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Bifurcation of Limit Cycles for a Perturbed Piecewise Quadratic Differential Systems

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Abstract In this paper, the bifurcation of limit cycles for planar piecewise smooth systems is studied which is separated by a straight line. We give a new form of Abelian integrals for piecewise smooth systems which is simpler than before. In application, for piecewise quadratic system the existence of 10 limit cycles and 12 small-amplitude limit cycles is proved respectively.

Keywords Piecewise system, limit cycle, Abelian integral

MR(2010) Subject Classification 34C07, 34C23, 37G15

1 Introduction

During the past few decades, a large number of problems raised from practical physics and engineering have been described in the form of discontinuous systems which can exhibit more complex dynamical phenomena in the real world [2, 10–12]. Piecewise differential systems are discontinuous systems which have different definitions in different regions. For example, the planar differential system

$$(\dot{x}, \dot{y}) = \begin{cases} (P^+(x,y), Q^+(x,y)), & h(x,y) > 0, \\ (P^-(x,y), Q^-(x,y)), & h(x,y) < 0, \end{cases} \tag{1.1}$$

is a piecewise differential system, also called switching system, where $P^{\pm}(x,y)$ and $Q^{\pm}(x,y)$ are analytic functions in $D^{\pm} = \{(x,y) \in \mathbb{R}^2 : \pm h(x,y) \geq 0\}$, respectively. In many papers, authors deal with the simplest case: $h(x,y) = x$.

In piecewise smooth differential systems, many people considered the number of limit cycles and showed that there are more limit cycles than smooth systems. For example, there is no limit cycle in linear systems, but in piecewise smooth linear systems the existence of limit cycles is possible, see [1, 5, 13, 14, 16, 17, 20, 27, 28, 30] for instance. For piecewise linear systems with two zones separated by a straight line, an example with 3 limit cycles was firstly detected numerically in [17] by Huan and Yang. Later, it was analytically proved by Llibre and Ponce in [27]. Up to now, it is still an open problem whether 3 is the maximum number of limit cycles that piecewise linear differential systems with two zones separated by a straight line can have.

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For piecewise differential systems, there are many techniques to compute the bifurcation of limit cycles, such as Lyapunov constants, Averaging theory and Milnikov function. Using Lyapunov constants, the authors in [7, 15, 29] studied the number of limit cycles bifurcating from the center of piecewise systems. Chen and Du in [7] obtained 9 small-amplitude limit cycles around a center in a switching Bautin system. Tian and Yu in [29] showed the existence of 10 small-amplitude limit cycles bifurcating from a center in quadratic switching system. Recently Guo et al. in [15] obtained 18 limit cycles on a class of $\mathbb{Z}_2$-equivariant cubic switching systems. Averaging theory is developed in [9, 18, 23, 24] to studying the periodic solutions of piecewise systems. Llibre and Mereu in [23] obtained 5 limit cycles by the first order of Averaging theory. Recently, for a quadratic vector field with the curve of discontinuity $y + \sqrt{3}x = 0$ in [9], the authors predicted that there are at most 7 limit cycles by the first order averaging method and at least 16 limit cycles can exist in neighborhood of the origin by the second order averaging method. Higher-order Averaging method can be seen in [25, 26] and Buică proved the equivalence of the Averaging method and the Melnikov function method in [3].

In this paper, we focus on Melnikov function method to study limit cycles of piecewise differential systems which bifurcate from a period annulus. For Melnikov function method, many works have been made in [21, 22] and reference therein. Liu and Han in [22] considered the following system

$$(\dot{x}, \dot{y}) = \begin{cases} (H_y^+(x,y) + \epsilon f^+(x,y), -H_x^+(x,y) + \epsilon g^+(x,y)), & x > 0, \\ (H_y^-(x,y) + \epsilon f^-(x,y), -H_x^-(x,y) + \epsilon g^-(x,y)), & x < 0, \end{cases} \tag{1.2}$$

where $f^{\pm}(x,y), g^{\pm}(x,y), H^{\pm}(x,y)$ are analytic functions. The unperturbed system $(1.2)|_{\epsilon=0}$ has a family of periodic orbits and satisfies the following assumptions:

H1. There exists an open interval (α, β), and two points $A(h) = (0, r(h)), C(h) = (0, \tilde{r}(h))$ such that for $h \in (\alpha, \beta)$, $H^+(A(h)) = H^+(C(h)) = h, H^-(A(h)) = H^-(C(h))$, where $r(h) \neq \tilde{r}(h)$.

H2. When $x > 0$, system $(1.2)|_{\epsilon=0}$ has an orbital arc Γ_h^+ starting from $A(h)$ and ending at $C(h)$ defined by $H^+(x,y) = h$. When $x \leq 0$, system $(1.2)|_{\epsilon=0}$ has an orbital arc Γ_h^- starting from $C(h)$ and ending at $A(h)$ defined by $H^-(x,y) = H^-(A(h))$. See Figure 1.

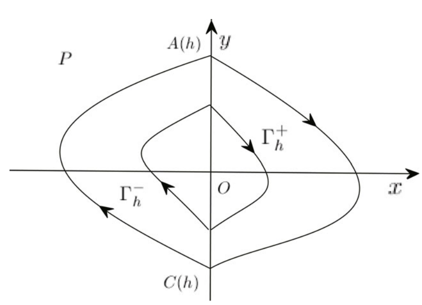

Figure 1 A family of orbits $\{\Gamma_h : \Gamma_h = \Gamma_h^+ \cup \Gamma_h^-, h \in (\alpha, \beta)\}$ of system $(1.2)_{\epsilon=0}$

Under assumptions **H1** and **H2**, $\{\Gamma_h : \Gamma_h = \Gamma_h^+ \cup \Gamma_h^-, h \in (\alpha, \beta)\}$ is a family of periodic orbits of system $(1.2)|_{\epsilon=0}$ which is called a period annulus. Without loss of generality, suppose Γ_h has a clockwise orientation. The authors obtained the first order of Melnikov function, known as Abelian integral, to study the number of limit cycles. For sufficiently small $|\epsilon| > 0$, the Abelian integral of system (1.2) is

$$I(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))}\left(\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))}\int_{\Gamma_h^+} g^+(x,y)dx - f^+(x,y)dy + \int_{\Gamma_h^-} g^-(x,y)dx - f^-(x,y)dy\right).$$

Similar to smooth systems, for sufficiently small $|\epsilon| > 0$, the number of isolated zeros of $I(h)$ (taking into account the multiplicities) gives an upper bound of the number of limit cycles of system (1.2) (taking into account the multiplicities) in any compact domain of period annulus. In [21], Li et al. generalized system (1.2) to

$$(\dot{x},\dot{y}) = \begin{cases} \left(\dfrac{H_y^+(x,y)}{M^+(x,y)} + \epsilon f^+(x,y), -\dfrac{H_x^+(x,y)}{M^+(x,y)} + \epsilon g^+(x,y)\right), & x > 0, \\ \left(\dfrac{H_y^-(x,y)}{M^-(x,y)} + \epsilon f^-(x,y), -\dfrac{H_x^-(x,y)}{M^-(x,y)} + \epsilon g^-(x,y)\right), & x < 0, \end{cases} \tag{1.3}$$

where $M^{\pm}(x,y)$ are the integrating factors and system (1.3) satisfies **H1**, **H2**. For sufficiently small $|\epsilon| > 0$, the authors gave the Abelian integral

$$\begin{aligned} I(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))}\bigg(\frac{H_y^-(A_1(h))}{H_y^+(A_1(h))}\int_{\Gamma_h^+} M^+(x,y)g^+(x,y)dx - M^+(x,y)f^+(x,y)dy \\ + \int_{\Gamma_h^-} M^-(x,y)g^-(x,y)dx - M^-(x,y)f^-(x,y)dy\bigg). \end{aligned} \tag{1.4}$$

The conditions **H1** and **H2** are the standard ones to have a period annulus. And the expression for the function $I(h)$ is (more or less) the usual one, maybe only difference with a multiplicative coefficient. Motivated by the above works, we consider system (1.1) with $h(x,y) = x$ and always suppose that system (1.1) has a center at $(0,0)$ and $\mathcal{P}$ is the period annulus. Furthermore, we make the following assumption **H3** and define $\mathcal{P}^{\pm} = D^{\pm} \cap \mathcal{P}$.

H3. The closed orbit in $\mathcal{P}$ transversely intersects the line $x = 0$.

Theorem 1.1 *Suppose that the assumption* **H3** *holds for system* (1.1)*, then system* (1.1) *has a first integral on* $\mathcal{P}$ *as the following form*

$$H(x,y) = \begin{cases} H^+(x,y), & x > 0, \\ H^-(x,y), & x < 0, \end{cases}$$

where

(1) $H^{\pm}(x,y)$ are analytic in $\mathcal{P}^{\pm} \setminus \{(0,0)\}$ respectively;

(2) $\lim_{x\to 0^+} H^+(x,y) = \lim_{x\to 0^-} H^-(x,y)$ on $\mathcal{P}$.

Remark 1.2 Though $P^{\pm}(x,y)$ and $Q^{\pm}(x,y)$ are analytic in $D^{\pm}$, $H^{\pm}(x,y)$ may not be analytic in $\mathcal{P}^{\pm}$ respectively. For example, system

$$(\dot{x},\dot{y}) = \begin{cases} (x-y, x+y), & x > 0, \\ (-x-y, x-y), & x < 0 \end{cases}$$

satisfies the condition of Theorem 1.1 and hence has a first integral $H(x,y)$ as the above form. If $H^+(x,y)$ of system $(\dot{x},\dot{y}) = (x-y, x+y)$, $x > 0$, is analytic in point $(0,0)$, then $H^+(x,y)$ is also an analytic first integral of the smooth system $(\dot{x},\dot{y}) = (x-y, x+y)$ in a neighborhood of $(0,0)$. Notice that $(0,0)$ is focus of the smooth system. For arbitrary orbit $(x(t),y(t))$ near $(0,0)$, $\lim_{t\to-\infty} H^+(x,y)=H^+(0,0)$ which is a contradiction to the definition of first integral.

Under the assumption **H3**, system (1.1) has a first integral $H(x,y)$ and $M^\pm(x,y)$ are the integrating factors responding to $H^\pm(x,y)$ respectively. Then system (1.1) can be written as the following form on $\mathcal{P}$ which is system $(1.3)|_{\epsilon=0}$

$$(\dot{x},\dot{y})=\begin{cases}\left(\dfrac{H_y^+(x,y)}{M^+(x,y)},-\dfrac{H_x^+(x,y)}{M^+(x,y)}\right), & x>0,\\ \left(\dfrac{H_y^-(x,y)}{M^-(x,y)},-\dfrac{H_x^-(x,y)}{M^-(x,y)}\right), & x<0.\end{cases}$$

The perturbed system of (1.1) is defined as $(1.1)_\epsilon$. Next, under the assumption **H3** we consider system $(1.1)_\epsilon$ which is actually equivalent to studying system (1.3). Denote

$$M(x,y)=\begin{cases}M^+(x,y), & x>0,\\ M^-(x,y), & x<0,\end{cases}\quad f(x,y)=\begin{cases}f^+(x,y), & x>0,\\ f^-(x,y), & x<0,\end{cases}\quad g(x,y)=\begin{cases}g^+(x,y), & x>0,\\ g^-(x,y), & x<0.\end{cases}$$

Theorem 1.3 *Suppose that the assumption* **H3** *holds for system* (1.1), *then for sufficiently small* $|\epsilon|>0$,

(1) *the Abelian integral of system* $(1.1)_\epsilon$ *can be expressed as*

$$I(h)=\int_{\Gamma_h} M(x,y)[g(x,y)dx-f(x,y)dy], \tag{1.5}$$

(2) *if* $I(h^*)=0$ *and* $I'(h^*)\neq 0$ *hold for* $h^*\in(\alpha,\,\beta)$, *that is,* $I(h)$ *has a simple zero* h^* *with* $h^*\in(\alpha,\,\beta)$, *then system* $(1.1)_\epsilon$ *has a limit cycle near* Γ_{h^*}.

(3) *if* $I(h)$ *has* k *simple zeros with* $h\in(\alpha,\,\beta)$, *then system* $(1.1)_\epsilon$ *has at least* k *limit cycles.*

It is worth pointing out that the form of Abelian integral (1.5) of piecewise smooth systems is same as that of smooth systems. And our form of Abelian integral here is more concise than that in [21, 22].

As an application, we consider a class of quadratic systems in the following form

$$(\dot{x},\dot{y})=\begin{cases}(y(1+a_1x+a_2y)+\epsilon f^+(x,y),-x(1+a_1x+a_2y)+\epsilon g^+(x,y)), & x>0,\\ (y(1+b_1x+b_2y)+\epsilon f^-(x,y),(x+1)(1+b_1x+b_2y)+\epsilon g^-(x,y)), & x<0,\end{cases} \tag{1.6}$$

where $|\epsilon|>0$ is a small parameter and

$$f^+(x,y)=\sum_{i+j=0}^{2} p_{ij}x^iy^j,\quad g^+(x,y)=\sum_{i+j=0}^{2} q_{ij}x^iy^j,$$
$$f^-(x,y)=\sum_{i+j=0}^{2} s_{ij}x^iy^j,\quad g^-(x,y)=\sum_{i+j=0}^{2} t_{ij}x^iy^j.$$

Theorem 1.4 *For sufficiently small* $|\epsilon|>0$, *there exists a system* (1.6) *with* 10 *limit cycles bifurcating from the period annulus of system* $(1.6)_{\epsilon=0}$.

In Theorem 1.4, we obtain 10 limit cycles which bifurcate from the period annulus. If we consider small-amplitude limit cycles bifurcating from the period annulus near the center, more limit cycles can be obtained.

Theorem 1.5 *For sufficiently small* $|\epsilon| > 0$*, there exists a system* (1.6) *with* 12 *small-amplitude limit cycles bifurcating from the period annulus near the center of system* $(1.6)_{\epsilon=0}$.

2 Preliminary

In this section we present some results which shall be used to investigate the piecewise system (1.6).

Lemma 2.1 *Consider* $p+1$ *linearly independent analytic functions* $f_i : U \subset R \to R$, $i = 0, 1, \ldots, p$.

(1) *Given* p *arbitrary values* $x_i \in U$, $i = 1, 2, \ldots, p$, *there exist* $p+1$ *constants* C_i, $i = 0, 1, \ldots, p$ *such that*

$$f(x) := \sum_{i=0}^{p} C_i f_i(x), \tag{2.1}$$

is not the zero function and $f(x_i) = 0$ *for* $i = 1, 2, \ldots, p$.

(2) *Furthermore, there exists* $f(x)$ *in* (2.1) *such that it has at least* p *simple zeroes in* U.

Lemma 4.5 in [6] proved the same conclusions, but they need an extra condition "there exists $j \in \{0, 1, \ldots, p\}$ such that $f_j|_U$ has constant sign".

Proof For part (1), from expression (2.1), by imposing $f(x_i) = 0$ for these p arbitrary values x_i, we get a homogeneous linear system in the variables C_i, $i = 0, 1, \ldots, p$. This system has solutions different from the zero solution. Furthermore the independence of the functions $f_i(x)$, $i \in \{0, 1, \ldots, p\}$ proves that the function $f(x)$ is not a zero function.

For part (2), the analytical function $f_1(x)$ is not a zero function. Otherwise, it contradicts the assumption of independence. In a compact subset $K \subset U$, choose p values x_i, $i = 1, 2, \ldots, p$, such that $f_1(x_i) \neq 0$. From part (1), we can regard x_i as the zero point of $f(x)$. Define:

(1) p_1 the number of zeroes of $f(x)$ on which $f(x)$ has a local minimum and $f_1(x) > 0$;

(2) p_2 the number of zeroes of $f(x)$ on which $f(x)$ has a local minimum and $f_1(x) < 0$;

(3) p_3 the number of zeroes of $f(x)$ on which $f(x)$ has a local maximum and $f_1(x) > 0$;

(4) p_4 the number of zeroes of $f(x)$ on which $f(x)$ has a local maximum and $f_1(x) < 0$;

(5) p_5 the number of zeroes of $f(x)$ on which $f(x)$ has a simple zero;

(6) p_6 the number of zeroes of $f(x)$ on which $f'(x)$ vanishes but where $f(x)$ has neither a local minimum nor a local maximum.

Obviously, $p = \sum_{i=1}^{6} p_i$. Without loss of generality, we suppose $p_1 + p_4 \geq p_2 + p_3$.

When x_1 is a zero of $f(x)$ where $f(x)$ has a local minimum and $f_1(x_1) > 0$, then x_1 is a m-order zero of $f(x)$ where m is an even number. We have

$$f(x) = a(x - x_1)^m + O(x - x_1)^{m+1}, \quad x \in V,$$

and

$$f_1(x) := \sum_{i=0}^{m} b_i (x - x_1)^i + O(x - x_1)^{m+1}, \quad x \in V,$$

where $V \subset K$ is a neighborhood of x_1 and $ab_0 > 0$. Define

$$f_\epsilon(x) = f(x) - \epsilon f_1(x), \quad x \in V.$$

For a small $\epsilon > 0$, the function $F(\epsilon, z) = f_\epsilon(z + x_1)$ is an analytic function such that $F(0, 0) = 0$ and $\frac{\partial F}{\partial \epsilon}(0, 0) \neq 0$. By the implicit function theorem, in a neighborhood of $(0, 0)$ we can write

$\epsilon = \epsilon(z) = \frac{a}{b_0} z^m + O(z^m)$ such that $F(\epsilon(z), z) = 0$. From the equality $\epsilon = \epsilon(z)$ we obtain $z = z(\epsilon) = \pm \sqrt[m]{\frac{b_0}{a}} \epsilon^{\frac{1}{m}} + O(\epsilon^\alpha)$, $\alpha > \frac{1}{m}, \alpha \in \mathbb{R}$, which assure two simple zeros of $f_\epsilon(x)$ in V.

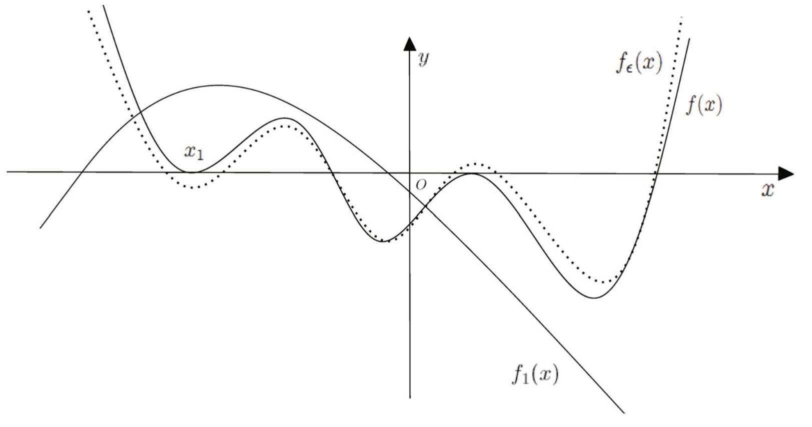

Figure 2 The point x_1 on which $f(x)$ has a local minimum and $f_1(x_1) > 0$

When x_1 is a zero of $f(x)$ where $f(x)$ has a local maximum and $f_1(x) < 0$, two simple zeros can be also obtained in the above process.

For a zero of $f(x)$ where $f(x)$ has neither a local minimum nor a local maximum, it is easy to prove that one simple zero of $f_\epsilon(x)$ exists for a small enough $\epsilon > 0$ in the above process.

At last, the number of simple zeroes of $f_\epsilon(x)$ is at least $2(p_1 + p_4) + p_5 + p_6$. On account of $p_1 + p_4 \geq p_2 + p_3$, then $2(p_1 + p_4) + p_5 + p_6 \geq \sum_{i=1}^{6} p_i = p$. So the proof is completed. □

Next, the following lemma gives sufficient conditions for the existence of small-amplitude limit cycles.

Lemma 2.2 *Suppose that* $c = (c_1, c_2, \ldots, c_k)$ *and Abelian integral* $I(h) = \sum_{i=1}^{\infty} A_i(c) h^i$ *such that* $A_i(c^*) = 0$, $i = 1, 2, \ldots, N-1$, $A_N(c^*) \neq 0$. *If there exists* $1 \leq i_1 < i_2 < \ldots < i_k < N$ *satisfying*

$$\operatorname{rank}\left(\frac{\partial(A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_k}(c))}{\partial(c_1, c_2, \ldots, c_k)} \mid_{c^*} \right) = k,$$

and $A_j(c) = O(|A_{i_1}(c), \ldots, A_{i_l}(c)|)$ *for any* $i_l < j < i_{l+1}$, *then* $I(h) = 0$ *can have* k *distinct real positive roots near* $h = 0$.

Proof Indeed, Lemma 2.2 is the same as Lemma 3.2 in [7] and Theorem 2.1 in [8]. For convenience to readers, we prove it here.

By the assumption, we can select $A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_k}(c)$ with $1 \leq i_1 < i_2 < \cdots < i_k < N$ such that

$$\operatorname{rank}\left(\frac{\partial(A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_k}(c))}{\partial(c_1, c_2, \ldots, c_k)} \bigg|_{c^*} \right) = k, \tag{2.2}$$

and for $i_1 \leq i_p < j < i_{p+1} \leq i_k$,

$$\begin{aligned} &\operatorname{rank}\left(\frac{\partial(A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_p}(c))}{\partial(c_1, c_2, \ldots, c_k)} \bigg|_{c^*} \right) \\ &= \operatorname{rank}\left(\frac{\partial(A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_p}(c), A_j(c))}{\partial(c_1, c_2, \ldots, c_k)} \bigg|_{c^*} \right). \end{aligned} \tag{2.3}$$

From (2.3), in a neighborhood of c^*, the linear part of $A_j(c)$ can be expressed by the linear parts of $A_{i_1}(c), A_{i_2}(c), \ldots, A_{i_p}(c)$, i.e., $A_j(c) = \sum_{l=i_1}^{i_p} a_l A_l(c) + O(|c - c^*|)$. Based on (2.2), $A_{i_l}(c)$, $l = 1, \ldots, k$ are independent in a neighborhood of c^*. Since $(A_{i1}(c^*), \ldots, A_{ik}(c^*)) = (0, \ldots, 0)$, in a small neighbourhood of c^* we can perform a change of coordinates such that $A_{i_1}(c) = \lambda_1, A_{i_2}(c) = \lambda_2, \ldots, A_{i_k}(c) = \lambda_k$ where $\lambda_1, \lambda_2, \ldots, \lambda_k$ can be chosen independently. Taking $\lambda_p = m_p \epsilon^{N-i_p}$ $(1 \le p \le k)$ where m_p can be chosen independently, we have

$$I(h) = \sum_{p=1}^{k} m_p \epsilon^{N-i_p} h^{i_p} + A_N h^N + \Phi(h, \epsilon),$$

where $\Phi(h, \epsilon)$ contains only terms of order greater than N in h and ϵ. It is easy to prove that $A_j(c)h^j$ discussed above fells in $\Phi(h, \epsilon)$. Define

$$\mathbb{I}(h) = \sum_{p=1}^{k} m_p \epsilon^{N-i_p} h^{i_p} + A_N h^N.$$

Suppose $h = \epsilon \bar{h}$ and divide both sides of the equations $I(\epsilon\bar{h})$ and $\mathbb{I}(\epsilon\bar{h})$ by ϵ^N, respectively. Define

$$\bar{I}(\bar{h}, \epsilon) \triangleq \frac{I(\epsilon\bar{h})}{\epsilon^N} = \sum_{p=1}^{k} m_p \bar{h}^{ip} + A_N \bar{h}^N + o(\epsilon), \quad \bar{\mathbb{I}}(\bar{h}) \triangleq \frac{\mathbb{I}(\epsilon\bar{h})}{\epsilon^N} = \sum_{p=1}^{k} m_p \bar{h}^{ip} + A_N \bar{h}^N.$$

With the independence of $m_1, \ldots, m_k$, the function $\bar{\mathbb{I}}(\bar{h})$ can have k distinct positive roots $\bar{h}_p$, $p = 1, \ldots, k$ such that $\bar{\mathbb{I}}'(\bar{h}_p) \neq 0$. Thus, $\bar{I}(\bar{h}_p, 0) = 0$ and $\bar{I}'(\bar{h}_p, 0) \neq 0$. By the implicit function theorem, $\bar{I}(\bar{h}, \epsilon)$ can have k distinct positive roots $\bar{h}_p(\epsilon)$, $p = 1, \ldots, k$ such that $\bar{h}_p(\epsilon) \to \bar{h}_p$ as $\epsilon \to 0$.

Then $I(h)$ can have k distinct positive roots. □

3 Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1 When $x \geq 0$, $\forall (x_1, y_1) \in \mathcal{P}^+ \setminus \{(0, 0)\}$, there exists an orbit

$$\Gamma^+ : \{(x, y) : x = x(t, x_1, y_1), y = y(t, x_1, y_1)\}$$

through (x_1, y_1) where $x(t, x_1, y_1)$ and $y(t, x_1, y_1)$ are analytic depending on (x_1, y_1). Denote $(0, y^*)$ and $(0, y_*)$ which are two intersection points of Γ^+ and $x = 0$ with $y^* > 0 > y_*$. Then there exists $t = t_*$ such that

$$x(t_*, x_1, y_1) = 0, \quad y(t_*, x_1, y_1) = y_*.$$

Since the assumption **H3** holds, then $\frac{dx(t,x_1,y_1)}{dt}|_{t=t^*} \neq 0$. By the implicit function theorem, we obtain $t_* = t_*(x_1, y_1)$. Define $H^+(x_1, y_1) = y_*$, then $H^+(x_1, y_1) = y(t_*(x_1, y_1), x_1, y_1)$ is analytic in $\mathcal{P}^+ \setminus \{(0, 0)\}$.

When $x \leq 0$, $\forall (x_2, y_2) \in \mathcal{P}^- \setminus \{(0, 0)\}$, there exists $\Gamma^- : \{(x, y) : x = \tilde{x}(t, x_2, y_2), y = \tilde{y}(t, x_2, y_2)\}$ which intersects the line $x = 0$ at $(0, \tilde{y}^*)$ and $(0, \tilde{y}_*)$. By the same way, there exists $t = \tilde{t}_*$ such that $\tilde{x}(\tilde{t}_*, x_2, y_2) = 0$ and $\tilde{y}(\tilde{t}_*, x_2, y_2) = \tilde{y}_*$. Because the assumption **H3** holds, we obtain $\tilde{t}_* = \tilde{t}_*(x_2, y_2)$. Define $H^-(x_2, y_2) = \tilde{y}_*$, then $H^-(x_2, y_2) = y(\tilde{t}_*(x_2, y_2), x_2, y_2)$ is analytic in $\mathcal{P}^- \setminus \{(0, 0)\}$.

Because $\mathcal{P}$ is a period annulus, $\Gamma^+ \cup \Gamma^-$ is a periodic orbit if and only if $(0, y_*) = (0, \tilde{y}_*)$.

Define

$$H(x,y)=\begin{cases}H^+(x,y), & x>0,\\ H^-(x,y), & x<0.\end{cases}$$

Then, on $\Gamma^-\cup\Gamma^+$, $H(x,y)=y_*$ is a constant and $H(x,y)$ is not a constant on $\mathcal{P}$. Therefore, system (1.1) has a first integral $H(x,y)$. Obviously, $H^{\pm}(x,y)$ are analytic in $\mathcal{P}^{\pm}\setminus\{(0,0)\}$ respectively and $\lim_{x\to 0^+}H^+(x,y)=\lim_{x\to 0^-}H^-(x,y)$ on $\mathcal{P}$. □

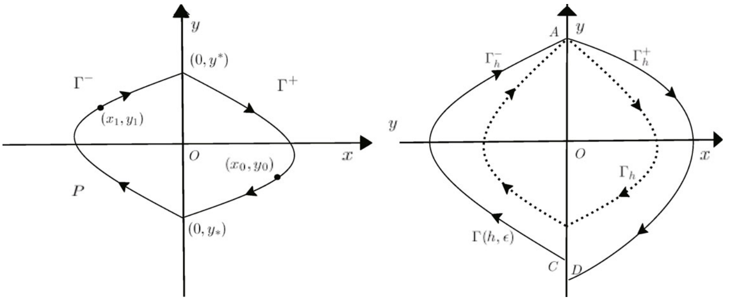

Figure 3 The graphs are trajectories of systems (1.1) and $(1.1)_\epsilon$

In fact, the first integral of a system is not unique. Therefore, we will select the appropriate first integration as required in the following discussions.

Proof of Theorem 1.3 According to the proof of Theorem 1.1 and the assumption **H3**, we denote

$$\Gamma_h=\begin{cases}\Gamma_h^+=\{(x,y)\,|\,H^+(x,y)=h\}, & x>0,\\ \Gamma_h^-=\{(x,y)\,|\,H^-(x,y)=h\}, & x<0,\end{cases}$$

as a orbit of system (1.1). $\{\Gamma_h, h\in(\alpha,\beta)\}$ is the period annulus P.

See the right graph of Figure 3. Suppose that the orbit $\Gamma_{(h,\epsilon)}$ of the system $(1.1)_\epsilon$ changed by Γ_h also passes through A. Next, we calculate the displacement function $d(h,\epsilon)\triangleq H^+(D)-H^+(C)$ which is similar to [21]. Based on Theorem 1.1, we obtain

$$H^+(D)-H^+(C)=[H^+(D)-H^+(A)]+[H^-(A)-H^-(C)].$$

First, we have

$$\begin{aligned}
&H^+(D)-H^+(A)\\
&=\int_{\Gamma^+_{(h,\epsilon)}}\frac{dH^+(x,y)}{dt}dt\\
&=\int_{\Gamma^+_{(h,\epsilon)}}\left[\frac{\partial H^+(x,y)}{\partial x}\cdot\left(\frac{H_y^+(x,y)}{M^+(x,y)}+\epsilon f^+(x,y)\right)+\frac{\partial H^+(x,y)}{\partial y}\cdot\left(\frac{-H_x^+(x,y)}{M^+(x,y)}+\epsilon g^+(x,y)\right)\right]dt\\
&=\epsilon\int_{\Gamma^+_{(h,\epsilon)}}\left[\frac{\partial H^+(x,y)}{\partial x}\cdot f^+(x,y)+\frac{\partial H^+(x,y)}{\partial y}\cdot g^+(x,y)\right]dt\\
&=\epsilon\bigg(\int_{\Gamma^+_h}\left[\frac{\partial H^+(x,y)}{\partial x}\cdot f^+(x,y)+\frac{\partial H^+(x,y)}{\partial y}\cdot g^+(x,y)\right]dt+O(\epsilon)\bigg)
\end{aligned}$$

$$= \epsilon\biggl(\int_{\Gamma_h^+}(M^+(x,y)g^+(x,y)dx - M^+(x,y)f^+(x,y)dy) + O(\epsilon)\biggr). \tag{3.1}$$

Similarly,

$$H^-(A) - H^-(C) = \epsilon\biggl(\int_{\Gamma_h^-}(M^-(x,y)g^-(x,y)dx - M^-(x,y)f^-(x,y)dy) + O(\epsilon)\biggr). \tag{3.2}$$

Combining (3.1) and (3.2),

$$\begin{aligned} d(h,\epsilon) &= \epsilon\biggl(\int_{\Gamma_h^+}(M^+(x,y)g^+(x,y)dx - M^+(x,y)f^+(x,y)dy) \\ &\quad + \int_{\Gamma_h^-}(M^-(x,y)g^-(x,y)dx - M^-(x,y)f^-(x,y)dy) + O(\epsilon)\biggr) \\ &= \epsilon\biggl(\int_{\Gamma_h} M(x,y)[g(x,y)dx - f(x,y)dy] + O(\epsilon)\biggr) \\ &= \epsilon(I(h) + O(\epsilon)). \end{aligned}$$

If $I(h^*) = 0$ and $I'(h^*) \neq 0$, $h^* \in (\alpha,\beta)$, then, for sufficiently small $|\epsilon| > 0$, $d(h,\epsilon)$ has a zero point $h(\epsilon)$ near h^* by the implicit function theorem and system $(1.1)_\epsilon$ has a limit cycle near Γ_{h^*}. If $I(h)$, $h \in (\alpha,\beta)$, has k simple zeros, system $(1.1)_\epsilon$ has at least k limit cycles. □

4 Applications

Let's rewrite system (1.6) as

$$(\dot{x},\dot{y}) = \begin{cases} (y(1+a_1x+a_2y) + \epsilon f^+(x,y), -x(1+a_1x+a_2y) + \epsilon g^+(x,y)), & x > 0, \\ (y(1+b_1x+b_2y) + \epsilon f^-(x,y), (x+1)(1+b_1x+b_2y) + \epsilon g^-(x,y)), & x < 0, \end{cases}$$

The unperturbed system $(1.6)_{\epsilon=0}$ has a period annulus around the origin and the closed orbit transversely intersects the line $x = 0$. According to Theorem 1.1, system $(1.6)_{\epsilon=0}$ has a first integral

$$H(x,y) = \begin{cases} H^+(x,y) = x^2 + y^2, & x > 0, \\ H^-(x,y) = -(x+1)^2 + y^2 + 1, & x < 0, \end{cases}$$

and an integrating factor

$$M(x,y) = \begin{cases} M^+(x,y) = \dfrac{1}{1+a_1x+a_2y}, & x > 0, \\ M^-(x,y) = \dfrac{1}{1+b_1x+b_2y}, & x < 0. \end{cases}$$

The periodic orbit is

$$\Gamma_h = \begin{cases} \Gamma_h^+ = \{(x,y) | H^+(x,y) = h\}, & x > 0, \\ \Gamma_h^- = \{(x,y) | H^-(x,y) = h\}, & x < 0. \end{cases}$$

Now we can obtain the algebraic structure of Abelian integral $I(h)$ of system (1.6).

Lemma 4.1 *For system (1.6), Abelian integral has the form $I(h) = \sum_{i=1}^{11} k_i I_i(h)$ where k_i is arbitrary and*

$$
\begin{aligned}
I_1(h) &= \int_{\Gamma_h^+} \frac{1}{1+a_1x+a_2y}dy, \quad I_2(h) = \int_{\Gamma_h^+} \frac{x}{1+a_1x+a_2y}dy,\\
I_3(h) &= \int_{\Gamma_h^+} \frac{x^2}{1+a_1x+a_2y}dy, \quad I_4(h) = \int_{\Gamma_h^+} \frac{1}{1+a_1x+a_2y}dx,\\
I_5(h) &= \int_{\Gamma_h^-} \frac{1}{1+b_1x+b_2y}dy, \quad I_6(h) = \int_{\Gamma_h^-} \frac{x}{1+b_1x+b_2y}dy,\\
I_7(h) &= \int_{\Gamma_h^-} \frac{x^2}{1+b_1x+b_2y}dy, \quad I_8(h) = \int_{\Gamma_h^-} \frac{1}{1+b_1x+b_2y}dx,\\
I_9(h) &= \int_{\Gamma_h^-} xdy, \quad I_{10}(h) = h^{\frac{1}{2}}, \quad I_{11}(h) = h.
\end{aligned}
$$

The proof of Lemma 4.1 can be found in Appendix.

Proof of Theorem 1.4 From Lemma 2.1 and Lemma 4.1, the proof of Theorem 1.4 only needs to prove that $I_i(h), i = 1, 2, \ldots, 11$ are linearly independent.

Next apply Maple to get the 15 order Taylor expansion of $I_i(h), i = 1, 2, \ldots, 11$, at $h = 0$. Since h is sufficiently small, we obtain $\frac{1}{1+a_1x+a_2y} = \sum_{n=0}^{+\infty}(-1)^n(a_1x + a_2y)^n$. For convenience, suppose $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, $b_2 = 5$.

On $\Gamma_h^+ = \{(x, y) : x^2 + y^2 = h\}$, let $x = \sqrt{h}\cos\theta$, $y = \sqrt{h}\sin\theta$.

$$
\begin{aligned}
I_1(h) &= \sum_{n=0}^{+\infty} h^{\frac{n+1}{2}} \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} (-1)^n(a_1\cos\theta + a_2\sin\theta)^n \cos\theta d\theta\\
&= -2h^{\frac{1}{2}} + \frac{\pi}{2}h - 4h^{\frac{3}{2}} + \frac{15\pi}{8}h^2 - \frac{208}{15}h^{\frac{5}{2}} + \frac{125\pi}{16}h^3 - \frac{1952}{35}h^{\frac{7}{2}} + \frac{4375\pi}{128}h^4 - \frac{15104}{63}h^{\frac{9}{2}}\\
&\quad + \frac{39375\pi}{256}h^5 - \frac{740864}{693}h^{\frac{11}{2}} + \frac{721875\pi}{1024}h^6 - \frac{14645248}{3003}h^{\frac{13}{2}} + \frac{6703125\pi}{2048}h^7\\
&\quad - \frac{145371136}{6435}h^{\frac{15}{2}} + O(h^8),\\
I_2(h) &= -\frac{1}{2}\pi h + \frac{4}{3}h^{\frac{3}{2}} - \frac{7}{8}\pi h^2 + \frac{64}{15}h^{\frac{5}{2}} - \frac{45}{16}\pi h^3 + \frac{1696}{105}h^{\frac{7}{2}} - \frac{1375}{128}\pi h^4 + \frac{20864}{315}h^{\frac{9}{2}} - \frac{11375}{256}\pi h^5\\
&\quad + \frac{2560}{9}h^{\frac{11}{2}} - \frac{196875}{1024}\pi h^6 + \frac{11337728}{9009}h^{\frac{13}{2}} - \frac{1753125}{2048}\pi h^7 + \frac{85348352}{15015}h^{\frac{15}{2}} + O(h^8),\\
I_3(h) &= -\frac{4}{3}h^{\frac{3}{2}} + \frac{3}{8}\pi h^2 - \frac{32}{15}h^{\frac{5}{2}} + \frac{17}{16}\pi h^3 - \frac{32}{5}h^{\frac{7}{2}} + \frac{475}{128}\pi h^4 - \frac{2432}{105}h^{\frac{9}{2}} + \frac{3675}{256}\pi h^5 - \frac{35328}{385}h^{\frac{11}{2}}\\
&\quad + \frac{60375}{1024}\pi h^6 - \frac{1150976}{3003}h^{\frac{13}{2}} + \frac{515625}{2048}\pi h^7 - \frac{74641408}{45045}h^{\frac{15}{2}} + O(h^8),\\
I_4(h) &= -\pi h + \frac{8}{3}h^{\frac{3}{2}} - \frac{15}{4}\pi h^2 + \frac{224}{15}h^{\frac{5}{2}} - \frac{125}{8}\pi h^3 + \frac{2624}{35}h^{\frac{7}{2}} - \frac{4375}{64}\pi h^4 + \frac{2560}{7}h^{\frac{9}{2}} - \frac{39375}{128}\pi h^5\\
&\quad + \frac{1223680}{693}h^{\frac{11}{2}} - \frac{721875}{512}\pi h^6 + \frac{1961984}{231}h^{\frac{13}{2}} - \frac{6703125}{1024}\pi h^7 + \frac{262627328}{6435}h^{\frac{15}{2}} + O(h^8).
\end{aligned}
$$

On $\Gamma_h^- = \{(x, y) : -(x+1)^2 + y^2 + 1 = h\}$, let $s = y^2 - h$ and we get $x = \sqrt{1+s} - 1$ which can be expressed as $x = \frac{1}{2}s - \frac{1}{8}s^2 + \frac{1}{16}s^3 - \frac{5}{128}s^4 + \frac{7}{256}s^5 - \frac{21}{1024}s^6 + \frac{33}{2048}s^7 - \frac{429}{32768}s^8 + \frac{715}{65536}s^9 + O(s^{10})$.

$$
I_5(h) = \int_{\Gamma_h^-} \sum_{n=0}^{+\infty}(-1)^n(b_1x + b_2y)^n dy = 2h^{\frac{1}{2}} + \frac{56}{3}h^{\frac{3}{2}} + \frac{1414}{5}h^{\frac{5}{2}} + 5060h^{\frac{7}{2}} + \frac{31016254}{315}h^{\frac{9}{2}}
$$

$$
\begin{aligned}
&+\frac{465435232}{231}h^{\frac{11}{2}}+\frac{42675916182}{1001}h^{\frac{13}{2}}+\frac{660577871444}{715}h^{\frac{15}{2}}+O(h^8),\\
I_6(h)&=-\frac{2}{3}h^{\frac{3}{2}}-\frac{64}{15}h^{\frac{5}{2}}-\frac{970}{21}h^{\frac{7}{2}}-\frac{22452}{35}h^{\frac{9}{2}}-\frac{7069862}{693}h^{\frac{11}{2}}-\frac{1590345496}{9009}h^{\frac{13}{2}}\\
&-\frac{145758941474}{45045}h^{\frac{15}{2}}+O(h^8),\\
I_7(h)&=\frac{4}{15}h^{\frac{5}{2}}+\frac{148}{105}h^{\frac{7}{2}}+\frac{3736}{315}h^{\frac{9}{2}}+\frac{30952}{231}h^{\frac{11}{2}}+\frac{1472140}{819}h^{\frac{13}{2}}+\frac{1212101404}{45045}h^{\frac{15}{2}}+O(h^8),\\
I_8(h)&=-\frac{10}{3}h^{\frac{3}{2}}-\frac{164}{3}h^{\frac{5}{2}}-\frac{6950}{7}h^{\frac{7}{2}}-\frac{408056}{21}h^{\frac{9}{2}}-\frac{25126810}{63}h^{\frac{11}{2}}-\frac{25391299660}{3003}h^{\frac{13}{2}}\\
&-\frac{236128145582}{1287}h^{\frac{15}{2}}+O(h^8),\\
I_9(h)&=-\frac{2}{3}h^{\frac{3}{2}}-\frac{2}{15}h^{\frac{5}{2}}-\frac{2}{35}h^{\frac{7}{2}}-\frac{2}{63}h^{\frac{9}{2}}-\frac{2}{99}h^{\frac{11}{2}}-\frac{2}{143}h^{\frac{13}{2}}-\frac{2}{195}h^{\frac{15}{2}}+O(h^8).
\end{aligned}
$$

Denote $I_i(h)\triangleq\sum_{j=1}^{15}C_{i,j}h^{\frac{j}{2}}+O(h^8),\ i=1,2,\ldots,11$, and $C=(C_{ij})_{11\times15}$. Using Maple, $\operatorname{rank}(C)=11$ is obtained which implies $I_i(h)$, $i=1,2,\ldots,11$, are linearly independent. So the proof is completed. □

Proof of Theorem 1.5 Let $a_1=1,b_1=3$. Using the same method of Theorem 1.4, we get the 20 order Taylor expansion of $I_i(h)$, $i=1,2,\ldots,11$, which are omitted here for brevity. Then the Abelian integral of system (1.6) has the following form:

$$
I(h)=\sum_{i=1}^{11}k_iI_i(h)=\sum_{i=0}^{20}A_ih^{\frac{i}{2}}+O(h^{\frac{20}{2}}),
$$

where k_i is arbitrary and A_i is a polynomial of a_2,b_2 and k_j, $j=1,2,\ldots,11$.

$$
\begin{aligned}
A_0&=0,\\
A_1&=-2\,k_1+2\,k_5+k_{10},\\
A_2&=\frac{1}{2}\pi k_1-\frac{1}{2}\pi k_2-\frac{1}{2}a_2\pi k_4+k_{11},\\
A_3&=\Big(-\frac{2}{3}a_2^2-\frac{4}{3}\Big)k_1+\frac{4}{3}k_2-\frac{4}{3}k_3+\frac{4}{3}a_2k_4+\Big(\frac{2}{3}b_2^2+2\Big)k_5-\frac{2}{3}k_6-\frac{2}{3}b_2k_8-\frac{2}{3}k_9,\\
A_4&=\Big(\frac{3}{8}a_2^2\pi+\frac{3}{8}\pi\Big)k_1+\Big(-\frac{1}{8}a_2^2\pi-\frac{3}{8}\pi\Big)k_2+\frac{3}{8}\pi k_3+\Big(-\frac{3}{8}a_2^3\pi-\frac{3}{8}a_2\pi\Big)k_4,\\
A_5&=\Big(-\frac{2}{5}a_2^4-\frac{8}{5}a_2^2-\frac{16}{15}\Big)k_1+\Big(\frac{4}{5}a_2^2+\frac{16}{15}\Big)k_2+\Big(-\frac{4}{15}a_2^2-\frac{16}{15}\Big)k_3+\Big(\frac{8}{5}a_2^3+\frac{16}{15}a_2\Big)k_4\\
&+\Big(\frac{2}{5}b_2^4+\frac{6}{5}b_2^2+\frac{14}{5}\Big)k_5+\Big(-\frac{2}{15}b_2^2-\frac{14}{15}\Big)k_6+\frac{4}{15}k_7+\Big(-\frac{2}{5}b_2^3-\frac{14}{15}b_2\Big)k_8-\frac{2}{15}k_9,\\
A_6&=\Big(\frac{5}{16}a_2^4\pi+\frac{5}{8}a_2^2\pi+\frac{5}{16}\pi\Big)k_1+\Big(-\frac{1}{16}a_2^4\pi-\frac{3}{8}a_2^2\pi-\frac{5}{16}\pi\Big)k_2+\Big(\frac{3}{16}a_2^2\pi+\frac{5}{16}\pi\Big)k_3\\
&+\Big(-\frac{5}{16}a_2^5\pi-\frac{5}{8}a_2^3\pi-\frac{5}{16}a_2\pi\Big)k_4,\\
A_7&=\Big(-\frac{32}{35}-\frac{12}{7}a_2^4-\frac{16}{7}a_2^2-\frac{2}{7}a_2^6\Big)k_1+\Big(\frac{4}{7}{a_2}^4+\frac{32}{21}a_2^2+\frac{32}{35}\Big)k_2+\Big(-\frac{4}{35}a_2^4-\frac{32}{35}a_2^2\\
&-\frac{32}{35}\Big)k_3+\Big(\frac{12}{7}a_2^5+\frac{16}{7}a_2^3+\frac{32}{35}a_2\Big)k_4+\Big(\frac{30}{7}+\frac{2}{7}b_2^6+\frac{6}{7}{b_2}^4+\frac{78}{35}b_2^2\Big)k_5+\Big(-\frac{2}{35}b_2^4
\end{aligned}
$$

$$-\frac{38}{105}b_2^2-\frac{10}{7}\Big)k_6+\Big(\frac{4}{105}b_2^2+\frac{16}{35}\Big)k_7+\Big(-\frac{26}{35}b_2^3-\frac{10}{7}b_2-\frac{2}{7}b_2^5\Big)k_8-\frac{2}{35}k_9,$$

......

Next, Lemma 2.2 is applied to prove that $I(h)$ can have 12 simple zero points. The first step starts $A_i=0$, $i=1,2,\ldots,6$, which yield k_i, $i=1,2,\ldots,6$, successively.

$$\begin{aligned}
k_1&=k_5+\frac{1}{2}k_{10},\\
k_2&=-a_2k_4+k_5+\frac{1}{2}k_{10}+\frac{2k_{11}}{\pi},\\
k_3&=-\frac{1}{2}(a_2^2-b_2^2-3)k_5-\frac{1}{2}k_6-\frac{1}{2}b_2k_8-\frac{1}{2}k_9-\frac{1}{4}a_2^2k_{10}+\frac{2k_{11}}{\pi},\\
k_4&=\frac{1}{8a_2^3}\Big((2a_2^2+6b_2^2+18)k_5-6k_6-6b_2k_8-6k_9+a_2^2k_{10}-\frac{8a_2^2k_{11}}{\pi}\Big),\\
k_5&=-\frac{1}{2M_5}\Big((-4a_2^2+4b_2^2+30)k_6-8k_7+(-4a_2^2b_2+12b_2^3+30b_2)k_8+(-4a_2^2+6)k_9\\
&\quad+(4a_2^4+a_2^2)k_{10}-\frac{8a_2^2k_{11}}{\pi}\Big),\\
M_5&=4a_2^4+2a_2^2b_2^2-6b_2^4+7a_2^2-19b_2^2-45,\\
k_6&=-\frac{1}{2M_6}\Big((12a_2^4-12a_2^2b_2^2-32a_2^2-4b_2^2-12)k_7+(18a_2^6b_2-18a_2^4b_2^3-36a_2^4b_2+36a_2^2b_2^3\\
&\quad-14a_2^2b_2+14b_2^3)k_8+(18a_2^6-18a_2^2b_2^4-48a_2^2b_2^2-6b_2^4-110a_2^2-16b_2^2-36)k_9\\
&\quad+(9a_2^6b_2^2-9a_2^4b_2^4+27a_2^6-24a_2^4b_2^2-3a_2^2b_2^4-54a_2^4-9a_2^2b_2^2-21a_2^2)k_{10}\\
&\quad+(16a_2^8+8a_2^6b_2^2-24a_2^4b_2^4+24a_2^6-96a_2^4b_2^2+24a_2^2b_2^4-240a_2^4+72a_2^2b_2^2+168a_2^2)\frac{k_{11}}{\pi}\Big),\\
M_6&=9a_2^6-3a_2^4b_2^2-6a_2^2b_2^4-18a_2^4+2a_2^2b_2^2-2b_2^4-7a_2^2+b_2^2.
\end{aligned}\tag{4.1}$$

Then $A_{2i}\equiv 0$, $i=4,5,\ldots,10$, are found under the above processes. Next, $A_{2i-1}=0$, $i=4,5,6,7$, yield k_7,k_8,k_9,k_{10} successively which can be seen in Appendix. Meanwhile, A_{15},A_{17} and A_{19} can be presented as

$$\begin{aligned}
A_{15}&=\frac{16}{6435}a_2^6k_{11}(a_2^2-1)^2(a_2^2-b_2^2)^3\frac{F_{15}}{K},\\
A_{17}&=\frac{16}{109395}a_2^6k_{11}(a_2^2-1)^2(a_2^2-b_2^2)^3\frac{F_{17}}{K},\\
A_{19}&=\frac{16}{138567}a_2^6k_{11}(a_2^2-1)^2(a_2^2-b_2^2)^3\frac{F_{19}}{K},\\
K&=\pi(3a_2^2+1)(1155a_2^8b_2^2-1050a_2^6b_2^4+175a_2^4b_2^6-2541a_2^8-2226a_2^6b_2^2+2849a_2^4b_2^4-306a_2^2b_2^6\\
&\quad+10080a_2^6-1715a_2^4b_2^2-2000a_2^2b_2^4+67b_2^6-12285a_2^4+4100a_2^2b_2^2+233b_2^4+4330a_2^2-866b_2^2),\\
F_{15}&=3465a_2^8b_2^2-2310a_2^6b_2^4+525a_2^4b_2^6-7623a_2^8-21252a_2^6b_2^2+11925a_2^4b_2^4-2202a_2^2b_2^6+32b_2^8\\
&\quad+58674a_2^6+46041a_2^4b_2^2-19704a_2^2b_2^4+2189b_2^6-169839a_2^4-39882a_2^2b_2^2+10281b_2^4\\
&\quad+218820a_2^2+10220b_2^2-105760,\\
F_{17}&=90090a_2^{10}b_2^2+45045a_2^8b_2^4-60060a_2^6b_2^6+17325a_2^4b_2^8-198198a_2^{10}-567567a_2^8b_2^2\\
&\quad-467610a_2^6b_2^4+355305a_2^4b_2^6-73242a_2^2b_2^8+1152b_2^{10}+1045044a_2^8+1649076a_2^6b_2^2
\end{aligned}$$

$$+ 1561545a_2^4b_2^4 - 698334a_2^2b_2^6 + 76109b_2^8 - 723294a_2^6 - 3304251a_2^4b_2^2 - 2050776a_2^2b_2^4$$
$$+ 448401b_2^6 - 4977912a_2^4 + 4392804a_2^2b_2^2 + 885108b_2^4 + 10962120a_2^2 - 2559160b_2^2 - 6610320,$$
$$F_{19} = 135135a_2^{12}b_2^2 + 120120a_2^{10}b_2^4 + 45045a_2^8b_2^6 - 90090a_2^6b_2^8 + 30030a_2^4b_2^{10} - 297297a_2^{12}$$
$$- 858858a_2^{10}b_2^2 - 937365a_2^8b_2^4 - 658944a_2^6b_2^6 + 582780a_2^4b_2^8 - 127644a_2^2b_2^{10} + 2112b_2^{12}$$
$$+ 1327326a_2^{10} + 2220504a_2^8b_2^2 + 2620332a_2^6b_2^4 + 2709882a_2^4b_2^6 - 1256826a_2^2b_2^8 + 136398b_2^{10}$$
$$- 812526a_2^8 - 1323036a_2^6b_2^2 - 3518886a_2^4b_2^4 - 4330056a_2^2b_2^6 + 891944b_2^8 - 723294a_2^6$$
$$- 8205231a_2^4b_2^2 + 3123204a_2^2b_2^4 + 2348761b_2^6 - 11524557a_2^4 + 20435754a_2^2b_2^2 - 1934317b_2^4$$
$$+ 31445436a_2^2 - 14143404b_2^2 - 21482352.$$

According to Lemma 2.2, define

$$J(k_1, \ldots, k_{10}, a_2, b_2) = \det\left(\frac{\partial(A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_9, A_{11}, A_{13}, A_{15}, A_{17})}{\partial(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, a_2, b_2)}\right).$$

When $A_i = 0$, $i = 1, \ldots, 7, 9, 11, 13$, we take k_j, $j = 1, \ldots, 10$ into $J(k_1, \ldots, k_{10}, a_2, b_2)$ one by one and obtain

$$J(a_2, b_2) \triangleq J(k_1, \ldots, k_{10}, a_2, b_2) = -\frac{32768a_2^{16}b_2^{12}k_{11}^2(a_2^2-1)^3(b_2^2-1)^2(a_2^2-b_2^2)^6}{1090227071049161357812 5}\frac{F_J}{K^2},$$

where F_J is a polynomial of degree 36 in a_2, b_2. Due to the size of F_J, it can be seen in Appendix.

The second step shows the existence of $(a_2, b_2) = (a_2^*, b_2^*)$ such that $F_{15} = F_{17} = 0$ and $F_{19} \times F_J \neq 0$.

First, to find the existence of a_2^*, b_2^* satisfying $F_{15} = F_{17} = 0$, we use the Maple built-in command $'RealRootIsolate'$ where the width of the interval is less than or equal $\frac{1}{2^{15}}$ and get

$$R_1 = \left[\left[a_2 = \left[\frac{27195}{16384}, \frac{54391}{32768}\right], b_2 = \left[-\frac{1517583}{1048576}, -\frac{758791}{524288}\right]\right],\right.$$
$$\left[a_2 = \left[-\frac{54391}{32768}, -\frac{27195}{16384}\right], b_2 = \left[-\frac{1517583}{1048576}, -\frac{758791}{524288}\right]\right],$$
$$\left[a_2 = \left[\frac{27195}{16384}, \frac{54391}{32768}\right], b_2 = \left[\frac{758791}{524288}, \frac{1517583}{1048576}\right]\right],$$
$$\left[a_2 = \left[-\frac{54391}{32768}, -\frac{27195}{16384}\right], b_2 = \left[\frac{758791}{524288}, \frac{1517583}{1048576}\right]\right],$$
$$\left[a_2 = \left[\frac{671}{512}, \frac{42945}{32768}\right], b_2 = \left[-\frac{241228922683}{549755813888}, -\frac{120614461341}{274877906944}\right]\right],$$
$$\left[a_2 = \left[-\frac{42945}{32768}, -\frac{671}{512}\right], b_2 = \left[-\frac{241228922683}{549755813888}, -\frac{120614461341}{274877906944}\right]\right],$$
$$\left[a_2 = \left[\frac{22375}{16384}, \frac{44751}{32768}\right], b_2 = \left[-\frac{17140589133}{34359738368}, -\frac{4285147283}{8589934592}\right]\right],$$
$$\left[a_2 = \left[-\frac{44751}{32768}, -\frac{22375}{16384}\right], b_2 = \left[-\frac{17140589133}{34359738368}, -\frac{4285147283}{8589934592}\right]\right],$$
$$\left[a_2 = \left[\frac{6741}{4096}, \frac{431431}{262144}\right], b_2 = \left[-\frac{7079261381833247403}{4611686018427387904}, -\frac{5663409105466597922 3}{36893488147419103232}\right]\right],$$
$$\left[a_2 = \left[-\frac{431431}{262144}, -\frac{6741}{4096}\right], b_2 = \left[-\frac{7079261381833247403}{4611686018427387904}, -\frac{56634091054665979223}{36893488147419103232}\right]\right],$$

$$\left[a_2 = \left[\frac{6741}{4096}, \frac{431431}{262144}\right], b_2 = \left[\frac{56634091054665979223}{36893488147419103232}, \frac{70792613818333247403}{46116860184273879040}\right]\right],$$
$$\left[a_2 = \left[-\frac{431431}{262144}, -\frac{6741}{4096}\right], b_2 = \left[\frac{56634091054665979223}{36893488147419103232}, \frac{70792613818333247403}{46116860184273879040}\right]\right],$$
$$\left[a_2 = \left[\frac{22375}{16384}, \frac{44751}{32768}\right], b_2 = \left[\frac{4285147283}{8589934592}, \frac{17140589133}{34359738368}\right]\right],$$
$$\left[a_2 = \left[-\frac{44751}{32768}, -\frac{22375}{16384}\right], b_2 = \left[\frac{4285147283}{8589934592}, \frac{17140589133}{34359738368}\right]\right],$$
$$\left[a_2 = \left[\frac{671}{512}, \frac{42945}{32768}\right], b_2 = \left[\frac{120614461341}{274877906944}, \frac{241228922683}{549755813888}\right]\right],$$
$$\left[a_2 = \left[-\frac{42945}{32768}, -\frac{671}{512}\right], b_2 = \left[\frac{120614461341}{274877906944}, \frac{241228922683}{549755813888}\right]\right]\Bigg],$$

where solutions of $F_{15} = F_{17} = 0$ are located. The theory and algorithms are described in [4, 19] and their references.

Second, we find a_2^*, b_2^* such that $F_{15} = F_{17} = F_{19} = 0$ and $F_{15} = F_{17} = F_J = 0$.

$$\begin{aligned}
&F_{151719}\\
&\quad = \text{Basis}([F_{15}, F_{17}, F_{19}], \text{plex}(a_2, b_2))\\
&\quad = [175b_2^8 - 800b_2^6 + 786b_2^4 + 928{b_2}^2 - 1409, -175b_2^6 + 415\, b_2^4 + 336a_2^2 + 71b_2^2 - 1287],\\
&F_{1517J}\\
&\quad = \text{Basis}([F_{15}, F_{17}, F_J], \text{plex}(a_2, b_2))\\
&\quad = [175b_2^8 - 800b_2^6 + 786b_2^4 + 928{b_2}^2 - 1409, -175b_2^6 + 415b_2^4 + 336a_2^2 + 71b_2^2 - 1287].
\end{aligned}$$

The existence of b_2^* satisfying $F_{15} = F_{17} = F_{19} = 0$ and $F_{15} = F_{17} = F_J = 0$ can be obtained from the Maple built-in command

$$\begin{aligned}
R_2 &= \text{realroot}\left(175\, b_2^8 - 800\, b_2^6 + 786\, b_2^4 + 928\, b_2^2 - 1409,\ \frac{1}{2^{15}}\right)\\
&= \left[\left[-\frac{1517583}{1048576}, -\frac{758791}{524288}\right], \left[\frac{758791}{524288}, \frac{1517583}{1048576}\right]\right].
\end{aligned}$$

where b_2^* falls into one of the above two intervals.

Third, we take the $(a_2^*, b_2^*) \in [[\frac{671}{512}, \frac{42945}{32768}], [-\frac{241228922683}{549755813888}, -\frac{120614461341}{274877906944}]]$ with $b_2^* \notin R_2$ such that $F_{15} = F_{17} = 0$ and $F_{19} \times F_J \neq 0$. Without loss of generality, we suppose k_{11} as a constant and it is easy to verify $A_{15} = A_{17} = 0$ and $A_{19} \times J(a_2, b_2) \neq 0$ when $(a_2, b_2) = (a_2^*, b_2^*)$.

The third step continues to prove $K \neq 0$ and $M_i \neq 0$, $i = 5, 6, \dots, 10$. It is proved by the same method in the second step and we omitted it here.

At last, we solve k_i^*, $i = 10, \dots, 1$ one by one according to the expressions of k_i in (4.1) and Appendix, which satisfy $(A_1, \dots, A_7, A_9, A_{11}, A_{13}) = (0, \dots, 0)$. For $(k_1^*, \dots, k_{10}^*, a_2^*, b_2^*)$, the results $(A_1, \dots, A_7, A_9, A_{11}, A_{13}, A_{15}, A_{17}) = (0, \dots, 0)$ and $A_{19} \times J(a_2^*, b_2^*) \neq 0$ hold. Here, $J(a_2^*, b_2^*) \neq 0$ means $J(k_1^*, \dots, k_{10}^*, a_2^*, b_2^*) \neq 0$. By Lemma 2.2, $I(h) = 0$ can have 12 simple positive roots near the origin.

System (1.6) can have at least 12 small-amplitude limit cycles bifurcating from the period annulus. □

Next we simulate an example to check numerically the existence of 12 limit cycles.

Let $a_1 = 1$, $b_1 = 3$. From the proof of Theorem 1.5, we know that $A_i = 0$, $i = 1, \dots, 6$ can be solved one by one using the 6 parameters: $k_1, \dots, k_6$. Here $A_{2i} \equiv 0$, $i = 4, \dots, 9$. Next $A_{2i-1} = 0$, $i = 4, \dots, 7$ can be solved one by one using the 4 parameters: $k_7, \dots, k_{10}$. At last, we solved $A_{15} = 0$ and $A_{17} = 0$ by a_2 and b_2. We select $a_2^* = 1.310568177\cdots$, $b_2^* = -4.387928541\cdots \times 10^{-1}$ and solve k_i^*, $i = 10, \dots, 1$ one by one, which satisfy $(A_1, \dots, A_7, A_9, A_{11}, A_{13}, A_{15}, A_{17}) = (0, \dots, 0)$ and $A_{19} \times J(k_1^*, \dots, k_{10}^*, a_2^*, b_2^*) \neq 0$.

Without loss of generality, suppose $k_{11} = 10000\pi$. By Lemma 2.2, it is enough to prove that

$$\mathbb{I}(h) = \sum_{i=1}^{6} A_i h^i + A_7 h^7 + A_9 h^9 + A_{11} h^{11} + A_{13} h^{13} + A_{15} h^{15} + A_{17} h^{17} + A_{19} h^{19}$$

has 12 distinct positive roots which are approximated amplitudes of the 12 limit cycles where the coefficients are functions with a_2, b_2 and k_j for $j = 1, \dots, 10$.

To prove that $\mathbb{I}(h)$ has 12 distinct positive roots, we do it in two steps.

First step is that we determine the perturbed values of the parameters a_2 and b_2 such that $h^{15}(A_{19} h^4 + A_{17} h^2 + A_{15})$ has 2 positive roots.

For sufficient small ϵ, suppose $a_2 = a_2^* + l_{11}\epsilon + l_{12}\epsilon^2$ and $b_2 = b_2^* + l_{21}\epsilon + l_{22}\epsilon^2$. Substituting them into A_{15} and A_{17} and expanding them in Taylor series, we have

$$A_{15} = 0 + E_{51}\epsilon + E_{52}\epsilon^2 + O(\epsilon^3), \quad A_{17} = 0 + E_{71}\epsilon + E_{72}\epsilon^2 + O(\epsilon^3).$$

Without loss of generality, we suppose $E_{51} = 0$, $E_{52} = -1$, $E_{71} = 10$, $E_{72} = 0$ and choose $\epsilon = 10^{-10}$. By a directly computation,

$$l_{11} = -4.343174836\cdots \times 10^{-2}, \quad l_{12} = 3.244135303\cdots \times 10^{-1}, \quad l_{21} = 1.688910736\dots,$$
$$l_{22} = -1.488390040\dots, \quad a_2 = 1.310568177\dots, \quad b_2 = -4.387928540\cdots \times 10^{-1},$$
$$A_{15} = -9.999999799\cdots \times 10^{-21}, \quad A_{17} = 1.000000000\cdots \times 10^{-9}, \quad A_{19} = -0.8049804384\cdots.$$

Then $h^{15}(A_{19} h^4 + A_{17} h^2 + A_{15})$ has two positive roots

$$h = 3.1751882638\cdots \times 10^{-6}, \quad 3.51024841218\cdots \times 10^{-5}.$$

The second step is select perturbed $k_i, i = 10, \dots, 1$ one by one. For example, we select $k_{10} = k_{10}(a_2, b_2) - 10^{-28}$ which yields $A_{13} = 9.686292015\cdots \times 10^{-33}$ and the truncated equation $h^{13}(A_{19} h^6 + A_{17} h^4 + A_{15} h^2 + A_{13})$ gives 3 positive roots

$$h = 1.042398565\cdots \times 10^{-6}, 2.997871756\cdots \times 10^{-6}, 3.510259794\cdots \times 10^{-5}.$$

Similarly, we can perturb $k_i, i = 9, \dots, 1$, as

$$\begin{aligned}
k_9 &= k_9(k_{10}, a_2, b_2) - 10^{-42},\\
k_8 &= k_8(k_9, k_{10}, a_2, b_2) + 10^{-56},\\
k_7 &= k_7(k_8, k_9, k_{10}, a_2, b_2) + 10^{-74},\\
k_6 &= k_6(k_7, k_8, k_9, k_{10}, a_2, b_2) + 10^{-83},\\
k_5 &= k_5(k_6, k_7, k_8, k_9, k_{10}, a_2, b_2) - 10^{-94},\\
k_4 &= k_4(k_5, k_6, k_7, k_8, k_9, k_{10}, a_2, b_2) - 10^{-105},\\
k_3 &= k_3(k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, a_2, b_2) + 10^{-117},
\end{aligned}$$

$$k_2 = k_2(k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, a_2, b_2) - 10^{-130},$$
$$k_1 = k_1(k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, a_2, b_2) + 10^{-144},$$

so that

$$A_{19} = -0.8049804384\ldots, \quad A_{17} = 1.000000000\cdots\times 10^{-9},$$
$$A_{15} = -9.999999799\cdots\times 10^{-21}, \quad A_{13} = 9.686292015\cdots\times 10^{-33},$$
$$A_{11} = -8.859349489\cdots\times 10^{-46}, \quad A_9 = 7.475397682\cdots\times 10^{-60},$$
$$A_7 = -1.782857998\cdots\times 10^{-75}, \quad A_6 = 1.676323576\cdots\times 10^{-84},$$
$$A_5 = -1.626379739\cdots\times 10^{-94}, \quad A_4 = 1.767944944\cdots\times 10^{-105},$$
$$A_3 = -1.333333333\cdots\times 10^{-117}, \quad A_2 = 1.570796326\cdots\times 10^{-130},$$
$$A_1 = -2.000000000\cdots\times 10^{-144},$$

and $\mathbb{I}(h)$ yields 12 positive roots

$$h = 1.4477383097\cdots\times 10^{-14}, \quad 1.2723535513\cdots\times 10^{-13},$$
$$6.6284774064\cdots\times 10^{-13}, \quad 1.1515969596\cdots\times 10^{-11},$$
$$9.5886045923\cdots\times 10^{-11}, \quad 8.3483520448\cdots\times 10^{-10},$$
$$1.5167863256\cdots\times 10^{-8}, \quad 9.5500039201\cdots\times 10^{-8},$$
$$3.0275184581\cdots\times 10^{-7}, \quad 9.8638310682\cdots\times 10^{-7},$$
$$2.9999761542\cdots\times 10^{-6}, \quad 3.5102597932\cdots\times 10^{-5}.$$

Meanwhile, we can verify $(k_1, \ldots, k_{10}, a_2, b_2)$ in a small neighbourhood of $(k_1^*, \ldots, k_{10}^*, a_2^*, b_2^*)$.

Remark 4.2 Naturally, we conjecture that 13 small-amplitude limit cycles can bifurcate from the period annulus of system $(1.6)_{\epsilon=0}$. Because of the huge calculation, we will study it in the future.

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APPENDIX

Proof of Lemma 4.1 According to Theorem 1.3, we have

$$I(h) = \int_{\Gamma_h^+} \frac{g^+(x,y)}{1+a_1x+a_2y}dx - \frac{f^+(x,y)}{1+a_1x+a_2y}dy + \int_{\Gamma_h^-} \frac{g^-(x,y)}{1+b_1x+b_2y}dx - \frac{f^-(x,y)}{1+b_1x+b_2y}dy, \qquad (1)$$

where

$$f^+(x,y) = \sum_{i+j=0}^{2} p_{ij}x^iy^j, \quad g^+(x,y) = \sum_{i+j=0}^{2} q_{ij}x^iy^j,$$

$$f^{-}(x,y)=\sum_{i+j=0}^{2}s_{ij}x^iy^j,\quad g^{-}(x,y)=\sum_{i+j=0}^{2}t_{ij}x^iy^j.$$

When $x>0$, $(\dot{x},\dot{y})=(y(1+a_1x+a_2y),-x(1+a_1x+a_2y))$, then $xdx=-ydy$.

$$\int_{\Gamma_h^+}\frac{p_{01}y}{1+a_1x+a_2y}dy=-\frac{p_{01}}{a_2}(I_1(h)+a_1I_2(h)+2I_{10}(h)),\tag{2}$$

$$\int_{\Gamma_h^+}\frac{p_{11}xy}{1+a_1x+a_2y}dy=-\frac{p_{11}}{a_2}(I_2(h)+a_1I_3(h)+\frac{\pi}{2}I_{11}(h)),\tag{3}$$

$$\int_{\Gamma_h^+}\frac{p_{02}y^2}{1+a_1x+a_2y}dy=\frac{p_{02}}{a_2^2}(I_1(h)+2a_1I_2(h)+a_1^2I_3(h)+2I_{10}(h)+\frac{\pi}{2}a_1I_{11}(h)),$$

$$\int_{\Gamma_h^+}\frac{q_{10}x}{1+a_1x+a_2y}dx=\frac{q_{10}}{a_2}(I_1(h)+a_1I_2(h)+2I_{10}(h)),$$

$$\int_{\Gamma_h^+}\frac{q_{20}x^2}{1+a_1x+a_2y}dx=\frac{q_{20}}{a_2}(I_2(h)+a_1I_3(h)+\frac{\pi}{2}I_{11}(h)),$$

$$\int_{\Gamma_h^+}\frac{q_{11}xy}{1+a_1x+a_2y}dx=-\frac{q_{11}}{a_2^2}(I_1(h)+2a_1I_2(h)+a_1^2I_3(h)+2I_{10}(h)+\frac{\pi}{2}a_1I_{11}(h)),$$

$$\int_{\Gamma_h^+}\frac{q_{01}y}{1+a_1x+a_2y}dx=-\frac{q_{01}}{a_2^2}(a_1I_1(h)+a_1^2I_2(h)+a_2I_4(h)+2a_1I_{10}(h)),$$

$$\begin{aligned}\int_{\Gamma_h^+}\frac{q_{02}y^2}{1+a_1x+a_2y}dx=&\frac{q_{02}}{a_2^3}(2a_1I_1(h)+3a_1^2I_2(h)+a_1^3I_3(h)+a_2I_4(h)+\frac{\pi}{2}(a_1^2+a_2^2)I_{11}(h)\\&+4a_1I_{10}(h)).\end{aligned}$$

When $x<0$, $(\dot{x},\dot{y})=(y(1+b_1x+b_2y),(x+1)(1+b_1x+b_2y))$, then $(x+1)dx=ydy$.

$$\int_{\Gamma_h^-}\frac{s_{11}xy}{1+b_1x+b_2y}dy=\frac{s_{11}}{b_2}(-I_6(h)-b_1I_7(h)+I_9(h)),\tag{4}$$

$$\int_{\Gamma_h^-}\frac{s_{01}y}{1+b_1x+b_2y}dy=\frac{s_{01}}{b_2}(-I_5(h)-b_1I_6(h)+2I_{10}(h)),$$

$$\int_{\Gamma_h^-}\frac{s_{02}y^2}{1+b_1x+b_2y}dy=\frac{s_{02}}{b_2^2}(I_5(h)+2b_1I_6(h)+b_1^2I_7(h)-b_1I_9(h)-2I_{10}(h)),$$

$$\int_{\Gamma_h^-}\frac{t_{10}x}{1+b_1x+b_2y}dx=\frac{t_{10}}{b_2}(-I_5(h)-b_1I_6(h)-b_2I_8(h)+2I_{10}(h)),$$

$$\int_{\Gamma_h^-}\frac{t_{01}y}{1+b_1x+b_2y}dx=\frac{t_{01}}{b_2^2}(b_1I_5(h)+b_1^2I_6(h)+(b_1-1)b_2I_8(h)-2b_1I_{10}(h)),$$

$$\int_{\Gamma_h^-}\frac{t_{20}x^2}{1+b_1x+b_2y}dx=\frac{t_{20}}{b_2}(I_5(h)+(b_1-1)I_6(h)-b_1I_7(h)+b_2I_8(h)+I_9(h)-2I_{10}(h)),$$

$$\begin{aligned}\int_{\Gamma_h^-}\frac{t_{11}xy}{1+b_1x+b_2y}dx=&\frac{t_{11}}{b_2^2}(-(b_1-1)I_5(h)-b_1(b_1-2)I_6(h)\\&+b_1^2I_7(h)-(b_1-1)b_2I_8(h)-b_1I_9(h)+2(b_1-1)I_{10}(h)),\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_h^-}\frac{t_{02}y^2}{1+b_1x+b_2y}dx=&\frac{t_{02}}{b_2^3}(b_1(b_1-2)I_5(h)+b_1^2(b_1-3)I_6(h)-b_1^3I_7(h)\\&+b_2(b_1-1)^2I_8(h)+(b_1^2-b_2^2)I_9(h)-2b_1(b_1-2)I_{10}(h)).\end{aligned}$$

From the above, the Abelian integral $I(h)$ can be expressed by $I_i(h)$, $i=1,2,\dots,11$. Thus, $I(h)$ has the form $I(h)=\sum_{i=1}^{11}k_iI_i(h)$. From (1), it is easy to verify that $p_{00},p_{10},p_{20},q_{00},s_{00},s_{10},s_{20}$ and t_{00} are only contained in k_i which imply the arbitrariness of k_i, $i=1,2,\dots,8$, respectively. From (4),

s_{11} only appears in k_6, k_7 and k_9 where we can obtain that k_9 is arbitrary. By the same way, k_{10} and k_{11} are also arbitrary which can be proved by (2) and (3) respectively. □

$$\begin{aligned}
k_7 = \frac{1}{4M_7}\Big(&(180a_2^8b_2^3 - 216a_2^6b_2^5 + 36a_2^4b_2^7 - 300a_2^6b_2^3 + 324a_2^4b_2^5 - 24a_2^2b_2^7 - 180a_2^4b_2^3 + 192a_2^2b_2^5 \\
&- 12b_2^7 - 20a_2^2b_2^3 + 20b_2^5)k_8 + (-90a_2^8b_2^2 + 54a_2^6b_2^4 + 36a_2^2b_2^8 - 540a_2^8 + 312a_2^6b_2^2 + 324a_2^2b_2^6 \\
&+ 12b_2^8 + 1116a_2^6 + 18a_2^4b_2^2 - 234a_2^2b_2^4 + 108b_2^6 + 108a_2^4 - 140a_2^2b_2^2 - 76b_2^4 - 108a_2^2 - 36b_2^2)k_9 \\
&+ (90a_2^8b_2^4 - 108a_2^6b_2^6 + 18a_2^4b_2^8 - 45a_2^8b_2^2 - 123a_2^6b_2^4 + 162a_2^4b_2^6 + 6a_2^2b_2^8 + 120a_2^6b_2^2 - 132a_2^4b_2^4 \\
&+ 66a_2^2b_2^6 - 45a_2^4b_2^2 - 27a_2^2b_2^4 - 30a_2^2b_2^2)k_{10} + (48a_2^{12} - 96a_2^{10}b_2^2 + 96a_2^8b_2^4 - 96a_2^6b_2^6 + 48a_2^4b_2^8 \\
&- 672a_2^{10} + 280a_2^8b_2^2 - 376a_2^6b_2^4 + 816a_2^4b_2^6 - 48a_2^2b_2^8 + 1200a_2^8 + 240a_2^6b_2^2 + 192a_2^4b_2^4 - 528a_2^2b_2^6 \\
&- 600a_2^4b_2^2 + 216a_2^2b_2^4 + 240a_2^2b_2^2)\frac{k_{11}}{\pi}\Big),
\end{aligned}$$

$$\begin{aligned}
k_8 = -\frac{1}{4M_8}\Big(&(-420a_2^{10}b_2^4 + 360a_2^8b_2^6 - 36a_2^6b_2^8 + 96a_2^2b_2^{12} - 210a_2^{10}b_2^2 + 1210a_2^8b_2^4 - 636a_2^6b_2^6 \\
&- 108a_2^2b_2^10 + 32b_2^{12} + 650a_2^8b_2^2 - 630a_2^6b_2^4 - 72a_2^4b_2^6 - 140a_2^2b_2^8 - 36b_2^{10} - 426a_2^6b_2^2 - 210a_2^4b_2^4 \\
&+ 72a_2^2b_2^6 - 48b_2^8 - 114a_2^4b_2^2 + 50a_2^2b_2^4 + 4b_2^6 + 36a_2^2b_2^2)k_9 + (1050a_2^{10}b_2^6 - 1260a_2^8b_2^8 + 162a_2^6b_2^{10} \\
&+ 48a_2^4b_2^{12} - 630a_2^{10}b_2^4 - 1450a_2^8b_2^6 + 2118a_2^6b_2^8 - 54a_2^4b_2^{10} + 16a_2^2b_2^{12} - 315a_2^{10}b_2^2 + 1725a_2^8b_2^4 \\
&- 1410a_2^6b_2^6 + 180a_2^4b_2^8 - 36a_2^2b_2^{10} + 975a_2^8b_2^2 - 759a_2^6b_2^4 + 216a_2^4b_2^6 - 222a_2^2b_2^8 - 639a_2^6b_2^2 \\
&- 297a_2^4b_2^4 + 162a_2^2b_2^6 - 171a_2^4b_2^2 + 57a_2^2b_2^4 + 54a_2^2b_2^2)k_{10} + (240a_2^{16} - 160a_2^{14}b_2^2 - 432a_2^{12}b_2^4 \\
&+ 784a_2^{10}b_2^6 - 704a_2^8b_2^8 + 144a_2^6b_2^{10} + 128a_2^4b_2^{12} - 1920a_2^{14} + 480a_2^{12}b_2^2 + 1104a_2^{10}b_2^4 - 4432a_2^8b_2^6 \\
&+ 5616a_2^6b_2^8 - 720a_2^4b_2^{10} - 128a_2^2b_2^{12} + 4992a_2^{12} - 312a_2^{10}b_2^2 + 1800a_2^8b_2^4 + 4080a_2^6b_2^6 - 8544a_2^4b_2^8 \\
&+ 288a_2^2b_2^{10} - 5184a_2^{10} + 1224a_2^8b_2^2 - 6408a_2^6b_2^4 + 3456a_2^4b_2^6 + 1776a_2^2b_2^8 + 1872a_2^8 - 3768a_2^6b_2^2 \\
&+ 4200a_2^4b_2^4 - 1296a_2^2b_2^6 + 3096a_2^4b_2^2 - 456a_2^2b_2^4 - 432a_2^2b_2^2)\frac{k_{11}}{\pi}\Big),
\end{aligned}$$

$$\begin{aligned}
k_9 = -\frac{a_2^2}{2b_2^2M_9}\Big(&(7560a_2^8b_2^8 - 5040a_2^6b_2^{10} + 360a_2^4b_2^{12} + 8505a_2^8b_2^6 - 19845a_2^6b_2^8 + 7680a_2^4b_2^{10} \\
&- 240a_2^2b_2^{12} - 43470a_2^8b_2^4 - 13545a_2^6b_2^6 + 7575a_2^4b_2^8 + 1968a_2^2b_2^{10} - 120b_2^{12} + 8505a_2^8b_2^2 \\
&+ 134925a_2^6b_2^4 - 2475a_2^4b_2^6 + 2865a_2^2b_2^8 - 384b_2^{10} - 29295a_2^6b_2^2 - 89875a_2^4b_2^4 + 2525a_2^2b_2^6 - 715b_2^8 \\
&+ 25335a_2^4b_2^2 - 23961a_2^2b_2^4 + 510b_2^6 + 1323a_2^2b_2^2 + 7533b_2^4 - 3564b_2^2)k_{10} + (14000a_2^{16}b_2^2 \\
&- 33040a_2^{14}b_2^4 + 14480a_2^{12}b_2^6 + 14800a_2^{10}b_2^8 - 10880a_2^8b_2^{10} + 640a_2^6b_2^{12} - 5600a_2^{16} \\
&- 79520a_2^{14}b_2^2 + 226720a_2^{12}b_2^4 - 81120a_2^{10}b_2^6 - 29440a_2^8b_2^8 + 14720a_2^6b_2^{10} + 320a_2^4b_2^{12} \\
&+ 43680a_2^{14} + 90000a_2^{12}b_2^2 - 559920a_2^{10}b_2^4 + 150600a_2^8b_2^6 - 23400a_2^6b_2^8 + 21120a_2^4b_2^{10} \\
&- 1920a_2^2b_2^{12} - 127200a_2^{12} + 142240a_2^{10}b_2^2 + 594384a_2^8b_2^4 - 116472a_2^6b_2^6 + 70952a_2^4b_2^8 \\
&- 28032a_2^2b_2^{10} + 960b_2^{12} + 184640a_2^{10} - 293608a_2^8b_2^2 - 60264a_2^6b_2^4 + 42520a_2^4b_2^6 - 45800a_2^2b_2^8 \\
&+ 3072b_2^{10} - 153408a_2^8 + 62040a_2^6b_2^2 - 519880a_2^4b_2^4 - 3880a_2^2b_2^6 + 5720b_2^8 + 76896a_2^6 \\
&+ 157896a_2^4b_2^2 + 432744a_2^2b_2^4 - 4080b_2^6 - 19008a_2^4 - 124632a_2^2b_2^2 - 60264b_2^4 + 28512b_2^2)\frac{k_{11}}{\pi}\Big),
\end{aligned}$$

$$\begin{aligned}
k_{10} = \frac{k_{11}}{\pi M_{10}}(&5880a_2^{20}b_2^6 - 19320a_2^{18}b_2^8 + 22960a_2^{16}b_2^{10} - 11760a_2^{14}b_2^{12} + 2520a_2^{12}b_2^{14} - 280a_2^{10}b_2^{16} \\
&- 6615a_2^{20}b_2^4 - 22155a_2^{18}b_2^6 + 113995a_2^{16}b_2^8 - 143465a_2^{14}b_2^{10} + 67200a_2^{12}b_2^{12} - 9520a_2^{10}b_2^{14} \\
&+ 560a_2^8b_2^{16} - 10290a_2^{20}b_2^2 + 78330a_2^{18}b_2^4 - 63490a_2^{16}b_2^6 - 187530a_2^{14}b_2^8 + 303940a_2^{12}b_2^{10}
\end{aligned}$$

$$
\begin{aligned}
&- 136360a_2^{10}b_2^{12} + 12600a_2^8b_2^{14} - 560a_2^6b_2^{16} + 2205a_2^{20} + 86625a_2^{18}b_2^2 - 381619a_2^{16}b_2^4 \\
&+ 480529a_2^{14}b_2^6 - 26726a_2^{12}b_2^8 - 246158a_2^{10}b_2^{10} + 125048a_2^8b_2^{12} - 16552a_2^6b_2^{14} + 2216a_2^4b_2^{16} \\
&- 21420a_2^{18} - 274512a_2^{16}b_2^2 + 1001868a_2^{14}b_2^4 - 1076212a_2^{12}b_2^6 + 321832a_2^{10}b_2^8 - 3172a_2^8b_2^{10} \\
&- 24704a_2^6b_2^{12} + 23336a_2^4b_2^{14} - 2984a_2^2b_2^{16} + 85638a_2^{16} + 356902a_2^{14}b_2^2 - 1527973a_2^{12}b_2^4 \\
&+ 1218855a_2^{10}b_2^6 - 207109a_2^8b_2^8 + 128263a_2^6b_2^{10} - 79216a_2^4b_2^{12} - 11896a_2^2b_2^{14} + 536b_2^{16} \\
&- 179844a_2^{14} + 4266a_2^{12}b_2^2 + 1343986a_2^{10}b_2^4 - 747562a_2^8b_2^6 - 132942a_2^6b_2^8 - 25648a_2^4b_2^{10} \\
&+ 72472a_2^2b_2^{12} + 792b_2^{14} + 208605a_2^{12} - 508455a_2^{10}b_2^2 - 627105a_2^8b_2^4 + 271395a_2^6b_2^6 \\
&+ 184520a_2^4b_2^8 - 62680a_2^2b_2^{10} - 10120b_2^{12} - 126360a_2^{10} + 501344a_2^8b_2^2 + 119128a_2^6b_2^4 \\
&- 105208a_2^4b_2^6 - 29552a_2^2b_2^8 + 15720b_2^{10} + 31176a_2^8 - 155880a_2^6b_2^2 + 34640a_2^2b_2^6 - 6928b_2^8),
\end{aligned}
$$

$$
\begin{aligned}
M_7 &= (3a_2^2 + 1)(15a_2^6 - 3a_2^4b_2^2 - 2a_2^2b_2^4 - 10b_2^6 - 36a_2^4 + 6b_2^4 + 9a_2^2 + 3b_2^2), \\
M_8 &= b_2^3(3a_2^2 + 1)(a_2 - b_2)^2(175a_2^6b_2^2 - 35a_2^4b_2^4 - 8a_2^2b_2^6 - 70a_2^6 - 400a_2^4b_2^2 + 26a_2^2b_2^4 + 8b_2^6 \\
&\quad + 195a_2^4 + 123a_2^2b_2^2 + 9b_2^4 - 114a_2^2 - 46b_2^2 + 9), \\
M_9 &= (3a_2^2 + 1)(2520a_2^8b_2^6 - 1680a_2^6b_2^8 + 120a_2^4b_2^{10} - 2835a_2^8b_2^4 - 5425a_2^6b_2^6 + 3200a_2^4b_2^8 \\
&\quad - 40a_2^2b_2^{10} - 4410a_2^8b_2^2 + 10360a_2^6b_2^4 + 910a_2^4b_2^6 - 800a_2^2b_2^8 - 80b_2^{10} + 945a_2^8 + 15015a_2^6b_2^2 \\
&\quad - 9675a_2^4b_2^4 + 891a_2^2b_2^6 + 336b_2^8 - 3570a_2^6 - 13740a_2^4b_2^2 + 1110a_2^2b_2^4 - 432b_2^6 + 4005a_2^4 \\
&\quad + 1983a_2^2b_2^2 + 176b_2^4 - 1188a_2^2), \\
M_{10} &= b_2^6(b_2 - 1)^2(b_2 + 1)^2(3a_2^2 + 1)(1155a_2^8b_2^2 - 1050a_2^6b_2^4 + 175a_2^4b_2^6 - 2541a_2^8 - 2226a_2^6b_2^2 \\
&\quad + 2849a_2^4b_2^4 - 306a_2^2b_2^6 + 10080a_2^6 - 1715a_2^4b_2^2 - 2000a_2^2b_2^4 + 67b_2^6 - 12285a_2^4 + 4100a_2^2b_2^2 \\
&\quad + 233b_2^4 + 4330a_2^2 - 866b_2^2), \\
F_J &= (11816106427125\,b_2^6 - 77986302419025\,b_2^4 + 172110030187095\,b_2^2 - 127006263939555)a_2^{30} \\
&\quad + (-32613109278750\,b_2^8 - 10235318818725\,b_2^6 + 1018959116580405\,b_2^4 \\
&\quad - 2959249854936375\,b_2^2 + 2451611942304525)a_2^{28} + (38757191232375\,b_2^{10} \\
&\quad + 322853906355750\,b_2^8 - 1317393970993485\,b_2^6 - 4908574564740693\,b_2^4 \\
&\quad + 22590408982071546\,b_2^2 - 21475552945687533)a_2^{26} + (-25600895397000\,b_2^{12} \\
&\quad - 466944916266675\,b_2^{10} - 783457952341680\,b_2^8 + 11584179718584165\,b_2^6 \\
&\quad + 6074054329114233\,b_2^4 - 100023539590817766\,b_2^2 + 112638986308670283)a_2^{24} \\
&\quad + (10139029621875\,b_2^{14} + 319129297398450\,b_2^{12} + 2245676316036615\,b_2^{10} \\
&\quad - 3492578501189316\,b_2^8 - 46894464240611526\,b_2^6 + 42038364363396093\,b_2^4 \\
&\quad + 279956015685334296\,b_2^2 - 392752565515717407)a_2^{22} + (-2409885843750\,b_2^{16} \\
&\quad - 123859317980475\,b_2^{14} - 1674968685562620\,b_2^{12} - 4952279746282491\,b_2^{10} \\
&\quad + 28822687702506684\,b_2^8 + 104442422275939734\,b_2^6 - 241151938125016527\,b_2^4 \\
&\quad - 498630966689040444\,b_2^2 + 954575438198020569)a_2^{20} + (321848690625\,b_2^{18} \\
&\quad + 28137226171950\,b_2^{16} + 639627421121325\,b_2^{14} + 4733681376992202\,b_2^{12} \\
&\quad + 1852774753200354\,b_2^{10} - 89286976360107198\,b_2^8 - 114697002396543138\,b_2^6 \\
&\quad + 625549477728238857\,b_2^4 + 509977348600290390\,b_2^2 - 1645307783712569727)a_2^{18} \\
&\quad + (-19100812500\,b_2^{20} - 3577414546125\,b_2^{18} - 136808148084300\,b_2^{16} \\
&\quad - 1807163045677773\,b_2^{14} - 7468906464133776\,b_2^{12} + 16481868407473734\,b_2^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ 1552661513710309998\, b_2^8 - 13363986212646462\, b_2^6 - 9681863265882506643\, b_2^4 \\
&- 1255697331521666658\, b_2^2 + 1997099352163026585) a_2^{16} + (212935912875\, b_2^{20} \\
&+ 16033423232610\, b_2^{18} + 357761668416795\, b_2^{16} + 2996268038670294\, b_2^{14} \\
&+ 5522018967967770\, b_2^{12} - 42739351147182006\, b_2^{10} - 156621163336510266\, b_2^8 \\
&+ 226322819257448529\, b_2^6 + 910503172340007984\, b_2^4 - 407556217314234639\, b_2^2 \\
&- 1641259490197938066) a_2^{14} + (-2116800000\, b_2^{22} - 890818611165\, b_2^{20} \\
&- 37136512352454\, b_2^{18} - 536887604767209\, b_2^{16} - 2839391849468118\, b_2^{14} \\
&+ 943248558997086\, b_2^{12} + 50032693437258870\, b_2^{10} + 76651543146954456\, b_2^8 \\
&- 316586849094069969\, b_2^6 - 446675367561873258\, b_2^4 + 587583438358166787\, b_2^2 \\
&+ 812223140717303814) a_2^{12} + (12018182400\, b_2^{22} + 1782903465591\, b_2^{20} \\
&+ 46375157456916\, b_2^{18} + 443322839068371\, b_2^{16} + 1235747352295218\, b_2^{14} \\
&- 5041928856213198\, b_2^{12} - 28523718231944361\, b_2^{10} + 5761061398267392\, b_2^8 \\
&+ 197956710060749247\, b_2^6 + 6185241010918992\, b_2^4 - 323332689361321488\, b_2^2 \\
&- 133817837749508160) a_2^{10} + (-38707200\, b_2^{24} - 22878173952\, b_2^{22} - 1696471416009\, b_2^{20} \\
&- 27560872480032\, b_2^{18} - 147079140569817\, b_2^{16} + 146772635654238\, b_2^{14} \\
&+ 3263798474444070\, b_2^{12} + 2919041379309957\, b_2^{10} - 25853716083733710\, b_2^8 \\
&- 30084679962464535\, b_2^6 + 114166077763754910\, b_2^4 + 12829753931213400\, b_2^2 \\
&- 92870765832353040) a_2^8 + (82169856\, b_2^{24} + 14485731840\, b_2^{22} + 492157105089\, b_2^{20} \\
&+ 1399510893902\, b_2^{18} - 45828708191743\, b_2^{16} - 365107743837017\, b_2^{14} - 126064506870772\, b_2^{12} \\
&+ 5568089224849855\, b_2^{10} + 8963410421857910\, b_2^8 - 34881074804678000\, b_2^6 \\
&- 45907614911525920\, b_2^4 + 74948349766797280\, b_2^2 + 59422465363367680) a_2^6 \\
&+ (-11907072\, b_2^{24} + 2171022336\, b_2^{22} + 298005901033\, b_2^{20} + 6707170710246\, b_2^{18} \\
&+ 53677234927051\, b_2^{16} + 110981058654881\, b_2^{14} - 676737148673066\, b_2^{12} \\
&- 2971126312895659\, b_2^{10} + 1383287697028450\, b_2^8 + 22013544628131920\, b_2^6 \\
&- 3101518573693760\, b_2^4 - 35163323813616000\, b_2^2 - 10803337492188800) a_2^4 \\
&+ (-20090880\, b_2^{24} - 4143796992\, b_2^{22} - 216924008483\, b_2^{20} - 3002590146821\, b_2^{18} \\
&- 15478441374301\, b_2^{16} - 1019120151887\, b_2^{14} + 265257687790508\, b_2^{12} \\
&+ 541132108023616\, b_2^{10} - 1299050882589280\, b_2^8 - 4866576728219360\, b_2^6 \\
&+ 5352468905154560\, b_2^4 + 5183582022790400\, b_2^2) a_2^2 + 7409664\, b_2^{24} \\
&+ 1039502592\, b_2^{22} + 41174368977\, b_2^{20} + 430794171213\, b_2^{18} + 1532504597673\, b_2^{16} \\
&- 3003136917537\, b_2^{14} - 32501401212102\, b_2^{12} - 26168155997160\, b_2^{10} \\
&+ 197103344224080\, b_2^8 + 369680430111360\, b_2^6 - 878680574736000\, b_2^4.
\end{aligned}
$$