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Weighted Composition Operators between Bergman Spaces with Exponential Weights

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Abstract In this paper, we study some properties of weighted composition operators on a class of weighted Bergman spaces A^p_φ with $0 < p \le \infty$ and $\varphi \in \mathcal{W}_0$. Also, we completely characterize the q-Carleson measure for A^p_φ in terms of the averaging function and the generalized Berezin transform with $0 < q < \infty$. As applications, the boundedness and compactness of weighted composition operators acting from one Bergman space A^p_φ to another A^q_φ are equivalently described and the Schatten class property of the weighted composition operator acting on A^2_φ are given. Our main results are expressed in terms of certain Berezin type integral transforms.

Keywords Weighted composition operator, boundedness, compactness, Schatten class, Bergman spaces

MR(2010) Subject Classification 47B33, 30H20

1 Introduction

Let $\mathbb{D}$ be the open unit disk and dA be the normalized Lebesgue area measure on $\mathbb{D}$. For a strictly subharmonic function φ on $\mathbb{D}$, the space L^p_φ consists of those Lebesgue measurable functions f such that

$$\|f\|_{p,\varphi} = \left\{ \int_{\mathbb{D}} |f(z)|^p \mathrm{e}^{-p\varphi(z)} dA(z) \right\}^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$
$$\|f\|_{\infty,\varphi} = \sup_{z\in\mathbb{D}} |f(z)| \mathrm{e}^{-\varphi(z)} < \infty, \quad p = \infty.$$

Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. The weighted Bergman space A^p_φ consists of those analytic functions in L^p_φ, that is,

$$A^p_\varphi = L^p_\varphi \cap H(\mathbb{D}).$$

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Write $\mathbb{R}$ as the set of the real numbers and C_0 as the space of all continuous functions ρ on $\mathbb{D}$ such that $\lim_{|z|\to 1}\rho(z)=0$. Define

$$\mathcal{L}=\left\{\rho:\mathbb{D}\to\mathbb{R}:\|\rho\|_{\mathcal{L}}=\sup_{z,w\in\mathbb{D},z\neq w}\frac{|\rho(z)-\rho(w)|}{|z-w|}<\infty,\ \rho\in C_0\right\}.$$

Let $\mathcal{L}_0$ denote the set of those $\rho\in\mathcal{L}$ with the property that for each $\varepsilon>0$, there exists a compact subset $E\subset\mathbb{D}$ such that $|\rho(z)-\rho(w)|\leq\varepsilon|z-w|$ whenever $z,w\in\mathbb{D}\setminus E$.

We denote $A\simeq B$ if there exist constants $C_1,C_2>0$ such that $A\leq C_1B$ (or $A\lesssim B$) and $A\geq C_2B$ (or $A\gtrsim B$).

The weight class $\mathcal{W}_0$, first introduced in [16], is defined as

$$\mathcal{W}_0=\left\{\varphi\in C^2:\Delta\varphi>0,\text{and }\exists\ \rho\in\mathcal{L}_0\text{ such that}\frac{1}{\sqrt{\Delta\varphi}}\simeq\rho\right\},$$

where Δ denotes the standard Laplace operator. We would like to mention two classes of weights related closely to $\mathcal{W}_0$ here. One is denoted by $\mathcal{OP}$, introduced by Oleinik and Perel'man [25, 26] and well studied in [3, 5, 19, 20]. The other one is denoted by $\mathcal{BDK}$, introduced by Borichev, Dhuez and Kellay [6] and widely considered in [2, 7, 8, 14, 29, 30]. The relationship of these three weights was given in [16] as $\mathcal{OP}\backslash\mathcal{W}_0\neq\emptyset$, $\mathcal{W}_0\backslash\mathcal{OP}\neq\emptyset$, $\mathcal{W}_0\backslash\mathcal{BDK}\neq\emptyset$ and $\mathcal{BDK}\subset\mathcal{W}_0$.

Let ψ be an analytic function on $\mathbb{D}$ and ϕ be an analytic self-map of $\mathbb{D}$. The weighted composition operator $W_{\psi,\phi}$ induced by ψ and ϕ is defined by $W_{\psi,\phi}f=\psi\cdot(f\circ\phi)$ for $f\in H(\mathbb{D})$. If $\phi(z)=z$, then the weighted composition operator $W_{\psi,\phi}$ becomes the multiplication operator M_ψ. When ψ is identically 1, $W_{\psi,\phi}$ reduces to the composition operator C_φ systematically studied in [10, 31, 32]. As a combination of pointwise multiplication operators and composition operators, the arise of weighted composition operators is of great significance. For example, isometric operators on Hardy spaces H^p and Bergman spaces A^p with $1\leq p<\infty$ and $p\neq 2$ are necessarily weighted composition operators of a certain kind, see [13, 18]. It has attracted great research interest to study the properties of the weighted composition operators on different function spaces with different kinds of weights, mainly focus on their boundedness, compactness, Schatten p-classes, compact differences, essential norms, spectra properties and so on. The readers may refer to [1, 4, 9, 11, 12, 17, 22–24, 27, 28] for details.

Motivated by the above work, in this paper, we study the bounded, compact and Schatten p-classes weighted composition operators $W_{\psi,\phi}$ on the weighted Bergman space A^p_φ with $\varphi\in\mathcal{W}_0$. More concretely, we answer the question when $W_{\psi,\phi}:A^p_\varphi\to A^q_\varphi$ is bounded or compact for $0<p\leq\infty$ and $0<q<\infty$. We also consider the question when $W_{\psi,\phi}$ on A^2_φ belongs to the Schatten p-classes. In the process of investigation, we still find the relationship between the two questions and Carleson measure on A^p_φ, and our main results are naturally expressed in terms of the generalized Berezin transform. Part of our proofs are inspired by the recent work around the large Fock spaces setting [4], but extra different work needs new methods and techniques in Bergman spaces setting.

Several auxiliary notations are needed to state our main results.

Recall from [16, Lemma 3.3] that the point evaluation at each $z\in\mathbb{D}$ is a bounded linear functional on A^2_φ. By the Riesz representation theorem, for every $z\in\mathbb{D}$, there exists a unique element $K_z\in A^2_\varphi$ such that $f(z)=\langle f,K_z\rangle_{A^2_\varphi}$ for all $f\in A^2_\varphi$, where the inner product is defined by

$$\langle f,g\rangle_{A^2_\varphi}=\int_{\mathbb{D}} f(w)\overline{g(w)}\mathrm{e}^{-2\varphi(w)}dA(w)$$

for $f,g\in A^2_\varphi$. We call $K_z(\cdot)=K(\cdot,z)$ the reproducing kernel at z.

For $0<p\le\infty$ and $z\in\mathbb{D}$, the normalized reproducing kernel of A^p_φ is given by $k_{p,z}=K_z/\|K_z\|_{p,\varphi}$. When $p=2$, we abbreviate the notation $k_{2,z}$ as k_z for simplicity.

Our main results will be expressed in terms of the following integral transform

$$B_{p,q,\phi}(|\psi|)(z)=\int_{\mathbb{D}}|k_{p,z}(\phi(w))|^q|\psi(w)|^q\mathrm{e}^{-q\varphi(w)}dA(w),\quad z\in\mathbb{D},$$

where $0<p\le\infty$, $0<q<\infty$, ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. For any Borel set E in $\mathbb{D}$, we define the pullback measure

$$\mu_{q,\phi}(E)=\int_{\phi^{-1}(E)}|\psi(z)|^q\mathrm{e}^{-q\varphi(z)}dA(z) \tag{1.1}$$

and set

$$d\nu_{q,\phi}(z)=\mathrm{e}^{q\varphi(z)}d\mu_{q,\phi}(z),\quad z\in\mathbb{D}. \tag{1.2}$$

For $1<p\le\infty$, we set $L^p=L^p(\mathbb{D},dA)$.

Now we state our main results.

Theorem 1.1 *Let $0<p\le q<\infty$ and $\varphi\in\mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}}\simeq\rho\in\mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. Then*

(a) *The weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ if and only if*

$$B_{p,q,\phi}(|\psi|)\in L^\infty. \tag{1.3}$$

Moreover,

$$\|W_{\psi,\phi}\|^q_{A^p_\varphi\to A^q_\varphi}\simeq\|B_{p,q,\phi}(|\psi|)\|_{L^\infty}. \tag{1.4}$$

(b) *The weighted composition operator $W_{\psi,\phi}$ is compact from A^p_φ to A^q_φ if and only if*

$$\lim_{|z|\to1^-}B_{p,q,\phi}(|\psi|)(z)=0. \tag{1.5}$$

In order to characterize the bounded and compact weighted composition operators $W_{\psi,\phi}$ from A^p_φ to A^q_φ when $0<q<p<\infty$, or $0<q<\infty$ and $p=\infty$, we need the measure $d\lambda_\rho$ defined by

$$d\lambda_\rho(z)=\frac{dA(z)}{\rho(z)^2},\quad z\in\mathbb{D}. \tag{1.6}$$

Theorem 1.2 *Let $0<q<p<\infty$ and $\varphi\in\mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}}\simeq\rho\in\mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent*:

(a) *The weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ.*

(b) *The weighted composition operator $W_{\psi,\phi}$ is compact from A^p_φ to A^q_φ.*

(c) *The integral transform $B_{p,q,\phi}(|\psi|)$ is in $L^{\frac{p}{p-q}}(\mathbb{D},d\lambda_\rho)$.*

Moreover,

$$\|W_{\psi,\phi}\|^q_{A^p_\varphi\to A^q_\varphi}\simeq\|B_{p,q,\phi}(|\psi|)\|_{L^{\frac{p}{p-q}}(\mathbb{D},d\lambda_\rho)}. \tag{1.7}$$

Theorem 1.3 *Let $0 < q < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent*:

(a) *The weighted composition operator $W_{\psi,\phi}$ is bounded from A^∞_φ to A^q_φ.*

(b) *The weighted composition operator $W_{\psi,\phi}$ is compact from A^∞_φ to A^q_φ.*

(c) *The integral transform $B_{\infty,q,\phi}(|\psi|)$ is in $L^1(\mathbb{D}, d\lambda_\rho)$.*

Moreover,

$$\|W_{\psi,\phi}\|^q_{A^\infty_\varphi \to A^q_\varphi} \simeq \|B_{\infty,q,\phi}(|\psi|)\|_{L^1(\mathbb{D},d\lambda_\rho)}. \tag{1.8}$$

Recall that if T is a compact operator on a separable Hilbert space H, then there exist orthonormal sets $\{\mathrm{e}_k\}_k$ and $\{\sigma_k\}_k$ in H such that

$$Tx = \sum_k \lambda_k \langle x, \mathrm{e}_k\rangle \sigma_k, \quad x \in H,$$

where λ_k is the k-th singular value of T.

For $0 < p < \infty$, let $\mathcal{S}_p(H)$ denote the Schatten p-class of operators on H. The class $\mathcal{S}_p(H)$ consists of all compact operators T on H with its singular value sequence $\{\lambda_k\}_k$ belonging to l^p, the p-summable sequence space. For more information about $\mathcal{S}_p(H)$, refer to [34, Chapter 1].

Theorem 1.4 *Let $0 < p < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. Then the weighted composition operator $W_{\psi,\phi} \in \mathcal{S}_p(A^2_\varphi)$ if and only if $B_{2,2,\phi}(|\psi|) \in L^{p/2}(\mathbb{D}, d\lambda_\rho)$.*

Throughout this paper, the notation $\|T\|_{A\to B}$ denotes the operator norm of T from A to B, and $D(z, r)$ denotes the Euclidean disc centered at z with radius $r > 0$. For convenience, we simply write $D^r(z)$ instead of $D(z, r\rho(z))$.

The paper is organized as follows. In Section 2, we present some technical lemmas that will be used in the subsequent sections. In Section 3, we formulate some geometric characterizations of Carleson measures for A^p_φ. The main proofs on the boundedness, compactness and Schatten p-class of the weighted composition operators are given in Section 4 and Section 5, respectively.

2 Preliminaries

In this section, we begin with an estimate on the function ρ, which behaves an analogue of Harnack's inequality for harmonic functions, see [16, Lemma 3.1].

Lemma 2.1 *Let $\rho \in \mathcal{L}$ be positive. Then there exist positive numbers α and C such that*

$$C^{-1}\rho(w) \le \rho(z) \le C\rho(w) \tag{2.1}$$

for any $z \in \mathbb{D}$ and $w \in D^\alpha(z)$.

We continue with a covering lemma given in [16, Lemma 3.2].

Lemma 2.2 *If $\rho \in \mathcal{L}$ is positive, then there exist positive constants α and s, depending only on $\|\rho\|_{\mathcal{L}}$, such that for $0 < r \le \alpha$ there is a sequence $\{a_k\}_k \subset \mathbb{D}$ satisfying*

(a) *$\mathbb{D} = \bigcup_k D^r(a_k)$.*

(b) *$D^{sr}(a_k) \cap D^{sr}(a_j) = \emptyset$ for $k \ne j$.*

(c) *$\{D^{2\alpha}(a_k)\}_k$ is a covering of $\mathbb{D}$ of finite multiplicity N.*

A sequence $\{a_k\}_k$ satisfying (a)–(c) in Lemma 2.2 is called a (ρ, r)-lattice.

The following result plays an important role in proving our main theorems and can be regarded as one type of generalized sub-mean property of $|f\mathrm{e}^{-\varphi}|^p$, which describes the boundedness of the point evaluation functionals on A^p_φ for $0 < p < \infty$.

Lemma 2.3 *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ and $0 < p < \infty$. There exist positive constants α and C such that for $r \in (0, \alpha]$ and $f \in H(\mathbb{D})$,*

$$|f(z)|^p \mathrm{e}^{-p\varphi(z)} \le \frac{C}{\rho(z)^2} \int_{D^r(z)} |f(w)|^p \mathrm{e}^{-p\varphi(w)} dA(w).$$

Proof This is an easy consequence of [16, Lemma 3.3] and Lemma 2.1. □

As in [16], in what follows, we always assume α to be chosen such that the conclusions of Lemmas 2.1–2.3 are valid.

Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. For $z, w \in \mathbb{D}$, we define the distance

$$d_\rho(z, w) = \inf_\gamma \int_0^1 |\gamma'(t)| \frac{dt}{\rho(\gamma(t))},$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0, 1] \to \mathbb{D}$ with $\gamma(0) = z$ and $\gamma(1) = w$.

The next lemma gives the upper bound estimate of the reproducing kernel $K(z, w)$ for all $z, w \in \mathbb{D}$ and the lower bound estimate of $K(z, w)$ near the diagonal, refer to [16, Theorem 3.2] for the details.

Lemma 2.4 *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. There are positive constants C_1, C_2, σ such that*

$$|K(z, w)| \le C_1 \frac{\mathrm{e}^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} \mathrm{e}^{-\sigma d_\rho(z,w)}, \quad z, w \in \mathbb{D}, \tag{2.2}$$

and

$$|K(z, w)| \ge C_2 \frac{\mathrm{e}^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)}, \quad w \in D^\alpha(z). \tag{2.3}$$

We also need the asymptotic estimates of the $\|\cdot\|_{p,\varphi}$ norm for the reproducing kernel, see [16, Corollary 3.2].

Lemma 2.5 *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ and $0 < p \le \infty$. Then for $z \in \mathbb{D}$, we have*

$$\|K_z\|_{p,\varphi} \simeq \mathrm{e}^{\varphi(z)} \rho(z)^{\frac{2}{p}-2}. \tag{2.4}$$

Lemma 2.6 *Let $\rho \in \mathcal{L}_0$, $-\infty < l < \infty$ and $\sigma > 0$. Then there exists some constant $C > 0$ such that*

$$\int_{\mathbb{D}} \rho(w)^l \mathrm{e}^{-\sigma d_\rho(z,w)} dA(w) \le C\rho(z)^{l+2}, \quad z \in \mathbb{D}.$$

Proof Taking $k = 0$ in [16, Corollary 3.1] we obtain the desired result. □

Lemma 2.7 *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0 < p \le \infty$ and $l \in \mathbb{R}$. Then for $z \in \mathbb{D}$, we have*

$$\int_{\mathbb{D}} |K(w, z)|^p \mathrm{e}^{-p\varphi(w)} \rho(w)^l dA(w) \simeq \mathrm{e}^{p\varphi(z)} \rho(z)^{2(1-p)+l}. \tag{2.5}$$

Proof On one hand, (2.2) and Lemma 2.6 imply that

$$\begin{aligned}
&\int_{\mathbb{D}} |K(w,z)|^p e^{-p\varphi(w)} \rho(w)^l dA(w) \\
&\quad \leq C_1 e^{p\varphi(z)} \rho(z)^{-p} \int_{\mathbb{D}} \rho(w)^{l-p} e^{-p\sigma d_\rho(z,w)} dA(w) \\
&\quad \leq C_2 e^{p\varphi(z)} \rho(z)^{2(1-p)+l} \qquad (2.6)
\end{aligned}$$

for positive constants C_1, C_2 and σ.

On the other hand, by (2.3) and Lemma 2.1, there exist positive constants C_3 and C_4 such that

$$\begin{aligned}
&\int_{\mathbb{D}} |K(w,z)|^p e^{-p\varphi(w)} \rho(w)^l dA(w) \\
&\quad \geq C_3 e^{p\varphi(z)} \rho(z)^{-p} \int_{D^\alpha(z)} \rho(w)^{l-p} dA(w) \\
&\quad \geq C_4 e^{p\varphi(z)} \rho(z)^{2(1-p)+l}.
\end{aligned}$$

This together with (2.6) gives (2.5). □

The following proposition was proved in [33, Proposition 2.2].

Proposition 2.8 *Let $0 < p < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. Then the set $\{k_{p,z} : z \in \mathbb{D}\}$ is bounded in A^p_φ and the normalized reproducing kernel $k_{p,z}$ tends to zero uniformly on compact subsets of $\mathbb{D}$ as $|z| \to 1^-$.*

Given a positive Borel measure μ on $\mathbb{D}$ and $r, t > 0$, the averaging function $\widehat{\mu}_r$ and the t-Berezin transform $\widetilde{\mu}_t$ with respect to μ are defined to be

$$\widehat{\mu}_r(z) = \frac{\mu(D^r(z))}{\rho(z)^2}, \quad z \in \mathbb{D},$$

and

$$\widetilde{\mu}_t(z) = \int_{\mathbb{D}} |k_{t,z}(w)|^t e^{-t\varphi(w)} d\mu(w), \quad z \in \mathbb{D},$$

respectively.

If $t = 2$, then the t-Berezin transform becomes the classical Berezin transform, see [34, Chapter 6] for example. In this case, we abbreviate the notation $\widetilde{\mu}_t$ as $\widetilde{\mu}$ for simplicity.

Proposition 2.9 *Let $0 < p < \infty$, $\delta \in (0, \alpha]$ and μ be a positive Borel measure on $\mathbb{D}$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f(z)|^p e^{-p\varphi(z)} d\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^p e^{-p\varphi(z)} \widehat{\mu}_\delta(z) dA(z)$$

for any $f \in H(\mathbb{D})$.

Proof Given $\delta \in (0, \alpha]$, in views of Lemma 2.1, there exists an $r \in (0, \alpha]$ such that

$$\chi_{D^r(z)}(w) \leq \chi_{D^\delta(w)}(z), \quad z, w \in \mathbb{D}. \qquad (2.7)$$

This, in combination with Lemmas 2.3, 2.1 and Fubini's theorem, concludes that for any $f \in H(\mathbb{D})$,

$$\int_{\mathbb{D}} |f(z)|^p e^{-p\varphi(z)} d\mu(z) \lesssim \int_{\mathbb{D}} \frac{1}{\rho(z)^2} \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} dA(w) d\mu(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{D}} \frac{1}{\rho(z)^2} \int_{\mathbb{D}} \chi_{D^r(z)}(w)|f(w)|^p \mathrm{e}^{-p\varphi(w)} dA(w) d\mu(z) \\
&\lesssim \int_{\mathbb{D}} |f(w)|^p \mathrm{e}^{-p\varphi(w)} dA(w) \int_{\mathbb{D}} \frac{\chi_{D^\delta(w)}(z)}{\rho(w)^2} d\mu(z) \\
&= \int_{\mathbb{D}} |f(w)|^p \mathrm{e}^{-p\varphi(w)} \widehat{\mu}_\delta(w) dA(w).
\end{aligned}
$$

This finishes the proof. □

Given a measurable function f, let $\widetilde{f}_t$ be the t-Berezin transform of f. If we set $d\mu = f dA$, then we write $\widetilde{f}_t = \widetilde{\mu}_t$.

Proposition 2.10 *Let $1 \le p \le \infty$ and $t > 0$. Then the operator $f \mapsto \widetilde{f}_t$ is bounded on L^p.*

Proof For $p = 1$, applying Fubini's theorem and using estimates (2.4) and (2.5), we obtain

$$
\begin{aligned}
\|\widetilde{f}_t\|_{L^1} &= \int_{\mathbb{D}} |\widetilde{f}_t(z)| dA(z) \\
&\le \int_{\mathbb{D}} \int_{\mathbb{D}} |k_{t,z}(w)|^t \mathrm{e}^{-t\varphi(w)} |f(w)| dA(w) dA(z) \\
&= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|K(w,z)|^t}{\|K_z\|^t_{t,\varphi}} \mathrm{e}^{-t\varphi(w)} |f(w)| dA(w) dA(z) \\
&\simeq \int_{\mathbb{D}} |f(w)| \mathrm{e}^{-t\varphi(w)} \Big(\int_{\mathbb{D}} |K(w,z)|^t \mathrm{e}^{-t\varphi(z)} \rho(z)^{2t-2} dA(z) \Big) dA(w) \\
&\simeq \int_{\mathbb{D}} |f(w)| dA(w) \\
&= \|f\|_{L^1}.
\end{aligned}
$$

For $p = \infty$, it is easy to check

$$
\begin{aligned}
|\widetilde{f}_t(z)| &\le \int_{\mathbb{D}} |k_{t,z}(w)|^t \mathrm{e}^{-t\varphi(w)} |f(w)| dA(w) \\
&\le \|f\|_{L^\infty} \int_{\mathbb{D}} |k_{t,z}(w)|^t \mathrm{e}^{-t\varphi(w)} dA(w) \\
&= \|f\|_{L^\infty}.
\end{aligned}
$$

Hence, we get $\|\widetilde{f}_t\|_{L^\infty} \le \|f\|_{L^\infty}$. It follows by interpolation that the operator $f \mapsto \widetilde{f}_t$ is bounded on L^p for all $1 \le p \le \infty$. □

Proposition 2.11 *Let $0 < p < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) $\widetilde{\mu}_t \in L^p$ *for some (or any)* $t > 0$.

(b) $\widehat{\mu}_\delta \in L^p$ *for some (or any)* $\delta \in (0, \alpha]$.

(c) *The sequence* $\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\}_k \in l^p$ *for some (or any)* (ρ, r)*-lattice* $\{a_k\}$ *with* $r \in (0, \alpha]$. *Moreover, we have*

$$
\|\widetilde{\mu}_t\|_{L^p} \simeq \|\widehat{\mu}_\delta\|_{L^p} \simeq \|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\}_k\|_{l^p}. \tag{2.8}
$$

Proof The equivalence (b) $\Leftrightarrow$ (c) was proved in [33, Proposition 2.5] with

$$
\|\widehat{\mu}_\delta\|_{L^p} \simeq \|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2/p}\}_k\|_{l^p}. \tag{2.9}
$$

It suffices to show the equivalence (a) $\Leftrightarrow$ (b) to complete the proof.

(a) $\Rightarrow$ (b). For $z \in \mathbb{D}$ and $\delta \in (0, \alpha]$, Lemmas 2.4 and 2.1 give

$$|K(w,z)| \simeq \frac{e^{\varphi(w)+\varphi(z)}}{\rho(z)^2}, \quad w \in D^\delta(z).$$

This combined with estimate (2.4) shows that

$$\begin{aligned}\widehat{\mu}_\delta(z) &= \frac{1}{\rho(z)^2}\int_{D^\delta(z)} d\mu(w)\\ &\simeq e^{-t\varphi(z)}\rho(z)^{2t-2}\int_{D^\delta(z)} |K(w,z)|^t e^{-t\varphi(w)} d\mu(w)\\ &\le e^{-t\varphi(z)}\rho(z)^{2t-2}\int_{\mathbb{D}} |K(w,z)|^t e^{-t\varphi(w)} d\mu(w)\\ &\simeq \int_{\mathbb{D}} |k_{t,z}(w)|^t e^{-t\varphi(w)} d\mu(w)\\ &= \widetilde{\mu}_t(z), \end{aligned}\tag{2.10}$$

which yields statement (b) with

$$\|\widehat{\mu}_\delta\|_{L^p} \lesssim \|\widetilde{\mu}_t\|_{L^p}, \quad 0 < p < \infty. \tag{2.11}$$

(b) $\Rightarrow$ (a). For $1 \le p < \infty$, by taking $f = k_{t,z}$ and $p = t$ in Proposition 2.9, we obtain

$$\widetilde{\mu}_t(z) \lesssim (\widetilde{\widehat{\mu}_\delta})_t(z), \quad t > 0, \ z \in \mathbb{D}.$$

Hence, this together with Proposition 2.10 gives

$$\|\widetilde{\mu}_t\|_{L^p} \lesssim \|(\widetilde{\widehat{\mu}_\delta})_t\|_{L^p} \lesssim \|\widehat{\mu}_\delta\|_{L^p}, \quad 1 \le p < \infty. \tag{2.12}$$

For $0 < p < 1$, given any (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$, note that there exists some constant $B > 1$ such that

$$\bigcup_{w \in D^r(a_j)} D^r(w) \subset D^{Br}(a_j), \quad w \in \mathbb{D}, \tag{2.13}$$

see [16, Lemma 3.1 (B)] and its proof. Then we can divide the lattice $\{a_k\}_k$ into J subsequence $\{a_{j,k}\}_k$ $(j = 1, 2, \ldots, J)$ such that each sequence $\{a_{j,k}\}_k$ is a (ρ, Br)-lattice. It follows from estimate (2.9) that

$$\sum_k \widehat{\mu}_{Br}(a_k)^p \rho(a_k)^2 = \sum_{j=1}^J \sum_k \widehat{\mu}_{Br}(a_{j,k})^p \rho(a_{j,k})^2 \lesssim \|\widehat{\mu}_\delta\|_{L^p}^p \tag{2.14}$$

for $\delta \in (0, \alpha]$. By estimate (2.4), Proposition 2.9, Lemma 2.2, (2.13), estimate (2.2) and Lemma 2.1, we deduce

$$\begin{aligned}|\widetilde{\mu}_t(z)|^p &\simeq e^{-tp\varphi(z)}\rho(z)^{2tp-2p}\bigg(\int_{\mathbb{D}} |K(w,z)|^t e^{-t\varphi(w)} d\mu(w)\bigg)^p\\ &\lesssim e^{-tp\varphi(z)}\rho(z)^{2tp-2p}\bigg(\int_{\mathbb{D}} |K(w,z)|^t e^{-t\varphi(w)} \widehat{\mu}_r(w) dA(w)\bigg)^p\\ &\le e^{-tp\varphi(z)}\rho(z)^{2tp-2p}\bigg(\sum_j \int_{D^r(a_j)} |K(w,z)|^t e^{-t\varphi(w)} \widehat{\mu}_r(w) dA(w)\bigg)^p\\ &\lesssim e^{-tp\varphi(z)}\rho(z)^{2tp-2p}\bigg(\sum_j \widehat{\mu}_{Br}(a_j) \int_{D^r(a_j)} |K(w,z)|^t e^{-t\varphi(w)} dA(w)\bigg)^p \end{aligned}$$

$$
\begin{aligned}
&\le \mathrm{e}^{-tp\varphi(z)}\rho(z)^{2tp-2p}\sum_j \widehat{\mu}_{Br}(a_j)^p\bigg(\int_{D^r(a_j)}|K(w,z)|^t\mathrm{e}^{-t\varphi(w)}dA(w)\bigg)^p\\
&\lesssim \rho(z)^{tp-2p}\sum_j \widehat{\mu}_{Br}(a_j)^p\rho(a_j)^{2p}\sup_{w\in D^r(a_j)}\rho(w)^{-tp}\mathrm{e}^{-t\sigma pd_\rho(z,w)}\\
&\lesssim \rho(z)^{tp-2p}\sum_j \widehat{\mu}_{Br}(a_j)^p\rho(a_j)^{2p-tp}\sup_{w\in D^r(a_j)}\mathrm{e}^{-t\sigma pd_\rho(z,w)}.
\end{aligned}
$$

Integrating both sides above and using Lemma 2.6, we get

$$
\begin{aligned}
\|\widetilde{\mu}_t\|^p_{L^p} &\lesssim \sum_j \widehat{\mu}_{Br}(a_j)^p\sup_{w\in D^r(a_j)}\rho(a_j)^{2p-tp}\int_{\mathbb{D}}\rho(z)^{tp-2p}\mathrm{e}^{-t\sigma pd_\rho(z,w)}dA(z)\\
&\lesssim \sum_j \widehat{\mu}_{Br}(a_j)^p\rho(a_j)^{2p-tp}\sup_{w\in D^r(a_j)}\rho(w)^{tp-2p+2}\\
&\lesssim \sum_j \widehat{\mu}_{Br}(a_j)^p\rho(a_j)^2.
\end{aligned}
$$

Therefore, this combined with estimate (2.14) yields

$$
\|\widetilde{\mu}_t\|^p_{L^p} \lesssim \sum_j \widehat{\mu}_{Br}(a_j)^p\rho(a_j)^2 \lesssim \|\widehat{\mu}_\delta\|^p_{L^p}, \quad 0<p<1.
$$

This together with estimate (2.12) gives the implication (b) $\Rightarrow$ (a) with

$$
\|\widetilde{\mu}_t\|_{L^p} \lesssim \|\widehat{\mu}_\delta\|_{L^p}, \quad 0<p<\infty. \tag{2.15}
$$

The quantity equivalences (2.8) come from estimates (2.9), (2.11) and (2.15), which completes the proof of Proposition 2.11. □

3 Carleson Measures

For $0<p<\infty$ and a positive Borel measure μ on $\mathbb{D}$, let

$$
L^p_{\varphi,\mu} = \bigg\{f \text{ Lebesgue measurable} : \int_{\mathbb{D}}|f(z)|^p\mathrm{e}^{-p\varphi(z)}d\mu(z)<\infty\bigg\}.
$$

Let $0<p\le\infty$ and $0<q<\infty$. A positive Borel measure μ on $\mathbb{D}$ will be called a q-Carleson measure for A^p_φ if there is a positive constant C such that

$$
\bigg(\int_{\mathbb{D}}|f(z)|^q\mathrm{e}^{-q\varphi(z)}d\mu(z)\bigg)^{\frac{1}{q}} \le C\|f\|_{p,\varphi}
$$

for all f in A^p_φ. This means the identity operator $\mathrm{Id}: A^p_\varphi \to L^q_{\varphi,\mu}$ is bounded. Correspondingly, μ is said to be a vanishing q-Carleson measure for A^p_φ if

$$
\int_{\mathbb{D}}|f_k(z)|^q\mathrm{e}^{-q\varphi(z)}d\mu(z)\to 0,
$$

whenever $\{f_k\}_k$ is bounded sequence in A^p_φ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$.

To characterize the q-Carleson measures for A^p_φ with $0<p\le\infty$ and $0<q<\infty$, the following so-called partial atomic decomposition of A^p_φ is needed.

Proposition 3.1 *Let $0 < p \le \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. If $\{a_k\}_k$ is a (ρ, r)-lattice with $r \in (0, \alpha]$, then the function*

$$F(z) = \sum_k c_k k_{p,a_k}(z)$$

belongs to A^p_φ for any sequence $\{c_k\}_k \in l^p$. Moreover,

$$\|F\|_{p,\varphi} \lesssim \|\{c_k\}_k\|_{l^p}.$$

Proof The result for $0 < p < \infty$ was proved in [33, Proposition 2.1]. For $p = \infty$, it follows from (2.4) and Lemma 2.3 that

$$\begin{aligned}
\|F\|_{\infty,\varphi} &= \sup_{z\in\mathbb{D}} |F(z)|\mathrm{e}^{-\varphi(z)} \\
&\le \sup_{z\in\mathbb{D}} \sum_k |c_k||k_{\infty,a_k}(z)|\mathrm{e}^{-\varphi(z)} \\
&= \sup_{z\in\mathbb{D}} \sum_k |c_k| \frac{|K(z,a_k)|}{\|K_{a_k}\|_{\infty,\varphi}} \mathrm{e}^{-\varphi(z)} \\
&\simeq \sup_{z\in\mathbb{D}} \sum_k |c_k||K(z,a_k)|\rho(a_k)^2 \mathrm{e}^{-\varphi(a_k)-\varphi(z)} \\
&\lesssim \sup_{z\in\mathbb{D}} \sum_k |c_k| \int_{D^r(a_k)} |K(z,w)|\mathrm{e}^{-\varphi(w)} dA(w)\mathrm{e}^{-\varphi(z)} \\
&\lesssim \|\{c_k\}_k\|_{l^\infty}
\end{aligned}$$

for $\{c_k\}_k \in l^\infty$. This completes the proof. □

Theorem 3.2 *Let $0 < p \le q < \infty$, $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(a) *μ is a q-Carleson measure for A^p_φ.*

(b) *$\widetilde{\mu}_t \rho^{2-2q/p} \in L^\infty$ for some (or any) $t > 0$.*

(c) *$\widehat{\mu}_\delta \rho^{2-2q/p} \in L^\infty$ for some (or any) $\delta \in (0, \alpha]$ small enough.*

(d) *$\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k \in l^\infty$ for some (or any) (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$ small enough.*

Moreover, we have

$$\|\mathrm{Id}\|^q_{A^p_\varphi \to L^q_{\varphi,\mu}} \simeq \|\widetilde{\mu}_t \rho^{2-2q/p}\|_{L^\infty} \simeq \|\widehat{\mu}_\delta \rho^{2-2q/p}\|_{L^\infty} \simeq \|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k\|_{l^\infty}. \tag{3.1}$$

Proof The equivalences of (a) ⇔ (c) ⇔ (d) were proved in [33, Theorem 3.1] with

$$\|\mathrm{Id}\|^q_{A^p_\varphi \to L^q_{\varphi,\mu}} \simeq \|\widehat{\mu}_\delta \rho^{2-2q/p}\|_{L^\infty} \simeq \|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k\|_{l^\infty}. \tag{3.2}$$

We are to show the implications (b) ⇒ (c) and (d) ⇒ (b) to complete the proof of this theorem.

(b) ⇒ (c). This implication follows easily from estimate (2.10) with

$$\|\widehat{\mu}_\delta \rho^{2-2q/p}\|_{L^\infty} \lesssim \|\widetilde{\mu}_t \rho^{2-2q/p}\|_{L^\infty}. \tag{3.3}$$

(d) ⇒ (b). Suppose $\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k \in l^\infty$ for some (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$ taken small enough. For any $a \in \mathbb{D}$, Lemma 2.1 gives that

$$D^r(w) \subset D^{2\alpha}(a), \quad \text{for } w \in D^r(a). \tag{3.4}$$

Given $t>0$, let $s=tp/q$. Then

$$\sup_{w\in D^r(a)}|f(w)|^s\mathrm{e}^{-s\varphi(w)}\lesssim\frac{1}{\rho(a)^2}\int_{D^{2\alpha}(a)}|f(\zeta)|^s\mathrm{e}^{-s\varphi(\zeta)}dA(\zeta)\tag{3.5}$$

for arbitrary $f\in A^s_\varphi$ by combining Lemmas 2.3, 2.1 and (3.4).

In views of estimate (2.4), we have the following fact

$$|k_{t,z}(w)|^t\rho(z)^{2-2q/p}\simeq|k_{s,z}(w)|^t,\quad z,w\in\mathbb{D}.\tag{3.6}$$

This together with Lemma 2.2 and (3.5) yields

$$\begin{aligned}\widetilde{\mu}_t(z)\rho(z)^{2-2q/p}&=\int_{\mathbb{D}}|k_{t,z}(w)|^t\rho(z)^{2-2q/p}\mathrm{e}^{-t\varphi(w)}d\mu(w)\\&\simeq\int_{\mathbb{D}}|k_{s,z}(w)|^t\mathrm{e}^{-t\varphi(w)}d\mu(w)\\&\le\sum_k\int_{D^r(a_k)}|k_{s,z}(w)|^t\mathrm{e}^{-t\varphi(w)}d\mu(w)\\&\le\sum_k\mu(D^r(a_k))\Big(\sup_{w\in D^r(a_k)}|k_{s,z}(w)|^s\mathrm{e}^{-s\varphi(w)}\Big)^{q/p}\\&\lesssim\sum_k\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\Big(\int_{D^{2\alpha}(a_k)}|k_{s,z}(\zeta)|^s\mathrm{e}^{-s\varphi(\zeta)}dA(\zeta)\Big)^{q/p}.\end{aligned}\tag{3.7}$$

Notice that $q/p\ge1$, with the fact

$$\sum_k b_k^\theta\le\Big(\sum_k b_k\Big)^\theta,\quad b_k>0,\quad 1\le\theta<\infty\tag{3.8}$$

and Lemma 2.2, we deduce

$$\begin{aligned}\widetilde{\mu}_t(z)\rho(z)^{2-2q/p}&\lesssim\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\Big(\sum_k\int_{D^{2\alpha}(a_k)}|k_{s,z}(\zeta)|^s\mathrm{e}^{-s\varphi(\zeta)}dA(\zeta)\Big)^{q/p}\\&\lesssim N^{q/p}\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\|k_{s,z}\|^t_{s,\varphi}\\&\lesssim\sup_k\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p},\end{aligned}$$

where N is chosen as in Lemma 2.2. This gives (b) with

$$\|\widetilde{\mu}_t\rho^{2-2q/p}\|_{L^\infty}\lesssim\|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k\|_{l^\infty}.$$

This estimate together with (3.2) and (3.3) gives (3.1), and the proof is complete here. □

Theorem 3.3 *Let $0<p\le q<\infty$, $\varphi\in\mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}}\simeq\rho\in\mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) *μ is a vanishing q-Carleson measure for A^p_φ.*

(b) $\lim_{|z|\to1^-}\widetilde{\mu}_t(z)\rho(z)^{2-2q/p}=0$ *for some (or any) $t>0$.*

(c) $\lim_{|z|\to1^-}\widehat{\mu}_\delta(z)\rho(z)^{2-2q/p}=0$ *for some (or any) $\delta\in(0,\alpha]$ small enough.*

(d) $\lim_{k\to\infty}\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}=0$ *for some (or any) (ρ,r)-lattice $\{a_k\}_k$ with $r\in(0,\alpha]$ small enough.*

Proof The equivalences of (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) were proved in [33, Theorem 3.2]. The proof of (b) $\Rightarrow$ (c) is similar to the corresponding part of Theorem 3.2. It is enough to show (d) $\Rightarrow$ (b) to complete the proof.

(d) $\Rightarrow$ (b). Suppose $\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p} \to 0$ as $k \to \infty$ for some (ρ, r)-lattice $\{a_k\}$ with $r \in (0, \alpha]$ taken small enough. Then for arbitrary $\varepsilon > 0$, there exists some positive integer K such that $\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p} < \varepsilon$ whenever $k > K$. Since $\bigcup_{k=1}^K \overline{D^{2\alpha}(a_k)}$ is compact in $\mathbb{D}$, Proposition 2.8 gives that $\{k_{s,z}\}_{z\in\mathbb{D}} \subseteq A^s_\varphi$ uniformly converges to 0 on $\bigcup_{k=1}^K \overline{D^{2\alpha}(a_k)}$ as $|z| \to 1^-$, where $s = tp/q$. From estimates (3.7), (3.5), (3.8) and Lemma 2.2, as $|z|$ is sufficiently close to 1^-, we deduce

$$
\begin{aligned}
&\widetilde{\mu}_t(z)\rho(z)^{2-2q/p} \\
&\quad\simeq \int_{\mathbb{D}} |k_{s,z}(w)|^t \mathrm{e}^{-t\varphi(w)} d\mu(w) \\
&\quad\le \int_{\bigcup_{k=1}^K \overline{D^{2\alpha}(a_k)}} |k_{s,z}(w)|^t \mathrm{e}^{-t\varphi(w)} d\mu(w) \\
&\qquad + \sum_{k=K+1}^{\infty} \mu(D^r(a_k))\Big(\sup_{w\in D^r(a_k)} |k_{s,z}(w)|^s \mathrm{e}^{-s\varphi(w)}\Big)^{q/p} \\
&\quad\lesssim \varepsilon + \sum_{k=K+1}^{\infty} \widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\Big(\int_{D^{2\alpha}(a_k)} |k_{s,z}(\zeta)|^s \mathrm{e}^{-s\varphi(\zeta)} dA(\zeta)\Big)^{q/p} \\
&\quad\lesssim \varepsilon + \sup_{k\ge K+1} \widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\Big(\sum_{k=K+1}^{\infty}\int_{D^{2\alpha}(a_k)} |k_{s,z}(\zeta)|^s \mathrm{e}^{-s\varphi(\zeta)} dA(\zeta)\Big)^{q/p} \\
&\quad< \varepsilon + N^{q/p}\|k_{s,z}\|_{s,\varphi}^t \varepsilon \\
&\quad\lesssim \varepsilon,
\end{aligned}
$$

where N is chosen as in Lemma 2.2. This means $\widetilde{\mu}_t(z)\rho(z)^{2-2q/p} \to 0$ as $|z| \to 1^-$. □

The following theorem is a direct consequence of Proposition 2.11 and [33, Theorem 3.3].

Theorem 3.4 *Let $0 < q < p < \infty$, $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) *μ is a vanishing q-Carleson measure for A^p_φ.*

(b) *μ is a q-Carleson measure for A^p_φ.*

(c) *$\widetilde{\mu}_t \in L^{\frac{p}{p-q}}$ for some (or any) $t > 0$.*

(d) *$\widehat{\mu}_\delta \in L^{\frac{p}{p-q}}$ for some (or any) $\delta \in (0, \alpha]$ small enough.*

(e) *$\big\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\big\}_k \in l^{\frac{p}{p-q}}$ for some (or any) (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$.*

Moreover, we have

$$\|Id\|^q_{A^p_\varphi \to L^q_{\varphi,\mu}} \simeq \|\widetilde{\mu}_t\|_{L^{\frac{p}{p-q}}} \simeq \|\widehat{\mu}_\delta\|_{L^{\frac{p}{p-q}}} \simeq \|\{\widehat{\mu}_r(a_k)\rho(a_k)^{2-2q/p}\}_k\|_{l^{\frac{p}{p-q}}}. \tag{3.9}$$

In order to characterize the q-Carleson measure for A^∞_φ with $0 < q < \infty$, we will use the approach given by Luecking [21] via Khinchine's equality. Recall that the Rademacher functions

r_n are defined by

$$r_0(t) = \begin{cases} 1, & 0 \le t - [t] < \dfrac{1}{2}, \\ -1, & \dfrac{1}{2} \le t - [t] < 1, \end{cases}$$
$$r_n(t) = r_0(2^n t), \quad n \ge 1,$$

where $[t]$ denotes the largest integer less than or equal to t. The Khinchine's inequality is given in the following.

Khinchine′s inequality. For $0 < p < \infty$, there exist positive constants C_1 and C_2 depending only on p such that, for all natural numbers m and all complex numbers $b_1, b_2, \dots, b_m$,

$$C_1\Big(\sum_{j=1}^m |b_j|^2\Big)^{\frac{p}{2}} \le \int_0^1 \Big|\sum_{j=1}^m b_j r_j(t)\Big|^p dt \le C_2\Big(\sum_{j=1}^m |b_j|^2\Big)^{\frac{p}{2}}.$$

Theorem 3.5 *Let $0 < q < \infty$, $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) *μ is a vanishing q-Carleson measure for A^∞_φ.*

(b) *μ is a q-Carleson measure for A^∞_φ.*

(c) *$\widetilde{\mu}_t \in L^1$ for some (or any) $t > 0$.*

(d) *$\widehat{\mu}_\delta \in L^1$ for some (or any) $\delta \in (0, \alpha]$ small enough.*

(e) *$\{\mu(D^r(a_k))\}_k \in l^1$ for some (or any) (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$.*

Moreover, we have

$$\|Id\|^q_{A^\infty_\varphi \to L^q_{\varphi,\mu}} \simeq \|\widetilde{\mu}_t\|_{L^1} \simeq \|\widehat{\mu}_\delta\|_{L^1} \simeq \|\{\mu(D^r(a_k))\}_k\|_{l^1}. \tag{3.10}$$

Proof The equivalence of (c) $\Leftrightarrow$ (d) follows from Proposition 2.11 with

$$\|\widetilde{\mu}_t\|_{L^1} \simeq \|\widehat{\mu}_\delta\|_{L^1}. \tag{3.11}$$

The implication (a) $\Rightarrow$ (b) is trivial. It remains to show (b) $\Rightarrow$ (e), (e) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a).

(b) $\Rightarrow$ (e). Assume that μ is a q-Carleson measure for A^∞_φ. For any (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$, let

$$F(z) = \sum_k c_k k_{\infty, a_k}(z),$$

where the sequence $\{c_k\}_k$ is in l^∞. According to our assumption and Proposition 3.1, we get

$$\begin{aligned}\int_{\mathbb{D}} \Big|\sum_k c_k k_{\infty,a_k}(z)\Big|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) &= \int_{\mathbb{D}} |F(z)|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) \\ &\lesssim \|F\|^q_{\infty,\varphi} \|\mathrm{Id}\|^q_{A^\infty_\varphi \to L^q_{\varphi,\mu}} \\ &\lesssim \|\{c_k\}_k\|^q_{l^\infty} \|\mathrm{Id}\|^q_{A^\infty_\varphi \to L^q_{\varphi,\mu}}.\end{aligned}$$

Replacing c_k with $r_k(t)c_k$ in the above inequality and then integrating with respect to t from 0 to 1, we obtain

$$\int_0^1 \int_{\mathbb{D}} \Big|\sum_k r_k(t) c_k k_{\infty,a_k}(z)\Big|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) dt \lesssim \|\{c_k\}_k\|^q_{l^\infty} \|\mathrm{Id}\|^q_{A^\infty_\varphi \to L^q_{\varphi,\mu}}. \tag{3.12}$$

Applying Fubini's theorem and Khinchine's inequality, we get

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{D}} \Big| \sum_k r_k(t) c_k k_{\infty,a_k}(z) \Big|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) dt \\
&= \int_{\mathbb{D}} \int_0^1 \Big| \sum_k r_k(t) c_k k_{\infty,a_k}(z) \Big|^q dt \mathrm{e}^{-q\varphi(z)} d\mu(z) \\
&\gtrsim \int_{\mathbb{D}} \Big(\sum_k |c_k k_{\infty,a_k}(z)|^2 \Big)^{\frac{q}{2}} \mathrm{e}^{-q\varphi(z)} d\mu(z) \\
&\simeq \int_{\mathbb{D}} \Big(\sum_k |c_k|^2 \chi_{D^r(a_k)}(z) \Big)^{\frac{q}{2}} d\mu(z), \qquad (3.13)
\end{aligned}$$

where the last step is from the following observation given by Lemmas 2.1, 2.4, 2.5

$$|k_{\infty,a_k}(z)|^2 = \frac{|K(z,a_k)|^2}{\|K_{a_k}\|_{\infty,\varphi}^2} \simeq \mathrm{e}^{2\varphi(z)}, \quad z \in D^r(a_k).$$

By using the fact (3.8) when $q \geq 2$ or Hölder's inequality when $q < 2$, and with estimates (3.12) and (3.13), we conclude that

$$\begin{aligned}
\sum_k |c_k|^q \mu(D^r(a_k)) &= \int_{\mathbb{D}} \sum_k |c_k|^q \chi_{D^r(a_k)}(z) d\mu(z) \\
&\lesssim \int_{\mathbb{D}} \Big(\sum_k |c_k|^2 \chi_{D^r(a_k)}(z) \Big)^{\frac{q}{2}} d\mu(z) \\
&\lesssim \|\{c_k\}_k\|_{l^\infty}^q \|\mathrm{Id}\|_{A_\varphi^\infty \to L_{\varphi,\mu}^q}^q.
\end{aligned}$$

Since $\{c_k\}_k$ is an arbitrary sequence in l^∞, we may in particular take $c_k = 1$ for all k in the above inequality and then get $\{\mu(D^r(a_k))\}_k \in l^1$ with

$$\|\{\mu(D^r(a_k))\}_k\|_{l^1} \lesssim \|\mathrm{Id}\|_{A_\varphi^\infty \to L_{\varphi,\mu}^q}^q. \qquad (3.14)$$

(e) $\Rightarrow$ (d). Assume that $\{\mu(D^r(a_k))\}_k \in l^1$ for some (ρ, r)-lattice $\{a_k\}_k$ with $r \in (0, \alpha]$. For $\delta \in (0, \alpha]$ small enough, we have $D^\delta(z) \subset D^{2\alpha}(a_k)$ whenever $z \in D^r(a_k)$, it follows that

$$\begin{aligned}
\int_{\mathbb{D}} \widehat{\mu}_\delta(z) dA(z) &\leq \sum_k \int_{D^r(a_k)} \widehat{\mu}_\delta(z) dA(z) \\
&= \sum_k \int_{D^r(a_k)} \frac{\mu(D^\delta(z))}{\rho(z)^2} dA(z) \\
&\lesssim \sum_k \mu(D^{2\alpha}(a_k)) \\
&\lesssim \sum_k \mu(D^r(a_k)),
\end{aligned}$$

where the last step is from Lemma 2.2. Thus, we have $\widehat{\mu}_\delta \in L^1$ with

$$\|\widehat{\mu}_\delta\|_{L^1} \lesssim \|\{\mu(D^r(a_k))\}_k\|_{l^1}. \qquad (3.15)$$

(d) $\Rightarrow$ (a). Suppose $\widehat{\mu}_\delta \in L^1$ for $\delta \in (0, \alpha]$ small enough, we aim to show that μ is a vanishing q-Carleson measure for A_φ^∞, that is, if $\{f_k\}_k$ is a bounded sequence in A_φ^∞ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, then $\lim_{k\to\infty} \|f_k\|_{L_{\varphi,\mu}^q} = 0$.

Since $\rho \in \mathcal{L}$, there exists a positive constant s such that $\rho(z) \le s(1-|z|)$. Since $\delta \in (0,\alpha]$ is taken small enough, we fix $\delta < \frac{1}{2s}$ and then have

$$D^{\frac{\delta}{2}}(z) \subset \left\{w \in \mathbb{D}: \ |w| > \frac{r}{2}\right\}, \quad \text{as } |z| > r \tag{3.16}$$

for $r > \frac{1}{3}$. It follows from Lemma 2.3 that

$$|f_k(z)|^q \mathrm{e}^{-q\varphi(z)} \lesssim \frac{1}{\rho(z)^2} \int_{D^{\frac{\delta}{2}}(z)} |f_k(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w).$$

Integrating both sides on the above formula over the annulus $\{z \in \mathbb{D}: \ |z| > r\}$ with respect to the measure μ, using Fubini's theorem, (3.16) and Lemma 2.1, we obtain

$$\begin{aligned}
&\int_{\{z\in\mathbb{D}:\ |z|>r\}} |f_k(z)|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) \\
&\quad \lesssim \int_{\{z\in\mathbb{D}:\ |z|>r\}} \frac{1}{\rho(z)^2} \int_{D^{\frac{\delta}{2}}(z)} |f_k(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) d\mu(z) \\
&\quad \lesssim \int_{\{w\in\mathbb{D}:\ |w|>\frac{r}{2}\}} |f_k(w)|^q \mathrm{e}^{-q\varphi(w)} \left(\int_{D^{\delta}(w)} \frac{1}{\rho(z)^2} d\mu(z) \right) dA(w) \\
&\quad \lesssim \int_{\{w\in\mathbb{D}:\ |w|>\frac{r}{2}\}} |f_k(w)|^q \mathrm{e}^{-q\varphi(w)} \widehat{\mu}_\delta(w) dA(w).
\end{aligned}$$

Moreover, noting that $\widehat{\mu}_\delta \in L^1$, and then for arbitrary fixed $\varepsilon > 0$, there exists $r_0 \in (0,1)$ such that

$$\int_{\{w\in\mathbb{D}:\ |w|>\frac{r_0}{2}\}} \widehat{\mu}_\delta(w) dA(w) < \varepsilon,$$

which implies

$$\begin{aligned}
\int_{\{z\in\mathbb{D}:\ |z|>r_0\}} |f_k(z)|^q \mathrm{e}^{-q\varphi(z)} d\mu(z) &\lesssim \|f_k\|^q_{A^\infty_\varphi} \left(\int_{\{w\in\mathbb{D}:\ |w|>\frac{r_0}{2}\}} \widehat{\mu}_\delta(w) dA(w) \right) \\
&\lesssim \varepsilon.
\end{aligned} \tag{3.17}$$

On the other hand, since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \int_{|z|\le r_0} |f_k(z)|^q \, \mathrm{e}^{-q\varphi(z)} d\mu(z) = 0.$$

This together with (3.17) yields

$$\lim_{k\to\infty} \|f_k\|_{L^q_{\varphi,\mu}} = 0,$$

which proves (a). The equivalences of norm estimates in (3.10) come from (3.11), (3.14), (3.14) and the proof of (d) $\Rightarrow$ (a). This completes the proof. □

Remark 3.6 In fact, the special case of the above conclusion as $t = 2$ was proved in [15, Theorem 2.9].

4 Bounded and Compact Weighted Composition Operators

Proof of Theorem 1.1 We prove part (a) of Theorem 1.1 firstly. Assume that $W_{\psi,\phi}: A^p_\varphi \to A^q_\varphi$ is bounded, where $0 < p \le q < \infty$. For any $z \in \mathbb{D}$, we have

$$B_{p,q,\phi}(|\psi|)(z) = \int_{\mathbb{D}} |k_{p,z}(\phi(w))|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w)$$

$$
\begin{aligned}
&= \|W_{\psi,\phi}k_{p,z}\|_{q,\varphi}^q \\
&\le \|W_{\psi,\phi}\|_{A_\varphi^p\to A_\varphi^q}^q. \qquad (4.1)
\end{aligned}
$$

Taking the supremum over $z \in \mathbb{D}$, we obtain (1.3) with

$$\|B_{p,q,\phi}(|\psi|)\|_{L^\infty} \le \|W_{\psi,\phi}\|_{A_\varphi^p\to A_\varphi^q}^q. \tag{4.2}$$

Conversely, suppose that $B_{p,q,\phi}(|\psi|)$ is in L^∞. By setting $t = q$, $s = p$ in (3.6), we get

$$|k_{q,z}(w)|^q \rho(z)^{2-2q/p} \simeq |k_{p,z}(w)|^q, \quad z, w \in \mathbb{D}.$$

This together with (1.1) and (1.2) yields that

$$
\begin{aligned}
(\widetilde{\nu_{q,\phi}})_q(z)\rho(z)^{2-2q/p} &= \int_{\mathbb{D}} |k_{q,z}(w)|^q \mathrm{e}^{-q\varphi(w)} d\nu_{q,\phi}(w)\rho(z)^{2-2q/p} \\
&\simeq \int_{\mathbb{D}} |k_{p,z}(w)|^q d\mu_{q,\phi}(w) \\
&= \int_{\mathbb{D}} |k_{p,z}(\phi(w))|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) \\
&= B_{p,q,\phi}(|\psi|)(z) \qquad (4.3)
\end{aligned}
$$

and then $(\widetilde{\nu_{q,\phi}})_q\rho^{2-2q/p}$ is bounded on $\mathbb{D}$ by our assumption. An application of Theorem 3.2 implies that $\nu_{q,\phi}$ is a q-Carleson measure for A_φ^p. It follows from (3.1) and (4.3) that

$$\int_{\mathbb{D}} |f(w)|^q \mathrm{e}^{-q\varphi(w)} d\nu_{q,\phi}(w) \lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^\infty} \|f\|_{p,\varphi}^q, \quad \forall f \in A_\varphi^p.$$

Hence, using (1.1) and (1.2) again, we obtain

$$
\begin{aligned}
\|W_{\psi,\phi}f\|_{q,\varphi}^q &= \int_{\mathbb{D}} |f(\phi(w))|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) \\
&= \int_{\mathbb{D}} |f(w)|^q d\mu_{q,\phi}(w) \\
&= \int_{\mathbb{D}} |f(w)|^q \mathrm{e}^{-q\varphi(w)} d\nu_{q,\phi}(w) \\
&\lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^\infty} \|f\|_{p,\varphi}^q. \qquad (4.4)
\end{aligned}
$$

Therefore, $W_{\psi,\phi}$ is bounded from A_φ^p to A_φ^q with

$$\|W_{\psi,\phi}\|_{A_\varphi^p\to A_\varphi^q}^q \lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^\infty}.$$

This together with (4.2) gives the norm estimate in (1.4).

We proceed to prove part (b) of Theorem 1.1.

Assume that the weighted composition operator $W_{\psi,\phi}$ is compact from A_φ^p to A_φ^q with $0 < p \le q < \infty$. For any $z \in \mathbb{D}$, since $k_{p,z}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ by Proposition 2.8, we get $\|W_{\psi,\phi}k_{p,z}\|_{q,\varphi} \to 0$ as $|z| \to 1^-$. Note that

$$B_{p,q,\phi}(|\psi|)(z) = \|W_{\psi,\phi}k_{p,z}\|_{q,\varphi}^q$$

from (4.1), and this makes the establishment of (1.5).

Conversely, if

$$\lim_{|z|\to 1^-} B_{p,q,\phi}(|\psi|)(z) = 0,$$

then $(\widetilde{\nu_{q,\phi}})_q(z)\rho(z)^{2-2q/p}$ converges to zero as $|z| \to 1^-$ in views of estimate (4.3), and $\nu_{q,\phi}$ is a vanishing q-Carleson measure for A^p_φ by Theorem 3.3. This implies that

$$\int_{\mathbb{D}} |f_k(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) \to 0, \tag{4.5}$$

whenever $\{f_k\}_k$ is a bounded sequence in A^p_φ that converges to 0 on compact subsets of $\mathbb{D}$. Consequently,

$$\|W_{\psi,\phi} f_k\|_{q,\varphi} = \left(\int_{\mathbb{D}} |f_k(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) \right)^{\frac{1}{q}} \to 0 \tag{4.6}$$

as $k \to \infty$ by replacing f by f_k in (4.4), and this shows $W_{\psi,\phi}$ is compact from A^p_φ to A^q_φ. The proof is complete here. $\square$

In fact, the proof of Theorem 1.1 reveals the following corollary.

Corollary 4.1 *Let $0 < p \le q < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. Then*

(a) *The weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ if and only if the measure $\nu_{q,\phi}$ defined in* (1.2) *is a q-Carleson measure for A^p_φ.*

(b) *The weighted composition operator $W_{\psi,\phi}$ is compact from A^p_φ to A^q_φ if and only if $\nu_{q,\phi}$ is a vanishing q-Carleson measure for A^p_φ.*

The other corollary of Theorem 1.1 gives a more useful necessary condition for the boundedness of the weighted composition operator.

Corollary 4.2 *Let $0 < p \le q < \infty$ and $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. Assume that ψ is an analytic function on $\mathbb{D}$ and ϕ is an analytic self-map of $\mathbb{D}$. If the weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ, then*

$$\sup_{z\in\mathbb{D}} \frac{\rho(z)^{2/q}}{\rho(\phi(z))^{2/p}} \frac{\mathrm{e}^{\varphi(\phi(z))}}{\mathrm{e}^{\varphi(z)}} |\psi(z)| < \infty.$$

Proof Assume that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ for $0 < p \le q < \infty$. In views of Theorem 1.1, we have $\sup_{z\in\mathbb{D}} B_{p,q,\phi}(|\psi|)(z) < \infty$. In particular,

$$\sup_{z\in\mathbb{D}} B_{p,q,\phi}(|\psi|)(\phi(z)) < \infty, \tag{4.7}$$

since ϕ is an analytic self-map of $\mathbb{D}$. Applying Lemmas 2.3, 2.4 and 2.5, we deduce

$$\begin{aligned} B_{p,q,\phi}(|\psi|)(\phi(z)) &= \int_{\mathbb{D}} |k_{p,\phi(z)}(\phi(w))|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) \\ &\ge \int_{D^r(z)} |k_{p,\phi(z)}(\phi(w))|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) \\ &\gtrsim \rho(z)^2 |k_{p,\phi(z)}(\phi(z))|^q |\psi(z)|^q \mathrm{e}^{-q\varphi(z)} \\ &= \rho(z)^2 \frac{|K(\phi(z),\phi(z))|^q}{\|K_{\phi(z)}\|^q_{p,\varphi}} |\psi(z)|^q \mathrm{e}^{-q\varphi(z)} \\ &\gtrsim \left(\frac{\rho(z)^{2/q}}{\rho(\phi(z))^{2/p}} \frac{\mathrm{e}^{\varphi(\phi(z))}}{\mathrm{e}^{\varphi(z)}} |\psi(z)| \right)^q. \end{aligned}$$

This together with (4.7) yields the desired result. $\square$

Now, we turn to prove Theorem 1.2.

Proof of Theorem 1.2 The implication (b) $\Rightarrow$ (a) is trivial. To finish our proof, it suffices to show the implications (a) $\Rightarrow$ (b) and (a) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b). Assume that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ with $0 < q < p < \infty$. For any $f \in A^p_\varphi$, we have

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) &= \|W_{\psi,\phi} f\|^q_{q,\varphi} \\ &\le \|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi} \|f\|^q_{p,\varphi}\end{aligned}$$

by (4.4). In views of Theorem 3.4, $\nu_{q,\phi}$ is a vanishing q-Carleson measure for A^p_φ. A similar discussion as (4.5) and (4.6) leads to

$$\lim_{k\to\infty} \|W_{\psi,\phi} f_k\|_{q,\varphi} = 0$$

for any sequence $\{f_k\}_k$ in A^p_φ that converges to 0 on compact subsets of $\mathbb{D}$ as $k \to \infty$, which implies the compactness of $W_{\psi,\phi}$.

(c) $\Rightarrow$ (a). Suppose that the integral transform $B_{p,q,\phi}(|\psi|) \in L^{\frac{p}{p-q}}(\mathbb{D}, d\lambda_\rho)$, where $0 < q < p < \infty$ and $d\lambda_\rho$ is given in (1.6). According to (4.3), we have

$$\begin{aligned}\int_{\mathbb{D}} (\widetilde{\nu_{q,\phi}})_q(w)^{\frac{p}{p-q}} dA(w) &= \int_{\mathbb{D}} ((\widetilde{\nu_{q,\phi}})_q(w)\rho(w)^{2-2q/p}t)^{\frac{p}{p-q}} d\lambda_\rho(w) \\ &\simeq \int_{\mathbb{D}} B_{p,q,\phi}(|\psi|)(w)^{\frac{p}{p-q}} d\lambda_\rho(w) \\ &< \infty. \end{aligned} \tag{4.8}$$

An application of Theorem 3.4 yields that $\nu_{q,\phi}$ is a q-Carleson measure for A^p_φ. Thus,

$$\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) \lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^{\frac{p}{p-q}}(\mathbb{D},d\lambda_\rho)} \|f\|^q_{p,\varphi}, \quad \forall f \in A^p_\varphi$$

by combining (3.9) and (4.8). This along with (4.4) implies that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ with

$$\|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi} \lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^{\frac{p}{p-q}}(\mathbb{D},d\lambda_\rho)}. \tag{4.9}$$

(a) $\Rightarrow$ (c). If the weighted composition operator $W_{\psi,\phi}$ is bounded from A^p_φ to A^q_φ with $0 < q < p < \infty$, then for arbitrary $f \in A^p_\varphi$,

$$\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) = \|W_{\psi,\phi} f\|^q_{p,\varphi} \lesssim \|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi} \|f\|^q_{p,\varphi}$$

by (4.4). This implies that $\nu_{q,\phi}$ is a q-Carleson measure for A^p_φ. According to Theorem 3.4, we have $(\widetilde{\nu_{q,\phi}})_q \in L^{\frac{p}{p-q}}$ with

$$\|(\widetilde{\nu_{q,\phi}})_q\|_{L^{\frac{p}{p-q}}} \lesssim \|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi}.$$

Combining this with (4.8) we conclude that $B_{p,q,\phi}(|\psi|) \in L^{\frac{p}{p-q}}(\mathbb{D}, d\lambda_\rho)$ with

$$\|B_{p,q,\phi}(|\psi|)\|_{L^{\frac{p}{p-q}}(\mathbb{D},d\lambda_\rho)} \lesssim \|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi}.$$

This together with (4.9) gives (1.7), which completes the proof of Theorem 1.2. $\square$

Theorem 1.3 can be obtained by following a similar approach as in the proof of Theorem 1.2. We include it here for completeness.

Proof of Theorem 1.3 The implication (b) $\Rightarrow$ (a) is trivial. It is enough to show the implications (a) $\Rightarrow$ (b) and (a) $\Leftrightarrow$ (c) to complete this proof.

(a) $\Rightarrow$ (b). Assume that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^∞_φ to A^q_φ with $0<q<\infty$. According to (4.4), we have

$$\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) = \|W_{\psi,\phi} f\|^q_{q,\varphi} \le \|W_{\psi,\phi}\|^q_{A^\infty_\varphi \to A^q_\varphi} \|f\|^q_{\infty,\varphi}$$

for any $f \in A^\infty_\varphi$. Thus $\nu_{q,\phi}$ is a vanishing q-Carleson measure for A^∞_φ in views of Theorem 3.5. It follows by a similar discussion of (4.5) and (4.6) that

$$\lim_{k\to\infty} \|W_{\psi,\phi} f_k\|_{q,\varphi} = 0$$

for any sequence $\{f_k\}_k$ in A^∞_φ that converges to 0 on compact subsets of $\mathbb{D}$ as $k\to\infty$, which gives that $W_{\psi,\phi}$ is compact from A^∞_φ to A^q_φ.

(c) $\Rightarrow$ (a). Suppose the integral transform $B_{\infty,q,\phi}(|\psi|) \in L^1(\mathbb{D}, d\lambda_\rho)$ for $0<q<\infty$. By using (2.4), we have

$$\begin{aligned} |k_{\infty,z}(w)|^q &= \frac{|K_z(w)|^q}{\|K_z\|^q_{\infty,\varphi}} \simeq \frac{|K_z(w)|^q}{\mathrm{e}^{q\varphi(z)}\rho(z)^{-2q}} \\ &= \frac{|K_z(w)|^q}{\mathrm{e}^{q\varphi(z)}\rho(z)^{(\frac{2}{q}-2)q}}\rho(z)^2 \\ &\simeq |k_{q,z}(w)|^q \rho(z)^2. \end{aligned}$$

This along with (1.1) and (1.2) gives

$$\begin{aligned} (\widetilde{\nu_{q,\phi}})_q(z)\rho(z)^2 &= \int_{\mathbb{D}} |k_{q,z}(w)|^q \mathrm{e}^{-q\varphi(w)} d\nu_{q,\phi}(w)\rho(z)^2 \\ &\simeq \int_{\mathbb{D}} |k_{\infty,z}(w)|^q d\nu_{q,\phi}(w) \\ &= \int_{\mathbb{D}} |k_{\infty,z}(w)|^q |\psi(w)|^q \mathrm{e}^{-q\varphi(w)} dA(w) \\ &= B_{\infty,q,\phi}(|\psi|)(z). \end{aligned}$$

Hence, according to the definition of $d\lambda_\rho$ in (1.6) and our assumption, we obtain

$$\begin{aligned} \int_{\mathbb{D}} (\widetilde{\nu_{q,\phi}})_q(z) dA(z) &= \int_{\mathbb{D}} (\widetilde{\nu_{q,\phi}})_q(z)\rho(z)^2 d\lambda_\rho(z) \\ &= \int_{\mathbb{D}} B_{\infty,q,\phi}(|\psi|)(z) d\lambda_\rho(z) \\ &< \infty. \end{aligned} \tag{4.10}$$

An application of Theorem 3.5 yields that $\nu_{q,\phi}$ is a q-Carleson measure for A^∞_φ. Then

$$\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) \lesssim \|B_{p,q,\phi}(|\psi|)\|_{L^1(\mathbb{D},d\lambda_\rho)} \|f\|^q_{\infty,\varphi}, \quad \forall f \in A^\infty_\varphi$$

by using (3.10) and (4.10). This along with (4.4) shows that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^∞_φ to A^q_φ with

$$\|W_{\psi,\phi}\|^q_{A^p_\varphi \to A^q_\varphi} \lesssim \|B_{\infty,q,\phi}(|\psi|)\|_{L^1(\mathbb{D},d\lambda_\rho)}. \tag{4.11}$$

(a) $\Rightarrow$ (c). Assume that the weighted composition operator $W_{\psi,\phi}$ is bounded from A^∞_φ to A^q_φ for $0<q<\infty$, with the fact in (4.4), we deduce

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^q \mathrm{e}^{-q\varphi(z)} d\nu_{q,\phi}(z) &= \|W_{\psi,\phi} f\|^q_{q,\varphi}\\ &\lesssim \|W_{\psi,\phi}\|^q_{A^\infty_\varphi \to A^q_\varphi} \|f\|^q_{\infty,\varphi}\end{aligned}$$

for any $f \in A^\infty_\varphi$. This implies that $\nu_{q,\phi}$ is a q-Carleson measure for A^∞_φ. According to Theorem 3.5, we have $(\widetilde{\nu_{q,\phi}})_q \in L^1$ with

$$\|(\widetilde{\nu_{q,\phi}})_q\|_{L^1} \lesssim \|W_{\psi,\phi}\|^q_{A^\infty_\varphi \to A^q_\varphi}.$$

Combining this with (4.10) we conclude that $B_{p,q,\phi}(|\psi|) \in L^1(\mathbb{D}, d\lambda_\rho)$ with

$$\|B_{p,q,\phi}(|\psi|)\|_{L^1(\mathbb{D},d\lambda_\rho)} \lesssim \|W_{\psi,\phi}\|^q_{A^\infty_\varphi \to A^q_\varphi}.$$

This together with (4.11) gives (1.8), which completes the proof. □

5 Schatten Class Weighted Composition Operators

In this section, we aim to prove the Schatten p-class weighted composition operators on A^2_φ. To this end, we first introduce the notion of the Toeplitz operator T_μ induced by a positive measure μ as follows,

$$T_\mu f(z) = \int_{\mathbb{D}} f(w) K(z,w) \mathrm{e}^{-2\varphi(w)} d\mu(w), \quad z \in \mathbb{D}.$$

Proposition 5.1 ([33, Proposition 5.1]) *Let $0<p<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent*:

(a) $\widetilde{\mu} \in L^p(\mathbb{D}, d\lambda_\rho)$.

(b) $\widehat{\mu}_\delta \in L^p(\mathbb{D}, d\lambda_\rho)$ *for some (or any)* $\delta \in (0,\alpha]$.

(c) *The sequence* $\{\widehat{\mu}_r(w_k)\}_k \in l^p$ *for some (or any)* (ρ,r)*-lattice* $\{w_k\}$ *with* $r \in (0,\alpha]$.

Theorem 5.2 ([33, Theorem 5.1]) *Let $\varphi \in \mathcal{W}_0$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0<p<\infty$ and μ be a finite positive Borel measure on $\mathbb{D}$. Then the Toeplitz operator $T_\mu \in \mathcal{S}_p(A^2_\varphi)$ if and only if the function $\widehat{\mu}_\delta \in L^p(\mathbb{D}, d\lambda_\rho)$ for some (or any) $\delta \in (0,\alpha]$.*

We can now prove the last main theorem of this work.

Proof of Theorem 1.4 For any $f,g \in A^2_\varphi$, by (1.1) and (1.2), we deduce that

$$\begin{aligned}\langle W^*_{\psi,\phi} W_{\psi,\phi} f, g\rangle_{A^2_\varphi} &= \langle W_{\psi,\phi} f, W_{\psi,\phi} g\rangle_{A^2_\varphi}\\ &= \int_{\mathbb{D}} f(\phi(w))\overline{g(\phi(w))}|\psi(w)|^2 \mathrm{e}^{-2\varphi(w)} dA(w)\\ &= \int_{\mathbb{D}} f(w)\overline{g(w)} d\mu_{2,\phi}(w)\\ &= \int_{\mathbb{D}} f(w)\overline{g(w)} \mathrm{e}^{-2\varphi(w)} d\nu_{2,\phi}(w). \qquad (5.1)\end{aligned}$$

By applying Fubini's theorem and the reproducing property, we obtain

$$\begin{aligned}\langle T_{\nu_{2,\phi}} f, g\rangle_{A^2_\varphi} &= \int_{\mathbb{D}} \bigg(\int_{\mathbb{D}} f(w) K(z,w) \mathrm{e}^{-2\varphi(w)} d\nu_{2,\phi}(w)\bigg) \overline{g(z)} \mathrm{e}^{-2\varphi(z)} dA(z)\\ &= \int_{\mathbb{D}} f(w)\overline{g(w)} \mathrm{e}^{-2\varphi(w)} d\nu_{2,\phi}(w).\end{aligned}$$

This together with (5.1) gives

$$\langle W^*_{\psi,\phi}W_{\psi,\phi}f, g\rangle_{A^2_\varphi} = \langle T_{\nu_{2,\phi}}f, g\rangle_{A^2_\varphi}$$

for any $f, g \in A^2_\varphi$. It follows that

$$W^*_{\psi,\phi}W_{\psi,\phi} = T_{\nu_{2,\phi}}.$$

This, in combination with Theorem 1.26 in [34] implies that the weighted composition operator $W_{\psi,\phi} \in \mathcal{S}_p(A^2_\varphi)$ if and only if the Toeplitz operator $T_{\nu_{2,\phi}} \in \mathcal{S}_{p/2}(A^2_\varphi)$, which is equivalent to

$$(\widehat{\nu_{2,\phi}})_\delta \in L^{p/2}(\mathbb{D}, d\lambda_\rho), \quad \delta \in (0, \alpha]$$

according to Theorem 5.2. It is also equivalent to

$$(\widetilde{\nu_{2,\phi}})_2 \in L^{p/2}(\mathbb{D}, d\lambda_\rho)$$

by Proposition 5.1. We complete the proof by observing the fact

$$B_{2,2,\phi}(|\psi|)(z) = (\widetilde{\nu_{2,\phi}})_2(z), \quad z \in \mathbb{D}. \qquad \square$$

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References

[1] Aron, R., Lindström, M.: Spectra of weighted composition operators on weighted Banach spaces of analytic functions. *Israel J. Math.*, **141**, 263–276 (2004)

[2] Arroussi, H., Park, I., Pau, J.: Schatten class Toeplitz operators acting on large weighted Bergman spaces. *Studia Math.*, **229**(3), 203–221 (2015)

[3] Arroussi, H., Pau, J.: Reproducing kernel estimates, bounded projections and duality on large weighted Bergman spaces. *J. Geom. Anal.*, **25**(4), 2284–2312 (2015)

[4] Arroussi, H., Tong, C.: Weighted composition operators between large Fock spaces in several complex variables. *J. Funct. Anal.*, **277**(10), 3436–3466 (2019)

[5] Asserda, A., Hichame, A.: Pointwise estimate for the Bergman kernel of the weighted Bergman spaces with exponential weights. *C. R. Math. Acad. Sci. Paris.*, **352**(1), 13–16 (2014)

[6] Borichev, A., Dhuez, R., Kellay, K.: Sampling and interpolation in large Bergman and Fock spaces. *J. Funct. Anal.*, **242**(2), 563–606 (2007)

[7] Constantin, O., Peláez, J. Á.: Boundedness of the Bergman projection on L^p spaces with exponential weights. *Bull. Sci. Math.*, **139**(3), 245–268 (2015)

[8] Constantin, O., Peláez, J. Á.: Integral operators, embedding theorems and a Littlewood–Paley formula on weighted Fock spaces. *J. Geom. Anal.*, **26**(2), 1109–1154 (2016)

[9] Contreras, M. D., Hernández–Díaz, A. G.: Weighted composition operators between different Hardy spaces. *Integral Equations Operator Theory*, **46**(2), 165–188 (2003)

[10] Cowen, C., MacCluer, B.: Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995

[11] Čučković, Z., Zhao, R.: Weighted composition operators on the Bergman spaces. *J. Lond. Math. Soc.*, **70**(2), 499–511 (2004)

[12] Čučković, Z., Zhao, R.: Weighted composition operators between different weighted Bergman spaces and different Hardy spaces. *Illinois J. Math.*, **51**(2), 479–498 (2007)

[13] Forelli, F.: The isometries of H^p. *Can. J. Math.*, **16**, 721–728 (1964)

[14] Galanopoulos, P., Pau, J.: Hankel operators on large weighted Bergman spaces. *Ann. Acad. Sci. Fenn. Math.*, **37**(2), 635–648 (2012)

[15] He, Z., Wang, X., Liu, Y.: Toeplitz operators between Bergman spaces with exponential weights A^p_φ and A^∞_φ. *Acta Math. Sinica, Chin. Ser.*, **64**(3), 655–668 (2021)

[16] Hu, Z., Lv, X., Schuster, A.: Bergman spaces with exponential weights. *J. Funct. Anal.*, **276**(5), 1402–1429 (2019)

[17] Hyvärinen, O., Lindström, M., Nieminen, I., et al., Spectra of weighted composition operators with automorphic symbols. *J. Funct. Anal.*, **265**(8), 1749–1777 (2013)

[18] Kolaski, C. J.: Isometries of weighted Bergman spaces. *Can. J. Math.*, **34**(4), 910–915 (1982)

[19] Lin, P., Rochberg, R.: Hankel operators on the weighted Bergman spaces with exponential weights. *Integral Equations Operator Theory*, **21**(4), 460–483 (1995)

[20] Lin, P., Rochberg, R.: Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights. *Pacific J. Math.*, **173**(1), 127–146 (1996)

[21] Luecking, D. H.: Embedding theorems for spaces of analytic functions via Khinchine's inequality. *Michigan Math. J.*, **40**(2), 333–358 (1993)

[22] MacCluer, B. D., Zhao, R.: Essential norms of weighted composition operators between Bloch-type spaces. *Rocky Mountain J. Math.*, **33**(4), 1437–1458 (2003)

[23] Manhas, J. S.: Compact differences of weighted composition operators on weighted Banach spaces of analytic functions. *Integral Equations Operator Theory*, **62**(3), 419–428 (2008)

[24] Mengestie, T.: Carleson type measures for Fock–Sobolev spaces. *Complex Anal. Oper. Theory*, **8**(6), 1225–1256 (2014)

[25] Oleinik, V. L.: Embedding theorems for weighted classes of harmonic and analytic functions. *J. Sov. Math.*, **9**, 228–243 (1978)

[26] Oleinik, V. L., Perel'man, G. S.: Carleson's embedding theorems for a weighted Bergman spaces. *Math. Notes*, **7**(5–6), 577–581 (1990)

[27] Park, I.: The weighted composition operators on the large weighted Bergman spaces. *Complex Anal. Oper. Theory*, **13**(1), 223–239 (2019)

[28] Park, I.: Compact differences of composition operators on large weighted Bergman spaces. *J. Math. Anal. Appl.*, **479**(2), 1715–1737 (2019)

[29] Pau, J., Peláez, J. Á.: Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights. *J. Funct. Anal.*, **259**(10), 2727–2756 (2010)

[30] Pau, J., Peláez, J. Á.: Volterra type operators on Bergman spaces with exponential weights. *Contemp. Math.*, **561**(1), 239–252 (2012)

[31] Shapiro, J. H.: Composition Operators and Classical Function Theory, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993

[32] Singh, R. K., Manhas, J. S.: Composition Operators on Function Spaces, North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1993

[33] Zhang, Y., Wang, X., Hu, Z.: Toeplitz operators on Bergman spaces with exponential weights. *Complex Var. Elliptic Equ.*, (2022) https://www.webofscience.com/wos/alldb/full-record/WOS:000758715700001

[34] Zhu, K.: Operator Theory in Function Spaces, Second edition, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2007