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Maximizing the Minimum and Maximum Forcing Numbers of Perfect Matchings of Graphs

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Abstract Let G be a simple graph with $2n$ vertices and a perfect matching. The forcing number $f(G,M)$ of a perfect matching M of G is the smallest cardinality of a subset of M that is contained in no other perfect matching of G. Among all perfect matchings M of G, the minimum and maximum values of $f(G,M)$ are called the minimum and maximum forcing numbers of G, denoted by $f(G)$ and $F(G)$, respectively. Then $f(G) \leq F(G) \leq n-1$. Che and Chen (2011) proposed an open problem: how to characterize the graphs G with $f(G)=n-1$. Later they showed that for a bipartite graph G, $f(G)=n-1$ if and only if G is complete bipartite graph $K_{n,n}$. In this paper, we completely solve the problem of Che and Chen, and show that $f(G)=n-1$ if and only if G is a complete multipartite graph or a graph obtained from complete bipartite graph $K_{n,n}$ by adding arbitrary edges in one partite set. For all graphs G with $F(G)=n-1$, we prove that the forcing spectrum of each such graph G forms an integer interval by matching 2-switches and the minimum forcing numbers of all such graphs G form an integer interval from $\lfloor \frac{n}{2} \rfloor$ to $n-1$.

Keywords Perfect matching, minimum forcing number, maximum forcing number, forcing spectrum, complete multipartite graph

MR(2010) Subject Classification 05C70, 05C75, 05C35

1 Introduction

We only consider finite and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of vertices in G. A graph is *trivial* if it contains only one vertex. Otherwise, it is *non-trivial*. The *degree* of vertex v in G, written $d_G(v)$, is the number of edges incident to v. An *isolated vertex* is a vertex of degree 0. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If all vertices of G have degree k, then G is *k-regular*. A complete graph of order n is denoted by K_n. Let P_n be a path with n vertices and $\overline{P_n}$ be the complement of P_n.

For an edge subset F of G, we write $G-F$ for the subgraph of G obtained by deleting the edges in F. If $F=\{e\}$, we simply write $G-e$ instead of $G-\{e\}$. For a vertex subset T of G, we write $G-T$ for the subgraph of G obtained by deleting all vertices in T and their incident edges. If $T=\{v\}$ is a singleton, we write $G-v$ rather than $G-\{v\}$. For a vertex subset T of

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G, we write $G[T]$ for the subgraph $G - (V(G) \setminus T)$, induced by T. For a graph H, G is *H-free* if it contains no H as an induced subgraph.

A perfect matching of a graph G is a set of disjoint edges covering all vertices of G. A graph G is *factor-critical* if $G - u$ has a perfect matching for every vertex u of G. A graph G is *bicritical* if G contains an edge and $G - u - v$ has a perfect matching for every pair of distinct vertices u and v in G. A 3-connected bicritical graph is called a *brick*. For a nonnegative integer l, a connected graph G with at least $2l + 2$ vertices is *l-extendable* if G has a perfect matching and every matching of size l is contained in a perfect matching of G.

A perfect matching coincides with a Kekulé structure in organic chemistry or a dimer covering in statistic physics. Klein and Randić [13] proposed the "*innate degree of freedom*" of a Kekulé structure, i.e., the least number of double bonds can determine this entire Kekulé structure, which plays an important role in resonant theory. Afterwards, it was called the forcing number by Harary et al. [10]. A *forcing set* for a perfect matching M of G is a subset of M that is contained in no other perfect matching of G. The smallest cardinality of a forcing set of M is called the *forcing number* of M, denoted by $f(G, M)$.

Let G be a graph with a perfect matching M. A cycle of G is *M-alternating* if its edges appear alternately in M and $E(G)\setminus M$. If C is an M-alternating cycle of G, then the symmetric difference $M \oplus E(C) := (M \setminus E(C)) \cup (E(C) \setminus M)$ is another perfect matching of G. We use $V(S)$ to denote the set of all end vertices in an edge subset S of $E(G)$. An equivalent condition for a forcing set of a perfect matching was mentioned by Pachter and Kim as follows.

Lemma 1.1 ([18]) *Let G be a graph with a perfect matching M. Then a subset $S \subseteq M$ is a forcing set of M if and only if $G - V(S)$ contains no M-alternating cycles.*

Let $c(M)$ denote the maximum number of disjoint M-alternating cycles in G. Then $f(G, M) \geq c(M)$ by Lemma 1.1. For plane bipartite graphs, Pachter and Kim obtained the following minimax theorem.

Theorem 1.2 ([18]) *Let G be a plane bipartite graph. Then $f(G, M) = c(M)$ for any perfect matching M of G.*

The *minimum* (resp. *maximum*) *forcing number* of G is the minimum (resp. maximum) value of $f(G, M)$ over all perfect matchings M of G, denoted by $f(G)$ (resp. $F(G)$). Adams et al. [2] showed that determining a smallest forcing set of a perfect matching is NP-complete for bipartite graphs with the maximum degree 3. Using this result, Afshani et al. [3] proved that determining the minimum forcing number is NP-complete for bipartite graphs with the maximum degree 4. However, the computational complexity of the maximum forcing number of a graph is still an open problem [3].

Xu et al. [29] showed that the maximum forcing number of a hexagonal system is equal to its resonant number. The same result also holds for a polyomino graph [16, 39] and for a BN-fullerene graph [22]. In general, for 2-connected plane bipartite graphs, the resonant number can be computed in polynomial time (see Ref. [1] due to Abeledo and Atkinson). Hence, the maximum forcing numbers of such three classes of graphs can be computed in polynomial time.

Moreover, some minimax results have been obtained [38, 39]: for each perfect matching M of a hexagonal system G with $f(G, M) = F(G)$, there exist $F(G)$ disjoint M-alternating hexagons in G; for every perfect matching M of polyomino graphs G with $f(G,M)=F(G)$ or $F(G)-1$, $f(G,M)$ is equal to the maximum number of disjoint M-alternating squares in G.

Zhang and Li [32], and Hansen and Zheng [9] independently determined the hexagonal systems G with $f(G)=1$, and Zhang and Zhang [35] gave a generalization to plane bipartite graphs G with $f(G)=1$. For 3-connected cubic graphs G with $f(G)=1$, Wu et al. [28] showed that it can be generated from K_4 via $Y\to\Delta$-operation. For a convex hexagonal system $H(a_1,a_2,a_3)$ with a perfect matching, recently Zhang and Zhang [36] proved that its minimum forcing number is equal to $\min\{a_1,a_2,a_3\}$ by monotone path systems.

For n-dimensional hypercube Q_n, Pachter and Kim [18] conjectured that $f(Q_n)=2^{n-2}$ for integer $n\geq 2$. Later Riddle [21] confirmed it for even n by the trailing vertex method. Recently, Diwan [8] proved that the conjecture holds by linear algebra. Using well-known Van der Waerden theorem, Alon showed that $F(Q_n)>c2^{n-1}$ for any constant $0<c<1$ and sufficient large n (see [21]). There are also some researches about the minimum or maximum forcing numbers of other graphs, such as grids [3, 12, 14, 15, 18], fullerene graphs [11, 22, 23, 34], toroidal polyhexes [26], toroidal and Klein bottle lattices [12, 14, 21], etc.

We denote by $\mathcal{G}_{2n}$ the set of all graphs of order $2n$ and with a perfect matching. Let $G\in\mathcal{G}_{2n}$. Then each perfect matching of G has n edges and any $n-1$ edges among it form a forcing set. So we have that $f(G)\leq F(G)\leq n-1$. Che and Chen [6] proposed how to characterize the graphs G with $f(G)=n-1$. Afterwards, they [5] solved the problem for bipartite graphs and obtained the following result.

Theorem 1.3 ([5]) *Let G be a bipartite graph with $2n$ vertices. Then $f(G)=n-1$ if and only if G is complete bipartite graph $K_{n,n}$.*

In this paper, we solve the problem of Che and Chen. Let $\mathcal{K}^+_{n,n}$ be a family of graphs obtained by adding arbitrary additional edges in one partite set to complete bipartite graph $K_{n,n}$. In Section 2, we discuss some basic properties for graphs $G\in\mathcal{G}_{2n}$ with $F(G)=n-1$, and obtain that G is n-connected, 1-extendable except for graphs in $\mathcal{K}^+_{n,n}$. In particular, we give a characterization for a perfect matching M of G with $f(G,M)=n-1$. In Section 3, we answer the problem proposed by Che and Chen, and obtain that $f(G)=n-1$ if and only if G is either a complete multipartite graph with each partite set having size no more than n or a graph in $\mathcal{K}^+_{n,n}$. In Section 4, for all 1-extendable graphs G with $F(G)=n-1$, we determine which of them are not 2-extendable. Finally in Section 5 we show that $f(G)\geq\lfloor\frac{n}{2}\rfloor$ for any graph $G\in\mathcal{G}_{2n}$ with $F(G)=n-1$, and the minimum forcing numbers of all such graphs form an integer interval $[\lfloor\frac{n}{2}\rfloor,n-1]$. Also we prove that the forcing spectrum (set of forcing numbers of all perfect matchings) of each such graph G forms an integer interval.

2 Some Basic Properties of Graphs $G\in\mathcal{G}_{2n}$ with $F(G)=n-1$

Let $G\in\mathcal{G}_{2n}$ with $F(G)=n-1$. In this section, we will obtain some basic properties of graph G. By definition of forcing numbers, we obtain the following observation.

Observation 2.1 Let $G\in\mathcal{G}_{2n}$. Then

(i) $F(G)=n-1$ if and only if $f(G,M)=n-1$ for some perfect matching M of G, and

(ii) $f(G)=n-1$ if and only if $f(G,M)=n-1$ for every perfect matching M of G.

We call a graph $G\in\mathcal{G}_{2n}$ with $F(G)=n-1$ *minimal* if $F(G-e)\leq n-2$ for each edge e

of G. Next we give a characterization for a perfect matching M of G with $f(G,M)=n-1$.

Lemma 2.2 *Let $G \in \mathcal{G}_{2n}$ for $n \geq 2$. Then G has a perfect matching M such that $f(G,M)=n-1$ if and only if $G[V(\{e_i,e_j\})]$ contains an M-alternating cycle for any two distinct edges e_i and e_j of M. Moreover, G is minimal if and only if $G[V(\{e_i,e_j\})]$ is exactly an M-alternating 4-cycle for any two distinct edges e_i and e_j of M.*

Proof (1) Suppose that $f(G,M)=n-1$. Then $M \setminus \{e_i,e_j\}$ is not a forcing set of M for any two distinct edges e_i and e_j of M. By Lemma 1.1, $G[V(\{e_i,e_j\})]$ contains an M-alternating cycle. Conversely, for any subset S of M with size less than $n-1$, there are two distinct edges in $M \setminus S$. By the assumption, $G-V(S)$ contains an M-alternating cycle. By Lemma 1.1, S is not a forcing set of M, that is, $f(G,M) \geq n-1$.

(2) Suppose that G is minimal. Since $f(G,M)=n-1$, $G[V(\{e_i,e_j\})]$ contains an M-alternating 4-cycle for any two distinct edges e_i and e_j of M. Moreover, $G[V(\{e_i,e_j\})]$ is exactly an M-alternating 4-cycle. If not, then $G[V(\{e_i,e_j\})]$ is isomorphic to K_4 or a graph obtained from K_4 by deleting an edge, say u_iu_j, where $e_l=u_lv_l$ for $l=i,j$. Then $G[V(\{e_i,e_j\})]-v_iv_j$ contains an M-alternating cycle. So $F(G-v_iv_j)=f(G-v_iv_j,M)=n-1$, which contradicts the minimality of G.

Conversely, by the assumption, we have $f(G,M)=n-1$ where $M=\{e_l=u_lv_l \,|\, l=1,2,\ldots,n\}$ is a perfect matching of G. By Observation 2.1, $F(G)=n-1$. Next we are to prove that G is minimal. Let $G'=G-e$ where e is an arbitrary edge of G. We can show that G' has a perfect matching. If $e \notin M$, then M is a perfect matching of G'. If $e=e_i \in M$ for some $1 \leq i \leq n$, then $G[\{u_i,v_i,u_j,v_j\}]$ is exactly an M-alternating 4-cycle C for any $j \neq i$ and $1 \leq j \leq n$. Then $M \oplus E(C)$ is a perfect matching of G'. For any perfect matching M' of G', we will show that $f(G',M') \leq n-2$, and so G is minimal.

Without loss of generality, any edge e of G can be represented as either u_iv_i or u_iv_j where $j \neq i$ because any edge u_jv_j can be written as v_ju_j by switching two end vertices. Let $w \in \{u,v\}$.

Case 1 $e=u_iv_i$.

Then $\{u_iw_j, v_iw_k\} \subseteq M'$ for some integers j,k different from i. It follows that $v_iw_j \notin E(G') \subset E(G)$ since $G[\{u_i,v_i,u_j,v_j\}]$ is a 4-cycle by the assumption. Then $G[\{u_i,w_j,v_i,w_k\}]$ does not contain a 4-cycle of G' since v_i cannot be adjacent to either u_i or w_j in G'. By Lemma 1.1, $M' \setminus \{u_iw_j, v_iw_k\}$ is a forcing set of M'. So, $f(G',M') \leq n-2$.

Case 2 $e=u_iv_j$ where $j \neq i$.

Subcase 2.1: Both u_iv_i and u_jv_j are contained in M'. By the assumption, $G[\{u_i,v_i,u_j,v_j\}]$ is an M-alternating 4-cycle of G where $\{u_iv_i,u_jv_j\} \subseteq M$. So $G'[\{u_i,v_i,u_j,v_j\}]$ is just a path of length three since $G'=G-e=G-u_iv_j$. By Lemma 1.1, $M' \setminus \{u_iv_i,u_jv_j\}$ is a forcing set of M'. So, $f(G',M') \leq n-2$.

Subcase 2.2: At least one of u_iv_i and u_jv_j is not contained in M'. Without loss of generality, assume that $u_jv_j \notin M'$. Since $e=u_iv_j$ is not an edge of G', there exists a vertex $w_l \neq u_i$ such that $v_jw_l \in M'$. Note that v_iv_j cannot be an edge of $G' \subset G$ since $G[\{u_i,v_i,u_j,v_j\}]$ is an M-alternating 4-cycle of G. Then $w_l \neq v_i$. If $v_iw_l \notin E(G')$, then v_i is adjacent to neither w_l nor v_j in G'. So $G'[\{v_i,w_t,w_l,v_j\}]$ contains no M'-alternating cycles where $v_iw_t \in M'$. By Lemma 1.1, $M' \setminus \{v_iw_t, v_jw_l\}$ is a forcing set of M'. Thus, $f(G',M') \leq n-2$. If $v_iw_l \in E(G')$, then $u_iw_l \notin E(G') \subset E(G)$ since $G[\{v_i, u_i, u_l, v_l\}]$ is exactly an M-alternating 4-cycle by the assumption. Since $G' = G - e$, we have $e = u_iv_j \notin E(G')$. So u_i is adjacent to neither w_l nor v_j in G', and $G'[\{u_i, w_k, w_l, v_j\}]$ contains no M'-alternating cycles where $u_iw_k \in M'$. By Lemma 1.1, $M' \setminus \{u_iw_k, v_jw_l\}$ is a forcing set of M'. So, $f(G', M') \leq n - 2$. □

By Lemma 2.2, adding some extra edges to a minimal graph G maintains the same maximum forcing number as G. Clearly, $K_{n,n}$ is minimal for $n \geq 1$ by Lemma 2.2. For $n \geq 3$, we have other such minimal graphs shown in Figure 1.

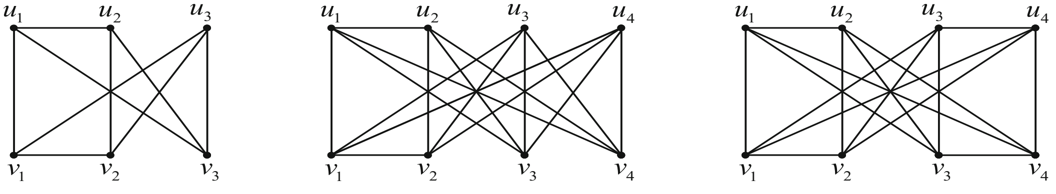

Figure 1 Some examples of minimal graphs with $n = 3$ and 4

Let G be a graph with a perfect matching. An edge e of G is called a *fixed double bond* if e is contained in all perfect matchings of G. The connectivity and edge connectivity of G are denoted by $\kappa(G)$ and $\lambda(G)$, respectively.

Lemma 2.3 *Assume that $G \in \mathcal{G}_{2n}$ has $F(G) = n - 1 > 0$. Then*

(i) *G has no fixed double bond, that is, $G - e$ has a perfect matching for each edge e of G.*

(ii) *$\kappa(G) \geq n$. Moreover, if G is minimal, then G is n-regular, and $\kappa(G) = \lambda(G) = n$.*

Proof (i) It has been implied in the sufficiency part of (2) in the proof of Lemma 2.2.

(ii) Since $F(G) = n - 1$, there exists a perfect matching M of G such that $f(G, M) = n - 1$ where $M = \{u_iv_i \mid i = 1, 2, \ldots, n\}$. For any $X \subseteq V(G)$ with $|X| < n$, there is at least one pair of vertices u_i and v_i not in X for some $1 \leq i \leq n$. Then for the other vertices not in X, say u_j (resp. v_j), either u_iu_j or v_iu_j (resp. u_iv_j or v_iv_j) is contained in $E(G)$ by Lemma 2.2. Hence $G - X$ is connected and so G is n-connected. Therefore, $\kappa(G) \geq n$.

If G is minimal, then for any $1 \leq i \leq n$, u_i (resp. v_i) is adjacent to exactly one of u_j and v_j for any $1 \leq j \leq n$ and $j \neq i$ by Lemma 2.2. Combining that u_iv_i is an edge, we obtain that u_i and v_i are of degree n. So G is n-regular. Combining that $n \leq \kappa(G) \leq \lambda(G) \leq \delta(G) = n$, we obtain that $\kappa(G) = \lambda(G) = n$. □

The number of odd components of G is denoted by $o(G)$. The following result gives an equivalent condition for a bicritical graph.

Lemma 2.4 ([17]) *A graph G is bicritical if and only if for any $X \subseteq V(G)$ and $|X| \geq 2$, $o(G - X) \leq |X| - 2$.*

Next we will show that $\mathcal{K}^+_{n,n}$ is a subclass of graphs $G \in \mathcal{G}_{2n}$ with $F(G) = n - 1$ such that G contains an independent set of size n.

Lemma 2.5 *Assume that $G \in \mathcal{G}_{2n}$ has $F(G) = n - 1$. Then G is a graph in $\mathcal{K}^+_{n,n}$ if and only if G contains an independent set of size n. Otherwise, G is a brick, and thus 1-extendable.*

Proof (1) The necessity is obvious. We prove sufficiency next. Since $F(G) = n-1$, there exists a perfect matching M of G such that $f(G, M) = n - 1$. Let $\{u_1, u_2, \ldots, u_n\}$ be an independent set of size n in G and let $M = \{u_iv_i \mid i = 1, 2, \ldots, n\}$. Then $u_iu_j \notin E(G)$ for any $1 \le i < j \le n$. By Lemma 2.2, $\{u_iv_j, v_iu_j\}$ is contained in $E(G)$. Hence, G is some graph in $\mathcal{K}^+_{n,n}$, for there may be some other edges with both end vertices in $\{v_1, v_2, \ldots, v_n\}$.

(2) If G is not a graph in $\mathcal{K}^+_{n,n}$, then we will prove that G is a brick. For $n \ge 3$, G is 3-connected by Lemma 2.3. For $n \le 2$, exactly one graph K_4 that is not in $\mathcal{K}^+_{n,n}$ is 3-connected. Next we prove that G is bicritical. Suppose that $X \subseteq V(G)$ with $|X| \ge 2$. If $|X| \le n-1$, then $G-X$ is connected by Lemma 2.3. Hence $o(G-X) \le 1 \le |X|-1$. Otherwise, we have $|X| \ge n$. Then $o(G-X) \le |V(G-X)| \le n$. Since G is not a graph in $\mathcal{K}^+_{n,n}$, G contains no independent set of size n. Hence $o(G-X) \le n-1 \le |X|-1$. Since G is of even order, $o(G-X)$ and $|X|$ are of the same parity. So $o(G-X) \le |X|-2$. By Lemma 2.4, G is bicritical. □

3 Graphs $G \in \mathcal{G}_{2n}$ with the Minimum Forcing Number $n-1$

In this section, we will determine all graphs $G \in \mathcal{G}_{2n}$ with $f(G) = n-1$ to completely solve the problem proposed by Che and Chen [6].

Tutte's theorem states that G has a perfect matching if and only if $o(G-S) \le |S|$ for any $S \subseteq V(G)$. By Tutte's theorem, Yu [30] obtained an equivalent condition for a connected graph with a perfect matching that is not l-extendable. Recently, for an $(l-1)$-extendable graph G with $l \ge 1$, Alajbegović et al. [4] obtained that G is not l-extendable if and only if there exists a subset $S \subseteq V(G)$ such that $G[S]$ contains l independent edges and $o(G-S) = |S| - 2l + 2$. In fact, S can be chosen so that each component of $G-S$ is factor-critical (see Theorem 2.2.3 in [7]). Combining these, we obtain the following result.

Lemma 3.1 *Let $l \ge 1$ be an integer and G be an $(l-1)$-extendable graph of order at least $2l+2$. Then G is not l-extendable if and only if there exists a subset $S \subseteq V(G)$ such that $G[S]$ contains l independent edges, all components of $G-S$ are factor-critical, and $o(G-S) = |S| - 2l + 2$.*

A *complete multipartite graph* is a graph whose vertices can be partitioned into sets so that u and v are adjacent if and only if u and v belong to different sets of the partition. We write $K_{n_1,n_2,\ldots,n_k}$ for the complete k-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_k$. In fact, a complete multipartite graph is a $\overline{P_3}$-free graph (see Exercise 5.2.2 in [27]).

Lemma 3.2 ([27]) *A graph is $\overline{P_3}$-free if and only if it is a complete multipartite graph.*

Theorem 3.3 *Let $G \in \mathcal{G}_{2n}$. Then $f(G) = n-1$ if and only if G is a complete multipartite graph with each partite set having size no more than n or G is some graph in $\mathcal{K}^+_{n,n}$.*

Proof Sufficiency. Suppose that G is some graph in $\mathcal{K}^+_{n,n}$. Then any perfect matching M of G is also a perfect matching of $K_{n,n}$. By Lemma 2.2, $f(G, M) = n-1$. By the arbitrariness of M, $f(G) = n-1$.

Suppose that $G = K_{n_1,n_2,\ldots,n_k}$ is a complete multipartite graph where $1 \le n_i \le n$ for $1 \le i \le k$ and $k \ge 2$. First we will show that G has a perfect matching. For any nonempty subset S of $V(G)$, if $G-S$ has at least two vertices from different partite sets of G, then $G-S$ is connected and $o(G-S) \le 1 \le |S|$. Otherwise, all vertices of $G-S$ belong to one partite set of G. Then $|V(G-S)| \le n$ and $|S| \ge n$. Hence $o(G-S) \le n \le |S|$. For $S = \emptyset$, $o(G-S) = o(G) = |S|$. By Tutte's theorem, G has a perfect matching.

Clearly, the result holds for $n = 1$. Next let $n \ge 2$. Suppose to the contrary that $f(G) \le$

$n-2$. Then there exists a perfect matching M of G and a minimum forcing set S of M such that $|S| = f(G, M) = f(G)$. By Lemma 1.1, $G - V(S)$ contains no M-alternating cycles. So there are two distinct edges $\{u_iv_i, u_jv_j\} \subseteq M \setminus S$ and $G[\{u_i, v_i, u_j, v_j\}]$ contains no M-alternating cycles. That is to say, neither $\{u_iu_j, v_iv_j\}$ nor $\{u_iv_j, v_iu_j\}$ is contained in $E(G)$. Without loss of generality, we assume that none of u_iu_j and u_iv_j belong to $E(G)$. Then $G[\{u_i, u_j, v_j\}]$ is isomorphic to $\overline{P_3}$, which contradicts Lemma 3.2.

Necessity. Suppose that $f(G) = n - 1$. If G is a complete multipartite graph, then each partite set of G has size no more than n for G has a perfect matching. If G is not a complete multipartite graph, then by Lemma 3.2, G contains an induced subgraph H isomorphic to $\overline{P_3}$. Note that the edge e of H is not in any perfect matching of G. Otherwise, there is a perfect matching M of G containing e. By Observation 2.1, $f(G, M) = n - 1$. Let v be the vertex of H except for both end vertices of e, and e' be the edge of M incident with v. Then $G[V(\{e, e'\})]$ contains no M-alternating cycles for v is not incident with any end vertices of e, which contradicts Lemma 2.2. So G is not 1-extendable. By Lemma 3.1, there exists $S \subseteq V(G)$ such that $G[S]$ contains an edge, all components of $G - S$ are factor-critical, and $o(G - S) = |S| \geq 2$.

We claim that all components of $G - S$ are singletons. Otherwise, assume that C_1 is a non-trivial component of $G - S$. Let M be a perfect matching of G. By Observation 2.1, $f(G, M) = n-1$. Since $o(G-S) = |S|$, M matches S to distinct components of $G-S$. Assume that e_1 is an edge in $M \cap E(C_1)$ and e_2 is an edge of M which connects a vertex of S and a vertex of another component C_2 of $G - S$. Then $G[V(\{e_1, e_2\})]$ contains no M-alternating cycles, which contradicts Lemma 2.2. Therefore, each component of $G - S$ is a singleton and so $|S| + o(G - S) = 2n$. It follows that $o(G - S) = n$ since we have shown that $o(G - S) = |S|$. So G contains an independent set of size n. By Lemma 2.5, G is a graph in $\mathcal{K}^+_{n,n}$. □

Taking $n = 3$ for example, $K_{3,3}, K_{3,2,1}, K_{3,1,1,1}, K_{2,2,2}, K_{2,2,1,1}, K_{2,1,1,1,1}, K_6$ are all complete multipartite graphs with each partite set having size no more than 3 and $K_{3,3} + e$ is the unique graph in $\mathcal{K}^+_{3,3}$ except for above complete multipartite graphs, where e is an edge connecting any two nonadjacent vertices of $K_{3,3}$.

4 Extendability of Graphs $G \in \mathcal{G}_{2n}$ with $F(G) = n - 1$

Let G be a graph in $\mathcal{G}_{2n}$ with $F(G) = n-1$ and be different from graphs in $\mathcal{K}^+_{n,n}$. By Lemma 2.5, G is 1-extendable. However, it is not necessarily 2-extendable. We know that an l-extendable graph is $(l - 1)$-extendable for an integer $l \geq 1$, and 2-extendable graphs are either bricks or braces (2-extendable bipartite graphs) [19]. In this section, we will determine which graphs in Lemma 2.5 are 1-extendable but not 2-extendable.

Theorem 4.1 *Let $G \in \mathcal{G}_{2n}$ with $F(G) = n - 1$ where $n \geq 3$. Then G is 1-extendable but not 2-extendable if and only if G has a perfect matching $M = \{u_iv_i \mid i = 1, 2, \ldots, n\}$ with $f(G, M) = n - 1$ so that one of the following conditions holds.*

(i) *$n \geq 4$, $G[\{v_1, v_2, \ldots, v_n\}]$ consists of one triangle and $n - 3$ isolated vertices, and $G[\{u_1, u_2, \ldots, u_n\}]$ has at least two independent edges (see an example in Figure 2 (a)).*

(ii) *$\{v_1, v_2, \ldots, v_{n-1}\}$ is an independent set, $G[\{u_1, u_2, \ldots, u_n, v_n\}]$ has at least two independent edges, and $\{v_iv_n, v_ju_n\} \subseteq E(G)$ for some i and j with $1 \leq i, j \leq n - 1$ (see examples*

(b) *and* (c) *in Figure* 2).

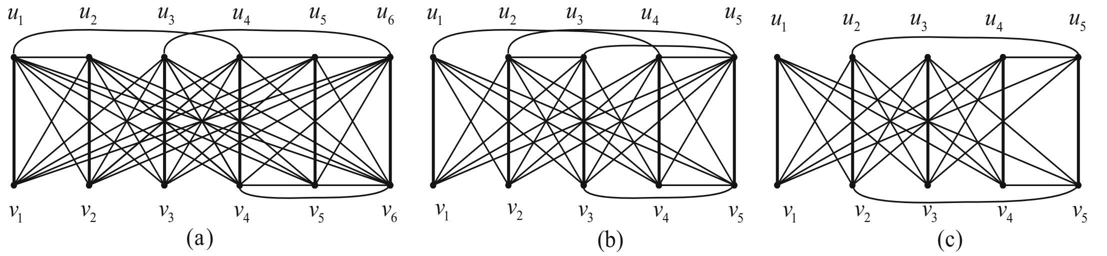

Figure 2 Three examples for non-2-extendable graphs

Proof Sufficiency. Since $f(G,M)=n-1$, by Lemma 2.5, G is 1-extendable or G contains an independent set of size n. We claim that G contains no independent set of size n. If we have done, then G is 1-extendable. Let $S=\{u_1,u_2,\ldots,u_n\}$ (resp. $\{u_1,u_2,\ldots,u_n,v_n\}$) be a subset of $V(G)$ corresponding to (i) (resp. (ii)). Then $G[S]$ contains two independent edges, all components of $G-S$ are factor-critical, and $o(G-S)=|S|-2$. By Lemma 3.1, G is not 2-extendable.

Now we prove the claim. Let I be any independent set of G. We will prove that $|I|\le n-1$. We consider the graphs G satisfying (i). Let $\{v_1,v_2,\ldots,v_{n-3}\}$ be the set of $n-3$ isolated vertices and $G[\{v_{n-2},v_{n-1},v_n\}]$ be the triangle of $G[\{v_1,v_2,\ldots,v_n\}]$. If $I\subseteq\{u_1,u_2,\ldots,u_n\}$, then $|I|\le n-2$ for $G[\{u_1,u_2,\ldots,u_n\}]$ has at least two independent edges. Otherwise, there exists $v_i\in I$ for some $1\le i\le n$. For any $1\le j\le n$ and $j\ne i$, $G[\{u_i,v_i,u_j,v_j\}]$ contains an M-alternating 4-cycle by Lemma 2.2. So if $v_iv_j\notin E(G)$ for some j then $v_iu_j\in E(G)$ and $u_j\notin I$. Hence, if $1\le i\le n-3$ then $u_j\notin I$ for any $1\le j\le n$ and $j\ne i$ since $v_iv_j\notin E(G)$. Hence $I\subseteq\{v_1,v_2,\ldots,v_n\}$ and $|I|\le n-2$. If $n-2\le i\le n$, then $u_j\notin I$ for any $1\le j\le n-3$ since $v_iv_j\notin E(G)$. Hence $I\subseteq\{v_1,v_2,\ldots,v_{n-3},v_i\}\cup(\{u_{n-2},u_{n-1},u_n\}\setminus\{u_i\})$. If $v_1\in I$, then $\{u_{n-2},u_{n-1},u_n\}\cap I=\emptyset$ by Lemma 2.2, so $|I|\le n-2$. Otherwise, $v_1\notin I$. Then $|I|\le n-1$.

We consider the graphs G satisfying (ii). If $I\subseteq\{u_1,u_2,\ldots,u_n,v_n\}$, then $|I|\le n-1$ for $G[\{u_1,u_2,\ldots,u_n,v_n\}]$ has at least two independent edges. Otherwise, there exists $v_k\in I$ for some $1\le k\le n-1$. Since $v_kv_l\notin E(G)$ for any $1\le l\le n-1$ and $l\ne k$, $v_ku_l\in E(G)$ since $G[\{u_k,v_k,u_l,v_l\}]$ contains an M-alternating 4-cycle by Lemma 2.2. So $u_l\notin I$ and $I\subseteq\{v_1,v_2,\ldots,v_n,u_n\}$. By the assumption, $\{v_ju_n,v_iv_n\}\subseteq E(G)$. If $i\ne j$, then $\{v_ju_n,v_iv_n\}$ are two independent edges and $|I|\le n-1$. Otherwise, $G[\{v_i,v_n,u_n\}]$ is a triangle and $|I|\le n-1$.

Necessity. By Lemma 3.1, there exists a subset $S\subseteq V(G)$ such that $G[S]$ contains two independent edges, all components of $G-S$ are factor-critical, and $o(G-S)=|S|-2$. Hence $|S|\ge 4$. Let C_i $(1\le i\le|S|-2)$ be all components of $G-S$. Since $F(G)=n-1$, there exists a perfect matching M of G so that $f(G,M)=n-1$. We claim that $G-S$ has at most one component containing exactly three vertices, and the others are trivial. Otherwise, $G-S$ has one component, say C_1, containing at least five vertices or $G-S$ has two components, say C_1 and C_2, containing exactly three vertices. Since $G-S$ has exactly $|S|-2$ factor-critical components, in either case there is one edge e_1 in M belonging to C_1 or C_2. Then none of end vertices of e_1 are adjacent to any vertex, say w, of another component. So $G[V(\{e_1,e_2\})]$ cannot contain an M-alternating cycle, where e_2 is the edge in M incident with w, which contradicting Lemma 2.2.

If one component of $G-S$ is a triangle and the other components are singletons, then $|S|=n\geq 4$. Let $V(C_i)=\{v_i\}$ for each $1\leq i\leq n-3$, $V(C_{n-2})=\{v_{n-2},v_{n-1},v_n\}$ and $u_iv_i\in M$ for each $1\leq i\leq n-3$. Since $f(G,M)=n-1$, each edge of C_{n-2} does not belong to M by Lemma 2.2. So we denote the remaining three edges of M by $u_{n-2}v_{n-2}, u_{n-1}v_{n-1}$ and u_nv_n. Thus $M=\{u_iv_i\,|\,i=1,2,\ldots,n\}$ and $S=\{u_1,u_2,\ldots,u_n\}$. Hence (i) holds.

If all components of $G-S$ are trivial, then $|S|=n+1$ and $o(G-S)=n-1$. Let $V(C_i)=\{v_i\}$ and $u_iv_i\in M$ for each $1\leq i\leq n-1$. Then the remaining edge of M is denoted by u_nv_n. So $\{v_1,v_2,\ldots,v_{n-1}\}$ is an independent set of G and $S=\{u_1,u_2,\ldots,u_n,v_n\}$ with $G[S]$ containing at least two independent edges. Since G is not a graph in $\mathcal{K}^+_{n,n}$, G contains no independent set of size n by Lemma 2.5. Combining that $\{v_1,v_2,\ldots,v_{n-1}\}$ is an independent set of G, there exists i and j with $1\leq i,j\leq n-1$ such that $\{u_nv_j,v_nv_i\}$ is contained in $E(G)$. So (ii) holds. □

Next we will determine all minimal non-2-extendable graphs in Theorem 4.1.

Corollary 4.2 *Let graph G in Theorem 4.1 be minimal. Then G is not 2-extendable if and only if* (ii) *in Theorem 4.1 holds where $i\neq j$, $\{u_1,u_2,\ldots,u_{n-1}\}$ is an independent set and $G[\{u_n,v_n,u_i,v_i\}]$ is just a 4-cycle for $1\leq i\leq n-1$ (see an example in Figure 2 (c)).*

Proof Sufficiency. It suffices to prove that G is minimal by Theorem 4.1. Since $f(G,M)=n-1$, by Lemma 2.2, $G[\{u_i,v_i,u_j,v_j\}]$ contains an M-alternating 4-cycle for any $1\leq i<j\leq n-1$. By the assumption, $\{u_1,u_2,\ldots,u_{n-1}\}$ and $\{v_1,v_2,\ldots,v_{n-1}\}$ are two independent sets. So neither v_iv_j nor u_iu_j is an edge of G. Hence $G[\{u_i,v_i,u_j,v_j\}]$ is exactly a 4-cycle. Combining that $G[\{u_n,v_n,u_i,v_i\}]$ is just a 4-cycle for $1\leq i\leq n-1$, we obtain that G is minimal by Lemma 2.2.

Necessity. By Theorem 4.1, G has a perfect matching $M=\{u_iv_i\,|\,i=1,2,\ldots,n\}$ with $f(G,M)=n-1$ so that (i) or (ii) holds. First we show that each graph G satisfying (i) is not minimal. Let $\{v_1,v_2,\ldots,v_{n-3}\}$ be the set of $n-3$ isolated vertices and $G[\{v_{n-2},v_{n-1},v_n\}]$ be the triangle of $G[\{v_1,v_2,\ldots,v_n\}]$. Then there is at least one vertex, say u_i for some $1\leq i\leq n-3$, incident with one of the two independent edges of $G[\{u_1,u_2,\ldots,u_n\}]$. Assume that u_iu_j is such an edge for some $1\leq j\leq n$. By Lemma 2.2, $G[\{u_i,v_i,u_j,v_j\}]$ contains an M-alternating 4-cycle. Since $v_iv_j\notin E(G)$, we have $\{u_iv_j,v_iu_j\}\subseteq E(G)$. But $u_iu_j\in E(G)$, $G[\{u_i,v_i,u_j,v_j\}]$ is not a 4-cycle. So G is not minimal by Lemma 2.2.

Next let G be a graph satisfying (ii). Since G is minimal, $G[\{u_k,v_k,u_l,v_l\}]$ is just a 4-cycle for $1\leq k<l\leq n$ by Lemma 2.2. Since $\{v_iv_n,v_ju_n\}\subseteq E(G)$ for some i and j with $1\leq i,j\leq n-1$, we have $i\neq j$. Since $\{v_1,v_2,\ldots,v_{n-1}\}$ is an independent set, $v_kv_l\notin E(G)$ for $1\leq k<l\leq n-1$. By Lemma 2.2, $G[\{u_k,v_k,u_l,v_l\}]$ is a 4-cycle $u_kv_lu_lv_ku_k$ and $u_ku_l\notin E(G)$. Hence $\{u_1,u_2,\ldots,u_{n-1}\}$ is an independent set. □

5 Minimum Forcing Numbers and Forcing Spectrum of Graphs G with $F(G)=n-1$

The *forcing spectrum* of a graph G is the set of forcing numbers of all perfect matchings of graph G. If the forcing spectrum of G is an integer interval, then we say it is *continuous* (or *consecutive*). Afshani et al. [3] showed that any finite subset of positive integers is the forcing spectrum of some graph. Besides, they [3] obtained that the forcing spectra of column continuous subgrids are continuous by matching 2-switches. Further, Zhang and Jiang [33] generalized their result to any polyomino with perfect matchings by applying the Z-transformation graph. Zhang and Deng [31] obtained that the forcing spectrum of any hexagonal system with a forcing edge form either the integer interval from 1 to its Clar number or with only the gap 2. For more researches on the forcing spectra of special graphs, see [2, 3, 11, 20, 24, 25, 37].

Let $G \in \mathcal{G}_{2n}$ with $F(G) = n-1$. In this section we will prove that $f(G) \geq \lfloor \frac{n}{2} \rfloor$, and find that all minimum forcing numbers of such graphs G form an integer interval $[\lfloor \frac{n}{2} \rfloor, n-1]$. Further, we will show that the forcing spectrum of each such graph G is continuous. Next we give a lemma obtained by Che and Chen.

Lemma 5.1 ([5]) *Let G be a k-connected graph with a perfect matching. Then $f(G) \geq \lfloor \frac{k}{2} \rfloor$.*

Combining Lemmas 2.3 and 5.1, we have the following result.

Corollary 5.2 *Let $G \in \mathcal{G}_{2n}$ with $F(G) = n-1$. Then $f(G) \geq \lfloor \frac{n}{2} \rfloor$.*

For $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, let $H_k \in \mathcal{G}_{2n}$ be a minimal graph with $f(H_k, M_0) = n-1$, where $M_0 = \{u_i v_i \mid i = 1, 2, \ldots, n\}$ is a perfect matching of H_k, so that $H_k[\{u_1, u_2, \ldots, u_n\}]$ contains exactly k edges $\{u_{2i-1}u_{2i} \mid i = 1, 2, \ldots, k\}$. Then $H_k[\{v_1, v_2, \ldots, v_n\}]$ contains exactly k edges $\{v_{2i-1}v_{2i} \mid i = 1, 2, \ldots, k\}$ and $H_k[\{u_{2k+1}, v_{2k+1}, \ldots, u_n, v_n\}]$ is isomorphic to a complete bipartite graph (see examples in Figure 3).

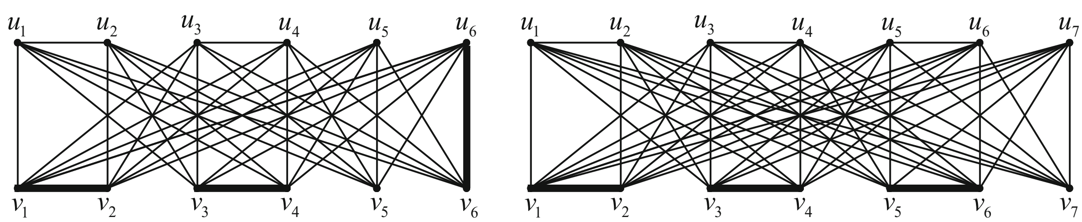

Figure 3 H_k with $k = \lfloor \frac{n-1}{2} \rfloor$ and $n = 6$ and 7

Remark 5.3 For $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $f(H_k) \leq n-k-1$. Especially, for $k = \lfloor \frac{n-1}{2} \rfloor$ we have $f(H_k) = \lfloor \frac{n}{2} \rfloor$, i.e., the lower bound of Corollary 5.2 is sharp.

Let $M = \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \mid i = 1, 2, \ldots, k\} \cup \{u_{2k+1}v_{2k+1}, u_{2k+2}v_{2k+2}, \ldots, u_n v_n\}$ be a perfect matching of H_k. It follows that $S = \{v_{2i-1}v_{2i} \mid i = 1, 2, \ldots, k\} \cup \{u_{2k+2}v_{2k+2}, u_{2k+3}v_{2k+3}, \ldots, u_n v_n\}$ is a forcing set of M by Lemma 1.1 (see examples in Figure 3, where bold lines form S). So $f(H_k) \leq f(H_k, M) \leq |S| = k + (n-2k-1) = n-k-1$.

Especially, for $k = \lfloor \frac{n-1}{2} \rfloor$ we have $f(H_k) \leq n-k-1 = \lfloor \frac{n}{2} \rfloor$. Combining Corollary 5.2, we obtain that $f(H_k) = \lfloor \frac{n}{2} \rfloor$.

Next we will prove that $f(H_k) = n-k-1$ for $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Here we give two simple facts that can be obtained from Lemma 1.1.

Fact 5.4 If G' is a spanning subgraph of G with $f(G', M) = |S|$ for some perfect matching M of G', then $f(G, M) \geq |S|$.

Fact 5.5 Let M be a perfect matching of G with $M = M_1 \cup M_2$, and let $G_i = G[V(M_i)]$ for $i = 1, 2$. Then $f(G, M) \geq f(G_1, M_1) + f(G_2, M_2)$.

Lemma 5.6 *For* $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $f(H_k) = n - k - 1$.

Proof By Remark 5.3, it suffices to prove that $f(H_k) \geq n - k - 1$. We proceed by induction on n. It is trivial for $n = 1$. Since $H_0 \cong K_{n,n}$, we have $f(H_0) = n - 1$ by Theorem 1.3. By Remark 5.3, for $k = \lfloor \frac{n-1}{2} \rfloor$ we have $f(H_k) = \lfloor \frac{n}{2} \rfloor = n - k - 1$. So next we suppose $n \geq 2$ and $1 \leq k \leq \frac{n-3}{2}$. Let M be any perfect matching of H_k. Then we have the following claims.

Claim 1 If $u_i v_i \in M$ for some $1 \leq i \leq 2k$, then $f(H_k, M) \geq n - k - 1$.

Let $H^2 = H_k - \{u_i, v_i\}$ and $M_2 = M \setminus \{u_i v_i\}$. Then H^2 is isomorphic to $H'_{k'}$ with $n' = n-1$ and $k' = k - 1$. By the induction hypothesis,

$$f(H^2, M_2) = f(H'_{k'}, M_2) \geq f(H'_{k'}) \geq n' - k' - 1 = n - k - 1.$$

By Fact 5.5, $f(H_k, M) \geq f(H^2, M_2) \geq n - k - 1$.

Claim 2 If H_k has an M-alternating 4-cycle containing exactly two edges of $\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \,|\, i = 1, 2, \ldots, k\}$, then $f(H_k, M) \geq n - k - 1$.

If $M \cap \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \,|\, i = 1, 2, \ldots, k\} = \emptyset$, then we assume that C is an M-alternating 4-cycle of H_k, and $\{u_{2i-1}u_{2i}, v_{2j-1}v_{2j}\}$ is contained in C for some $1 \leq i, j \leq k$. So $C = u_{2i-1}v_{2j-1}v_{2j}u_{2i}u_{2i-1}$ or $u_{2i-1}v_{2j}v_{2j-1}u_{2i}u_{2i-1}$. Let $H^2 = H_k - V(C)$ and $M_2 = M \cap E(H^2)$, $M_1 = M \setminus M_2$, $H^1 = H_k[V(M_1)]$. Then H^1 contains C and $f(H^1, M_1) \geq 1$ by Theorem 1.2. On the other hand, $H^2 - \{v_{2i-1}v_{2i}, u_{2j-1}u_{2j}\}$ is isomorphic to $H'_{k'}$ with $n' = n - 2$, $k' \leq k - 1$, and M_2 is a perfect matching of $H'_{k'}$. By the induction hypothesis and Fact 5.4,

$$f(H^2, M_2) \geq f(H'_{k'}, M_2) \geq f(H'_{k'}) \geq n' - k' - 1 \geq n - k - 2.$$

By Fact 5.5, $f(H_k, M) \geq f(H^1, M_1) + f(H^2, M_2) \geq 1 + (n - k - 2) = n - k - 1$.

If $M \cap \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \,|\, i = 1, 2, \ldots, k\} \neq \emptyset$, then the intersection of the two sets is denoted by M_1. By the structure of H_k, all vertices of $\{u_1, u_2, \ldots, u_n\} \setminus V(M_1)$ must match into all vertices of $\{v_1, v_2, \ldots, v_n\} \setminus V(M_1)$. So we have

$$|M_1 \cap \{u_{2j-1}u_{2j} \,|\, j = 1, 2, \ldots, k\}| = |M_1 \cap \{v_{2j-1}v_{2j} \,|\, j = 1, 2, \ldots, k\}|.$$

Let $M_2 = M \setminus M_1$ and $H^i = H_k[V(M_i)]$ for $i = 1, 2$. For any pair of edges $\{u_{2i-1}u_{2i}, v_{2j-1}v_{2j}\}$ of M_1 where $1 \leq i, j \leq k$, $H_k[\{u_{2i-1}, u_{2i}, v_{2j-1}, v_{2j}\}]$ is either a 4-cycle or K_4 by the structure of H_k. So $H_k[\{u_{2i-1}, u_{2i}, v_{2j-1}, v_{2j}\}]$ contains an M_1-alternating 4-cycle, and so H^1 contains $\frac{|M_1|}{2}$ disjoint M_1-alternating 4-cycles. By Lemma 1.1, $f(H^1, M_1) \geq \frac{|M_1|}{2}$. On the other hand,

$$H^2 - \{u_{2j-1}u_{2j} \,|\, v_{2j-1}v_{2j} \in M_1, j = 1, 2, \ldots, k\} \cup \{v_{2j-1}v_{2j} \,|\, u_{2j-1}u_{2j} \in M_1, j = 1, 2, \ldots, k\}$$

is isomorphic to $H'_{k'}$ with $n' = n - |M_1|$, $k' \leq k - \frac{|M_1|}{2}$. Since

$$k' \leq k - \frac{|M_1|}{2} \leq \frac{n - 3 - |M_1|}{2} < \frac{n - 2 - |M_1|}{2} \leq \left\lfloor \frac{n - 1 - |M_1|}{2} \right\rfloor = \left\lfloor \frac{n' - 1}{2} \right\rfloor,$$

by the induction hypothesis and Fact 5.4, we obtain that

$$f(H^2, M_2) \geq f(H'_{k'}, M_2) \geq f(H'_{k'}) \geq n' - k' - 1 \geq n - k - 1 - \frac{|M_1|}{2}.$$

By Fact 5.5, we obtain that

$$f(H_k, M) \geq f(H^1, M_1) + f(H^2, M_2) \geq \frac{|M_1|}{2} + \left(n - k - 1 - \frac{|M_1|}{2}\right) = n - k - 1.$$

By Claims 1 and 2, from now on we suppose that M contains no u_iv_i for any $1 \leq i \leq 2k$ and any M-alternating 4-cycle of H_k contains at most one edge of $\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \,|\, i = 1, 2, \ldots, k\}$. Particularly, M contains no edges of $\{u_{2i-1}u_{2i}, v_{2i-1}v_{2i} \,|\, i = 1, 2, \ldots, k\}$. So we may assume $\{v_1u_3, v_2u_5\}$ is contained in M. Next we are going to consider the edge of M incident with v_3, say v_3u_l. Then l has four possible values: $l \leq 2$, $l = 6$, $7 \leq l \leq 2k$ and $l \geq 2k+1$. Furthermore, if $7 \leq l \leq 2k$, then we suppose $l = 7$ and continue to consider the edge of M incident with v_5. Until we obtain an edge $v_hu_l \in M$ so that one of the other three cases ($l \leq h-1$, $l = h+3$, $l \geq 2k+1$) holds (since H_k is finite, such edge exists).

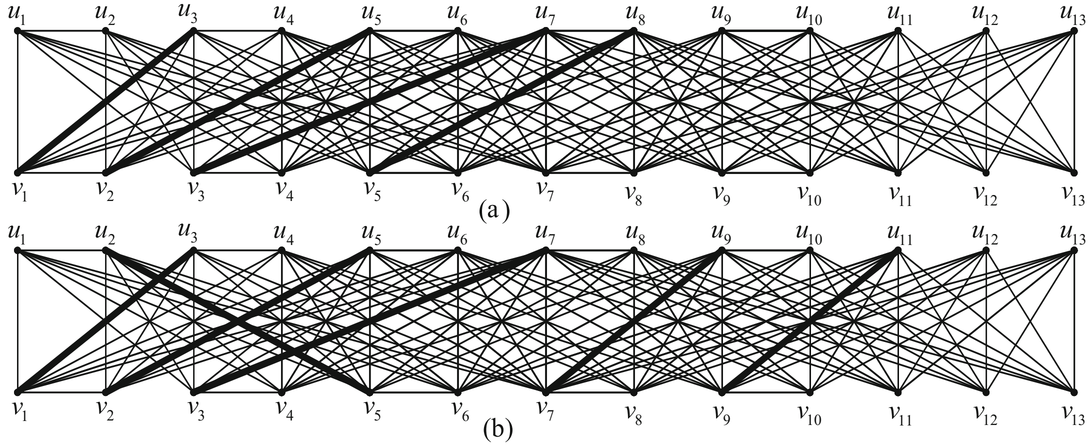

Figure 4 Illustration for the proof of Lemma 5.6, where bold lines form M_1

If $l = h+3$, then let $M_1 = \{v_1u_3, v_2u_5, \ldots, v_hu_{h+3}\}$ (see an example in Figure 4 (a) where $h = 5$), $M_2 = M \setminus M_1$, and $H^i = H_k[V(M_i)]$ for $i = 1, 2$. Then H^1 contains an M_1-alternating 4-cycle $v_1u_5v_2u_3v_1$. By Theorem 1.2, $f(H^1, M_1) \geq 1$. On the other hand, $H^2 - \{u_1u_2, v_{h+2}v_{h+3}\}$ is isomorphic to $H'_{k'}$ with $n' = n - |M_1|$. Since contribution of the edges v_1u_3, v_hu_{h+3} to k' is -2, 0, and that of each other edges in M_1 is -1, we have $k' = k - |M_1|$. Since

$$k' = k - |M_1| \leq \frac{n-3-2|M_1|}{2} < \frac{n-2-|M_1|}{2} \leq \left\lfloor \frac{n-1-|M_1|}{2} \right\rfloor = \left\lfloor \frac{n'-1}{2} \right\rfloor,$$

by the induction hypothesis and Fact 5.4,

$$f(H^2, M_2) \geq f(H'_{k'}, M_2) \geq f(H'_{k'}) \geq n' - k' - 1 = n - k - 1.$$

By Fact 5.5, $f(H_k, M) \geq f(H^1, M_1) + f(H^2, M_2) \geq 1 + n - k - 1 = n - k$.

If $l \in [2k+1, n] \cup [1, h-1]$ (see an example in Figure 4 (b) where $h = 5,\ l = 2$), then we continue to consider the edge of M incident with v_{h+2}, say $v_{h+2}u_i$. Then i has three possible values: $i \leq h+1$, $h+4 \leq i \leq 2k$ and $i \geq 2k+1$. Furthermore, if $h+4 \leq i \leq 2k$, then we suppose $i = h+4$ and continue to consider the edge of M incident with v_{h+4}. Until we obtain an edge of M, say v_ru_t with $t \in [2k+1, n] \cup [1, r-1]$ (By the finiteness of H_k, such edge exists) (see an example in Figure 4 (b) where $r = 9,\ t = 11$). Let $M_1 = \{v_1u_3, v_2u_5, \ldots, v_hu_l, \ldots, v_ru_t\}$, $M_2 = M \setminus M_1$, and $H^i = H_k[V(M_i)]$ for $i = 1, 2$. Then $H^2 - u_1u_2$ is isomorphic to $H'_{k'}$ with $n' = n - |M_1|$. Since contribution of the edges $v_1u_3,\ v_hu_l,\ v_ru_t$ to k' is -2, 0, 0 and that of each other edges in M_1 is -1, we have $k' = k - (|M_1| - 1)$. Since

$$k' = k - |M_1| + 1 \le \frac{n-1-2|M_1|}{2} \le \frac{n-2-|M_1|}{2} \le \left\lfloor \frac{n-1-|M_1|}{2} \right\rfloor = \left\lfloor \frac{n'-1}{2} \right\rfloor,$$

by the induction hypothesis and Fact 5.4,

$$f(H^2, M_2) \ge f(H'_{k'}, M_2) \ge f(H'_{k'}) \ge n' - k' - 1 = n - k - 2.$$

By Fact 5.5, $f(H_k, M) \ge f(H^1, M_1) + f(H^2, M_2) \ge 1 + (n - k - 2) = n - k - 1$.

By the arbitrariness of M, we obtain that $f(H_k) \ge n - k - 1$. □

Combining Lemma 5.6 and Corollary 5.2, we obtain the following result.

Theorem 5.7 *All minimum forcing numbers of graphs $G \in \mathcal{G}_{2n}$ with $F(G) = n - 1$ form an integer interval $[\lfloor \frac{n}{2} \rfloor, n - 1]$.*

Suppose that M is a perfect matching of G. If C is an M-alternating cycle of length 4, then $M \oplus E(C)$ is a *matching* 2*-switch* on M. Afshani et al. [3] obtained that a matching 2-switch on a perfect matching does not change the forcing number by more than 1.

Lemma 5.8 ([3]) *If M is a perfect matching of G and C is an M-alternating cycle of length 4, then*

$$|f(G, M \oplus E(C)) - f(G, M)| \le 1.$$

By Lemma 5.8, if $M_1, M_2, \ldots, M_s$ is a sequence of perfect matchings such that M_{i+1} is obtained from M_i by a matching 2-switch for $1 \le i \le s - 1$, then the integer interval $[\min\{f(G, M_1), f(G, M_s)\}, \max\{f(G, M_1), f(G, M_s)\}]$ is contained in the forcing spectrum of G.

Theorem 5.9 *If $G \in \mathcal{G}_{2n}$ with $F(G) = n - 1$, then its forcing spectrum is continuous.*

Proof Let $M_s = \{u_iv_i \mid i = 1, 2, \ldots, n\}$ be a perfect matching of G with $f(G, M_s) = n - 1$. Then we will prove that M_s can be obtained from any perfect matching M of G by repeatedly applying matching 2-switches. If we have done, then M_s can be obtained from a perfect matching of G with the minimum forcing number by repeatedly applying matching 2-switches. So the forcing spectrum of G is continuous by Lemma 5.8.

We proceed by induction on n. For $n = 1$, $G \cong K_2$ and the result is trivial. Next, for $n \ge 2$, $f(G, M_s) = n - 1 > 0$. Take any perfect matching M of G different from M_s. Then we have the following claims.

Claim 1 If $M_s \cap M \ne \emptyset$, then M_s can be obtained from M by repeatedly applying matching 2-switches.

Suppose $M_s \cap M$ contains an edge u_iv_i for some $1 \le i \le n$. Let $G' = G - \{u_i, v_i\}$ and $M' = M \setminus \{u_iv_i\}$, $M'_s = M_s \setminus \{u_iv_i\}$. Then G' has $2(n-1)$ vertices, M' and M'_s are two distinct perfect matchings of G'. By Lemma 2.2, $f(G', M'_s) = n - 2$. By the induction hypothesis, M'_s can be obtained from M' by repeatedly applying matching 2-switches. Hence, M_s is also obtained from M by the same series of matching 2-switches, and the claim holds.

Claim 2 If $M_s \oplus M$ contains a cycle of length 4, then M_s can be obtained from M by repeatedly applying matching 2-switches.

Assume that C is a cycle of length 4 contained in $M_s \oplus M$ and $V(C) = \{u_i, v_i, u_j, v_j\}$. Let $M' = M \oplus E(C)$. Then $\{u_iv_i, u_jv_j\} \subseteq M_s \cap M'$. By Claim 1, M_s can be obtained from M' by repeatedly applying matching 2-switches. Hence M_s can be obtained from M by repeatedly applying matching 2-switches, and the claim holds.

By Claims 1 and 2, from now on we suppose that $M_s \cap M = \emptyset$ and $M_s \oplus M$ contains no cycles of length 4. Then we may suppose that $v_1u_2 \in M$. So $u_1v_2 \notin M$. Without loss of generality, we assume that $u_1v_3 \in M$. If u_2v_3 is an edge of G, then $C_1 = u_1v_1u_2v_3u_1$ is an M-alternating 4-cycle. So $M' = M \oplus E(C_1)$ is a perfect matching of G and $M_s \cap M' = \{u_1v_1\}$. From Claim 1, we are done. Otherwise, u_2v_3 is not an edge of G. By Lemma 2.2, $\{u_2u_3, v_2v_3\} \subseteq E(G)$ (see Figure 5 (a)). Next we consider the following two cases according as whether v_2u_3 belongs to M or not.

Case 1 $v_2u_3 \in M$. If $v_1u_3 \in E(G)$, then $C_2 = v_1u_2v_2u_3v_1$ is an M-alternating 4-cycle. So $M' = M \oplus E(C_2)$ is a perfect matching of G and $M_s \cap M' = \{u_2v_2\}$. From Claim 1, we are done. Otherwise, $v_1u_3 \notin E(G)$. By Lemma 2.2, $\{u_1u_3, v_1v_3\} \subseteq E(G)$ (see Figure 5 (b)). Then $C_3 = u_1u_3v_2v_3u_1$ is an M-alternating 4-cycle and $C_4 = v_2v_3v_1u_2v_2$ is an $M \oplus E(C_3)$-alternating 4-cycle. So $M' = M \oplus E(C_3) \oplus E(C_4)$ is a perfect matching of G that is obtained from M by two matching 2-switches and $M_s \cap M' = \{u_2v_2\}$. From Claim 1, we are done.

Case 2 $v_2u_3 \notin M$. Without loss of generality, we can suppose that $v_2u_4 \in M$. If v_1u_4 is an edge of G, then $C_5 = v_1u_2v_2u_4v_1$ is an M-alternating 4-cycle. So $M' = M \oplus E(C_5)$ is a perfect matching of G and $M_s \cap M' = \{u_2v_2\}$. From Claim 1, we are done. Otherwise, v_1u_4 is not an edge of G. By Lemma 2.2, $\{u_1u_4, v_1v_4\} \subseteq E(G)$ (see Figure 5(c)). Next we distinguish the following two subcases according to $u_3v_4 \in M$ or not.

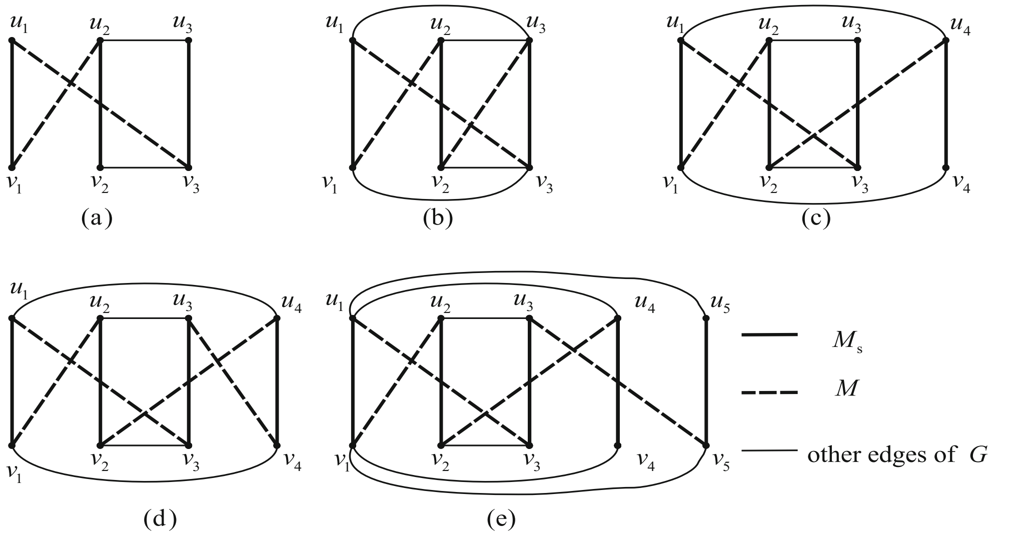

Figure 5 Illustration for the proof of Theorem 5.9

Subcase 2.1: $u_3v_4 \in M$ (see Figure 5 (d)). Then $C_6 = u_1v_3v_2u_4u_1$ and $C_7 = v_1v_4u_3u_2v_1$ are two M-alternating 4-cycles. By two matching 2-switches we have that $M' = M \oplus E(C_6) \oplus E(C_7)$ is a perfect matching of G. Hence, $M' = (M \setminus \{u_1v_3, v_2u_4, v_1u_2, u_3v_4\}) \cup \{v_2v_3, u_1u_4, u_2u_3, v_1v_4\}$. So $M_s \oplus M'$ contains a 4-cycle $u_2u_3v_3v_2u_2$. From Claim 2, M_s can be obtained from M' by repeatedly applying matching 2-switches, so we are done.

Subcase 2.2: $u_3v_4 \notin M$. Then we may suppose $u_3v_5 \in M$. If u_1v_5 is an edge of G, then $C_8 = u_1v_5u_3v_3u_1$ is an M-alternating 4-cycle. So $M' = M \oplus E(C_8)$ is a perfect matching of G and $M_s \cap M' = \{u_3v_3\}$. From Claim 1, we are done. So we may suppose $u_1v_5 \notin E(G)$. By Lemma 2.2, $\{v_1v_5, u_1u_5\} \subseteq E(G)$ (see Figure 5 (e)). Then $C_9 = u_2u_3v_5v_1u_2$ and C_6 are two M-alternating 4-cycles. By two matching 2-switches we have that $M' = M \oplus E(C_9) \oplus E(C_6)$ is a perfect matching of G. Hence, $M' = (M \setminus \{u_1v_3, v_2u_4, v_1u_2, u_3v_5\}) \cup \{v_2v_3, u_1u_4, u_2u_3, v_1v_5\}$. So $M_s \oplus M'$ contains a 4-cycle $u_2u_3v_3v_2u_2$. From Claim 2, we are done. □

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