \liminf_{n \to \infty} \widehat{\mathbb{V}}\left(\left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| < \epsilon \right)$$

$$= \inf_{\boldsymbol{x}: |\boldsymbol{x}-\boldsymbol{b}|<\epsilon} \varphi(\boldsymbol{x}) \to \varphi(\boldsymbol{b})$$

as $\epsilon \to 0$. By the arbitrariness of $\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}$,

$$\liminf_{n \to \infty} \widehat{\mathbb{E}}\left[\varphi\left(\frac{\boldsymbol{S}_n}{n} \right) \right] \geq \sup_{\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}} \varphi(\boldsymbol{b}).$$

The proof of (3.31) is completed. $\square$

Proof of Theorem 3.12   Let Q be a countable subset of $\mathbb{R}^d$ which is dense in $\mathbb{R}^d$. Then

$$\widehat{\mathbb{V}}^*\left(C\left\{ \frac{\boldsymbol{S}_n}{n} \right\} \not\subset \mathbb{M}_{\boldsymbol{X}} \right) = \widehat{\mathbb{V}}^*\left(\bigcup_{\boldsymbol{p} \in \mathbb{R}^d} \left\{ \limsup_{n \to \infty} \frac{\langle \boldsymbol{p}, \boldsymbol{S}_n \rangle}{n} > \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X}_1 \rangle] \right\} \right)$$

$$= \widehat{\mathbb{V}}^* \left(\bigcup_{\boldsymbol{p} \in Q} \left\{ \limsup_{n \to \infty} \frac{\langle \boldsymbol{p}, \boldsymbol{S}_n \rangle}{n} > \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X}_1 \rangle] \right\} \right)$$

$$\leq \sum_{\boldsymbol{p} \in Q} \widehat{\mathbb{V}}^* \left(\limsup_{n \to \infty} \frac{\langle \boldsymbol{p}, \boldsymbol{S}_n \rangle}{n} > \breve{\mathbb{E}}[\langle \boldsymbol{p}, \boldsymbol{X}_1 \rangle] \right)$$

$$= 0$$

by (3.2). And so, (3.24) is proved.

For (3.25), is is sufficient to show that

$$\mathbb{V}^{\mathscr{P}} \left(C \left\{ \frac{\boldsymbol{S}_n}{n} \right\} \supset \mathbb{M}_{\boldsymbol{X}} \right) = 1. \tag{4.14}$$

For any $\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}$ and $\epsilon > 0$, by Proposition 3.14, we have

$$\lim_{n \to \infty} \widehat{\mathbb{V}} \left(\left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| \leq \epsilon \right) = 1. \tag{4.15}$$

Let $\Theta = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots\}$ be a countable subset of $\mathbb{M}_{\boldsymbol{X}}$ which is dense in $\mathbb{M}_{\boldsymbol{X}}$. Let $\epsilon_k = 1/2^k$. By (4.15), there exists a sequence $\{n_k\}$ with $n_k \nearrow \infty$, $n_{k-1}/n_k^{1/p} \to 0$ such that

$$\widehat{\mathbb{V}} \left(\left| \frac{\boldsymbol{S}_{n_k} - \boldsymbol{S}_{n_{k-1}}}{n_k} - \boldsymbol{b}_j \right| \leq \epsilon_k \right) \geq 1/2, \quad j = 1, \ldots k.$$

Denote

$$A_{k,j} = \begin{cases} \left\{ \left| \frac{\boldsymbol{S}_{n_k} - \boldsymbol{S}_{n_{k-1}}}{n_k} - \boldsymbol{b}_j \right| \leq \epsilon_k \right\}, & j = 1, 2, \ldots, k \\ \emptyset, & j > k. \end{cases}$$

Then

$$\sum_{k=1}^{\infty} \widehat{\mathbb{V}}(A_{k,j}) = \sum_{k=j+1}^{\infty} \widehat{\mathbb{V}}(A_{k,j}) = \infty, \quad j = 1, 2, \ldots.$$

Notice that $A_{k,j}$s are closed sets of $\boldsymbol{X} = (X_1, X_2 \ldots)$. By the Borel–Cantelli Lemma (Lemma 2.5 (iii)),

$$\mathbb{V}^{\mathscr{P}} \left(\bigcap_{j=1}^{\infty} \{ A_{k,j} \ i.o. \} \right) = 1.$$

Notice that, on the event $A = \bigcap_{j=1}^{\infty} \{ A_{k,j} \ i.o. \}$ and $B = \{ C\{\frac{\boldsymbol{S}_n}{n}\} \subset \mathbb{M}_{\boldsymbol{X}} \}$, we have

$$\liminf_n \left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b}_j \right| \leq \liminf_k \left| \frac{\boldsymbol{S}_{n_k}}{n_k} - \boldsymbol{b}_j \right|$$

$$= \liminf_k \left| \frac{\boldsymbol{S}_{n_k} - \boldsymbol{S}_{n_{k-1}}}{n_k} - \boldsymbol{b}_j \right|$$

$$= 0, \quad \text{for all } \boldsymbol{b}_j \in \Theta.$$

Notice that Θ is dense in $\mathbb{M}_{\boldsymbol{X}}$. It follows that on A and B,

$$\liminf_n \left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| = 0, \quad \text{for all } \boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}}.$$

On the other hand, $\mathbb{V}^{\mathscr{P}}(B^c) = 0$ by (3.24). So, $\mathbb{V}^{\mathscr{P}}(AB) \geq \mathbb{V}^{\mathscr{P}}(A) - \mathbb{V}^{\mathscr{P}}(B^c) = 1$. It follows that

$$\mathbb{V}^{\mathscr{P}} \left(\liminf_n \left| \frac{\boldsymbol{S}_n}{n} - \boldsymbol{b} \right| = 0 \text{ for all } \boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}} \right) = 1.$$

Hence, (4.14) is proved.

Finally, we consider (3.26). Let $\boldsymbol{Y}_i = \boldsymbol{X}_i^{(i)}$, $\boldsymbol{T}_n = \sum_{i=1}^n \boldsymbol{Y}_i$, where $\boldsymbol{X}^{(c)} = (X_1^{(c)}, \ldots, X_d^{(c)})$ for $\boldsymbol{X} = (X_1, \ldots, X_d)$. Then

$$\sum_{n=1}^\infty \widehat{\mathbb{V}}(|\boldsymbol{X}_n| > n) \le \sum_{n=1}^\infty \widehat{\mathbb{V}}(|\boldsymbol{X}| > n/2) < \infty, \tag{4.16}$$

$$\frac{\sum_{i=1}^n \breve{\mathbb{E}}[|\boldsymbol{X}_i - \boldsymbol{Y}_i|]}{n} \le \frac{\sum_{i=1}^n \sum_{j=1}^d \breve{\mathbb{E}}[(|X_{i,j}| - i)^+]}{n} \to 0 \tag{4.17}$$

and

$$\sum_{i=1}^\infty \frac{\widehat{\mathbb{E}}[|\boldsymbol{Y}_i|^2]}{i^2} \le \sum_{i=1}^\infty \frac{\widehat{\mathbb{E}}[|\boldsymbol{X}_1|^2 \wedge (di)^2]}{i^2} < \infty, \tag{4.18}$$

by Lemma 4.1.

When $\boldsymbol{b} \notin \mathbb{M}_{\boldsymbol{X}}$, (3.26) is obvious by (3.24). Suppose

$$\boldsymbol{b} \in \mathbb{M}_{\boldsymbol{X}} = \widetilde{\mathbb{M}}_{\boldsymbol{X}_i} = \{E[\boldsymbol{X}_i] : E \in \mathscr{E}\}, \quad i = 1, 2, \ldots$$

There exists $E_i \in \mathscr{E}$ such that $\boldsymbol{b} = E_i[\boldsymbol{X}_i]$. Notice that each $\boldsymbol{Y}_n$ is tight. For linear operators E_i and the sequence $\{\boldsymbol{Y}_n; n \ge 1\}$, by Proposition 2.1 there exist a copy $\{\tilde{\boldsymbol{Y}}_n; n \ge 1\}$ on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ and a probability measure Q on $\widetilde{\Omega}$ such that $\{\tilde{\boldsymbol{Y}}_n; n \ge 1\}$ is a sequence of independent random variables under Q,

$$Q[\varphi(\tilde{\boldsymbol{Y}}_i)] = E_i[\varphi(\boldsymbol{Y}_i)] \text{ for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^d), \tag{4.19}$$

$$Q[\varphi(\tilde{\boldsymbol{Y}}_1, \ldots, \tilde{\boldsymbol{Y}}_p)] \le \widehat{\mathbb{E}}[\varphi(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_p)] \text{ for all } \varphi \in C_{b,\mathrm{Lip}}(\mathbb{R}^{d \times p}) \tag{4.20}$$

and

$$\widetilde{v}(B) \le Q(B) \le \widetilde{V}(B) \quad \text{for all } B \in \mathscr{B}(\tilde{\boldsymbol{Y}}_1, \tilde{\boldsymbol{Y}}_2, \ldots). \tag{4.21}$$

Similar to (4.9) and (4.10), we have $Q[\tilde{\boldsymbol{Y}}_i] = E_i[\boldsymbol{Y}_i]$ and $Q[|\tilde{\boldsymbol{Y}}_i|^2] \le \widehat{\mathbb{E}}[|\boldsymbol{Y}_i|^2]$. Then

$$\frac{1}{n} \sum_{i=1}^n |Q[\tilde{\boldsymbol{Y}}_i] - \boldsymbol{b}| = \frac{1}{n} \sum_{i=1}^n |E_i[\boldsymbol{Y}_i] - E_i[\boldsymbol{X}_i]| \le \frac{1}{n} \sum_{i=1}^n \breve{\mathbb{E}}[|\boldsymbol{Y}_i - \boldsymbol{X}_i|] \to 0 \tag{4.22}$$

by (4.20) and (4.17), and

$$\sum_{i=1}^\infty \frac{Q[|\tilde{\boldsymbol{Y}}_i|^2]}{i^2} \le \sum_{i=1}^\infty \frac{\widehat{\mathbb{E}}[|\boldsymbol{Y}_i|^2]}{i^2} < \infty,$$

by (4.18) and (4.20). With the same arguments as showing (4.11) we have

$$\sum_{k=1}^\infty \mathcal{V}^{\mathscr{P}}\left(\frac{\max_{n_k+1 \le n \le n_{k+1}} |\sum_{i=n_k+1}^n (\boldsymbol{Y}_i - Q[\tilde{\boldsymbol{Y}}_i])|}{n_k} > 2\epsilon_k\right) < \infty,$$

where $n_k = 2^k$, $\epsilon_k \searrow 0$, which, similarly to (4.7), implies

$$\mathbb{V}^{\mathscr{P}}\left(\lim_{n\to\infty} \frac{\sum_{i=1}^n (\boldsymbol{Y}_i - Q[\tilde{\boldsymbol{Y}}_i])}{n} = 0\right) = 1.$$

On the other hand, by (4.16) and the Borel–Cantelli lemma, we have $\widehat{\mathbb{V}}^*(\boldsymbol{X}_n \ne \boldsymbol{Y}_n \ i.o.) = 0$. It follows that

$$\mathbb{V}^{\mathscr{P}}\left(\lim_{n\to\infty} \frac{\boldsymbol{S}_n}{n} = \boldsymbol{b}\right) = 1.$$

(3.26) is proved. $\square$

Conflict of Interest   The authors declare no conflict of interest.

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