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Weakly Compact Sets and Riesz Representation Theorem in Musielak Sequence Spaces

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In memory of Professor Henryk Hudzik, a great man, our teacher and colleague

Abstract In this work, we give some criteria of the weakly compact sets and a representation theorem of Riesz's type in Musielak sequence spaces using the ideas and techniques of sequence spaces and Musielak function. Finally, as an immediate consequence of the criteria considered in this paper, the criteria of the weakly compact sets of Orlicz sequence spaces are deduced.

Keywords Compact set, weak topology, Banach space, dual space, Musielak sequence spaces

MR(2010) Subject Classification 46E30, 46B20

1 Introduction

Since the inception of the study of Banach space, one of the main topics has been compactness. A set A in a topological space X is said to be compact if any open cover of A has a finite subcover. A is said to be sequentially compact if any sequence of A has a convergent subsequence. A is said to be countably compact if any countable subset of A has a cluster point in A [18]. The three types of compactness coincide if the topology is metrizable.

These types of compactness play indispensable roles not only in theory study but also in practical applications. In the 1880s, Arzela–Ascoli's criterion was given for a compact set in continuous function space [3]. Kolmogorov's criterion and Riesz's criterion of the compactness rose in Riesz function spaces and Orlicz function spaces, respectively [1, 20]. In 1912, Brouwer gave a fixed point theorem in compact settings [5], which led to impressive developments [7, 8]. From then on, the Brouwer theorem has been one powerful tool of research in numerous

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theoretical and applied branches [9, 14, 19, 21]. It is a mark contribution made by Eberlein and Smulian that they proved that the three types of weak compactness coincide over a normed space [11, 26]. James gave one marvelous criterion of a weakly compact set in Banach spaces related with attainable functional and reflexivity [17].

Orlicz spaces are the extensions of Riesz spaces and have been adopted broadly in recent years, especially in nonlinear problems [22]. In 1962, Ando gave the criteria of weakly $\sigma(L_M, L_N)$-compact sets in Orlicz function spaces [2]. In 1982, Wu studied the compactness of normed topology and weak topology in Orlicz function spaces for general sense [29]. In 1997, Zhang gave one criterion of normed compact sets in Orlicz sequence spaces [30]. In 2009, Fabian investigated the weak compactness in L_1 [12]. For an outline about the development and applications, we refer the reader for a survey to ([4, 10, 22**?** , 23]). The paper adds to the literature on the criterion for the weak compactness of Musielak sequence spaces. Anisotropy is a general phenomenon of appearance in the real world. Musielak spaces broader than Orlicz spaces are perfect settings for anisotropic nonlinear problems in particular [15].

2 Preliminaries

Let R be the set of all real numbers. Let X be a real Banach space and let $B(X)$, $S(X)$ and X^* be the closed unit ball, the unit sphere, and the dual of X, respectively. A map $\Phi : R \to [0, \infty]$, where ∞ value may be possible, is called an *Orlicz* function if Φ is vanishing and continuous at zero, convex, even, left continuous on $(0, \infty)$ and not identically equal to zero on $(-\infty, \infty)$. $\phi^-(u)$ and $\phi(u)$ denote the left-hand derivative and right-hand derivative of $\Phi(u)$ respectively. For an Orlicz function Φ we define the complementary function $\Psi : R \to [0, \infty]$ in the Young's sense by the formula

$$\Psi(v) = \sup\{u|v| - \Phi(u) : u \geq 0\}.$$

The complementary function Ψ is also an Orlicz function. We define a subdifferential $\partial\Phi(u)$ of $\Phi(u)$ at $u \geq 0$ as follows:

$$\partial\Phi(u) = \{v \geq 0 : \Phi(u) + \Psi(v) = uv\}.$$

Set $\alpha_\Phi = \sup\{u \geq 0 : \Phi(u) = 0\}$ and $\beta_\Phi = \inf\{u \geq 0 : \Phi(u) = \infty\}$. Then we get

(1) if $u \in [0, \beta_\Phi)$, then $\partial\Phi(u) = [\phi^-(u), \phi(u)]$,

(2) if $u = \beta_\Phi$ and $\beta_\Phi < \infty$, then $\partial\Phi(\beta_\Phi) = [\phi^-(\beta_\Phi), \infty)$,

(3) if either ($u = \beta_\Phi$ and $\beta_\Phi = \infty$) or $u > \beta_\Phi$, then $\partial\Phi(u) = \emptyset$.

Analogous situation we get for v and $\Phi(\partial\Phi(v))$.

A sequence Orlicz function $\Phi = \{\Phi_i\}_{i=1}^\infty$ is called a Musielak function if Φ_i is an Orlicz function for each i. For a given Musielak function Φ and a scalar sequence $u = (u(1), u(2), \dots)$ we define a convex function, called a module of u, by the formula

$$\rho_\Phi(u) = \rho_\Phi(|u|) = \sum_{i=1}^\infty \Phi_i(|u(i)|)$$

where $|u| = (|u(1)|, |u(2)|, \dots, |u(n)|, \dots)$. Let

$$l_\Phi = \{u : \exists\ \lambda > 0,\ \text{s.t. } \rho_\Phi(\lambda u) < \infty\},$$

this family is linear and is usually equipped with one of the two following equivalent norms: the Luxemburg norm defined by:

$$\|u\|_{(\Phi)} = \inf\left\{\lambda > 0 : \rho_{\Phi}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

or the Orlicz norm defined that equals the Amemiya norm by:

$$\|u\|_{\Phi} = \sup_{\rho_{\Psi}(v)\leq 1} \sum_{i=1}^{\infty} u(i)v(i) = \inf_{k>0} \frac{1}{k}(1 + \rho_{\Phi}(ku)),$$

it forms a Banach space which is called a Musielak sequence space, denoted by

$$l_{(\Phi)} = (l_{\Phi}, \|\cdot\|_{(\Phi)}), \qquad l_{\Phi} = (l_{\Phi}, \|\cdot\|_{\Phi}).$$

Let $h_0(X) = \{u = (u(1), \ldots, u(i), 0, \ldots) : i = 1, 2, \ldots\}$, the closure of h_0 in $l_{(\Phi)}$ or l_{Φ} is denoted by $h_{(\Phi)}$ or h_{Φ}, respectively. If for all i, $\Phi_i(u) = M(u)$, an Orlicz function with its complementary $N(v)$, we call it an Orlicz sequence space denoted by $l_{(M)}$ or l_M. For details, please see [13, 25].

Below we recall the basic facts of the space that will be used for the present paper. The proof can be referred to as [25].

Lemma 2.1 ([25]) *For $u \in l_{\Phi}$, $\|u\|_{\Phi} \leq 1$ or $\|u\|_{(\Phi)} \leq 1$, we have $\rho_{\Phi}(u) \leq \|u\|_{(\Phi)} \leq \|u\|_{\Phi}$.*

Lemma 2.2 ([25]) *In a Musielak sequence space, there hold $\|u\|_{(\Phi)} \leq \|u\|_{\Phi} \leq 2\|u\|_{(\Phi)}$ for all $u \in l_{\Phi}$,, that says, $l_{(\Phi)}$ is isomorphic to l_{Φ}.*

Lemma 2.3 ([25]) *Hölder's inequality. In Musielak sequence spaces, there hold*

$$\begin{aligned}
\sum_{i=1}^{\infty} |u(i)v(i)| &\leq \|u\|_{(\Phi)}\|v\|_{\Psi} \quad (u \in l_{(\Phi)}, v \in l_{\Psi}).\\
\sum_{i=1}^{\infty} |u(i)v(i)| &\leq \|u\|_{\Phi}\|v\|_{(\Psi)} \quad (u \in l_{\Phi}, v \in l_{(\Psi)}).
\end{aligned}$$

Lemma 2.4 ([20, 25]) *For $v \in l_{\psi}$, there hold*

$$\|v\|_{\psi} = \sup_{\rho_{\Phi}(u)\leq 1} \sum_{i=1}^{\infty} |u(i)||v(i)| = \inf_{k>0} \frac{1}{k}(1 + \rho_{\Psi}(ku)).$$

For the convenience of the readers, we give the following lemmas with slight proofs.

Lemma 2.5 *For any sequence $v = (v(1), v(2), \ldots)$, denote $\|v\| = \sup\{\sum_{i=1}^{\infty} u(i)v(i) : \forall\, u \in l_{(\Phi)}$ with $\rho_{\Phi}(u) \leq 1\}$. If $\|v\| \leq 1$, then $\rho_{\Psi}(v) \leq \|v\|$.*

Proof If $\rho_{\Phi}(\psi(v)) = \sum_{i=1}^{\infty} \Phi_i(\psi_i(\|v(i)\|)) \leq 1$, we have

$$\begin{aligned}
\rho_{\Psi}(v) &= \sum_{i=1}^{\infty} \Psi_i(|v(i)|)\\
&\leq \sum_{i=1}^{\infty} \Phi_i(\psi_i(|v(i)|)) + \sum_{i=1}^{\infty} \Psi_i(|v(i)|)\\
&= \sum_{i=1}^{\infty} \psi_i(|v(i)|)|v(i)|
\end{aligned}$$

$$\leq \|v\|.$$

Hence, it is enough to show that if $\|v\| \leq 1$, $\sum_{i=1}^{\infty} \Phi_i(\psi_i(\|v(i)\|)) \leq 1$. If, suppose

$$\sum_{i=1}^{\infty} \Phi_i(\psi_i(|v(i)|)) > 1.$$

Take a natural number n and a positive number D such that

$$1 < \sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|)) < D < \infty.$$

By the convexity of an Orlicz function, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n} \Phi_i\left(\frac{\psi_i(|v(i)|)}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))}\right) \\
&\quad \leq \sum_{i=1}^{n} \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \Phi_i(\psi_i(|v(i)|)) \\
&\quad = \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|)) \\
&\quad = 1.
\end{aligned}$$

thus

$$\begin{aligned}
\|v\| &\geq \sum_{i=1}^{n} \frac{\psi_i(|v(i)|)}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} |v(i)| \\
&= \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} \psi_i(|v(i)|)|v(i)| \\
&= \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} (\Phi_i(\psi_i(|v(i)|)) + \Psi_i(|v(i)|)) \\
&= \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|)) + \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} \Psi_i(|v(i)|) \\
&= 1 + \frac{1}{\sum_{i=1}^{n} \Phi_i(\psi_i(|v(i)|))} \sum_{i=1}^{n} \Psi_i(|v(i)|) \\
&> 1.
\end{aligned}$$

This is contrary to the fact that $\|v\| \leq 1$. It ends the proof. □

Lemma 2.6 *For any real sequence $v = (v(1), v(2), \ldots)$, if for any $u \in l_{(\Phi)}, \langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ is convergent, then $\|v\| < \infty$.*

Proof If, suppose $\|v\| = \infty$. For each natural number n, by the definition of $\|v\|$, we have $u_n \in l_{(\Phi)}$ such that $\rho_{\Phi}(u_n) \leq 1$, $\sum_{i=1}^{\infty} u_n(i)v(i)$ converges and $\sum_{i=1}^{\infty} u_n(i)v(i) > 2^n$, and more, $u_n(i)v(i) \geq 0$ for all natural numbers i, due to the symmetry of $l_{(\Phi)}$. Let $\tilde{u}_n = \sum_{k=1}^{n} \frac{u_k}{2^k}$. We deduce

$$\rho_{\Phi}(\tilde{u}_n) = \sum_{i=1}^{\infty} \Phi_i(|\tilde{u}_n(i)|)$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{\infty} \Phi_i\bigg(\sum_{k=1}^{n}\bigg|\frac{u_k(i)}{2^k}\bigg|\bigg)\\
&= \sum_{k=1}^{n}\sum_{i=1}^{\infty}\frac{\Phi_i(|u_k(i)|)}{2^k}\\
&= \sum_{k=1}^{n}\frac{1}{2^k}\sum_{i=1}^{\infty}\Phi_i(|u_k(i)|)\\
&\leq \sum_{k=1}^{n}\frac{1}{2^k}\\
&\leq 1,
\end{aligned}
$$

so $\tilde{u}_n \in l_{(\Phi)}$. By the definition of Luxemburg norm, $\|\tilde{u}_n\|_{(\Phi)} \leq 1$. Set $\tilde{u} = \sum_{k=1}^{\infty} \frac{u_k}{2^k}$. From $\sum_{k=1}^{\infty} \frac{\|u_k\|_{(\Phi)}}{2^k} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, thanks to $l_{(\Phi)}$ is a Banach space, we get $\tilde{u}(i) \in l_{(\Phi)}$. Hence, by the given condition,

$$\sum_{i=1}^{\infty} \tilde{u}(i)v(i) < \infty.$$

On the other hand, compute

$$
\begin{aligned}
\sum_{i=1}^{\infty}\tilde{u}_n(i)v(i) &= \sum_{i=1}^{\infty}\bigg(\sum_{k=1}^{n}\frac{u_k(i)}{2^k}\bigg)v(i\rangle\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{n}\frac{1}{2^k}u_k(i)v(i)\\
&= \sum_{k=1}^{n}\frac{1}{2^k}\sum_{i=1}^{\infty}u_k(i)v(i)\\
&> \sum_{k=1}^{n}\frac{1}{2^k}2^k\\
&= n,
\end{aligned}
$$

so by Levi's lemma, it leads to a contradiction:

$$
\begin{aligned}
\infty &> \sum_{i=1}^{\infty}\tilde{u}(i)v(i)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{u_k(i)}{2^k}v(i)\\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\frac{u_k(i)}{2^k}v(i)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{i=1}^{\infty}\frac{u_k(i)}{2^k}v(i)\\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{k=1}^{n}\frac{u_k(i)}{2^k}v(i)\\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}\tilde{u}_n(i)v(i)
\end{aligned}
$$

$$\geq \lim_{n\to\infty} n = \infty.$$

It shows $\|v\| < \infty$. It ends the proof. $\square$

From Lemma 2.6, we have

Proposition 2.7 *For any real sequence $v = (v(1), v(2), \dots,)$, if for all $u \in l_{(\Phi)}$, $\langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ is convergent, then $v \in l_\Psi$.*

Proof From Lemma 2.5, $\rho_\Psi(\frac{v}{\|v\|}) \leq 1$, we see $v \in l_\Psi$ by the definition. $\square$

3 Riesz Expression of Dual Space

Using ideas and techniques of anisotropy and atom, we give a direct proof for the representation theorem of Riesz's type that applies from isotropic cases to anisotropic ones (please confer [16, 27]).

Theorem 3.1 *Representation of Riesz's Type. In Musielak sequence space, there hold*

$$h^*_{(\Phi)} \cong l_\Psi, \quad h^*_\Phi \cong l_{(\Psi)}$$

where $\langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ for all $u \in h_{(\Phi)}, v \in l_\Psi / u \in h_\Phi, v \in l_{(\Psi)}$.

Proof $h^*_{(\Phi)} \cong l_\Psi$. We set a mapping $T : l_\Psi \to h^*_{(\Phi)}$, where $T_v = f$ for each $v \in l_\Psi$, defined by $f(u) = \langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ for all $u \in h_{(\Phi)}$. By Hölder's inequality, it follows $\sum_{i=1}^{\infty} |u(i)v(i)| \leq \|u\|_{(\Phi)}\|v\|_\Psi$. Obviously, f is linear and $\|f\| \leq \|v\|_\Psi$, that says, $T_v = f \in h^*_{(\Phi)}$. Now, T is well defined. For each $f \in h^*_{(\Phi)}$, let $v = (f(e_1), f(e_2), \dots)$ where $e_i = (0, \dots, 0, 1, 0, \dots)$. We claim $v \in l_\Psi$. In fact, for each $u \in l_{(\Phi)}$, we take $\tilde{u}$ as that $\tilde{u}(i) = |u(i)|\mathrm{sign} f(e_i)$. Clearly, $|\tilde{u}(i)| = |u(i)|$, so $\tilde{u} \in l_\Phi$. Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} |u(i)v(i)| &= \lim_{n\to\infty} \sum_{i=1}^{n} |u(i)v(i)| \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} |u(i)| f(i) \mathrm{sign} f(e_i) \\
&\leq \lim_{n\to\infty} \left\langle \sum_{i=1}^{n} \tilde{u}(i), f \right\rangle \\
&\leq \lim_{n\to\infty} \left\| \sum_{i=1}^{n} \tilde{u}(i) \right\|_{(\Phi)} \|f\| \\
&\leq \|\tilde{u}\|_{(\Phi)} \|f\| \\
&= \|u\|_{(\Phi)} \|f\| \\
&< \infty.
\end{aligned}$$

By Proposition 2.7, $v \in l_\Psi$. Also from this inequality, we see $\|v\|_\Psi \leq \|f\|$. Combining $\|f\| \leq \|v\|_\Psi$, we get $\|f\| = \|v\|_\Psi$. That says, T is an isometric isomorphism where $T_v = f$ defined by $f(u) = \langle u, v\rangle = \sum_{i=1}^{\infty} u(i)v(i)$ for all $u \in h_{(\Phi)}, v \in l_\Psi$.

$h^*_\Phi \cong l_{(\Psi)}$. Replacing that $\|u\|_{(\Phi)}$ and $\|v\|_\Psi$ by $\|u\|_\Phi$ and $\|v\|_{(\Psi)}$ respectively, we get the conclusion by repeating the above arguments. It ends the proof. $\square$

4 Weak Compactness of Musielak Sequence Spaces

4.1 A Criterion of Weak Compactness

We say that a set A is relatively (sequentially, countably) compact in a topological space if the closure of A is (sequentially, countably) compact. Indeed, we may assume $A \neq \{\theta\}$ avoiding the trivial case of a singleton set.

Theorem 4.1 *Given a set A in a Musielak sequence space $l_{(\Phi)}$, A admits the relatively sequentially weak $\sigma(l_{(\Phi)}, l_\Psi)$-compactness if, and only if there hold*

(1) *A is normed bounded*;

(2) *for each $v \in l_\Psi$, $\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0$.*

Proof Sufficiency. By Theorem 3.1, $l_{(\Psi)}$ is the dual space of h_Φ, we have that a normed bounded A is w^* compact thanks to Banach–Alaoglu Theorem, so A is relatively weakly $\sigma(l_{(\Phi)}, h_\Psi)$-compact. Since h_Ψ is normed separable [20, 25], the w^* topology, i.e., $\sigma(l_{(\Phi)}, h_\Psi)$, on A can be metrizable, then A is relatively sequentially weakly $\sigma(l_{(\Phi)}, h_\Psi)$-compact. Thus for any sequence $u_n \in A$ there exist $u \in l_{(\Phi)}$ and subsequence u_{n_k}, still written as u_n, such that $u_n \to u$ $\sigma(l_{(\Phi)}, h_\Psi)$-weakly, in particular, $u_n(i) \to u(i)$ for all natural numbers i.

For any $v \in l_\Psi$ and any positive number ε, from the given condition, we get a natural number I_0 such that

$$\sup_{u\in A} \sum_{i=I_0}^{\infty} |u(i)v(i)| < \frac{\varepsilon}{4}.$$

By Hölder's inequality, $\sum_{i\geq I_1} |u(i)v(i)| \leq \|u\|_{(\Phi)}\|v\|_\Psi < \infty$, there exists a natural number $I_1 \geq I_0$ such that

$$\sum_{i=I_1}^{\infty} |u(i)v(i)| < \frac{\varepsilon}{4}.$$

From $u_n(i) \to u(i)$ for all natural numbers i, we get a natural number n_0 such that for all $n \geq n_0$

$$\sum_{i=1}^{I_1} |(u_n - u)(i)||v(i)| < \frac{\varepsilon}{4}.$$

Therefore, for all $n \geq n_0$

$$\begin{aligned}
|\langle v, u_n - u\rangle| &= \left|\sum_{i=1}^{\infty}(u_n(i) - u(i))v(i)\right| \\
&= \left|\sum_{i=1}^{I_1}(u_n - u)(i)v(i) + \sum_{i=I_1+1}^{\infty}(u_n - u)(i)v(i)\right| \\
&\leq \left|\sum_{i=1}^{I_1}(u_n - u)(i)v(i)\right| + \left|\sum_{i=I_1+1}^{\infty}(u_n - u)(i)v(i)\right| \\
&\leq \sum_{i=1}^{I_1}|(u_n - u)(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|(u_n - u)(i)||v(i)| \\
&\leq \sum_{i=1}^{I_1}|(u_n - u)(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|u_n(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|u(i)||v(i)|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{I_1} |(u_n - u)(i)||v(i)| + \sup_{u\in A}\sum_{i\ge I_0}|u(i)v(i)| + \sum_{i=I_1+1}^{\infty}|u(i)||v(i)|\\
&\le \frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\\
&< \varepsilon.
\end{aligned}
$$

This shows that for all $v \in l_\Psi$, $\langle v, u_n - u\rangle \to 0$. Hence A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact.

Necessity. Since A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact A is $\sigma(l_{(\Phi)}, l_\Psi)$-bounded. By the Banach uniformly bounded principle, we get (1) A is normed bounded.

Next, we will prove (2) for all $v \in l_\Psi$

$$\lim_{I\to\infty}\sup_{u\in A}\sum_{i=I}^{\infty}|u(i)||v(i)| = 0.$$

Otherwise, for some $v \in l_\Psi$ and positive ε_0 such that there exists a strictly increasing sequence of natural number $\{I_n\}$ satisfying

$$\sup_{u\in A}\sum_{i=I_n}^{\infty}|u(i)v(i)| > \varepsilon_0.$$

We take $u_n \in A$ such that

$$\sum_{i=I_n}^{\infty}|u_n(i)v(i)| > \varepsilon_0.$$

Since A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact, we deduce that the sequence $\{u_n\}$ has a subsequence, still written as $\{u_n\}$ for simplicity, and for some $u \in l_{(\Phi)}$ such that $u_n \to u$ $\sigma(l_{(\Phi)}, l_\Psi)$-weakly. Subsequently, $u_n \to u$ $\sigma(l_{(\Phi)}, h_\Psi)$-weakly. By Hölder's inequality $\sum_{i=1}^{\infty}|u(i)||v(i)| \le \|u\|_{(\Phi)}\|v\|_\Psi < \infty$, there exists a natural number I' such that

$$\sum_{i=I'}^{\infty}|u(i)||v(i)| \le \frac{\varepsilon_0}{2}.$$

Thus, for all $I_n \ge I'$,

$$
\begin{aligned}
\sum_{i=I_n}^{\infty}|(u_n(i)-u(i))v(i)| &\ge \sum_{i=I_n}^{\infty}|u_n(i)v(i)| - |u(i)v(i)|\\
&= \sum_{i=I_n}^{\infty}|u_n(i)v(i)| - \sum_{i=I_n}^{\infty}|u(i)v(i)|\\
&> \varepsilon_0 - \frac{\varepsilon_0}{2}\\
&= \frac{\varepsilon_0}{2}.
\end{aligned}
$$

For simplicity, we still written $u_n - u$ as u_n. Then for all natural number n,

$$\sum_{i=I_n}^{\infty}|u_n(i)v(i)| \ge \frac{\varepsilon_0}{2},$$

and $u_n \to \theta$ $\sigma(l_{(\Phi)}, l_\Psi)$-weakly, especially, $u_n(i) \to 0$ for all natural numbers i.

By Hölder's inequality $\sum_{i=1}^{\infty}|u_1(i)||v(i)| < \infty$, we write $I_1 = 0$, there exists a natural number $I_1' > I_1$ such that $\sum_{i=I_1'+1}^{\infty}|u_1(i)||v(i)| < \frac{\varepsilon_0}{8}$. We write $u_{n_1} = u_1$, so we have

$$\sum_{i=I_1+1}^{I_1'}|u_{n_1}(i)||v(i)| = \sum_{i=I_1+1}^{\infty}|u_1(i)||v(i)| - \sum_{i=I_1'+1}^{\infty}|u_1(i)||v(i)| \geq \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{8} = \frac{3\varepsilon_0}{8}.$$

Since $u_n(i) \to 0$ for all natural numbers i, there exists a natural number n_2 such that $I_{n_2} > I_1'$, $\sum_{i=1}^{I_1'}|u_{n_2}(i)||v(i)| < \frac{\varepsilon_0}{8}$.

By Hölder's inequality $\sum_{i=1}^{\infty}|u_{n_2}(i)||v(i)| < \infty$, we take a natural number $I_{n_2}' > I_{n_2}$ such that $\sum_{i=I_{n_2}'+1}^{\infty}|u_{n_2}(i)||v(i)| < \frac{\varepsilon_0}{8}$. Then we have

$$\begin{aligned}
\sum_{i=I_{n_1}'+1}^{I_{n_2}'}|u_{n_2}(i)v(i)| &= \sum_{i=I_{n_1}'+1}^{I_{n_2}}|u_{n_2}(i)v(i)| + \sum_{i=I_{n_2}+1}^{I_{n_2}'}|u_{n_2}(i)v(i)| \\
&\geq \sum_{i=I_{n_2}+1}^{I_{n_2}'}|u_{n_2}(i)v(i)| \\
&= \sum_{i=I_{n_2}+1}^{\infty}|u_{n_2}(i)v(i)| - \sum_{i=I_{n_2}'+1}^{\infty}|u_{n_2}(i)v(i)| \\
&\geq \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{8} \\
&= \frac{3\varepsilon_0}{8}.
\end{aligned}$$

In induction procedure, for each nature number k, by $u_n(i) \to 0$ for all natural numbers i, there exists a natural number n_k such that $I_{n_k} > I_{n_{k-1}}'$,

$$\sum_{i=1}^{I_{n_{k-1}'}}|u_{n_k}(i)||v(i)| < \frac{\varepsilon_0}{8}.$$

By Hölder's inequality $\sum_{i=1}^{\infty}|u_{n_k}(i)||v(i)| < \infty$, we take a natural number $I_{n_k}' > I_{n_k}$ such that

$$\sum_{i>I_{n_k}'}|u_{n_k}(i)||v(i)| < \frac{\varepsilon_0}{8}.$$

Thus, we have

$$\begin{aligned}
\sum_{i=I_{n_{k-1}}'+1}^{I_{n_k}'}|u_{n_k}(i)v(i)| &= \sum_{i=I_{n_{k-1}}'+1}^{I_{n_k}}|u_{n_k}(i)v(i)| + \sum_{i=I_{n_k}+1}^{I_{n_k}'}|u_{n_k}(i)v(i)| \\
&\geq \sum_{i=I_{n_k}+1}^{I_{n_k}'}|u_{n_k}(i)v(i)| \\
&= \sum_{i=I_{n_k}+1}^{\infty}|u_{n_k}(i)v(i)| - \sum_{i>I_{n_k}'}|u_{n_k}(i)v(i)| \\
&\geq \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{8} \\
&= \frac{3\varepsilon_0}{8}.
\end{aligned}$$

We set $\tilde{v}(i) = |v(i)| \operatorname{sign} u_{n_k}(i)$ as $I'_{n_{k-1}} < i \le I'_{n_k}$ where $I'_{n_0} = 0$. Obviously, $|\tilde{v}(i)| = |v(i)|$ for all i. Since l_Ψ is symmetry, we get $\tilde{v} \in l_\Psi$. But for all k

$$\begin{aligned}
\langle \tilde{v}, u_{n_k} \rangle &= \sum_{i=1}^{\infty} u_{n_k}(i)\tilde{v}(i) \\
&= \sum_{i=1}^{I'_{n_{k-1}}} u_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} u_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_k}+1}^{\infty} u_{n_k}(i)\tilde{v}(i) \\
&= \sum_{i=1}^{I'_{n_{k-1}}} u_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |u_{n_k}(i)||v(i)| + \sum_{i=I'_{n_k}+1}^{\infty} u_{n_k}(i)\tilde{v}(i) \\
&\ge -\sum_{i=1}^{I'_{n_{k-1}}} |u_{n_k}(i)\tilde{v}(i)| + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |u_{n_k}(i)||v(i)| - \sum_{i=I'_{n_k}+1}^{\infty} |u_{n_k}(i)\tilde{v}(i)| \\
&= \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |u_{n_k}(i)||v(i)| - \sum_{i=1}^{I'_{n_{k-1}}} |u_{n_k}(i)\tilde{v}(i)| - \sum_{i=I'_{n_k}+1}^{\infty} |u_{n_k}(i)v(i)| \\
&\ge \frac{3\varepsilon_0}{8} - \frac{\varepsilon_0}{8} - \frac{\varepsilon_0}{8} \\
&= \frac{\varepsilon_0}{8}.
\end{aligned}$$

This is a contradiction with that $\langle \tilde{v}, u_{n_k} \rangle \to 0$, as a result of $u_n \to \theta$ $\sigma(l_\Phi, l_\Psi)$-weakly. It ends the proof. □

A set A is sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact ensures that A is $\sigma(l_{(\Phi)}, l_\Psi)$-closed. We immediately have

Corollary 4.2 *Given a set A in a Musielak sequence space $l_{(\Phi)}$, A admits the sequentially weak $\sigma(l_{(\Phi)}, l_\Psi)$-compactness if, and only if there hold*

(1) *A is $\sigma(l_{(\Phi)}, l_\Psi)$-closed,*

(2) *A is normed bounded,*

(3) *for each $v \in l_\Psi$*

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0.$$

4.2 A Modular Criterion of Compactness with Limitation Expression

We give a modular criterion which is gotten rid of elements of l_Ψ of Theorem 4.1.

Theorem 4.3 *Given $\lim_{u\to 0} \frac{\Phi_i(u)}{u} = 0$ for all i, a set A in a Musielak sequence space $l_{(\Phi)}$ admits the relatively sequentially weak $\sigma(l_{(\Phi)}, l_\Psi)$-compactness if, and only if there holds*

$$\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_\Phi(\xi u)}{\xi} = 0.$$

Proof Sufficiency. From Theorem 4.1, it is enough to show that Condition (1) and (2) of Theorem 4.1 hold. By $\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_\Phi(\xi u)}{\xi} = 0$, there exist ξ_1, $0 < \xi_1 \le 1$ such that $\sup_{u\in A} \frac{\rho_\Phi(\xi_1 u)}{\xi_1} \le 1$. Thus $\sup_{u\in A} \rho_\Phi(\xi_1 u) \le \xi_1 \le 1$, by Lemma 2.1, we have $\sup_{u\in A} \|\xi_1 u\|_{(\Phi)} \le 1$, i.e., $\sup_{u\in A} \|u\|_{(\Phi)} \le \frac{1}{\xi_1}$. That is, (1) A is normed bounded.

Next, for each $v \in l_\Psi$, by the definition, there exists a positive number λ with $\rho_{\Psi}(\lambda v) < \infty$. For any $\varepsilon > 0$, by the given condition, there exists a positive number ξ such that

$$\sup_{u\in A} \frac{\rho_{\Phi}(\xi u)}{\xi} < \frac{\lambda\varepsilon}{2}.$$

We take a natural number I_0 such that $\sum_{i>I_0} \Psi_i(\lambda v(i)) < \frac{\lambda\xi\varepsilon}{2}$. Then for all $u \in A$ and for all natural numbers $I > I_0$

$$\begin{aligned}
\sum_{i=I}^{\infty} |u(i)v(i)| &\leq \sum_{i=I_0}^{\infty} |u(i)v(i)| \\
&= \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \xi|u(i)|\lambda|v(i)| \\
&\leq \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} [\Phi_i(\xi|u(i)|) + \Psi_i(\lambda|v(i)|)] \\
&= \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \Phi_i(\xi u(i)) + \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \Psi_i(\lambda v(i)) \\
&\leq \frac{1}{\xi\lambda} \sum_{i=1}^{\infty} \Phi_i(\xi u(i)) + \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \Psi_i(\lambda v(i)) \\
&= \frac{1}{\xi\lambda} \rho_{\Phi}(\xi u(i)) + \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \Psi_i(\lambda v(i)) \\
&< \frac{1}{\lambda}\frac{\lambda\varepsilon}{2} + \frac{1}{\xi\lambda}\frac{\xi\lambda\varepsilon}{2} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

that says

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)v(i)| = 0.$$

i.e., (2) of Theorem 4.1 holds. Combining (1) and (2), by Theorem 4.1, we get that A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact.

Necessity. Since A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact, A is $\sigma(l_{(\Phi)}, l_\Psi)$-bounded. So A is normed bounded thanks to Banach uniformly bounded principle. Moreover, A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact if, and only if λA is for each $\lambda > 0$. Then, without loss of generality, assume $\sup_{u\in A} \|u\|_{(\Phi)} \leq 1$, we will show

$$\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_{\Phi}(\xi u)}{\xi} = 0.$$

Otherwise, for some positive ε_0

$$\inf_{\xi>0} \sup_{u\in A} \frac{\rho_{\Phi}(\xi u)}{\xi} = \lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_{\Phi}(\xi u)}{\xi} > \varepsilon_0,$$

where the identity holds because $\frac{\Phi(u)}{u}$ is nondecreasing. For each natural number n, we take

$u_n \in l_\Phi$ such that

$$\frac{\rho_\Phi\left(\frac{1}{2^{n+1}}u_n\right)}{\frac{1}{2^{n+1}}} > \varepsilon_0.$$

From Young's inequality [20], we see that for each i

$$\Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u_n(t)\right)\right) \le \Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u_n(t)\right)\right) + \Phi_i\left(\frac{1}{2^{n+1}}u_n(t)\right) \le \Phi_i\left(2\frac{1}{2^{n+1}}u_n(t)\right).$$

By Lemma 2.1, we get that for each n

$$\begin{aligned}
\rho_\Psi\left(\phi\left(\frac{1}{2^{n+1}}u_n\right)\right) &:= \sum_{i=1}^{\infty}\Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\right) \\
&\le \sum_{i=1}^{\infty}\Phi_i\left(2\frac{1}{2^{n+1}}u_n(i)\right) \\
&= \rho_\Phi\left(2\frac{1}{2^{n+1}}u_n\right) \\
&\le 2\frac{1}{2^{n+1}}\rho_\Phi(u_n) \\
&\le 2\frac{1}{2^{n+1}} \\
&= \frac{1}{2^n}.
\end{aligned}$$

We set $v(i) = \sup_n \phi_i(\frac{1}{2^{n+1}}u_n(i))$ $i = 1, 2, \ldots$. From the left continuity of an Orlicz function, we have

$$\begin{aligned}
\rho_\Psi(v) &= \sum_{i=1}^{\infty}\Psi_i(v(i)) \\
&= \sum_{i=1}^{\infty}\Psi_i\left(\sup_n \phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\right) \\
&= \sum_{i=1}^{\infty}\sup_n \Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\right) \\
&\le \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\right) \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\Psi_i\left(\frac{2}{2^{n+1}}u_n(i)\right) \\
&\le \sum_{n=1}^{\infty}\frac{2}{2^{n+1}}\rho_\Psi(u_n) \\
&\le \sum_{n=1}^{\infty}\frac{1}{2^n} \\
&= 1.
\end{aligned}$$

Now, v is well defined and $v \in l_\Psi$. Again since A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact, by Theorem 4.1, there exists a natural number I such that

$$\sup_{u\in A}\sum_{i=I}^{\infty}|u(i)v(i)| \le \frac{\varepsilon_0}{4}.$$

Note, for all i $|u(i)| \leq c_i$ where $c_i = \inf\{t > 0 : \Phi_i(t) > 1\} < \infty$ for all $u \in A$, from $\lim_{u\to 0} \frac{\Phi_i(u)}{u} = 0$, we obtain that for n large enough

$$\sum_{i=1}^{I} \frac{\Phi_i(\frac{1}{2^{n+1}}u_n(i))}{\frac{1}{2^{n+1}}} \leq \sum_{i=1}^{I} \frac{\Phi_i(\frac{1}{2^{n+1}}c_i)}{\frac{1}{2^{n+1}}} \leq \frac{\varepsilon_0}{4}.$$

It leads a contradiction:

$$\begin{aligned}
\varepsilon_0 > \frac{\rho_\Phi(\frac{1}{2^{n+1}}u_n)}{\frac{1}{2^{n+1}}} &= \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} \Phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&\leq \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} \Phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&\quad + \sum_{i=I+1}^{\infty} \Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u(i)\right)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&\leq \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}c_i\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} \Phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&\quad + \sum_{i=I+1}^{\infty} \Psi_i\left(\phi_i\left(\frac{1}{2^{n+1}}u(i)\right)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&= \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}c_i\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} \frac{1}{2^{n+1}}|u_n(i)|\phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right)\frac{1}{\frac{1}{2^{n+1}}} \\
&\leq \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}c_i\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} |u_n(i)|\phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right) \\
&\leq \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}c_i\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} |u_n(i)|\sup_n \phi_i\left(\frac{1}{2^{n+1}}u_n(i)\right) \\
&= \sum_{i=1}^{I} \Phi_i\left(\frac{1}{2^{n+1}}c_i\right)\frac{1}{\frac{1}{2^{n+1}}} + \sum_{i=I+1}^{\infty} |u_n(i)||v(t)| \\
&\leq \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} \\
&= \frac{\varepsilon_0}{2}.
\end{aligned}$$

It ends the proof. □

Due to the same reason mentioned before Corollary 4.2, we have

Corollary 4.4 *Given* $\lim_{u\to 0} \frac{\Phi_i(u)}{u} = 0$ *for all* i*, a set* A *in a Musielak sequence space* $l_{(\Phi)}$ *admits the sequentially weak* $\sigma(l_{(\Phi)}, l_\Psi)$*-compactness if, and only if there hold*

(1) A *is* $\sigma(l_{(\Phi)}, l_\Psi)$*-closed,*

(2) $\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_\Phi(\xi u)}{\xi} = 0.$

4.3 A Modular Criterion of Compactness without Limitation Expression

We give one criterion of a modular type which is gotten rid of the computation of limitation.

Definition 4.5 (cf. [2]) *For Musielak functions* $\tilde{\Phi}$ *and* Φ *over real field* R *we say* $\tilde{\Phi}$ *more rapid than* Φ *(write* $\tilde{\Phi} \succ \Phi$*) provided that for any positive number* κ*, there is positive number*

D such that for all $0 < u$, $\tilde{\Phi}_i(Du) \geq D\kappa\Phi_i(u)$ $i = 1, 2, \ldots$.

Theorem 4.6 *Given $\lim_{u\to 0}\frac{\Phi_i(u)}{u} = 0$ for all i, a set A in a Musielak sequence space $l_{(\Phi)}$ admits the relatively sequentially weak $\sigma(l_{(\Phi)}, l_\Psi)$-compactness if, and only if there exists a Musielak function $\tilde{\Phi}$ more rapid than Φ (write $\tilde{\Phi} \succ \Phi$) such that*

$$\sup_{u\in A}\rho_{\tilde{\Phi}}(u) \leq 1.$$

Proof Due to the reason mentioned in Theorem 4.3, we assume $\sup_{u\in A}\|u\|_{(\Phi)} \leq 1$ without loss of generality.

Sufficiency. By $\tilde{\Phi} \succ \Phi$, that says, for any positive number $\varepsilon < 1$, let $\kappa = \frac{2}{\varepsilon}$ in the condition of $\tilde{\Phi} \succ \Phi$, we have a positive number D such that for all $0 < u$, $\tilde{\Phi}_i(Du) \geq D\frac{2}{\varepsilon}\Phi_i(u)$, $i = 1, 2, \ldots$.

For each $u \in A$, we take a positive number $\xi \leq 1$ such that $\frac{1}{\xi} \geq D$. Then we have

$$\begin{aligned}
\frac{\rho_\Phi(\xi u)}{\xi} &= \frac{1}{\xi}\sum_{i=1}^{\infty}\Phi_i(\xi u(i)) \\
&= \sum_{i=1}^{\infty}\frac{\Phi_i(\xi u(i))}{\xi} \\
&\leq \sum_{i=1}^{\infty}\frac{1}{\xi}\tilde{\Phi}_i(D\xi u(i))\frac{\varepsilon}{2D} \\
&= \sum_{i=1}^{\infty}\tilde{\Phi}_i(D\xi u(i))\frac{\varepsilon}{2D\xi} \\
&\leq \sum_{i=1}^{\infty}\tilde{\Phi}_i(u(i))\frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2}\sum_{i=1}^{\infty}\tilde{\Phi}_i(u(i)) \\
&\leq \frac{\varepsilon}{2}\rho_{\tilde{\Phi}}(u) \\
&\leq \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

where the fifth inequality holds because the function $f(u) = \frac{\Phi(u)}{u}$ is nondecreasing. That is, $\lim_{\xi\to 0}\sup_{u\in A}\frac{\rho_\Phi(\xi u)}{\xi} = 0$. By Theorem 4.3, it follows that A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact.

Necessity. By Theorem 4.3, A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact ensures that $\lim_{\xi\to 0}\sup_{u\in A}\frac{\rho_\Phi(\xi u)}{\xi} = 0$. We take $1 > \xi_1 > \cdots > \xi_k \to 0$ such that

$$\sup_{u\in A}\frac{\rho_\Phi(\xi_k u)}{\xi_k} < \frac{1}{2^{2k}}.$$

Analogously to that of [2], for each i and any $u \in \Re$, we set an Orlicz function as follows:

$$\tilde{\Phi}_i(u) = \sum_{k=1}^{\infty}2^k\frac{\Phi_i(\xi_k u)}{\xi_k}.$$

Then $\tilde{\Phi}_i \succ \Phi_i$. Indeed, for any positive κ we take a natural number k' such that $2^{2k'} \geq \kappa$ and denote $D = \frac{1}{\xi_{k'}}$. Therefor, for all $u \in R$

$$\begin{aligned}
\tilde{\Phi}_i(Dv)_{v=\xi_{k'}u} &= \tilde{\Phi}_i\left(\frac{\xi_{k'}u}{\xi_{k'}}\right) \\
&= \tilde{\Phi}_i(u) \\
&= \sum_{k=1}^{\infty} 2^{2k}\frac{\Phi_i(\xi_k u)}{\xi_k} \\
&\geq 2^{2k'}\frac{\Phi_i(\xi_{k'} u)}{\xi_{k'}} \\
&= 2^{2k'}\frac{\Phi_i(v)}{\xi_{k'}} \\
&\geq \kappa D\Phi_i(v)
\end{aligned}$$

it holds for all $v > 0$ due to the arbitrariness of u. By the arbitrariness of i, it follows that $\tilde{\Phi} \succ \Phi$. Moreover, we have that for all $u \in A$,

$$\begin{aligned}
\rho_{\tilde{\Phi}}(u) &= \sum_{i=1}^{\infty} \tilde{\Phi}_i(u(i)) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} 2^{k}\frac{\Phi_i(\xi_k u(i))}{\xi_k} \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} 2^{k}\frac{\Phi_i(\xi_k u(i))}{\xi_k} \\
&= \sum_{k=1}^{\infty} 2^{k}\sum_{i=1}^{\infty}\frac{\Phi_i(\xi_k u(i))}{\xi_k} \\
&\leq \sum_{k=1}^{\infty} 2^{k}\frac{1}{2^{2k}} \\
&= \sum_{k=1}^{\infty}\frac{1}{2^{k}} \\
&= 1.
\end{aligned}$$

It ends the proof. □

Analogously to the above, we have

Corollary 4.7 *Given* $\lim_{u\to 0}\frac{\Phi_i(u)}{u} = 0$ *for all* i*, a set* A *in a Musielak sequence space* $l_{(\Phi)}$ *admits the sequentially weakly* $\sigma(l_{(\Phi)}, l_\Psi)$*-compactness if, and only if there hold*

(1) A *is* $\sigma(l_{(\Phi)}, l_\Psi)$*-closed,*

(2) *there exists a Musielak function* $\tilde{\Phi}$ *more rapid than* Φ *(write* $\tilde{\Phi} \succ \Phi$*) such that*

$$\sup_{u\in A}\rho_{\tilde{\Phi}}(u) \leq 1.$$

From Theorem 4.1, 4.3, and 4.6, we see

Remark 4.8 In $l_{(\Phi)}$, a set A is relatively sequentially weakly $\sigma(l_{(\Phi)}, l_\Psi)$-compact if, and only if $|A|$ is, where $|A| = \{|u| : u \in A\}$.

By Lemma 2.2, $l_{(\Phi)}$ is isomorphic to l_Φ, we have

Remark 4.9 All results obtained in the main results hold in a Musielak sequence space l_Φ with Orlicz norm since the sequentially weak compactness is invariant under an isomorphism.

5 Applications to Orlicz Sequence Spaces

Let us notice that in the case of $\Phi_i = M$, $i = 1, 2, \ldots$, Musielak sequences spaces become the well-known Orlicz sequence spaces. Based on the results obtained in the previous parts of this paper we easily get respective criteria for Orlicz sequence spaces $l_{(M)}$.

Corollary 5.1 *Given a set A in an Orlicz sequence space $l_{(M)}$, A admits relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if, and only if there hold*

(1) *A is normed bounded,*

(2) *for each $v \in l_N$*

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0.$$

Corollary 5.2 *Given a set A in an Orlicz sequence space $l_{(M)}$, A admits sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if, and only if there hold*

(1) *A is $\sigma(l_{(M)}, l_N)$-closed,*

(2) *A is normed bounded,*

(3) *for each $v \in l_N$*

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0.$$

Corollary 5.3 *Given $\lim_{u\to 0} \frac{M(u)}{u} = 0$, a set A in an Orlicz sequence space $l_{(M)}$ admits relatively sequentially weak $\sigma(l_{(M)}, l_N)$-compact if, and only if $\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_\Phi(\xi u)}{\xi} = 0$.*

Corollary 5.4 *Given $\lim_{u\to 0} \frac{M(u)}{u} = 0$, a set A in an Orlicz sequence space $l_{(M)}$ admits sequentially weak $\sigma(l_{(M)}, l_N)$-compact if, and only if there hold*

(1) *A is $\sigma(l_{(M)}, l_N)$-closed,*

(2) $\lim_{\xi\to 0} \sup_{u\in A} \frac{\rho_\Phi(\xi u)}{\xi} = 0$.

Definition 5.5 (cf. [2]) *For Orlicz functions $\tilde{M}$ and M over real field R we say $\tilde{M}$ more rapid than M for small u (write $\tilde{M} \succ M$) provided that for any positive number κ, there are positive numbers D and d such that for $0 < u$ with $M(u) \le d$, $\tilde{M}(Du) \ge D\kappa M(u)$.*

Corollary 5.6 *Given $\lim_{u\to 0} \frac{M(u)}{u} = 0$, a set A in an Orlicz sequence space $l_{(M)}$ admits relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if, and only if there exists an Orlicz function $\tilde{M}$ more rapid than M (write $\tilde{M} \succ M$) such that*

$$\sup_{u\in A} \rho_{\tilde{M}}(u) \le 1.$$

Corollary 5.7 *Given $\lim_{u\to 0} \frac{M(u)}{u} = 0$, a set A in an Orlicz sequence space $l_{(M)}$ admits sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if, and only if*

(1) *A is $\sigma(l_{(M)}, l_N)$-closed,*

(2) *there exists an Orlicz function $\tilde{M}$ more rapid than M (write $\tilde{M} \succ M$) such that*

$$\sup_{u\in A} \rho_{\tilde{M}}(u) \le 1.$$

Remark 5.8 All results obtained in the main results hold in an Orlicz sequence space l_M with Orlicz norm since the sequentially weak compactness is invariant under an isomorphism.

Conflict of Interest The authors declare no conflict of interest.

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