Acta Mathematica Sinica, English Series
Jul., 2022, Vol. 38, No. 7, pp. 1241–1254
Published online: July 15, 2022
https://doi.org/10.1007/s10114-022-1474-y
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Applying the Theory of Numerical Radius of Operators to Obtain Multi-observable Quantum Uncertainty Relations

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Abstract Quantum uncertainty relations are mathematical inequalities that describe the lower bound of products of standard deviations of observables (i.e., bounded or unbounded self-adjoint operators). By revealing a connection between standard deviations of quantum observables and numerical radius of operators, we establish a universal uncertainty relation for k observables, of which the formulation depends on the even or odd quality of k. This universal uncertainty relation is tight at least for the cases $k = 2$ and $k = 3$. For two observables, the uncertainty relation is a simpler reformulation of Schrödinger's uncertainty principle, which is also tighter than Heisenberg's and Robertson's uncertainty relations.

Keywords Numerical radius of operators, quantum uncertainty principle, quantum observables, quantum deviations

MR(2010) Subject Classification 47A12, 47A63, 81Q10, 81P05

1 Introduction

In the mathematical framework of quantum mechanics, a quantum system can be simulated in a complex Hilbert space H with the inner product $\langle\cdot|\cdot\rangle$ and a pure state is described by a unit vector $|x\rangle$. Quantum observables for a state $|x\rangle$ are bounded or unbounded self-adjoint operators on H with domain containing $|x\rangle$ (Ref. [25]). The mean value of observable A for the pure state $|x\rangle$ is $\langle A\rangle = \langle x|A|x\rangle$. The uncertainty principle, discovered first by Heisenberg in 1927 (Ref. [10]), is often considered as one of the most important topics of quantum theory (Ref. [11, 25]) and can be linked to quantum entanglement and other important topics (Ref. [2, 20]). Heisenberg's uncertainty principle says that

$$\sigma_q\sigma_p \geq \frac{\hbar}{2}, \tag{1.1}$$

Received September 16, 2021, accepted December 15, 2021
Supported by National Natural Science Foundation of China (Grant Nos. 11771011, 12071336)
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where σ_q and σ_p denote standard deviations of the position operator $\hat{q}$ and momentum operator $\hat{p}$ respectively, $\hbar$ is the reduced Planck constant. In 1929, Robertson generalized Heisenberg's uncertainty principle, which says that, for observables A, B and pure state $|x\rangle$,

$$\sigma_A\sigma_B \geq \frac{1}{2}|\langle[A,B]\rangle|, \tag{1.2}$$

where $[A,B] = AB - BA$ is the Lie product of A and B,

$$\sigma_A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} \quad \text{and} \quad \sigma_B = \sqrt{\langle B^2\rangle - \langle B\rangle^2}$$

are the standard deviations of A and B, respectively [22]. Schrödinger gave a uncertainty principle, which is sharper than Robertson's and asserts that

$$\sigma_A\sigma_B \geq \sqrt{\frac{1}{4}|\langle[A,B]\rangle|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2}, \tag{1.3}$$

where $\{A,B\} = AB + BA$ is the Jordan product of A and B [23]. Schrödinger's uncertainty principle holds for mixed state, too. Recall that a mixed state ρ is a positive operator on H with trace 1. Then the mean value of observable A for the state ρ is $\langle A\rangle = \mathrm{Tr}(A\rho)$ and the standard deviation of A is $\sigma_A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{\mathrm{Tr}(A^2\rho) - \mathrm{Tr}(A\rho)^2}$. Here we assume that both $\mathrm{Tr}(A\rho)$ and $\mathrm{Tr}(A^2\rho)$ are finite. Furthermore, many other ways to formulate uncertainty relations were put forward, such as in terms of entropies (Ref. [4, 26]) and even for the multi-observable case (Ref. [12]), or based on other approaches, such as majorization and skew information and so on (Ref. [5, 6] and their references). Furthermore, variance-based sum uncertainty relations also be focused on (Ref. [3, 16, 18, 19]). Recently, some new technologies of improving state-independent uncertainty relations based on joint numerical ranges were found in [7, 24]. In the paper, we pay attention to refine the original uncertainty relations based on product form of deviations of observables.

What happens for the multi-observable case?

There is a natural way to get uncertainty relation from the uncertainty principles in (1.2) or (1.3). For example, let A, B, C be three observables, then by applying (1.2) one gets

$$\sigma_A^2\sigma_B^2\sigma_C^2 \geq \frac{1}{8}|\langle[A,B]\rangle\langle[B,C]\rangle\langle[A,C]\rangle|. \tag{1.4}$$

But (1.4) is not sharp enough.

Let $A = \hat{q}, B = \hat{p}$ and $C = \hat{r} = -\hat{p} - \hat{q}$; then $[p,q] = [q,r] = [r,p] = \frac{\hbar}{i}$, and thus (1.4), together with (1.1), gives

$$\sigma_q^2\sigma_p^2\sigma_r^2 \geq \left(\frac{\hbar}{2}\right)^3.$$

However, in [14], a tight uncertainty relation is given that

$$\sigma_q^2\sigma_p^2\sigma_r^2 \geq \left(\tau\frac{\hbar}{2}\right)^3 \tag{1.5}$$

with $\tau = \frac{2}{\sqrt{3}} > 1$.

This also happens for Pauli matrices X, Y, Z. As $[X,Y] = 2\mathrm{i}Z$, one has $\frac{1}{2}|\langle[X,Y]\rangle| = |\langle Z\rangle|$. Similarly $\frac{1}{2}|\langle[X,Z]\rangle| = |\langle Y\rangle|$ and $\frac{1}{2}|\langle[Y,Z]\rangle| = |\langle X\rangle|$. Thus by (1.4)

$$\sigma_X^2\sigma_Y^2\sigma_Z^2 \geq |\langle X\rangle\langle Y\rangle\langle Z\rangle|.$$

Also it was announced in [17] that

$$\sigma_X^2\sigma_Y^2\sigma_Z^2 \geq \frac{8}{3\sqrt{3}}|\langle X\rangle\langle Y\rangle\langle Z\rangle|. \tag{1.6}$$

This inequality is also tight and achieves "=" at $\rho = \frac{1}{2}(I + \frac{1}{\sqrt{3}}X + \frac{1}{\sqrt{3}}Y + \frac{1}{\sqrt{3}}Z)$.

Therefore, to obtain multi-observable uncertainty relations that are sharp enough, one needs new approaches. Let $A_1, A_2, \ldots, A_k$ be any k observables of a quantum system. The purpose of this paper is to establish a lower bound of $\sigma_{A_1}\sigma_{A_2}\cdots\sigma_{A_k}$ in terms of $\langle A_iA_j\rangle$, $\langle A_i\rangle\langle A_j\rangle$ and $\langle A_j^2\rangle$. For the case when $k = 2$, the uncertainty relation is equivalent to Schrödinger's uncertainty principle that is tight and has a simpler representation. For the case when $k = 3$, we show that the uncertainty relation is tight by taking Pauli matrices as observables.

2 Multi-observable Uncertainty Relations

Our main idea is based on the following observation, which establishes a formula to connect the standard deviation of a quantum observable A of a state $|x\rangle$ to the norm as well as the numerical radius of $[A, |x\rangle\langle x|]$, the Lie product of A and the rank one projection $|x\rangle\langle x|$ (or written equivalently as $x \otimes x$).

Let us recall some notions in mathematics. Let T be a bounded linear operator acting on a complex Hilbert space H. The numerical range of T is the set $W(T) = \{\langle x|T|x\rangle : |x\rangle \in H, \||x\rangle\| = 1\}$, and the numerical radius of T is $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$. The topic of numerical range and numerical radius plays an important role in mathematics and is applied into many areas (Ref. [8, 9, 13]). Denote by $\|T\|$ the operator norm of T. In the following lemma, we show a connection between the standard deviation and numerical radius.

Lemma 2.1 *Let $|x\rangle$ be a pure state and A an observable for it. Then*

$$\sigma_A = \|[A, |x\rangle\langle x|]\| = w([A, |x\rangle\langle x|]).$$

Proof Let H be the associated Hilbert space for the pure state $|x\rangle$ and the observable A. Write $A|x\rangle$ in the form $A|x\rangle = \alpha|x\rangle + \beta|y\rangle$, where normalized $|y\rangle$ is orthogonal to $|x\rangle$. Since A is self-adjoint we have $\alpha = \langle x|A|x\rangle \in \mathbb{R}$. Moreover, by self-adjointness of A, the Lie product of A and the rank one projection $|x\rangle\langle x|$ is represented by the following matrix relative to decomposition $H = [x] \oplus [y] \oplus \{x, y\}^\perp$, here $[x] = \mathrm{span}\{x\}$. Then

$$[A, |x\rangle\langle x|] = \begin{pmatrix} 0 & -\bar{\beta} \\ \beta & 0 \end{pmatrix} \oplus 0.$$

Note that $[A, |x\rangle\langle x|]$ is a skew self-adjoint operator because

$$[A, |x\rangle\langle x|]^\dagger = -[A, |x\rangle\langle x|].$$

Thus its numerical range $W([A, |x\rangle\langle x|]) = \mathrm{i}[-|\beta|, |\beta|]$, and hence $w([A, |x\rangle\langle x|]) = \|[A, x \otimes x]\| = |\beta|$. It follows that

$$\begin{aligned} w([A, |x\rangle\langle x|])^2 &= \|[A, |x\rangle\langle x|]\| = |\beta|^2 \\ &= \|A|x\rangle - \langle x|A|x\rangle\, |x\rangle\|^2 \\ &= \langle x|(A - \langle x|A|x\rangle)^2|x\rangle \end{aligned}$$

$$= \langle x|A^2|x\rangle - (\langle x|A|x\rangle)^2$$
$$= \langle A^2\rangle - \langle A\rangle^2 = \sigma_A^2.$$

Therefore, $\sigma_A = w([A, |x\rangle\langle x|]) = \|[A, |x\rangle\langle x|]\|$. We complete the proof. □

Lemma 2.1 is helpful for exploring new quantum uncertainty relations. Indeed, by Lemma 2.1, for any observables $A_1, A_2, \dots, A_k$ for a pure state $|x\rangle$, as the numerical radius is less than or equal to the operator norm (see Ref. [9]), we have

$$\prod_{j=1}^{k} \sigma_{A_j} = \prod_{j=1}^{k} \|[A_j, |x\rangle\langle x|]\| \geq \left\| \prod_{j=1}^{k} [A_j, |x\rangle\langle x|] \right\| \geq w\bigg(\prod_{j=1}^{k} [A_j, |x\rangle\langle x|] \bigg). \tag{2.1}$$

Note that, the value of $\prod_{j=1}^{k} \sigma_{A_j}$ does not depend on the order arrange of observables but $w(\prod_{j=1}^{k} [A_j, |x\rangle\langle x|])$ does. Therefore, the inequality (2.1) can be sharped to

$$\prod_{j=1}^{k} \sigma_{A_j} \geq \max_{\pi} w\bigg(\prod_{j=1}^{k} [A_{\pi(j)}, |x\rangle\langle x|] \bigg), \tag{2.2}$$

where the maximum is over all permutations π of $(1, 2, \dots, k)$. Thus the question of establishing an uncertainty relation for k observables is reduced to the question of calculating the numerical radius of the operator

$$D_k^{(\pi)} = \prod_{j=1}^{k} [A_{\pi(j)}, |x\rangle\langle x|], \tag{2.3}$$

which is an operator of rank ≤ 2.

The exact value of $w(D_k^{(\pi)})$ is computable and we can establish a multi-observable uncertainty relation by (2.2). For simplicity, and with no loss of generality, we state our results only for $\pi = \mathrm{id}$.

One may ask why not work on the stronger inequality

$$\prod_{j=1}^{k} \sigma_{A_j} \geq \left\| \prod_{j=1}^{k} [A_j, |x\rangle\langle x|] \right\|.$$

In fact, we will show from Eq. (2.17) that, this inequality does not lead to a stronger uncertainty relation. So the numerical radius is the better choice.

The following is our main result, here we agree on $\prod_{j\in\Lambda} a_j = 1$ if $\Lambda = \emptyset$. It is surprising that our uncertainty relation for any k observables has different formulation depending on the even or odd quality of the integer k.

Theorem 2.2 *Let $A_1, A_2, \dots, A_k$ with $k \geq 2$ be observables.*

(1) *If $k = 2n$, then*

$$\prod_{j=1}^{2n} \sigma_{A_j} \geq \frac{1}{2}\bigg(\prod_{j=1}^{n-1} |\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle| \bigg)(|\langle A_1A_{2n}\rangle - \langle A_1\rangle\langle A_{2n}\rangle| + \sigma_{A_1}\sigma_{A_{2n}}). \tag{2.4}$$

(2) *If $k = 2n+1$, then, identifying $2n+2$ with $[(2n+2) \bmod (2n+1)] = 1$,*

$$\prod_{j=1}^{2n+1} \sigma_{A_j} \geq \frac{1}{2}\bigg[2 \prod_{j=1}^{2n+1} |\langle A_jA_{j+1}\rangle - \langle A_j\rangle\langle A_{j+1}\rangle|$$

$$+\sigma_{A_1}^2 \prod_{j=1}^{n} |\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|^2$$

$$+\sigma_{A_{2n+1}}^2 \prod_{j=1}^{n} |\langle A_{2j-1}A_{2j}\rangle - \langle A_{2j-1}\rangle\langle A_{2j}\rangle|^2\Bigg]^{\frac{1}{2}}. \tag{2.5}$$

Next we show the proof of Theorem 2.2, which can be considered as the new proof of Schrödinger's uncertainty principle in [23]. Before start the proof of Theorem 2.2, we need a lemma.

Lemma 2.3 *Let*

$$E_1 = \begin{pmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & a \\ c & 0 \end{pmatrix}.$$

Then

$$w(E_1) = \frac{1}{2}\sqrt{|b|^2 + (|a|+|c|)^2}$$

and

$$w(E_2) = \frac{1}{2}(|a|+|c|).$$

Proof With $ac = |ac|\mathrm{e}^{2\mathrm{i}\theta}$, $\sigma(E_1) = \{\pm\sqrt{|ac|}\mathrm{e}^{\mathrm{i}\theta}, 0\}$. It is easily checked that E_1 satisfies the conditions of Theorem 2.3 and 2.4 of Ref. [15], and hence the numerical range $W(E_1)$ of E_1 is an elliptic disc with foci $\{\pm\sqrt{|ac|}\mathrm{e}^{\mathrm{i}\theta}\}$. Thus the numerical radius $w(E_1)$ is the half length of major axis of the ellipse.

Let $F = \mathrm{e}^{-\mathrm{i}\theta}E_1$. Then $w(F) = w(E_1)$. As $\sigma(F) = \{\pm\sqrt{|ac|}, 0\}$, we see that

$$w(F) = \|\mathrm{Re}(F)\|.$$

Note that

$$\sigma(\mathrm{Re}(F)) = \left\{0, \pm\frac{1}{2}\sqrt{|b|^2 + |a\mathrm{e}^{-\mathrm{i}\theta} + \bar{c}\mathrm{e}^{\mathrm{i}\theta}|^2}\right\}.$$

A simple computation shows that

$$|a\mathrm{e}^{-\mathrm{i}\theta} + \bar{c}\mathrm{e}^{\mathrm{i}\theta}|^2 = (|a|+|c|)^2.$$

Therefore, we have

$$w(E_1) = w(F) = \frac{1}{2}\sqrt{|b|^2 + (|a|+|c|)^2}.$$

It is also easily checked that

$$w(E_2) = \frac{1}{2}|a\mathrm{e}^{-\mathrm{i}\theta} + \bar{c}\mathrm{e}^{\mathrm{i}\theta}| = \frac{1}{2}(|a|+|c|).$$

We complete the proof. □

Proof of Theorem 2.2 According to the assumption of Theorem 2.2, it is enough to prove the theorem for the pure states.

For any given observables $A_1, A_2, \ldots, A_k$ with $k \geq 2$, let

$$D_k = \prod_{j=1}^{k} [A_j, |x\rangle\langle x|],$$

which has the form

$$D_k = a_k|x\rangle\langle x| + b_k|A_1x\rangle\langle x| + c_k|x\rangle\langle xA_k| + d_k|A_1x\rangle\langle xA_k|.$$

A direct computation gives

$$\begin{cases} a_2 = -\langle A_1A_2\rangle, \\ b_2 = \langle A_2\rangle, \\ c_2 = \langle A_1\rangle, \\ d_2 = -1. \end{cases} \tag{2.6}$$

For any $k \geq 3$, since $D_k = D_{k-1}[A_k, |x\rangle\langle x|]$, one may take

$$\begin{cases} a_k = a_{k-1}\langle A_k\rangle + c_{k-1}\langle A_{k-1}A_k\rangle, \\ b_k = b_{k-1}\langle A_k\rangle + d_{k-1}\langle A_{k-1}A_k\rangle, \\ c_k = -a_{k-1} - c_{k-1}\langle A_{k-1}\rangle, \\ d_k = -b_{k-1} - d_{k-1}\langle A_{k-1}\rangle. \end{cases} \tag{2.7}$$

Take unitors vectors $|y\rangle, |z\rangle$ so that $\{|x\rangle, |y\rangle, |z\rangle\}$ is orthogonal and

$$\begin{cases} |A_1x\rangle = \langle A_1\rangle|x\rangle + \sigma_{A_1}|y\rangle, \\ |A_kx\rangle = \langle A_k\rangle|x\rangle + \beta'|y\rangle + \gamma'|z\rangle. \end{cases}$$

Then

$$\beta' = \sigma_{A_1}^{-1}(\langle A_1A_k\rangle - \langle A_1\rangle\langle A_k\rangle), \quad \sigma_{A_k} = \sqrt{|\beta'|^2 + |\gamma'|^2}. \tag{2.8}$$

and

$$\begin{aligned} D_k &= (a_k + b_k\langle A_1\rangle + c_k\langle A_k\rangle + d_k\langle A_1\rangle\langle A_k\rangle)|x\rangle\langle x| + (c_k + d_k\langle A_1\rangle)\bar{\beta}'|x\rangle\langle y| \\ &\quad + (c_k + d_k\langle A_1\rangle)\bar{\gamma}'|x\rangle\langle z| + (b_k + d_k\langle A_k\rangle)\sigma_{A_1}|y\rangle\langle x| \\ &\quad + d_k\sigma_{A_1}\bar{\beta}'|y\rangle\langle y| + d_k\sigma_{A_1}\bar{\gamma}'|y\rangle\langle z| \\ &= f_{11}^{(k)}|x\rangle\langle x| + f_{12}^{(k)}|x\rangle\langle y| + f_{13}^{(k)}|x\rangle\langle z| \\ &\quad + f_{21}^{(k)}|y\rangle\langle x| + f_{22}^{(k)}|y\rangle\langle y| + f_{23}^{(k)}|y\rangle\langle z|. \end{aligned}$$

Note that, by (2.7), we have

$$\begin{aligned} f_{11}^{(k)} &= c_{k-1}(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle) + d_{k-1}(\langle A_1\rangle\langle A_{k-1}A_k\rangle - \langle A_1\rangle\langle A_{k-1}\rangle\langle A_k\rangle) \\ &= -f_{11}^{(k-2)}(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle) \end{aligned}$$

as

$$\begin{aligned} c_k + d_k\langle A_1\rangle &= -c_{k-2}(\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle) \\ &\qquad - d_{k-2}(\langle A_1\rangle\langle A_{k-2}A_{k-1}\rangle - \langle A_1\rangle\langle A_{k-2}\rangle\langle A_{k-1}\rangle) \\ &= -f_{11}^{(k-1)}, \\ d_k = -b_{k-1} - d_{k-1}\langle A_{k-1}\rangle &= d_{k-2}(\langle A_{k-2}\rangle\langle A_{k-1}\rangle - \langle A_{k-2}A_{k-1}\rangle) \end{aligned}$$

and

$$b_k + d_k\langle A_k\rangle = d_{k-1}(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle) = -d_{k+1},$$

which reveal that

$$\begin{aligned}
f_{11}^{(k)} &= -f_{11}^{(k-2)}(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle),\\
f_{12}^{(k)} &= -f_{11}^{(k-1)}\bar{\beta}',\\
f_{13}^{(k)} &= -f_{11}^{(k-1)}\bar{\gamma}',\\
f_{21}^{(k)} &= -d_{k+1}\sigma_{A_1} = (\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle)f_{21}^{(k-2)},\\
f_{22}^{(k)} &= d_k\sigma_{A_1}\bar{\beta}' = (\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle)d_{k-2}\sigma_{A_1}\bar{\beta}',\\
f_{23}^{(k)} &= d_k\sigma_{A_1}\bar{\gamma}' = (\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle)d_{k-2}\sigma_{A_1}\bar{\gamma}'.
\end{aligned} \tag{2.9}$$

It is easily checked that

$$D_2 = \begin{pmatrix} \langle A_1\rangle\langle A_2\rangle - \langle A_1A_2\rangle & 0 & 0\\ 0 & -\sigma_{A_1}\beta' & -\sigma_{A_1}\gamma'\\ 0 & 0 & 0 \end{pmatrix} \oplus 0$$

and $D_3 = D_3' \oplus 0$, where D_3' is

$$\begin{pmatrix} 0 & (\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle)\beta' & (\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle)\gamma'\\ (\langle A_2\rangle\langle A_3\rangle - \langle A_2A_3\rangle)\sigma_{A_1} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}$$

if $\dim H \geq 3$;

$$D_2 = \begin{pmatrix} \langle A_1\rangle\langle A_2\rangle - \langle A_1A_2\rangle & 0\\ 0 & -\sigma_{A_1}\sigma_{A_2} \end{pmatrix}$$

and

$$D_3 = \begin{pmatrix} 0 & (\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle)\sigma_{A_2}\\ (\langle A_2\rangle\langle A_3\rangle - \langle A_2A_3\rangle)\sigma_{A_1} & 0 \end{pmatrix}$$

if $\dim H = 2$. Hence, by identify the case of $\dim H = 2$ with the case $\gamma' = 0$, we may agree that D_2 and D_3 have respectively the block matrix

$$\begin{pmatrix} * & 0 & 0\\ 0 & * & *\\ 0 & 0 & 0 \end{pmatrix} \oplus 0 \tag{2.10}$$

and

$$\begin{pmatrix} 0 & * & *\\ * & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \oplus 0. \tag{2.11}$$

Then Eq. (2.9) implies that D_{2n} has the matrix (2.10) if $k = 2n$ is even and D_{2n+1} has the matrix (2.11) if $k = 2n+1$ is odd.

Let us first calculate $w(D_2)$. It is easily checked that

$$w(D_2) = \max\left\{|\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle|, \frac{1}{2}|\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle| + \frac{\sigma_{A_1}\sigma_{A_2}}{2}\right\}.$$

Thus we have

$$\sigma_{A_1}\sigma_{A_2} \geq w(D_2) \geq |\langle A_1\rangle\langle A_2\rangle - \langle A_1A_2\rangle| \tag{2.12}$$

and

$$w(D_2) = \frac{1}{2}(|\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle| + \sigma_{A_1}\sigma_{A_2}).$$

If $k = 2n$ is even, then it follows from (2.9) that

$$|f_{11}^{(2n)}| = |(\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle)\cdots(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle)|. \tag{2.13}$$

Since D_{2n} has the form (2.10), by (2.9) and (2.12), we have

$$\|D_{2n}\| = \max\left\{|f_{11}^{(2n)}|, \sqrt{|f_{22}^{(2n)}|^2 + |f_{23}^{(2n)}|^2}\right\} = |d_{2n}|\sigma_{A_1}\sigma_{A_{2n}}$$

and

$$\begin{aligned} w(D_{2n}) &= \max\left\{|f_{11}^{(2n)}|, \frac{1}{2}\left(|f_{22}^{(2n)}| + \sqrt{|f_{22}^{(2n)}|^2 + |f_{23}^{(2n)}|^2}\right)\right\} \\ &= \frac{1}{2}|d_{2n}|(|\langle A_1A_{2n}\rangle - \langle A_1\rangle\langle A_{2n}\rangle| + \sigma_{A_1}\sigma_{A_{2n}}). \end{aligned}$$

It needs to check that the second "=" holds true in the calculation of $w(D_{2n})$. To see this, by the definition of d_{k-2}, β' and $f_{11}^{(2n)}$,

$$\begin{aligned} &\frac{1}{2}(|f_{22}^{(2n)}| + \sqrt{|f_{22}^{(2n)}|^2 + |f_{23}^{(2n)}|^2}) \\ &\quad = |(\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle)d_{k-2}\sigma_{A_1}|\left(\frac{1}{2}(|\beta'| + \sqrt{|\beta'|^2 + |\gamma'|^2})\right) \\ &\quad \geq |(\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle)d_{k-2}\sigma_{A_1}||\beta'| \\ &\quad = |(\langle A_{k-2}A_{k-1}\rangle - \langle A_{k-2}\rangle\langle A_{k-1}\rangle)d_{k-2}\sigma_{A_1}\sigma_{A_2}| \\ &\quad \geq |(\langle A_1A_2\rangle - \langle A_1\rangle\langle A_2\rangle)\cdots(\langle A_{k-1}A_k\rangle - \langle A_{k-1}\rangle\langle A_k\rangle)| = |f_{11}^{(2n)}|. \end{aligned}$$

So the maximal value of $w(D_{2n})$ is obtained on the second item, and the equality holds true.

Now $d_2 = -1$ and

$$|d_{2n}| = |d_{2n-2}| \cdot |\langle A_{2n-2}A_{2n-1}\rangle - \langle A_{2n-2}\rangle\langle A_{2n-1}\rangle|$$

entail that

$$\begin{aligned} w(D_{2n}) = \frac{1}{2}&\left(\prod_{j=1}^{n-1} |\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|\right) \\ &\cdot (|\langle A_1A_{2n}\rangle - \langle A_1\rangle\langle A_{2n}\rangle| + \sigma_{A_1}\sigma_{A_{2n}}). \end{aligned} \tag{2.14}$$

As $\prod_{j=1}^{2n}\sigma_{A_j} \geq w(D_{2n})$, this completes the proof of (2.4).

If $k = 2n+1$ is odd, then D_{2n+1} has the patten (2.11). Applying Lemma 2.3 gives

$$\begin{aligned} w(D_{2n+1}) &= \frac{1}{2}\sqrt{(|f_{12}^{(2n+1)}| + |f_{21}^{(2n+1)}|)^2 + |f_{13}^{(2n+1)}|^2} \\ &= \frac{1}{2}\sqrt{(|f_{11}^{(2n)}\beta'| + |d_{(2n+2)}|\sigma_{A_1})^2 + |f_{11}^{(2n)}\gamma'|^2} \\ &= \frac{1}{2}\sqrt{(2|f_{11}^{(2n)}d_{(2n+2)}\sigma_{A_1}\beta'| + |d_{(2n+2)}|^2\sigma_{A_1}^2 + |f_{11}^{(2n)}|^2(|\beta'|^2 + |\gamma'|^2)}. \end{aligned}$$

Therefore,

$$w(D_{2n+1}) = \frac{1}{2}\sqrt{2\pi_1\pi_2|\langle A_1A_{2n+1}\rangle - \langle A_1\rangle\langle A_{2n+1}\rangle| + \pi_2^2\sigma_{A_1}^2 + \pi_1^2\sigma_{A_{2n+1}}^2}, \tag{2.15}$$

where

$$\pi_1 = \prod_{j=1}^{n} |\langle A_{2j-1}A_{2j}\rangle - \langle A_{2j-1}\rangle\langle A_{2j}\rangle|$$

and

$$\pi_2 = \prod_{j=1}^{n} |\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|.$$

As $\prod_{j=1}^{2n+1}\sigma_{A_j} \geq w(D_{2n+1})$, we complete the proof of (2.5) by (2.15). $\square$

Going through the proof of Theorem 2.2, it is straightforward to verify that "=" holds if and only if

$$\prod_{j=1}^{k}\|[A_j, |x\rangle\langle x|]\| = \left\|\prod_{j=1}^{k}[A_j, |x\rangle\langle x|]\right\| = w\left(\prod_{j=1}^{k}[A_j, |x\rangle\langle x|]\right). \tag{2.16}$$

Thus the uncertainty relation is tight if Eq. (2.16) holds for some observerbles $A_1, A_2, \ldots, A_k$ and some state. This is the case as will be illustrated in Section 4.

We remark that Theorem 2.2 holds for any state ρ with $|\mathrm{Tr}(A_j\rho)| < \infty$ and $\mathrm{Tr}(A_j^2\rho) < \infty$, $j = 1, 2, \ldots, k$. To see this, denote by $\mathcal{C}_2(H)$ be the Hilbert–Schmidt class in H, which is a Hilbert space with inner product $\langle T, S\rangle = \mathrm{Tr}(T^\dagger S)$. Then, a positive operator ρ is a state if and only if $\sqrt{\rho}$ is a unit vector in $\mathcal{C}_2(H)$. For a self-adjoint operator A on H, define a linear operator L_A on $\mathcal{C}_2(H)$ by $L_AT = AT$ if $\mathrm{Tr}(T^\dagger A^2 T) < \infty$. It is clear that L_A is self-adjoint as $L_A^\dagger = L_{A^\dagger} = L_A$. Note that

$$\langle A\rangle = \mathrm{Tr}(A\rho) = \langle\sqrt{\rho}|L_A|\sqrt{\rho}\rangle = \langle L_A\rangle$$

and thus

$$\sigma_A = \sigma_{L_A}, \quad \langle L_AL_B\rangle = \langle L_{AB}\rangle = \langle AB\rangle.$$

Then, Theorem 2.2 is true by applying (2.1) to $L_{A_1}, L_{A_2}, \ldots, L_{A_k}$ and the pure state $|\sqrt{\rho}\rangle$.

Finally we will explain why the sharper inequality

$$\prod_{j=1}^{k}\sigma_{A_j} \geq \|D_k\| \tag{2.17}$$

cannot achieve sharper uncertainty relations than the weaker inequality $\prod_{j=1}^{k}\sigma_{A_j} \geq w(D_k)$ can.

If $k = 2n$ is even, then by (2.9) and (2.10) one has

$$\|D_{2n}\| = \sigma_{A_1}\sigma_{A_{2n}}\prod_{j=1}^{n-1}|\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|.$$

This together with (2.17) gives

$$\prod_{j=2}^{2n-1}\sigma_{A_j} \geq \prod_{j=1}^{n-1}|\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|,$$

which is weaker than the inequality (2.4).

If $k = 2n+1$ is odd, then by (2.9) and (2.11) one has

$$\begin{aligned}\|D_{2n+1}\| &= \max\Big\{\sqrt{|f_{12}^{(2n+1)}|^2 + |f_{13}^{(2n+1)}|^2}, |f_{21}^{(2n+1)}|Big\} \\ &= \max\{|f_{11}^{(2n)}|\sigma_{A_{2n+1}}, |d_{2n+2}|\sigma_{A_1}\},\end{aligned}$$

which gives

$$\prod_{j=1}^{2n} \sigma_{A_j} \geq \prod_{j=1}^{n} |\langle A_{2j-1}A_{2j}\rangle - \langle A_{2j-1}\rangle\langle A_{2j}\rangle|$$

or

$$\prod_{j=2}^{2n+1} \sigma_{A_j} \geq \prod_{j=1}^{n} |\langle A_{2j}A_{2j+1}\rangle - \langle A_{2j}\rangle\langle A_{2j+1}\rangle|,$$

again weaker than (2.4).

3 The Case of $k = 2$: A Reformulation of Schrödinger's Principle

Before to see the uncertainty relations presented by Theorem 2.2 is sharper than those obtained by Heisenberg's uncertainty principle (1.2) and Schrödinger's uncertainty principle (1.3), we illustrate some application of Theorem 2.2 for the cases $k = 2$. Applying Theorem 2.2 (1) to the case when $k = 2$, the following result is immediate.

Theorem 3.1 *Let A and B be observables for a state. Then*

$$\sigma_A\sigma_B \geq |\langle AB\rangle - \langle A\rangle\langle B\rangle|, \tag{3.1}$$

which is equivalent to Schrödinger's uncertainty principle.

The expression of inequality is quite simpler than that of Schrödinger's uncertainty principle. We show that (3.1) is in fact equivalent to Schrödinger's uncertainty principle (1.3).

To check it, write $\langle A\rangle\langle B\rangle = r$ and $\langle AB\rangle = s + \mathrm{i}t$, where $s, t, r \in \mathbb{R}$. Then $\langle BA\rangle = s - \mathrm{i}t$. A simple computation gives

$$\begin{aligned}&|\langle AB\rangle - \langle A\rangle\langle B\rangle| = \sqrt{(s-r)^2 + t^2}, \\ &\sqrt{\frac{1}{4}|\langle [A,B]\rangle|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2} = \sqrt{(s-r)^2 + t^2}\end{aligned}$$

and

$$\frac{1}{2}|\langle [A,B]\rangle| = |t|.$$

So, we get

$$\begin{aligned}\sigma_A\sigma_B &\geq |\langle A\rangle\langle B\rangle - \langle AB\rangle| \\ &= \sqrt{\frac{1}{4}|\langle [A,B]\rangle|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2} \\ &\geq \frac{1}{2}|\langle [A,B]\rangle|.\end{aligned} \tag{3.2}$$

Now we are at a position to show that Theorem 2.2 is sharper than the uncertainty relations obtained by the approach mentioned in the introduction section.

Let $A_1, A_2, \ldots, A_k$ be observables.

If $k = 2n$ is even, by the inequality (3.2) one has

$$\begin{aligned}\prod_{j=1}^{k}\sigma_{A_j} =&\Big(\prod_{j=1}^{n-1}(\sigma_{A_{2j}}\sigma_{A_{2j+1}})\Big)(\sigma_{A_1}\sigma_{A_{2n}})\\ \geq&\Big(\prod_{j=1}^{n-1}|\langle A_{2j}A_{2j+1}\rangle-\langle A_{2j}\rangle\langle A_{2j+1}\rangle|\Big)|\langle A_1A_{2n}\rangle-\langle A_1\rangle\langle A_{2n}\rangle|\\ \geq&\frac{1}{2^n}\Big(\prod_{j=1}^{n-1}|\langle[A_{2j},A_{2j+1}]\rangle|\Big)|\langle[A_1,A_{2n}]\rangle|,\end{aligned} \tag{3.3}$$

which is weaker than the inequality (2.4) since $\sigma_{A_1}\sigma_{A_{2n}} \geq |\langle A_1A_{2n}\rangle - \langle A_1\rangle\langle A_{2n}\rangle|$.

If $k = 2n+1$ is odd, by (3.2) again we have

$$\begin{aligned}\prod_{j=1}^{k}\sigma_{A_j}^2 =& \Big(\prod_{j=1}^{n}(\sigma_{A_{2j-1}}\sigma_{A_{2j}})\Big)\Big(\prod_{j=1}^{n}(\sigma_{A_{2j}}\sigma_{A_{2j+1}})\Big)(\sigma_{A_1}\sigma_{A_{2n+1}})\\ \geq& \Big(\prod_{j=1}^{n}|(\langle A_{2j-1}A_{2j}\rangle-\langle A_{2j-1}\rangle\langle A_{2j}\rangle)(\langle A_{2j}A_{2j+1}\rangle-\langle A_{2j}\rangle\langle A_{2j+1}\rangle)|\Big)\cdot\\ &\cdot|\langle A_1A_{2n+1}\rangle-\langle A_1\rangle\langle A_{2n+1}\rangle|\\ \geq& \frac{1}{2^{2(2n+1)}}\Big(\prod_{j=1}^{n}(|\langle[A_{2j-1},A_{2j}]\rangle\langle[A_{2j},A_{2j+1}]\rangle|)\Big)|\langle[A_1,A_{2n+1}]\rangle|,\end{aligned} \tag{3.4}$$

which is clearly weaker than the inequality (2.5) as $a^2+b^2 \geq 2ab$ and $\sigma_{A_1}\sigma_{A_{2n+1}} \geq |\langle A_1A_{2n+1}\rangle - \langle A_1\rangle\langle A_{2n+1}\rangle|$.

4 Uncertainty Relations for Three or Four Observables

By Theorem 2.2, one gets an uncertainty relation for any three observables like the following.

Theorem 4.1 *Let A, B, C be three observables for a state ρ in a state space H, then*

$$\begin{aligned}\sigma_A^2\sigma_B^2\sigma_C^2 \geq&\frac{1}{4}(\sigma_C^2|\langle AB\rangle-\langle A\rangle\langle B\rangle|^2+\sigma_A^2|\langle BC\rangle-\langle B\rangle\langle C\rangle|^2)\\ &+\frac{1}{2}|(\langle AB\rangle-\langle A\rangle\langle B\rangle)(\langle BC\rangle-\langle B\rangle\langle C\rangle)(\langle AC\rangle-\langle A\rangle\langle C\rangle)|.\end{aligned} \tag{4.1}$$

Particularly, for the case when $\sigma_A\sigma_C = |\langle AC\rangle - \langle A\rangle\langle C\rangle|$ or $\langle AB\rangle = \langle A\rangle\langle B\rangle$ or $\dim H = 2$,

$$\sigma_A\sigma_B\sigma_C \geq \frac{1}{2}(\sigma_A|\langle BC\rangle-\langle B\rangle\langle C\rangle|+\sigma_C|\langle AB\rangle-\langle A\rangle\langle B\rangle|). \tag{4.2}$$

It is mentioned that, indeed, when $\dim H = 2$, the D_3 in the proofs of Lemma 2.1 and Theorem 2.2 is has the simple form. We have

$$\begin{aligned}\sigma_A\sigma_B\sigma_C \geq w(D_3) =&w\left(\begin{pmatrix} 0 & (\langle AB\rangle-\langle A\rangle\langle B\rangle)\sigma_B\\ (\langle B\rangle\langle C\rangle-\langle BC\rangle)\sigma_A & 0\end{pmatrix}\right)\\ =&\frac{1}{2}(\sigma_A|\langle BC\rangle-\langle B\rangle\langle C\rangle|+\sigma_C|\langle AB\rangle-\langle A\rangle\langle B\rangle|).\end{aligned}$$

The inequalities (4.1) and (4.2) are tight as illustrated by applying to Pauli matrices.

Example 4.2 Uncertainty relations for Pauli matrices.

Let X, Y, Z be Pauli matrices, that is,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that, for any dense matrix $\rho \in M_2(\mathbb{C})$, ρ has a representation

$$\rho = \frac{1}{2}(I_2 + r_1 X + r_2 Y + r_3 Z)$$

with Bloch vector $(r_1, r_2, r_3)^t \in \mathbb{R}^3$ and $r_1^2+r_2^2+r_3^2 \leq 1$; and ρ is pure if and only if $r_1^2+r_2^2+r_3^2 = 1$. Recall also that $XY = \mathrm{i}Z$, $YZ = \mathrm{i}X$ and $\sigma_A^2 = 1 - \langle A\rangle^2$ for $A \in \{X, Y, Z\}$, $\langle X^2\rangle = \langle Y^2\rangle = \langle Z^2\rangle = 1$ and $(\langle X\rangle, \langle Y\rangle, \langle Z\rangle) = (r_1, r_2, r_3)$.

Applying the inequality (4.2) of Theorem 4.1 to X, Y, Z we get

$$\begin{aligned}\sigma_X\sigma_Y\sigma_Z &\geq \frac{1}{2}(\sigma_X|\mathrm{i}\langle X\rangle - \langle Y\rangle\langle Z\rangle| + \sigma_Z|\mathrm{i}\langle Z\rangle - \langle X\rangle\langle Y\rangle|) \\ &= \frac{1}{2}(\sqrt{(1-\langle X\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2)} + \sqrt{(1-\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)}\ ). \qquad (4.3)\end{aligned}$$

The inequality (4.3) is tight and "=" holds if the Bloch vector satisfies $|r_1| = |r_3| = \frac{1}{\sqrt{2}}$ and $r_2 = 0$.

This illustrates that Theorem 2.2 is tight for three observables. Since Schrödinger's uncertainty principle (1.3) is tight and our uncertainty relation is equivalent to Schrödinger's uncertainty principle by (3.2), Theorem 2.2 is also tight for two oberables.

Moreover, by Theorem 3.1,

$$(1-\langle Z\rangle^2)(1-\langle X\rangle^2) \geq \langle Y\rangle^2 + \langle X\rangle^2\langle Z\rangle^2,$$

hence we have

$$\begin{aligned}\sigma_X^2\sigma_Y^2\sigma_Z^2 &\geq \frac{1}{4}[(1-\langle X\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2) + (1-\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)] \\ &\quad + \frac{1}{2}\sqrt{(1-\langle Z\rangle^2)(1-\langle X\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)} \\ &\geq \frac{1}{4}[(1-\langle X\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2) + (1-\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)] \\ &\quad + \frac{1}{2}\sqrt{(\langle Y\rangle^2 + \langle X\rangle^2\langle Z\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)} \\ &\geq \sqrt{(\langle Y\rangle^2 + \langle X\rangle^2\langle Z\rangle^2)(\langle X\rangle^2 + \langle Y\rangle^2\langle Z\rangle^2)(\langle Z\rangle^2 + \langle X\rangle^2\langle Y\rangle^2)} \\ &\geq \sqrt{(2\langle Y\rangle\langle X\rangle\langle Z\rangle)(2\langle X\rangle\langle Y\rangle\langle Z\rangle)(2\langle Z\rangle\langle X\rangle\langle Y\rangle)} \\ &= 2\sqrt{2}|\langle X\rangle\langle Y\rangle\langle Z\rangle|^{\frac{3}{2}}. \qquad (4.4)\end{aligned}$$

Particularly, one has

$$\sigma_X^2\sigma_Y^2\sigma_Z^2 \geq 2\sqrt{2}|\langle X\rangle\langle Y\rangle\langle Z\rangle|^{\frac{3}{2}}. \qquad (4.5)$$

Observe that we always have

$$1 \geq \sigma_X^2\sigma_Y^2\sigma_Z^2 = (1-r_1^2)(1-r_2^2)(1-r_3^2) \geq \frac{8}{27}$$

since the function $(1-r_1^2)(1-r_2^2)(1-r_3^2)$ has its minimum value $\frac{8}{27}$ at $|r_1|=|r_2|=|r_3|=\frac{1}{\sqrt{3}}$ and the maximum value 1 at $\rho=\frac{1}{2}I_2$. Moreover, $|r_1r_2r_3|$ achieves simultaneously its maximum value $\frac{1}{3\sqrt{3}}$ at $|r_1|=|r_2|=|r_3|=\frac{1}{\sqrt{3}}$. Thus the inequality (4.5) can be sharped to

$$\sigma_X^2\sigma_Y^2\sigma_Z^2 \geq \frac{8\sqrt[4]{3}}{3}|\langle X\rangle\langle Y\rangle\langle Z\rangle|. \tag{4.6}$$

Comparing (4.3) with (1.6), (4.6) and even Eq. (6) of [21], although these inequalities are all tight, one of the remarkable advantage of (4.3) is that, even if some of $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$ are zero, we still may get a positive lower bound of $\sigma_X\sigma_Y\sigma_Z$. For instance, saying $\langle Y\rangle=0$, we have

$$\sigma_X\sigma_Y\sigma_Z \geq \frac{1}{2}(\sqrt{(1-\langle X\rangle^2)\langle X\rangle^2}+\sqrt{(1-\langle Z\rangle^2)\langle Z\rangle^2});$$

saying $\langle Y\rangle=\langle Z\rangle=0$, we have

$$\sigma_X\sigma_Y\sigma_Z \geq \frac{1}{2}\sqrt{(1-\langle X\rangle^2)\langle X\rangle^2},$$

while we cannot get any information from (1.6) and (4.6).

Before conclusion we state the uncertainty relation from Theorem 2.2 for four observations, which has a relatively simple expression.

Theorem 4.3 *Let A_1, A_2, A_3, A_4 be observables. Then*

$$\sigma_{A_1}\sigma_{A_2}\sigma_{A_3}\sigma_{A_4} \geq \frac{1}{2}|\langle A_2A_3\rangle-\langle A_2\rangle\langle A_3\rangle|(|\langle A_1A_4\rangle-\langle A_1\rangle\langle A_4\rangle|+\sigma_{A_1}\sigma_{A_4}). \tag{4.7}$$

The inequality (4.7) is tight. For example, Consider bipartite continuous-variable system. Let $(A_1, A_4, A_2, A_3)=(\hat{q}_1,\hat{p}_1,\hat{q}_2,\hat{p}_2)$, where $\hat{q}_i, \hat{p}_i$ are the position and momentum in the ith mode satisfying the canonical commutation relation. As Heisenberg's uncertainty principle (1.1) is tight, we say that

$$\sigma_{q_1}\sigma_{p_1}\sigma_{q_2}\sigma_{p_2} \geq \frac{1}{2}|\langle \hat{q}_2\hat{p}_2\rangle-\langle \hat{q}_2\rangle\langle \hat{p}_2\rangle|(|\langle \hat{q}_1\hat{p}_1\rangle-\langle \hat{q}_1\rangle\langle \hat{p}_1\rangle|+\sigma_{q_1}\sigma_{p_1}). \tag{4.8}$$

is tight, the "=" is attained at $\rho=e$, where $e=e_1\otimes e_2$, e_1 and e_2 are the states on which the equalities of Heisenbergs uncertainty principles of $\{\hat{p}_1,\hat{q}_1\}$ and $\{\hat{p}_2,\hat{q}_2\}$ respectively.

Similarly, considering the positions and momentums $(\hat{q}_1,\hat{p}_1,\ldots,\hat{q}_n,\hat{p}_n)$ in an n-partite continuous variable system, one sees that the uncertainty relation (2.4) in Theorem 2.2 is tight. However we do not know whether the uncertainty relation (2.5) is tight for odd $k=2n+1\geq 5$.

5 Conclusion

Uncertainty relations discover lower bounds of the product of standard deviations of several observables. Larger the lower bound is, more powerful the corresponding uncertainty relation is. There are no known uncertainty relations that valid for arbitrary k observables. By finding the equality of deviation and the norm of the Lie product of the observable and the pure state, we reduce the question of establishing multi-observable uncertainty relation to the question of computing the numerical radius of an operator of rank ≤ 2. This enables us establish a universal uncertainty relation for any k observables, of which, the formulation depends on the even or odd quality of k. For two observables, our uncertainty relation is exactly a simpler reformulation of Schrödinger's uncertainty principle. The uncertainty relation provided in this paper is tight, at least for the cases of two and three observables, as illustrated by examples.

Acknowledgements We thank referees for helping to improve this work.

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