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Chaotic Dynamics of Monotone Twist Maps

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Abstract For a monotone twist map, under certain non-degenerate condition, we showed the existence of infinitely many homoclinic and heteroclinic orbits between two periodic neighboring minimal orbits with the same rotation number, which indicates chaotic dynamics. Our results also apply to geodesics of smooth Riemannian metrics on the two-dimension torus.

Keywords Monotone twist maps, homoclinic and heteroclinic orbits, topological entropy

MR(2010) Subject Classification 37C29, 37E40, 37B40

1 Introduction

The study of area preserving maps is a classical topic in dynamical system and goes back at least to the works of Poincaré and Birkhoff. These maps usually arise as Poincaré maps of Hamiltonian systems with two degrees or one and half degrees of freedom. Among them a special class called *monotone twist maps* has been studied intensively (see [12–16] and [5]). Here we define a monotone twist map as an orientation preserving C^1-diffeomorphism $f : S^1 \times [a,b] \to S^1 \times [a,b]$ of an annulus, which preserves each end of the annulus and admits a lift $\tilde{f} : \mathbb{R} \times [a,b] \to \mathbb{R} \times [a,b]$; $\tilde{f}(x_0, y_0) = (x_1, y_1)$ with the following properties: (a) $\tilde{f}$ preserves area; (b) twist condition: $\frac{\partial x_1}{\partial y_0} > 0$.

For a monotone twist map f as above with $\tilde{f}(x_0, y_0) = (x_1, y_1)$, there is a C^2 *generating function* $h : \mathbb{R}^2 \to \mathbb{R}$ (up to a constant) given by

$$dh(x_0, x_1) = y_1 dx_1 - y_0 dx_0.$$

This is equivalent to

$$y_0 = -\partial_1 h(x_0, x_1); \quad y_1 = \partial_2 h(x_0, x_1).$$

Such an h is usually referred as the *variational principle* associated with f, as it allows us study the dynamics of f using variational method and this is usually known as *Aubry–Mather theory*. We give a brief introduction in the following, for details see [5, 16] or [17].

Definition 1.1 *A continuous function $h : \mathbb{R}^2 \to \mathbb{R}$ will be called a* ***variational principle*** *if it satisfies the following conditions*:

- $H_1 : h(\xi + 1, \xi' + 1) = h(\xi, \xi')$ *for all* $\xi, \xi' \in \mathbb{R}$;
- $H_2 : \lim_{|\zeta| \to +\infty} h(\xi, \xi + \zeta) = +\infty$, *uniformly in* ξ;

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• *H_3 : $h(\xi,\zeta')+h(\zeta,\xi')>h(\xi,\xi')+h(\zeta,\zeta')$, if $\xi<\zeta$ and $\xi'<\zeta'$;*

• *H_4 : If $(\xi,\eta,\zeta)\neq(\xi',\eta,\zeta')$ both are minimal, i.e., $h(\xi,\eta)+h(\eta,\zeta)\leq h(\xi,\eta')+h(\eta',\zeta)$ and $h(\xi',\eta)+h(\eta,\zeta')\leq h(\xi',\eta')+h(\eta',\zeta')$ for any $\eta'\in\mathbb{R}$, then $(\xi-\xi')(\zeta-\zeta')<0$.*

• *H_5 : There exists a positive continuous function ρ on $\mathbb{R}^2$ such that*

$$h(\xi,\zeta')+h(\zeta,\xi')-h(\xi,\xi')-h(\zeta,\zeta')>\int_\xi^\zeta\int_{\xi'}^{\zeta'}\rho$$

if $\xi<\zeta$ and $\xi'<\zeta'$;

• *H_6 : There is a $\theta>0$, such that $\xi\mapsto\theta\xi^2/2-h(\xi,\xi')$ is convex, for any ξ', and $\xi'\mapsto\theta{\xi'}^2/2-h(\xi,\xi')$ is convex, for any ξ.*

Remark 1.2 Bangert [5] only required conditions H_1 to H_4. Conditions H_5 and H_6 were added by Mather [14] and will be needed for our results. One should notice that H_5 implies H_3, and H_5 and H_6 together imply H_4, for details see [14].

Remark 1.3 Although for a monotone twist map, the associated variational principle is usually C^2. We assume weaker regularity condition in the above definition, so that our result can be applied to finite composition of monotone twist maps as well. For these maps the associated variational principle may not be C^1 (for details see [14] or [16]).

For any $\{n_0<n_1\}\subset\mathbb{Z}\cup\{\pm\infty\}$, let $\prod_{i=n_0}^{n_1}\mathbb{R}$ be the configuration space with product topology. We extend the domain of h to all finite configuration spaces $\prod_{i=n_0}^{n_1}\mathbb{R}$, i.e., both n_0 and n_1 are finite, by setting

$$h(x_{n_0},\dots,x_{n_1})=\sum_{i=n_0}^{n_1-1}h(x_i,x_{i+1}),\quad\forall\{x_i\}_{i=n_0}^{n_1}\in\prod_{i=n_0}^{n_1}\mathbb{R}.$$

Definition 1.4 *A finite configuration $\{x_i\}_{i=n_0}^{n_1}$ will be called **minimal**, if*

$$h(x_{n_0},\dots,x_{n_1})\leq h(x^*_{n_0},\dots,x^*_{n_1}),$$

for any $\{x^*_i\}_{i=n_0}^{n_1}\in\prod_{i=n_0}^{n_1}\mathbb{R}$ with $x^*_{n_0}=x_{n_0}$ and $x^*_{n_1}=x_{n_1}$.

*A configuration, finite or not, will be called **minimal**, if any of its finite sub-configuration is minimal.*

*We say a configuration is **locally minimal**, if there is an open neighborhood of it in the corresponding configuration space, such that the above are true.*

Although we do not assume h is C^1, when $x=\{x_i\}_{i=n_0}^{n_1}$ is a locally minimal configuration, for any $n_0<i<n_1$, both $\partial_2h(x_{i-1},x_i)$ and $\partial_1h(x_i,x_{i+1})$ exist and satisfy

$$\partial_2h(x_{i-1},x_i)+\partial_1h(x_i,x_{i+1})=0,$$

as showed by Mather [14].

Now if h is a variational principle of a monotone twist map or finite composition of monotone twist maps f and $x=\{x_i\}_{i=n_0}^{n_1}$ is a locally minimal configuration, $\{(x_i,y_i)\}_{i=n_0+1}^{n_1-1}$ with $y_i=-\partial_1h(x_i,x_{i+1})=\partial_2h(x_{i-1},x_i)$ is an orbit of $\tilde{f}$. We will call locally minimal configurations *stationary configurations*. Now we can transfer the study of a monotone twist map to the study of stationary configurations of a variational principle.

Given any two infinite configurations x,y, we write $x(\pm\infty)=y$, if $\lim_{i\to\pm\infty}x_i-y_i=0$, or $x(\pm\infty)=u$, if y is a constant configuration with $y_i\equiv u\in\mathbb{R}$, $\forall i\in\mathbb{Z}$.

If $x \in \prod_{-\infty}^{+\infty} \mathbb{R}$ satisfies $x(-\infty) = y^1$ and $x(+\infty) = y^2$, we say x is a *heteroclinic configuration* (between y^1 and y^2), when $y^1 \neq y^2$, or a *homoclinic configuration* (between y^1 and y^2), if $y^1 = y^2$. Moreover if such an x is a stationary configuration, we will call it a *heteroclinic connection* or *homoclinic connection*.

Definition 1.5 *$\alpha \in \mathbb{R}$ will be called the* rotation number *of $x = \{x_i\}_{i\in\mathbb{Z}}$, if*

$$\lim_{|i|\to+\infty} x_i/i \text{ exists and equal to } \alpha.$$

Remark 1.6 In general it is possible the above sequence may have different limits as i goes to positive or negative infinity. However in this paper we will only consider those x's, such that the limits are the same.

Given a variational principle h, every minimal configuration has a rotation number. For any rotation number $\alpha \in \mathbb{R}$, there is a non-empty compact set $\mathcal{M}_\alpha$ consisting of minimal configurations with rotation number α.

For any $(m, n) \in \mathbb{Z}^2$, define an operator $T_{(m,n)}$ on $\prod_{-\infty}^{+\infty} \mathbb{R}$ by setting

$$(T_{(m,n)}x)_i = x_{i+n} - m, \quad \forall i.$$

If $T_{(m,n)}x = x$, we say x is (m, n)-periodic.

Given a rational number $\alpha \in \mathbb{Q}$, there is a unique pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}^+$ with p and q relatively prime and $\alpha = p/q$ (when $\alpha = 0$, we assume $(p, q) = (0, 1)$). From now on when we say a rotation number $\alpha = p/q \in \mathbb{Q}$, (p, q) will always be such a unique pair of integers.

By the Aubry–Mather theory, if $\alpha = p/q \in \mathbb{Q}$ and $x \in \mathcal{M}_\alpha$ is (kp, kq)-periodic, for some $k \in \mathbb{Z}^+$, it must be (p, q)-periodic. We define

$$\mathcal{M}_\alpha^{\text{per}} := \{x \in \mathcal{M}_\alpha : T_{(p,q)}x = x\} \neq \emptyset.$$

This is an ordered set, i.e., if $x \neq y \in \mathcal{M}_\alpha^{\text{per}}$, either $x < y$ ($x_i < y_i$, $\forall i \in \mathbb{Z}$) or $x > y$ ($x_i > y_i$, $\forall i \in \mathbb{Z}$).

Given a pair of (p, q)-periodic minimal configurations $x^0 \neq x^1$. Without loss of generality let's assume $x^0 < x^1$, we say they are *neighboring*, if there is no other $x \in \mathcal{M}_\alpha^{\text{per}}$ satisfying $x^0 < x < x^1$, and we define

$$\mathcal{M}_\alpha^+(x^0, x^1) := \{x \in \mathcal{M}_\alpha : x(-\infty) = x^0 \text{ and } x(+\infty) = x^1\},$$

$$\mathcal{M}_\alpha^-(x^0, x^1) := \{x \in \mathcal{M}_\alpha : x(-\infty) = x^1 \text{ and } x(+\infty) = x^0\}.$$

Again by Aubry–Mather theory, both of the above sets are non-empty.

Let $\mathcal{M}_\alpha^+$ be the union of $\mathcal{M}_\alpha^+(x^0, x^1)$ over all pairs of neighboring configurations in $\mathcal{M}_\alpha^{\text{per}}$, and $\mathcal{M}_\alpha^-$ similarly. Then $\mathcal{M}_\alpha$ is a disjoint union of $\mathcal{M}_\alpha^{\text{per}}, \mathcal{M}_\alpha^+$ and $\mathcal{M}_\alpha^-$. Both $\mathcal{M}_\alpha^{\text{per}} \cup \mathcal{M}_\alpha^+$ and $\mathcal{M}_\alpha^{\text{per}} \cup \mathcal{M}_\alpha^-$ are ordered sets.

After the above review, we are now ready to state our main results.

Theorem 1.7 *Given a rational number $\alpha \in \mathbb{Q}$ and a pair of neighboring minimal configurations $x^0 < x^1 \in \mathcal{M}_\alpha^{\text{per}}$. Let*

$$I_\alpha^+(x^0, x^1) := \{x_0 : x = \{x_i\}_{i\in\mathbb{Z}} \in \mathcal{M}_\alpha^+(x^0, x^1)\};$$

$$I_\alpha^-(x^0, x^1) := \{x_0 : x = \{x_i\}_{i\in\mathbb{Z}} \in \mathcal{M}_\alpha^-(x^0, x^1)\},$$

and (x_0^0, x_0^1) *the open interval between* x_0^0 *and* x_0^1*. If*

$$I_\alpha^+(x^0, x^1) \neq (x_0^0, x_0^1) \quad \text{and} \quad I_\alpha^-(x^0, x^1) \neq (x_0^0, x_0^1), \tag{gap}$$

then for every $\hat{\delta} > 0$ *small enough, there is an* $m = m(\hat{\delta}) \in \mathbb{N}$ *such that for every sequence of integers* $q = \{q_i \in \mathbb{Z}\}_{i=-\infty}^{+\infty}$ *with* $q_{i+1} - q_i \geq 4m$ *and for every* $j, k \in \mathbb{Z}$ *with* $j < k$*, there is a homoclinic or heteroclinic connection* x *satisfying*

1. $x_i^0 < x_i < x_i^1$ *for all* $i \in \mathbb{Z}$;
2. $|x_{q_i-m} - x_{q_i-m}^i| \leq \hat{\delta}$ *and* $|x_{q_i+m} - x_{q_i+m}^{i+1}| \leq \hat{\delta}$ *for all* $i = j, \ldots, k$;
3. $x(-\infty) = x^j$ *and* $x(+\infty) = x^{k+1}$.

For any $j \in \mathbb{Z}$*,* $x^j = x^0$*, if* j *is even and* $x^j = x^1$*, if* j *is odd.*

We will try to explain the meaning of the above theorem. Let $f : S^1 \times [a, b] \to S^1 \times [a, b]$ be the corresponding monotone twist map. Then $O^i = \{(\pi(x_j^i), y_j = \partial_2 h(x_{j-1}^i, x_j^i))\}_{j\in\mathbb{Z}}$, $i = 0, 1$ are two periodic orbits of f, where $\pi : \mathbb{R} \to S^1$. Theorem 1.7 shows the existence of infinitely many multibump homoclinic and heteroclinic orbits between O^0 and O^1.

Under some classical conditions that O^i, $i = 0, 1$, are hyperbolic and the associated stable and unstable manifolds intersect transversally, it is possible to get the above result using geometric and perturbative approaches. Meanwhile in our result, these conditions are not required. The existence of these multibump homoclinic and heterclinic orbits indicates chaotic dynamics of these maps. More precisely when the (gap) condition holds, Theorem 1.7 implies the topological entropy of the map must be positive. To see this, consider the sequence $\{4im\}_{i=-\infty}^{\infty}$, by choosing $q = \{q_i\}_{i=-\infty}^{+\infty}$ as different subsequences of the previous sequence, Theorem 1.7 can give us different stationary configurations which satisfies arbitrary choices between the following two conditions,

$$|x_{2m+4im} - x_{2m+4im}^0| \leq \hat{\delta}, \quad |x_{2m+4im} - x_{2m+4im}^1| \leq \hat{\delta}$$

for any $j < i < k$.

Since x^i, $i = 0, 1$, correspond to the two periodic orbits O^i, this means, for any $j < k$, we can find an initial condition such that whether its f^{2m+4im}'s $(j < i < k)$ image is close to O^0 or O^1 can be given arbitrarily. As a result, the number of orbit segments distinguishable with arbitrary fine but finite precision grows exponentially, so the topological entropy of f is positive.

Using this we recover a result first obtained in [2].

Corollary 1.8 *Let* α_1 *be the rotation number of* $f|_{S^1\times\{a\}}$ *and* α_2 *the rotation number of* $f|_{S^1\times\{b\}}$*. If the topological entropy of* f *vanishes,* f *must have a* (*homotopically*) *non-trivial invariant circle of rotation number* α*, for any* $\alpha \in (\alpha_1, \alpha_2)$.

Proof For any $\alpha \in (\alpha_1, \alpha_2) \cap \mathbb{Q}$, by the above explanation, the corresponding (gap) condition in Theorem 1.7 can not hold, as otherwise the topological entropy is positive. Therefore $\mathcal{M}_\alpha$ must foliate the whole configuration space and the corresponding orbits of f form a (homotopically) non-trivial invariant circle in the cylinder with rotation number α.

Because the set of non-trivial invariant circles is closed and so is the set of rotation numbers for which such non-trivial invariant circles exist, see [2] and [11], for every irrational $\alpha \in (\alpha_1, \alpha_2)$, f must have a non-trivial invariant circle as well. □

Now we explain how to apply our results to the geodesics of smooth Riemannian metrics on the two-dimension torus. The idea of this approach goes back to the work of Morse [18] and Hedlund [10]. In [5] Bangert explained how the variational principle can be defined for a smooth Riemannian metric on $\mathbb{T}^2$ and the stationary configurations give rise to geodesics. Although only conditions H_1 to H_4 are verified in [5], It was claimed by Mather in [14] that such a variational principle also satisfies H_5 and H_6.

As a result we have the following result as in direct corollary of Theorem 1.7.

Corollary 1.9 *If the topological entropy of a geodesic flow on $\mathbb{T}^2$ vanishes, then for every $\alpha \in \mathbb{Q}$, minimal geodesics with rotation number α is a foliation of $\mathbb{T}^2$.*

Related results about geodesic flows on $\mathbb{T}^2$ with vanishing topological entropy can be found in [8, 9].

Furthermore we would like to mention an interesting paper [7] by Bolotin and Rabinowitz, where a similar variational method was applied directly to the geodesics on $\mathbb{T}^2$ to show the existence of chaotic geodesics. [7, Theorem 2.3] says that under certain geometric condition, there are infinitely many homoclinic and heteroclinic geodesics between two periodic neighboring minimal geodesics.

Although the geometric condition posted in [7] is stronger than our (gap) condition, their result is also stronger than ours. Namely by our result the geodesics have to spend large enough time between every bump, while in [7] this is not necessary. There are other interesting homoclinic and heteroclinic geodesics in their paper beyond the reach of our result.

Our proof uses a variational method similar to those used in [6] and [20], where existence of multibump homoclinic and heteroclinic orbits of one-dimensional periodic forced pendulum was proved in the same spirit.

The paper is organized as follows: in Section 2 we introduce a normalized function J, which is the base of our variational method; in Section 3, we give an alternative proof of the existence of minimal heteroclinic connections; in Section 4 the existence of infinitely many homoclinic and heteroclinic connections between a pair of periodic neighboring minimal configurations with rotation number 0 will be shown; in Section 5, we generalize the previous result to periodic neighboring minimal configurations with any rational rotation number; in the Appendix, we give the proofs of several technical lemmas.

2 Preliminary

In this section we define a normalized function J following [6] and [20]. It gives us a convenient way to determine the asymptotic behaviors of the configurations.

Fix an arbitrary variational principle $h : \mathbb{R}^2 \to \mathbb{R}$ for the rest of the section. We define an associated function $\bar{h} : \mathbb{R} \to \mathbb{R}$ as $\bar{h}(\xi) = h(\xi, \xi)$.

By Condition H_1, there exists a finite number $c = \min\{\bar{h}(\xi) : \xi \in \mathbb{R}\}$. If $\bar{h}(u) = c$, we say u a *minimizer* of $\bar{h}$. When there is no confusion u will also be used to represent the constant configuration $x = \{x_i = u\}_{i \in \mathbb{Z}}$. The following result is well known and a proof can be found in [5] or [17].

Proposition 2.1 *If $u \in \mathbb{R}$ is a minimizer of $\bar{h}$, then as a configuration $u \in \mathcal{M}_0^{\text{per}}$, i.e., it is a minimal configuration.*

Assume $u_0 < u_1$ are two different minimizers of $\bar{h}$ and $\bar{h}(u) > \bar{h}(u_0)$, $\forall u \in (u_0, u_1)$. We call u_0 and u_1 a pair of *neighboring minimizers.*

Definition 2.2 *Set* $U := \{u_0, u_1\}$. *For any positive integer* n, *we define*

$$X(n) := X(n; U) := \{x = \{x_i\}_{i=0}^n : x_i \in [u_0, u_1]\};$$
$$\hat{X}(n) := \hat{X}(n; U) := \{x = \{x_i\}_{i=0}^n : x \in X(n) \text{ and } x_0 = x_n\};$$
$$X := X(U) := \{x = \{x_i\}_{i=-\infty}^{+\infty} : x_i \in [u_0, u_1]\}.$$

Here $[u_0, u_1]$ *is the closed interval between* u_0 *and* u_1.

Definition 2.3 *We define a normalized function* J *on* X *by setting*

$$J(x) = \sum_{i=-\infty}^{+\infty} a_i(x), \quad \text{where } a_i(x) = h(x_i, x_{i+1}) - c, \ \forall i \in \mathbb{Z}.$$

Remark 2.4 Although c is a minimizer of $\bar{h}$, when $\xi \neq \eta$, it is possible $h(\xi, \eta) - c < 0$. This means $J(x)$ may not have a lower bound. However we will prove there is a finite constant B independent of n, such that $\sum_{-n}^{n} a_i(x) \geq B$, $\forall x \in X$.

Moreover we will prove J is a well-defined function from X to $\mathbb{R} \cup \{+\infty\}$, in the sense that, for any $x \in X$, either the limit $\lim_{n\to+\infty} \sum_{i=-n}^{n} a_i(x)$ exists and converges to a finite number, or diverges to infinity.

For simplicity, we define translation operators $T_k : X \to X$ by $T_k x = T_{(0,k)} x$ for any $k \in \mathbb{Z}$. It is easy to see J is invariant under T_k.

Lemma 2.5 *For any* $n \in \mathbb{Z}^+$ *and* $x \in \hat{X}(n)$,

$$\sum_{i=0}^{n-1} a_i(x) = h(x_0, \ldots, x_n) - nc \geq 0,$$

the above inequality is an equality iff $x_i \equiv u_0$ *or* $x_i \equiv u_1$ *for all* $0 \leq i \leq n$.

Proof For a proof, see [5, Theorem 3.3]. □

To refine the above result, we introduce the following definition.

Definition 2.6 *For any* $x \in X(n)$, *we set*

$$d(x, U) := \max_{0 \leq i \leq n} \min_{j \in \{0,1\}} |x_i - u_j|.$$

Lemma 2.7 *For any* $\delta > 0$, *let*

$$\phi(\delta) := \inf_{n \in \mathbb{Z}^+} \inf \left\{ \sum_{i=0}^{n-1} a_i(x) : x \in \hat{X}(n) \text{ and } d(x, U) \geq \delta \right\}. \tag{2.1}$$

Then ϕ *is a continuous function satisfying* $\phi(\delta) > 0$, *if* $\delta > 0$; $\phi(\delta) = 0$, *if* $\delta = 0$. *It increase monotonically with respect to* δ. *Moreover, if* $x \in \hat{X}(n)$ *satisfies*

$$\min_{j \in \{0,1\}} |x_i - u_j| \geq \delta, \quad \forall i = 1, \ldots, n-1, \tag{2.2}$$

then $\sum_{i=0}^{n-1} a_i(x) \geq n\phi(\delta)$.

Proof First notice that $\phi(\delta) = 0$ if $\delta = 0$. From now on we assume $\delta > 0$.

For a fixed $n \in \mathbb{Z}^+$, by the compactness of $\{x \in \hat{X}(n) : d(x,U) \geq \delta\}$,

$$\phi(\delta, n) := \min\left\{\sum_{i=0}^{n-1} a_i(x) : x \in \hat{X}(n) \text{ with } d(x,U) \geq \delta\right\} > 0.$$

By induction on n, we will show $\phi(\delta,n) \geq \phi(\delta,1)$, $\forall n \in \mathbb{N}$, which then implies $\phi(\delta) = \phi(\delta,1) > 0$.

Now let us assume $\phi(\delta,k) \geq \phi(\delta,1)$, for all $k = 1,\dots,n-1$. Given an arbitrary $x \in \hat{X}(n)$ with $d(x,U) \geq \delta$, if there is a $j \in \{1,\dots,n-1\}$ satisfying $x_j = x_0 = x_n$, then $\{x_i\}_{i=0}^{j} \in \hat{X}(j)$ and $\{x_i\}_{i=j}^{n} \in \hat{X}(n-j)$. Moreover at least one of $d(\{x_i\}_{i=0}^{j}, U) \geq \delta$ and $d(\{x_i\}_{i=j}^{n}, U) \geq \delta$ must be true. Therefore

$$\sum_{i=0}^{n-1} a_i(x) \geq \min\{\phi(\delta,j), \phi(\delta,n-j)\} \geq \phi(\delta,1).$$

On the other hand, if $x_j \neq x_0$, $\forall j = 1,\dots,n-1$, there must be a $1 \leq k \leq n-1$ satisfying $(x_k - x_{k-1})(x_k - x_{k+1}) \geq 0$. By Condition H_3,

$$h(x_{k-1},x_k) + h(x_k,x_{k+1}) \geq h(x_{k-1},x_{k+1}) + h(x_k,x_k).$$

Let $x^1 = (x_k,x_k)$ and $x^{n-1} = (x_0,\dots,x_{k-1},x_{k+1},\dots,x_n)$. Then $x^1 \in \hat{X}(1), x^{n-1} \in \hat{X}(n-1)$, and at least one of $d(x^{n-1},U) \geq \delta$ and $d(x^1,U) \geq \delta$ must be true, so

$$\sum_{i=0}^{n-1} a_i(x) \geq h(x_0,\dots,x_{k-1},x_{k+1},\dots,x_n) - (n-1)c + h(x_k,x_k) - c \geq \phi(\delta,1).$$

Since x is arbitrary, $\phi(\delta,n) \geq \phi(\delta,1)$. Hence $\phi(\delta) = \phi(\delta,1)$. By the definition of $\phi(\delta,1)$, it is not hard to see it is continuous and monotonically increasing with respect to δ.

For the rest of the lemma, again by induction we assume the corresponding result holds for $k = 1,\dots,n-1$, if $x \in \hat{X}(n)$ satisfies (2.2). We divide the proof to two different cases.

Case 1 There is a $j \in \{1,\dots,n-1\}$, such that $x_j = x_0 = x_n$. Then $\{x_i\}_{i=0}^{j} \in \hat{X}(j)$ satisfies (2.2) with n replaced by j, and $\{x_i\}_{i=j}^{n} \in \hat{X}(n-j)$ satisfies (2.2) with n replaced by $n-j$. By the induction assumption

$$\sum_{i=0}^{n-1} a_i(x) = \sum_{i=0}^{j-1} a_i(x) + \sum_{i=j}^{n-1} a_i(x) \geq j\phi(\delta) + (n-j)\phi(\delta) = n\phi(\delta).$$

Case 2 $x_j \neq x_0$, $\forall j \in \{1,\dots,n-1\}$. Then there is a $1 \leq k \leq n-1$ satisfying $(x_k - x_{k-1})(x_k - x_{k+1}) \geq 0$. Let x^1 and x^{n-1} be defined as above. Then x^{n-1} satisfies (2.2) with n replaced by $n-1$. As a result,

$$\sum_{i=0}^{n-1} a_i(x) \geq h(x_0,\dots,x_{k-1},x_{k+1},\dots,x_n) - (n-1)c + h(x_k,x_k) - c \geq n\phi(\delta). \qquad \square$$

By the previous lemma, we can show that $\sum_{i=0}^{n-1} a_i(x)$ has a uniform lower bound, for any $x \in X(n)$. For this we set $C := \text{Lip}(h)$ for the rest of the paper (recall that h is Lipschitz continuous). Then for any real numbers ξ, ξ', ζ, and ζ',

$$|h(\xi,\xi') - h(\zeta,\zeta')| \leq C(|\xi - \zeta| + |\xi' - \zeta'|).$$

Lemma 2.8 *For any $\delta > 0$ and $n \in \mathbb{N}$, if $x \in X(n)$ satisfies $d(x, U) \geq \delta$,*

$$\sum_{i=0}^{n-1} a_i(x) \geq \phi(\delta) - C|x_n - x_0| \geq -C|x_n - x_0|$$

with $\phi(\delta)$ defined by (2.1). Moreover there is a constant $B \in \mathbb{R}$, such that

$$\sum_{i=0}^{n-1} a_i(x) = h(x_0, \dots, x_n) - nc \geq B, \quad \forall x \in X(n) \text{ and } n \in \mathbb{N}.$$

Proof First let us assume $d(x, U) = \min_{j \in \{0,1\}}\{|x_k - u_j|\}$ for some $k \neq 0$. Now we define a $\hat{x} \in \hat{X}(n)$ as

$$\hat{x}_i = \begin{cases} x_n, & \text{if } i = 0, \\ x_i, & \text{if } i = 1, \dots, n. \end{cases}$$

Obviously $d(\hat{x}, U) \geq \delta$, which implies $\sum_{i=0}^{n-1} a_i(\hat{x}) \geq \phi(\delta)$. Meanwhile

$$\left|\sum_{i=0}^{n-1} a_i(x) - \sum_{i=0}^{n-1} a_i(\hat{x})\right| = |h(x_0, x_1) - h(x_n, x_1)| \leq C|x_n - x_0|.$$

As a result,

$$\sum_{i=0}^{n-1} a_i(x) \geq \sum_{i=0}^{n-1} a_i(\hat{x}) - C|x_n - x_0| \geq \phi(\delta) - C|x_n - x_0|.$$

Now let us assume $d(x, U) = \min\{|x_0 - u_0|, |x_0 - u_1|\}$. Then we set

$$\hat{x}_i = \begin{cases} x_i, & \text{if } i = 0, \dots, n-1, \\ x_0, & \text{if } i = n, \end{cases}$$

and repeat the previous argument. This proves the first part of the lemma.

For any $n \in \mathbb{Z}^+$ and $x \in X(n)$, by what we just proved

$$\sum_{i=0}^{n-1} a_i(x) \geq \phi(d(x, U)) - C|x_n - x_0| \geq \phi(0) - C|x_n - x_0|$$
$$\geq -C|x_n - x_0| \geq -C(u_1 - u_0) =: B.$$

This proves the second part of the lemma. □

In general, we can not expect B in the above lemma to be non-negative. With the above lemma, following an approach given in [6], we get the next proposition, which plays a key role in our proof of the main result. We postpone its proof to Section 6.

Proposition 2.9 *J is a well-defined function from X to $\mathbb{R} \cup \{+\infty\}$. Moreover, if $x \in X$ and $J(x) < +\infty$, then $x(\pm\infty) = u_0$ or u_1.*

Again we can not except the lower bound of J to be non-negative. However if we restrict ourselves to a class of homoclinic configurations, it does have a non-negative lower bound, as shown by the following lemma.

Lemma 2.10 *If $x \in X$ satisfying $x(+\infty) = x(-\infty) = u_0$ (or $x(+\infty) = x(-\infty) = u_1$) and $x_i \neq u_0$ (or $x_i \neq u_1$) for some $i \in \mathbb{Z}$, then $J(x) > 0$.*

Proof First there is an $i_0 \in \mathbb{Z}$, such that $0 < \delta = |x_{i_0} - u_0| < (u_1 - u_0)/2$. For an integer $N > 0$ large enough, we have $|x_n - u_0| \leq \frac{\phi(\delta)}{2C}$, $\forall |n| \geq N$. Therefore $|x_n - x_{-n}| \leq \frac{\phi(\delta)}{2C}$, $\forall n > N$. By Lemma 2.8,

$$\sum_{i=-n}^{n-1} a_i(x) \geq \phi(\delta) - C|x_N - x_{-N}| \geq \frac{\phi(\delta)}{2} > 0.$$

Since this is true for any $n \geq N$, $J(x) > 0$. □

The multi-bump homoclinic and heteroclinic connections will be found as local minimizers of J. For this, we need to make sure each component of them does not equal to u_0 or u_1. For this, the next two lemmas will be needed.

Lemma 2.11 *For any $\delta \in (0, u_1 - u_0]$, if $\{x_i\}_{i=0}^2$ satisfies*

(a) $x_i \in [u_0, u_1]$ *for all* $i = 0, 1, 2$;

(b) $x_1 \in [u_1 - \delta, u_1]$, *and* $x_0 \neq u_1$ *or* $x_2 \neq u_1$;

(c) $h(x_0, x_1, x_2) \leq h(x_0, \xi, x_2)$, $\forall \xi \in [u_1 - \delta, u_1]$,

then $x_1 \neq u_1$. The statement still holds if we replace every u_1 by u_0 and every $[u_1 - \delta, u_1]$ by $[u_0, u_0 + \delta]$.

Lemma 2.12 *If a finite configuration $x = \{x_i\}_{i=n_0}^{n_1}$ satisfies*

(a) $x_i \in [u_0, u_1]$ *for all* $i = n_0, \ldots, n_1$;

(b) $h(x_{n_0}, \ldots, x_{n_1}) \leq h(y_{n_0}, \ldots, y_{n_1})$, *for any* $\{y_i\}_{i=n_0}^{n_1}$ *satisfying* $y_{n_0} = x_{n_0}$, $y_{n_1} = x_{n_1}$ *and* $y_i \in [u_0, u_1]$,

then x is a minimal configuration. Moreover, if x also satisfies $x_{n_0} \notin \{u_0, u_1\}$ or $x_{n_1} \notin \{u_0, u_1\}$, then $x_i \notin \{u_0, u_1\}$ for all $i = n_0 + 1, \ldots, n_1 - 1$.

The proofs of the above two lemmas can be found in Section 7.

We finish this section by a comparison lemma, which will be needed later.

Definition 2.13 *For any $j \in \{0, 1\}$ and $k \in \mathbb{Z}$, we define the following operators, $G_j^{\pm}(k) : X \to X$, as*

$$(G_j^+(k)x)_i = \begin{cases} x_i, & \text{if } i \leq k, \\ u_j, & \text{if } i > k \end{cases} \quad \text{and} \quad (G_j^-(k)x)_i = \begin{cases} u_j, & \text{if } i < k, \\ x_i, & \text{if } i \geq k. \end{cases}$$

By Definition 2.13 and the Lipschitz continuity of h, we get

Lemma 2.14 *For any $x \in X$,*

$$\left| J(G_j^+(k)x) - \sum_{i=-\infty}^{k-1} a_i(x) \right| \leq C|u_j - x_k|;$$

$$\left| J(G_j^-(k)x) - \sum_{i=k}^{+\infty} a_i(x) \right| \leq C|u_j - x_k|.$$

3 Minimal Heteroclinic Connections

In this section we prove the existence of minimal heteroclinic connections from u_0 to u_1 and from u_1 to u_0. The result is not new. We include it here because it illuminates some of the ideas that will be needed in the next section. The strategy is to consider a class of configurations with the desired asymptotic behaviors, and show that J has at least one minimizer in it, which is the desired minimal heteroclinic connection.

Definition 3.1

$$\begin{aligned} X^0 &:= X^0(u_0,u_1) := \{x \in X : x(-\infty) = u_0 \ \textit{and}\ x(+\infty) = u_1\}; \\ X^1 &:= X^1(u_0,u_1) := \{x \in X : x(-\infty) = u_1 \ \textit{and}\ x(+\infty) = u_0\}; \\ c_0 &:= \inf\{J(x) : x \in X^0\}, \quad c_1 := \inf\{J(x) : x \in X^1\}. \end{aligned}$$

Remark 3.2 It is easy to see that c_0 and c_1 are finite constants.

Definition 3.3

$$\begin{aligned} \mathcal{M}^0 &:= \mathcal{M}^0(u_0,u_1) := \{x \in X^0 : J(x) = c_0\}; \\ \mathcal{M}^1 &:= \mathcal{M}^1(u_0,u_1) := \{x \in X^1 : J(x) = c_1\}. \end{aligned}$$

Theorem 3.4 *$\mathcal{M}^0$ (resp., $\mathcal{M}^1$) is a non-empty set. If $x \in \mathcal{M}^0$ (resp., $x \in \mathcal{M}^1$), then x is a minimal configuration, furthermore it is a heteroclinic connection from u_0 to u_1 (resp., from u_1 to u_0).*

Proof Choose a $\delta \in (0,(u_1-u_0)/2)$ small enough. Let $\{x^n\}_{n\in\mathbb{N}} \subset X^0$ be a minimizing sequence of J, i.e., $\lim_{n\to+\infty} J(x^n) = c_0$. Since $J(x^n)$ are invariant under translation operators T_k, $\forall k \in \mathbb{Z}$, we may assume

$$\delta \le |x_1^n - u_0| \le \frac{u_1-u_0}{2} \quad \text{and} \quad |x_i^n - u_0| \le \delta, \quad \forall i \le 0, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

By the compactness of X and lower semi-continuity of J, x^n (passing to a subsequence if necessary) converges to an $x \in X$ with $J(x) \le c_0$. Then Proposition 2.9 tells us $x(\pm\infty) \in U$. Since every x^n satisfies (3.1), we must have $x(-\infty) = u_0$. To prove $x \in X^0$, we only need to show $x(+\infty) = u_1$.

By a contradiction argument, assume $x(+\infty) = u_0$. Choose $0 < \varepsilon \le \phi(\delta)/4C$, there is an N large enough such that $x_i \le u_0 + \varepsilon/2$, for all $i \ge N$.

For n large enough, we have $x_N^n \le u_0 + \varepsilon$. Because $x^n(-\infty) = u_0$, there exists $-k_n$ large enough, such that $x_k^n \le u_0 + \varepsilon$, $\forall k \le k_n$. Since $d((x_k^n,\dots,x_N^n),U) \ge |x_1^n - u_0| \ge \delta$,

$$\sum_{i=k}^{N-1} a_i(x^n) \ge \phi(\delta) - C|x_N^n - x_{k_n}^n| \ge \phi(\delta) - C\varepsilon,$$

As this holds for all $k \le k_n$, $\sum_{i=-\infty}^{N-1} a_i(x^n) \ge \phi(\delta) - C\varepsilon$.

Now set $\bar{x}^n = G_0^-(N)x^n$. Obviously $\bar{x}^n \in X^0$, and by Lemma 2.14

$$\begin{aligned} J(\bar{x}^n) &\le \sum_{i=N}^{+\infty} a_i(x^n) + C\varepsilon = J(x^n) - \sum_{i=-\infty}^{N-1} a_i(x^n) + C\varepsilon \\ &\le J(x^n) - \phi(\delta) + C\varepsilon + C\varepsilon \\ &\le J(x^n) - \phi(\delta)/2. \end{aligned}$$

This implies $\liminf_{n\to+\infty} J(\bar{x}^n) \le \liminf_{n\to+\infty} J(x^n) - \phi(\delta)/2 < c_0$, which is absurd. Hence $x(+\infty) = u_1$ and $x \in X^0$ and $J(x) \ge c_0$. As a result, $J(x) = c_0$ and $x \in \mathcal{M}^0$.

Although x is just a minimizer of J among configurations in X^0. By Lemmas 2.11 and 2.12, $x_i \notin U$ for any $i \in \mathbb{Z}$ and in fact it is a (global) minimal configuration, so x is a heteroclinic connection from u_0 to u_1. □

By Theorem 3.4,

$$\mathcal{M}^0 \subset \mathcal{M}_0^+(u_0, u_1), \quad \mathcal{M}^1 \subset \mathcal{M}_0^-(u_0, u_1). \tag{3.2}$$

Next proposition shows these minimal configurations are monotone.

Proposition 3.5 *If $x \in \mathcal{M}^0$ (resp., $x \in \mathcal{M}^1$), then x is strictly monotonically increasing (resp., decreasing), i.e., $x_i < x_{i+1}$ (resp., $x_i > x_{i+1}$), for all $i \in \mathbb{Z}$.*

Proof First we show $x_k \neq x_{k+1}$, $\forall k \in \mathbb{Z}$. Proof by contradiction. Let's assume $x_k = x_{k+1}$ for some $k \in \mathbb{Z}$. From the proof of Theorem 3.4, $x_i \in (u_0, u_1)$, $\forall i \in \mathbb{Z}$. Then $a_k(x) = h(x_k, x_k) - c > 0$. Set $\bar{x} = (\dots, x_{k-1}, x_{k+1}, \dots)$, then $J(\bar{x}) = J(x) - (h(x_k, x_{k+1}) - c) < c_0$, which is absurd.

Next assume there is a $k \in \mathbb{Z}$ satisfying $(x_k - x_{k-1})(x_k - x_{k+1}) > 0$. By Condition H_3,

$$h(x_{k-1}, x_k) + h(x_k, x_{k+1}) > h(x_{k-1}, x_{k+1}) + h(x_k, x_k).$$

Set $\bar{x} = (\dots, x_{k-1}, x_{k+1}, \dots)$, then

$$c_0 = J(x) > J(\bar{x}) + h(x_k, x_k) - c \geq J(\bar{x}).$$

Since $\bar{x} \in X^0$, we get a contradiction. Therefore x must be strictly monotonic and the asymptotic behaviors of x guarantee that it must be increasing. □

4 Multi-bump Homoclinic and Heteroclinic Connections

In this section, we prove Theorem 1.7 for $x^0 = u_0$ and $x^1 = u_1$, namely for a pair of $(0,1)$-periodic neighboring minimal configurations. First we find minimizers of J on a class of configurations with desired asymptotic and oscillating behaviors. Then we show these minimizers are stationary configurations.

If we replace x^0, x^1 and x_0^0, x_0^1 in Theorem 1.7 by u_0, u_1 correspondingly, then by (3.2), the condition (gap) implies

$$I_0 \neq (u_0, u_1) \quad \text{and} \quad I_1 \neq (u_0, u_1), \tag{$*$}$$

where $I_j := \{x_0 : x \in \mathcal{M}^j\}$ for $j = 0, 1$.

We assume $(*)$ holds for the rest of this section.

Proposition 4.1 *For any $\hat{\delta} > 0$, there are $\delta_i \in (0, \hat{\delta})$, $i = 0, \dots, 3$, and positive constants $e_0 = e_0(\delta_0, \delta_2)$, $e_1 = e_1(\delta_1, \delta_3)$, such that*

$$\inf\{J(x) : x \in X^0,\ x_0 = u_0 + \delta_0,\ \text{or}\ x_0 = u_1 - \delta_2\} \geq c_0 + e_0;$$

$$\inf\{J(x) : x \in X^1,\ x_0 = u_1 - \delta_1,\ \text{or}\ x_0 = u_0 + \delta_3\} \geq c_1 + e_1.$$

Proof We give a detailed proof of the first inequality. The proof of the other is similar and will be omitted.

By $(*)$ there is a $u \in (u_0, u_1) \setminus I_0$. From the ordering structure of $\mathcal{M}^0$ inherited from $\mathcal{M}_0^+(u_0, u_1)$, there is a pair of minimal configurations $y < z$ from $\mathcal{M}^0$ satisfying $u \in (y_0, z_0)$ and no other configuration from $\mathcal{M}^0$ lies between y and z.

By Proposition 3.5, there is a $-n_0 \in \mathbb{Z}^+$ large enough such that

$$u_0 < y_{n_0} < z_{n_0} < u_0 + \hat{\delta}.$$

Choose a $\delta_0 \in (y_{n_0} - u_0, z_{n_0} - u_0)$. Then $\delta_0 \in (u_0, u_1) \setminus I_0$. Let

$$Y_{n_0}^0 = \{x \in X^0 : x_{n_0} = u_0 + \delta_0\} \quad \text{and} \quad b_0 = \inf\{J(x) : x \in Y_{n_0}^0\}.$$

Obviously b_0 is finite and $\delta_0 \in (u_0, u_1) \setminus I_0$ implies

$$b_0 > c_0. \tag{4.1}$$

Since $J(T_{n_0}x) = J(x)$, this implies

$$\inf\{J(x) : x \in Y_0^0\} = b_0 > c_0, \quad \text{where } Y_0^0 = \{x \in X_0 : x_0 = u_0 + \delta_0\}.$$

Similarly there is an $n_1 \in \mathbb{Z}^+$ large enough, such that $u_1 - \hat{\delta} < y_{n_1} < z_{n_1} < u_1$. Then we can find a $\delta_2 \in (y_{n_1}, z_{n_1})$, such that

$$b_2 = \{J(x) : x \in X_0,\, x_0 = u_1 - \delta_2\} > c_0.$$

Letting $e_0 = \min\{b_0 - c_0, b_2 - c_0\}$, we get the desired result. □

For the rest of this section, we choose a constant $\bar{\delta} \in (0, (u_1 - u_0)/2)$ and set

$$\varepsilon^* := \min\{\rho(\xi, \zeta) : \xi, \zeta \in [u_0, u_1]\}; \tag{4.2}$$

$$\bar{\varepsilon} := \min\{\phi(\bar{\delta}), \quad \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2\}. \tag{4.3}$$

Obviously both ε^* and $\bar{\varepsilon}$ are positive.

From now on we assume $\hat{\delta}$ in Proposition 4.1 also satisfies

$$0 < \hat{\delta} < \bar{\delta} \text{ and } \hat{\delta} \leq \frac{1}{4C}\bar{\varepsilon}. \tag{4.4}$$

Definition 4.2 *For an $m \in \mathbb{N}$, we set*

$$Z_0 := \{x \in X : x_{-m} \leq u_0 + \delta_0, x_m \geq u_1 - \delta_2\};$$

$$Z_1 := \{x \in X : x_{-m} \geq u_1 - \delta_1, x_m \leq u_0 + \delta_3\},$$

where δ_i, $i = 0, \ldots, 3$, are those given in Proposition 4.1.

Obviously Z_0, Z_1 depend on the choice of m, so it will be fixed in all our results. The precise requirement of m will be given later. For the moment, we just assume it is large enough, so that both $\mathcal{M}^0 \cap Z_0$ and $\mathcal{M}^1 \cap Z_1$ are non-empty. Choose two minimal configurations $y^0 \in \mathcal{M}^0 \cap Z_0$ and $y^1 \in \mathcal{M}^1 \cap Z_1$.

Remark 4.3 For the sake of simplicity, from now on we make the following agreement that when we label c, e, u, y, Z and $G^{\pm}$ by an index, that index is considered mod 2.

Let m be a positive integer and $q = \{q_i\}_{i \in \mathbb{Z}}$ a bi-infinite sequence of integers satisfying $q_{i+1} - q_i \geq 4m$, $\forall i \in \mathbb{Z}$.

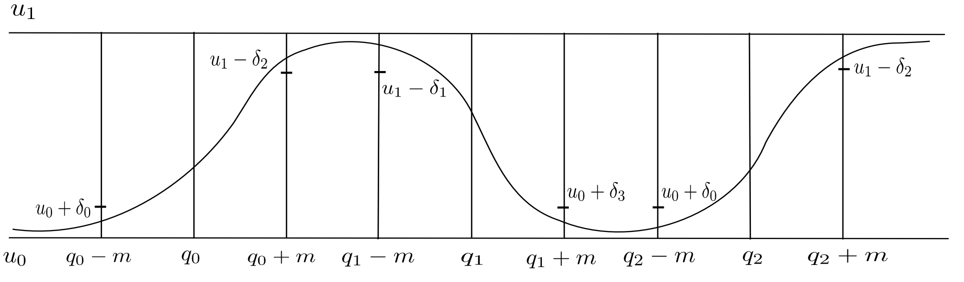

Figure 1 The Aubry graph of an $x \in Z(0, 2)$

Definition 4.4 *For any pair of integers $j \leq k$, we define*

$$Z(j,k) := \{x \in X : x(-\infty) = u_j,\ x(+\infty) = u_{k+1} \text{ and } T_{q_l}x \in Z_l,\ \forall l \in \{j, \dots, k\}\},$$

and $c(j,k) := \inf\{J(x) : x \in Z(j,k)\}$.

Proposition 4.5 *For any two integers $j \leq k$, there exists at least one minimizer $x^{jk} \in Z(j,k)$ satisfying $J(x^{jk}) = c(j,k)$.*

Proof The proposition is trivial when $j = k$, since in this case $c(j,k) = c_j$ and $T_{-q_j}y^j \in Z(j,k)$ with $J(T_{-q_j}y^j) = J(y^j) = c_j$.

When $j < k$, let $\{x^n\}_{n\in\mathbb{Z}} \subset Z(j,k)$ be a minimizing sequence, $\lim_{n\to+\infty} J(x^n) = c(j,k)$. Similar to the proof of Theorem 3.4, we may assume x^n converges to an $x \in X$, such that

$$\liminf_{n\to+\infty} J(x^n) = c(j,k) \geq J(x).$$

It is easy to see $T_{q_l}x \in Z_l$ for $j \leq l \leq k$, since $T_{q_l}x^n \in Z_l$ and x_i^n converges to x_i as n goes to infinity, $\forall i \in \mathbb{Z}$. To prove $x \in Z(j,k)$, we need to show that $x(-\infty) = u_j$ and $x(+\infty) = u_{k+1}$. Since $J(x)$ is finite, Proposition 2.9 implies $x(\pm\infty) \in U$. Therefore it is enough to show that $x(-\infty) \neq u_{j+1}$ and $x(+\infty) \neq u_k$. For k being even, we give a detailed proof of $x(+\infty) \neq u_k$. The other cases can be proven similarly.

By contradiction, let's assume $x(+\infty) = u_k = u_0$ (since we are considering the case that k is even). Then there is an N large enough ($N > q_k + m$), such that

$$x_N \leq u_0 + \delta_2/2,$$

where δ_2 is the same as defined in Proposition 4.1. Then for n large enough,

$$x_N^n \leq u_0 + \delta_2.$$

Meanwhile since $x^n \in Z(j,k)$, $x^n(+\infty) = u_{k+1} = u_1$. Hence for each x^n, there is a $p_n > N$ large enough, such that $x_p^n \geq u_1 - \delta_2$, $\forall p \geq p_n$.

Now for each $\{x_i^n\}_{i=q_k+m}^p$, there are two possibilities.

Case 1 There is a $j \in [q_k+m, p]\cap\mathbb{Z}$, such that $x_j^n \in [u_0+\bar{\delta}, u_1-\bar{\delta}]$. Then $d(\{x_i^n\}_{i=q_k+m}^p, U) \geq \bar{\delta}$. By Lemma 2.8,

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) \geq \phi(\bar{\delta}) - C|x_p^n - x_{q_k+m}^n| \geq \phi(\bar{\delta}) - C\delta_2. \tag{4.5}$$

The second inequality follows from the fact that x_p^n and $x_{q_k+m}^n$ belong to $[u_1 - \delta_2, u_1]$.

Case 2 For any $j \in [q_k+m, p]\cap\mathbb{Z}$, $x_j^n \notin [u_0+\bar{\delta}, u_1-\bar{\delta}]$. Because $x_{q_k+m}^n \in [u_1-\delta_2, u_1]$, $x_N^n \in [u_0, u_0+\delta_2]$ and $x_p^n \in [u_1-\delta_2, u_1]$, there exist two integers $j_0 \leq j_1$ from $[q_k+m+1, p-1]\cap\mathbb{Z}$ satisfying

$$x_{j_0}^n \in [u_1-\bar{\delta}, u_1], \quad x_{j_0+1}^n \in [u_0, u_0+\bar{\delta}],$$

$$x_{j_1}^n \in [u_0, u_0+\bar{\delta}], \quad x_{j_1+1}^n \in [u_1-\bar{\delta}, u_1].$$

By condition H_5,

$$h(x_{j_0}^n, x_{j_0+1}^n) + h(x_{j_1}^n, x_{j_1+1}^n) \geq h(x_{j_1}^n, x_{j_0+1}^n) + h(x_{j_0}^n, x_{j_1+1}^n) + \int_{x_{j_1}^n}^{x_{j_0}^n}\int_{x_{j_0+1}^n}^{x_{j_1+1}^n} \rho$$

$$\geq h(x^n_{j_1}, x^n_{j_0+1}) + h(x^n_{j_0}, x^n_{j_1+1}) + \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2. \tag{4.6}$$

At the same time,

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) = h(x^n_{q_k+m}, \dots, x^n_{j_0}) + h(x^n_{j_0+1}, \dots, x^n_{j_1}) + h(x^n_{j_1+1}, \dots, x^n_p) \\ - (p - q_k - m)c + h(x^n_{j_0}, x^n_{j_0+1}) + h(x^n_{j_1}, x^n_{j_1+1}). \tag{4.7}$$

Combine (4.6) and (4.7),

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) \geq h(x^n_{q_k+m}, \dots, x^n_{j_0}) + h(x^n_{j_0}, x^n_{j_1+1}) + h(x^n_{j_1+1}, \dots, x^n_p) \\ + h(x^n_{j_1}, x^n_{j_0+1}) + h(x^n_{j_0+1}, \dots, x^n_{j_1}) - (p - q_k - m)c \\ + \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2.$$

By Lemma 2.7,

$$h(x^n_{j_1}, x^n_{j_0+1}) + h(x^n_{j_0+1}, \dots, x^n_{j_1}) - (j_1 - j_0)c \geq \phi(0) \geq 0.$$

Then Lemma 2.8 implies

$$h(x^n_{q_k+m}, \dots, x^n_{j_0}) + h(x^n_{j_0}, x^n_{j_1+1}) + h(x^n_{j_1+1}, \dots, x^n_p) - (p - q_k - m - j_1 + j_0)c \\ \geq \phi(0) - C\delta_2 \geq -C\delta_2.$$

As a result,

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) \geq \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2 - C\delta_2. \tag{4.8}$$

This finishes our discussion of the two different cases.

Recall that $\bar{\varepsilon} := \min\{\phi(\bar{\delta}), \varepsilon^*(u_1 - u_0 - 2\bar{\delta})^2\} > 0$. By (4.5) and (4.8),

$$\sum_{i=q_k+m}^{p-1} a_i(x^n) \geq \bar{\varepsilon} - C\delta_2.$$

Since the above inequality holds for all $p \geq p_n$, the following also holds

$$\sum_{i=q_k+m}^{+\infty} a_i(x^n) \geq \bar{\varepsilon} - C\delta_2.$$

Consider the sequence $\{G_1^+(q_k + m)x^n\}_{n\in\mathbb{N}} \subset Z(j,k)$. By Lemma 2.14 and our assumption on $\hat{\delta}$,

$$J(G_1^+(q_k + m)x^n) \leq J(x^n) - \sum_{i=q_k+m}^{+\infty} a_i(x^n) + C\delta_2 \leq J(x^n) - \bar{\varepsilon} + C\delta_2 + C\delta_2 \\ \leq J(x^n) - \bar{\varepsilon} + 2C\hat{\delta} \leq J(x^n) - \bar{\varepsilon}/2.$$

This then implies

$$\liminf_{n\to+\infty} J(G_1^+(q_k + m)x^n) \leq \liminf_{n\to+\infty} J(x^n) - \bar{\varepsilon}/2 \leq c(j,k) - \bar{\varepsilon}/2 < c(j,k),$$

which is a contradiction. This finishes our proof. □

In the rest of this section we will show minimizers obtained in Proposition 4.5 are stationary configurations.

Lemma 4.6 *Every minimizer $x^{jk} \in Z(j,k)$ with $J(x^{jk}) = c(j,k)$ satisfies $x^{jk} \notin \{u_0, u_1\}$ for all $i \in \mathbb{Z}$.*

Proof This lemma is an immediate consequence of Lemmas 2.11 and 2.12, once the readers notice that $x^{jk} \neq u_0$ and $x^{jk} \neq u_1$. □

Lemma 4.7 *Every $x^{jk} \in Z(j,k)$ with $J(x^{jk}) = c(j,k)$ satisfies*

$$x_{i-1} < x_i, \quad \forall i \leq q_j - m, \ \text{if } j (\mathrm{mod}\ 2) = 0;$$

$$x_{i-1} > x_i, \quad \forall i \leq q_j - m, \ \text{if } j (\mathrm{mod}\ 2) = 1;$$

$$x_{i+1} > x_i, \quad \forall i \geq q_k + m, \ \text{if } k (\mathrm{mod}\ 2) = 0;$$

$$x_{x+1} < x_i, \quad \forall i \geq q_k + m, \ \text{if } k (\mathrm{mod}\ 2) = 1.$$

Proof The proof is the same as the proof of Proposition 3.5. □

For the rest of the section, we assume

$$0 < \varepsilon < \frac{\min\{e_1, e_2\}}{4C}, \tag{4.9}$$

and fix an $m \in \mathbb{Z}^+$ satisfying the following conditions:

$$m \geq \frac{2C\hat{\delta}}{\phi(\varepsilon)}; \tag{4.10}$$

$$y^0_{-2m} < u_0 + \delta_3 \quad \text{and} \quad y^0_{2m} > u_1 - \delta_1; \tag{4.11}$$

$$y^1_{-2m} > u_1 - \delta_2 \quad \text{and} \quad y^1_{2m} < u_0 + \delta_0, \tag{4.12}$$

where δ_i, $i = 0, \ldots, 3$, are those given in Proposition 4.1.

Lemma 4.8 *$c(j,k) < c(j,l) + c(l+1,k)$, for any $j \leq l < k$.*

Proof We give a detailed proof for l being odd. The other is similar.

We claim it is possible to find two minimizers $x \in Z(j,l)$ and $y \in Z(l+1,k)$ ($J(x) = c(j,l)$ and $J(y) = c(l+1,k)$) satisfying

$$x_{q_{l+1}-m} < u_0 + \delta_0, \quad y_{q_l+m} < u_0 + \delta_3. \tag{4.13}$$

If $l = j$, set $x = T_{-q_l} y^1$. Then by the conditions required for m,

$$x_{q_{l+1}-m} < x_{q_{l+2m}} = y^1_{2m} < u_0 + \delta_0.$$

If $j < l < k$, set $\bar{x} = T_{-q_l} y^1$. Then $\bar{x}_{q_{l-1}-m}$ is close to u_1 and $x_{q_{l-1}-m}$ is close to u_0. Hence

$$\bar{x}_{q_{l-1}-m} > x_{q_{l-1}-m}. \tag{4.14}$$

At the same time, by (4.12),

$$\bar{x}_{q_{l+1}-m} < u_0 + \delta_0.$$

Assume the first inequality in (4.13) does not hold. Then

$$x_{q_{l+1}-m} > \bar{x}_{q_{l+1}-m}. \tag{4.15}$$

By (4.14) and (4.15), $(x_{q_{l-1}-m}, \ldots, x_{q_{l+1}-m})$ and $(\bar{x}_{q_{l-1}-m}, \ldots, \bar{x}_{q_{l+1}-m})$ have at least one intersection (see Figure 2, where $Au(x)$ is denoted by the solid curve and $Au(\bar{x})$ by the dashed curve).

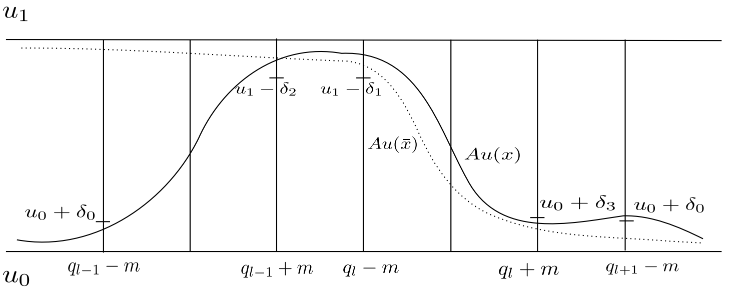

Figure 2 Aubry graphs $Au(x)$ and $Au(\bar{x})$

Set

$$x^+ = \{x_i^+\}_{i=-\infty}^{+\infty}, \quad \text{where } x_i^+ = \max\{x_i, \bar{x}_i\},\ \forall i \in \mathbb{Z};$$
$$x^- = \{x_i^-\}_{i=-\infty}^{+\infty}, \quad \text{where } x_i^- = \min\{x_i, \bar{x}_i\},\ \forall i \in \mathbb{Z}.$$

A simple application of condition H_5 gives

$$J(x^-) + J(x^+) \le J(x) + J(\bar{x}).$$

Since $x^- \in Z(j,l)$ and $T_{q_l}x^+ \in Z(l,l)$,

$$J(x^-) \le J(x^-) + J(x^+) - c_1 \le J(x) + J(\bar{x}) - c_1 = c(j,l).$$

Therefore x^- is also a minimizer of J in $Z(j,l)$, but satisfies

$$x^-_{q_{l+1}-m} < u_0 + \delta_0.$$

So we can simply rename x^- to x.

A similar argument as above can be applied to y as well.

With the claim justified, by Lemma 4.7, the Aubry graphs of x and y must intersect at least once between $q_l - m$ and $q_{l+1} + m$ (see Figure 3, where $Au(x)$ is denoted by the solid curve and $Au(y)$ by the dashed curve).

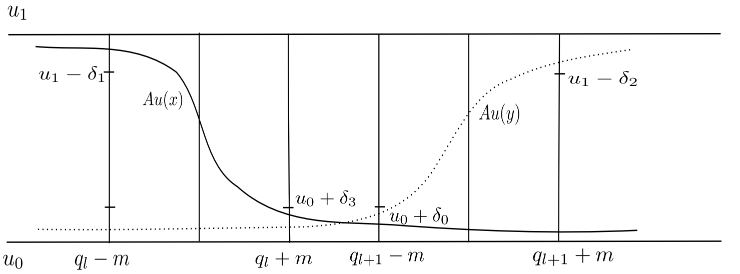

Figure 3 Aubry graphs $Au(x)$ and $Au(y)$

Set

$$z^+ = \{z_i^+\}_{i=-\infty}^{+\infty}, \quad \text{where } z_i^+ = \max\{x_i, y_i\},\ \forall i \in \mathbb{Z};$$

$$z^- = \{z_i^-\}_{i=-\infty}^{+\infty}, \quad \text{where } z_i^- = \min\{x_i, y_i\},\ \forall i \in \mathbb{Z}.$$

Then $z^+ \in Z(j,k)$ and $z^- \in X$ with $z^-(\pm\infty) = u_0$. Lemmas 2.10 and 4.6 imply $J(z^-) > 0$. Therefore

$$c(j,k) \le J(z^+) < J(z^+) + J(z^-) \le J(x) + J(y) = c(j,l) + c(l+1,k). \qquad \square$$

Now we are ready to prove the x^{jk}'s are stationary.

Theorem 4.9 *Under the assumption* $(*)$*, for any two different integers* $j < k$*, if* $x^{jk} \in Z(j,k)$ *satisfies* $J(x^{jk}) = c(j,k)$*, it is a stationary configuration, and therefore a homoclinic or heteroclinic connection from* u_j *to* u_{k+1}*.*

Proof For simplicity, set $x = x^{jk}$. By Lemma 4.6, $x_i \notin \{u_0, u_1\}$, so it is enough to show for $l = j, \dots, k$,

$$x_{q_l - m} < u_0 + \delta_0, \quad x_{q_l + m} > u_1 - \delta_2, \quad \text{when } l \text{ is even}, \tag{4.16}$$

$$x_{q_l - m} > u_1 - \delta_1, \quad x_{q_l + m} < u_0 + \delta_3, \quad \text{when } l \text{ is odd}. \tag{4.17}$$

Define a finite configuration $x^* = \{x_i^*\}_{i=q_l+m}^{q_{l+1}-m}$ as

$$x_i^* = \begin{cases} x_i, & \text{if } i = q_l + m \text{ or } q_{l+1} - m; \\ u_{l+1}, & \text{if } q_l + m < i < q_{l+1} - m. \end{cases}$$

Since x is a minimizer of J in $Z(j,k)$,

$$\sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) \le \sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x^*) \le 2C\hat{\delta}. \tag{4.18}$$

Set $I_l = [q_l + m, q_{l+1} - m] \cap \mathbb{Z}$. We claim there is a $p_l \in I_l$, such that

$$|x_{p_l} - u_{l+1}| < \varepsilon, \quad \forall l \in \{j, \dots, k-1\}. \tag{4.19}$$

Before proving the claim, we first show $|x_i - u_{l+1}|$ is uniformly bounded by $\bar{\delta}$,

$$|x_i - u_{l+1}| \le \bar{\delta}, \quad \forall i \in I_l \text{ and } l \in \{j, \dots, k-1\}. \tag{4.20}$$

Notice that there is no x_i satisfying $\min\{|x_i - u_l|, |x_i - u_{l+1}|\} \ge \bar{\delta}$, as otherwise $d(\{x_i\}_{i \in I_l}, U) \ge \bar{\delta}$. By Lemma 2.8,

$$\sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) \ge \phi(\bar{\delta}) - C\hat{\delta} \ge \bar{\varepsilon} - C\hat{\delta} \ge 4C\hat{\delta} - C\hat{\delta} > 2C\hat{\delta}.$$

The two inequalities in the middle follow from (4.3), (4.4) on $\bar{\varepsilon}$ and $\hat{\delta}$. This is a contradiction to (4.18).

Now we will show there is no x_i satisfying $|x_i - u_l| \le \bar{\delta}$. If not, by

$$|u_{q_l+m} - u_{l+1}| \le \hat{\delta}, \quad |u_{q_{l+1}-m} - u_{l+1}| \le \hat{\delta},$$

and what we just showed, there exist two integers $j_0 < j_1 \in I_l$, such that

$$|x_{j_0} - u_{l+1}| \le \bar{\delta} \quad \text{and} \quad |x_{j_0+1} - u_l| \le \bar{\delta}; \tag{4.21}$$

$$|x_{j_1} - u_l| \le \bar{\delta} \quad \text{and} \quad |x_{j_1+1} - u_{l+1}| \le \bar{\delta}. \tag{4.22}$$

By the same argument of the proof of Proposition 4.5,

$$\sum_{i=q_l+m}^{q_{l+1}-m-1} a_i(x) \geq \varepsilon^*(u_1-u_0-2\bar{\delta})^2 - C\hat{\delta} \geq \bar{\varepsilon} - C\hat{\delta} > 2C\hat{\delta}.$$

This violates (4.18), which proves (4.20).

Since $0 < \bar{\delta} < (u_1-u_0)/2$,

$$\min\{|x_i-u_l|, |x_i-u_{l+1}|\} = |x_i-u_{l+1}|, \quad \forall i \in I_l \text{ and } l \in \{j,\dots,k\}.$$

Now we are ready to prove our claim. Given an arbitrary $l \in \{j,\dots,k-1\}$, let's assume $|x_i-u_{l+1}| \geq \varepsilon$, $\forall i \in I_l$. Define a finite configuration $\bar{x} = \{\bar{x}_i\}_{i=q_l+m-1}^{q_{l+1}-m+1}$ as

$$\bar{x}_i = \begin{cases} u_{l+1}, & \text{if } i = q_l+m-1 \text{ or } q_{l+1}-m+1; \\ x_i, & \text{if } i \in I_l. \end{cases}$$

Since $\bar{x} \in \hat{X}(q_{l+1}-q_l-2m+2; U)$, by Lemma 2.7,

$$\sum_{i=q_L+m-1}^{q_{l+1}-m+1} a_i(\bar{x}) \geq (q_{l+1}-q_l-2m+2)\phi(\varepsilon).$$

Combining with (4.18), we get

$$\begin{aligned} 2C\hat{\delta} \geq \sum_{i=q_l+m}^{q_{l+1}-m} a_i(x) &\geq \sum_{i=q_l+m-1}^{q_{l+1}-m} a_i(\bar{x}) - 2C\hat{\delta} \\ &\geq (q_{l+1}-q_l-2m+2)\phi(\varepsilon) - 2C\hat{\delta} > 2m\phi(\varepsilon) - 2C\hat{\delta}. \end{aligned}$$

This implies $m < 2C\hat{\delta}/\phi(\varepsilon)$, which violates (4.10). As a result, for any $l = j,\dots,k-1$, there is at least one $p_l \in I_l$ satisfying (4.19).

Finally we are ready to prove (4.16), (4.17). If $j < l < k$, set

$$x^+ = G_l^+(p_{l-1})x, \quad x^- = G_{l+1}^-(p_l)x, \quad x' = G_l^-(p_{l-1}) \circ G_{l+1}^+(p_l)x.$$

Notice that $x^+ \in Z(j, l-1)$, $x^- \in Z(l+1,k)$ and $x' \in Z_l$.

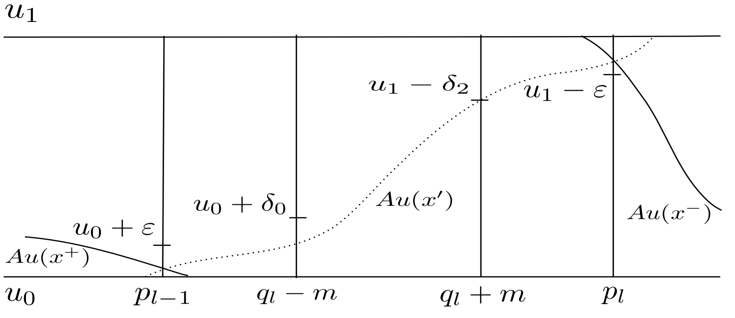

Figure 4 Aubry graphs of $Au(x^+), Au(x^-)$ and $Au(x')$

Assume x violates either (4.16) or (4.17), then so does x'. Figure 4 show the Aubry graphs of x^+, x^- and x' for an even l. By Proposition 4.1, $J(x') \geq c_l + e_l$. Then Lemmas 2.14 and 4.8 imply

$$c_l + e_l \leq J(x') \leq J(x) - J(x^+) - J(x^-) + 4C\varepsilon$$

$$\begin{aligned}&\le c(j,k) - c(j,l-1) - c(l+1,k) + 4C\varepsilon\\&< c(l,l) + 4C\varepsilon = c_l + 4C\varepsilon,\end{aligned}$$

which is a contradiction to (4.9).

If $l = j$, we can choose $-p_{l-1}$ large enough, such that $|x_{p_{l-1}} - u_l| < \varepsilon$. This is possible, since $x(-\infty) = u_j$, $\forall x \in Z(j,k)$. Now we just repeat the above argument with the only modification that $J(x^-) > 0$, because $x^- \in X$ and $x(\pm\infty) = u_j$. The proof for the case $l = k$ is similar. As a result, x satisfies (4.16) and (4.17), so it is a locally minimal configuration. As a result, x is a homoclinic or heteroclinic connection from u_j to u_{k+1}. □

Remark 4.10 Theorems 3.4 and 4.9 still hold even when $\{u_0, u_1\}$ is just a pair of local minimizers of $\bar{h}$ satisfying $\bar{h}(u) > \bar{h}(u_0) = \bar{h}(u_1)$, $\forall u \in (u_0, u_1)$. Results about heteroclinic connection between local minimizers can be found in [21] and [19].

5 Generalization to Configurations with Non-zero Rational Rotation Numbers

In this section, we will generalize our result from the previous section to periodic neighboring minimal configurations with non-zero rational rotation numbers and give a proof of Theorem 1.7.

Choose an arbitrary variational principle h and an arbitrary rational number $\alpha = p/q \neq 0$. Let $x^- < x^+$ be an arbitrary pair of (p,q)-periodic neighboring minimal configurations of h.

Definition 5.1

$$\begin{aligned}&X_\alpha(q) := X_\alpha(q; x^-, x^+) := \{x = \{x_i\}_{i=0}^q : x_i \in [x_i^-, x_i^+], i = 0, \dots, q\};\\&\hat{X}_\alpha(q) := \hat{X}_\alpha(q; x^-, x^+) := \{x \in X_\alpha(q) : x_q = x_0 + p\};\\&X_\alpha := X_\alpha(x^-, x^+) := \{x = \{x_i\}_{i=-\infty}^{+\infty} : x_i \in [x_i^-, x_i^+], \forall i \in \mathbb{Z}\}.\end{aligned}$$

To use results from the previous section, we define a new function $H : \mathbb{R}^2 \to \mathbb{R}^2$ associated to h as

$$H(\xi, \xi') := h^{*q}(\xi, \xi' + p), \quad \forall (\xi, \xi') \in \mathbb{R}^2.$$

Here $h^{*q} = h * \cdots * h$ is a q-fold *conjunction* of h with itself.

Definition 5.2 *Given two variational principles h_1 and h_2, we define $h * h' : \mathbb{R}^2 \to \mathbb{R}$ given below as their* ***conjunction***:

$$h_1 * h_2(\xi, \xi') = \min_{\zeta \in \mathbb{R}} \{h_1(\xi, \zeta) + h_2(\zeta, \xi')\}.$$

Remark 5.3 As showed by Mather [14], the conjunction of two variational principles h_1 and h_2 must be a variational principle as well. This means H is also a variational principle.

Lemma 5.4 *For any $i \in \mathbb{Z}$, $\{x_i^-, x_i^+\}$ is a pair of $(0,1)$-periodic neighboring minimal configurations of H.*

Proof We give a detailed proof for $i = 0$ (the others are similar). By a contradiction argument, let's assume there is a $\zeta \in (x_0^-, x_0^+)$, such that

$$H(\zeta, \zeta) \le H(x_0^-, x_0^-) = H(x_0^+, x_0^+),$$

then,

$$h^{*q}(\zeta, \zeta + p) \le h^{*q}(x_0^-, x_0^- + p) = h^{*q}(x_0^+, x_0^+ + p),$$

This is a contradiction to the fact that x^-, x^+ is a pair of Since x^-, x^+ is a pair of (p, q)-periodic neighboring minimal configurations of h. □

Definition 5.5 *For any* $y = \{y_i\}_{i=-\infty}^{+\infty} \in X(\{x_0^-, x_0^+\})$ *(see Definition 2.2), we define a corresponding* $x = \{x_i\}_{i=-\infty}^{+\infty}$ *as*

(a) $x_{iq} = y_i + ip$, $\forall i \in \mathbb{Z}$;

(b) $\{x_j\}_{j=iq}^{(i+1)q}$ *is a minimal configuration of* h, *i.e.,*

$$h(x_{iq}, \dots, x_{(i+1)q}) = H(x_{iq}, x_{(i+1)q}) = H(y_i, y_{i+1}), \quad \forall i \in \mathbb{Z}.$$

Notice that x defined above must belong to $\in X_\alpha(x^-, x^+)$.

Proposition 5.6 *Let* $y \in X(\{x_0^-, x_0^+\})$ *and* $x \in X_\alpha(x^-, x^+)$ *be defined as in Definition 5.5. If* y *is a stationary configuration of* H, x *must be a stationary configuration of* h. *Moreover if* $y(\pm\infty) = x_0^\pm$, *then* $x(\pm\infty) = x^\pm$ *correspondingly.*

First we explain the main result can be proven by the above proposition.

Proof of Theorem 1.7 It follows immediately from Proposition 5.6 and Theorem 4.9. □

The rest of the section is devoted to the proof of Proposition 5.6.

Definition 5.7 *For any* $x \in X_\alpha(q)$, *we define*

$$d_\alpha(x) := d_\alpha(x, \{x^-, x^+\}) := \max_{0 \le i \le q} \min\{|x_i - x_i^-|, |x_i - x_i^+|\}.$$

We set $\lambda_i := \frac{1}{2}|x_i^+ - x_i^-|$ for $i = 0, \dots, q$, and $\lambda := \min\{\lambda_i : i = 0, \dots, q\}$.

Lemma 5.8 *We set* $c_\alpha := h(x_0^-, \dots, x_q^-) = h(x_0^+, \dots, x_q^+)$. *For any* $0 \le \delta \le \lambda$, *let*

$$\phi_\alpha(\delta) := \inf\{h(x_0, \dots, x_q) - c_\alpha : x = \{x_i\}_{i=0}^q \in \hat{X}_\alpha(q),\ d_\alpha(x) \ge \delta\} \ge 0,$$

then ϕ_α *is a monotonically increasing and continuous function of* δ *satisfying*

$$\phi_\alpha(\delta) > 0, \text{ if } \delta > 0; \quad \phi_\alpha(\delta) = 0, \text{ if } \delta = 0.$$

The proof of this lemma is a simpler version of the proof of Lemma 2.7, so we omit it here.

Lemma 5.9 *Let* $C := \mathrm{Lip}(h)$. *Then for any* $0 \le \delta \le \lambda$, *if* $x = \{x_i\}_{i=0}^q \in X_\alpha(q)$ *with* $d_\alpha(x) \ge \delta$, *we have*

$$h(x_0, \dots, x_q) - c_\alpha \ge \phi_\alpha(\delta) - C|x_q - x_0 - p|.$$

Again the proof is similar to the proof of Lemma 2.8 and we omit it here.

Definition 5.10 *For* $0 \le \varepsilon \le \frac{\phi_\alpha(\lambda)}{4C+1}$, *we define*

$$\psi_\alpha(\varepsilon) := \inf\{0 \le \delta \le \lambda : \phi_\alpha(\delta) \ge (4C+1)\varepsilon\}.$$

By the continuity of ϕ_α,

$$\phi_\alpha(\psi_\alpha(\varepsilon)) \ge (4C+1)\varepsilon > 4C\varepsilon. \tag{5.1}$$

Because ϕ_α is a monotonically increasing continuous function, by simple calculation we see ψ_α is also a monotonically increasing continuous function w.r.t. ε and $\psi_\alpha(0) = 0$.

Let $y \in X(\{x_0^-, x_0^+\})$ and $x \in X_\alpha(x^-, x^+)$ be defined as in Definition 5.5, the key to the proof of Proposition 5.6 is to show that for any $i \in \mathbb{Z}$, $|x_j - x_j^\pm|, j = iq+1, \dots, (i+1)q-1$ is controlled by $|x_i - x_i^\pm|$ and this can be done by the following two lemmas.

Lemma 5.11 *For $0 \le \varepsilon \le \frac{\phi_\alpha(\lambda)}{4C+1}$, if $x \in X_\alpha(q; x^-, x^+)$ is a minimal configuration of h, which satisfies $|x_i - x_i^+| \le \varepsilon$, $i = 0, q$ or $|x_i - x_i^-| \le \varepsilon$, $i = 0, q$, then $d_\alpha(x) < \psi_\alpha(\varepsilon)$ for $\psi_\alpha(\varepsilon)$ defined as above.*

Proof We will give the detailed proof for the case $|x_i - x_i^+| \le \varepsilon, i = 0, q$, while the other is similar.

Assuming $d_\alpha(x) \ge \psi_\alpha(\varepsilon)$, then by Lemma 5.9,

$$h(x_0, \ldots, x_q) - c_\alpha \ge \phi_\alpha(\psi_\alpha(\varepsilon)) - C|x_q - x_0 - p| \ge \phi_\alpha(\psi_\alpha(\varepsilon)) - 2C\varepsilon. \tag{5.2}$$

The last inequality follows from

$$|x_q - (x_0 + p)| = |x_q - x_q^+ + x_0^+ + p - (x_0 + p)| \le |x_q - x_q^+| + |x_0^+ - x_0| \le 2\varepsilon.$$

Then the minimality of x and the Lipschitz continuity of h tell us

$$h(x_0, \ldots, x_q) \le h(x_0, x_1^+, \ldots, x_{q-1}^+, x_q) \le c_\alpha + 2C\varepsilon. \tag{5.3}$$

Combining (5.2) and (5.3), we have

$$\phi_\alpha(\psi_\alpha(\varepsilon)) \le 4C\varepsilon,$$

but this contradicts (5.1). Hence $d_\alpha(x) < \psi_\alpha(\varepsilon)$. □

Lemma 5.12 *There is a small enough $\varepsilon^* > 0$, such that if $x \in X_\alpha(q; x^-, x^+)$ is a minimal configuration and $|x_i - x_i^+| \le \varepsilon^*$, $i = 0, q$ (resp., $|x_i - x_i^-| \le \varepsilon^*$, $i = 0, q$), then $|x_i - x_i^+| < \lambda_i$ for all $i = 1, \ldots, q-1$ (resp., $|x_i - x_i^-| < \lambda_i$ for all $i = 1, \ldots, q-1$).*

Proof We will only show the proof for the case $|x_i - x_i^+| \le \varepsilon^*$, $i = 0, q$. Assume the lemma is not true, then there is a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}} \searrow 0$ and a sequence of minimal configurations $\{x^n\}_{n\in\mathbb{N}} \subset X_\alpha(q)$, such that

$$|x_i^n - x_i^+| \le \varepsilon_n, \quad i = 0, q, \quad \forall n \in \mathbb{N},$$

while there is at least one $i_n \in \{1, \ldots, q-1\}$ with

$$|x_{i_n}^n - x_{i_n}^+| \ge \lambda_{i_n}, \quad \forall n \in \mathbb{N}.$$

Replacing $\{x^n\}$ by a subsequence if necessary, we may assume there is a fixed $j \in \{1, \ldots, q-1\}$, such that

$$|x_j^n - x_j^+| \ge \lambda_j, \quad \forall n \in \mathbb{N}. \tag{5.4}$$

Passing to a subsequence if necessary, x^n converges to an $x = \{x_i\}_{i=0}^q \in X_\alpha(q)$. Since every x^n is minimal, so is x.

Then $x_i = \lim_{n\to+\infty} x_i^n = x_i^+$, for $i = 0, q$. Because x is minimal, we must have $x_i = x_i^+$ for all $i = 0, \ldots, q$.

On the other hand, since $x_j = \lim_{n\to+\infty} x_j^n$, by (5.4), $|x_j - x_j^+| \ge \lambda_j$, which is a contradiction. So the assumption we made is incorrect and we are done. □

Now we are ready to prove Proposition 5.6.

Proof of Proposition 5.6 If $y \in X(\{x_0^-, x_0^+\})$ is a stationary configuration of H, i.e., locally minimal w.r.t. H, by the way $x \in X(x^-, x^+)$ is defined, it is not hard to see x is locally minimal w.r.t. h, so x is a stationary configuration of h.

By the monotonicity of ψ_α, for ε^* satisfying Lemma 5.12, we can find a $0 < \hat{\varepsilon} \le \frac{\phi_\alpha(\lambda)}{4C+1}$, such that

$$0 < \psi_\alpha(\varepsilon) \le \varepsilon^* \quad \text{for } 0 < \varepsilon < \hat{\varepsilon}.$$

We will prove that $y(+\infty) = x_0^+$ implies $x(+\infty) = x^+$. The other cases are similar.

By the way x is defined, we have

$$\lim_{i\to+\infty} |x_{iq} - x_{iq}^+| = \lim_{i\to+\infty} |x_{iq} - (x_0^+ + ip)| = \lim_{i\to+\infty} |y_i - x_0^+| = 0.$$

Hence, for any $0 < \varepsilon < \hat{\varepsilon}$, there is an n_0 large enough, such that

$$|x_{iq} - x_{iq}^+| < \varepsilon, \quad i > n_0.$$

Because every $\{x_j\}_{j=iq}^{(i+1)q}$ is a minimal configuration of h, Lemma 5.11 tells us

$$d_\alpha(\{x_j\}_{j=iq}^{(i+1)q}) < \psi_\alpha(\varepsilon) \le \varepsilon^*, \quad \forall i > n_0.$$

Then by Lemma 5.12,

$$|x_j - x_j^+| < \lambda_{j\,(\mathrm{mod}\,q)}, \qquad \text{for } j = iq+1, \dots, (i+1)q-1, \quad \forall i > n_0.$$

Because of the periodicity of x^-, x^+, we have $\frac{|x_j^+ - x_j^-|}{2} = \lambda_{j\,(\mathrm{mod}\,q)}$ for any $j \in \mathbb{Z}$. Therefore,

$$|x_j - x_j^+| < \psi_\alpha(\varepsilon), \qquad \text{for } j = iq+1, \dots, (i+1)q-1, \quad \forall i > n_0.$$

Since $\psi(\varepsilon)$ goes to zero when ε goes to zero, we have $x(+\infty) = x^+$. □

6 Proof of Proposition 2.9

Lemma 6.1 *For any $x \in X$, if $\limsup_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty$, then*

$$J(x) = \lim_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty.$$

Proof By a contradiction argument, let's assume

$$\liminf_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = A_1 < +\infty.$$

Choose a constant $A_2 > A_1 + 1 - 2B$ with B given in Lemma 2.8. We can find two positive integers $n_0 < n_1$ satisfying

$$\sum_{i=-n_0}^{n_0-1} a_i(x) \ge A_2, \qquad \sum_{i=-n_1}^{n_1-1} a_i(x) \le A_1 + 1.$$

As a result,

$$\sum_{i=-n_1}^{-n_0-1} a_i(x) + \sum_{i=n_0}^{n_1-1} a_i(x) = \sum_{i=-n_1}^{n_1-1} a_i(x) - \sum_{i=-n_0}^{n_0-1} a_i(x) \le A_1 + 1 - A_2 < 2B,$$

This implies one of the following two inequalities must be true,

$$\sum_{i=-n_1}^{-n_0-1} a_i(x) < B; \qquad \sum_{i=n_0}^{n_1-1} a_i(x) < B.$$

This contradicts Lemma 2.8. □

Proof of Proposition 2.9 For the moment we assume J is well defined (this will be shown later), and will prove the second part of the proposition first. For this it is enough to show that if the limit of x_i does not exist or the limit exists but does not equal to u_0 or u_1, as $i \to \pm\infty$, then $J(x) = \lim_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty$.

We only give the details for $i \to +\infty$. The other case is similar. Choosing a proper $\delta > 0$, there is a sequence of positive integers $\{k_j \nearrow +\infty\}_{j\in\mathbb{N}}$, and another sequence of integers $\{i_j \in [k_j, k_{j+1})\}_{j\in\mathbb{N}}$, such that

$$\lim_{j\to+\infty} |x_{k_j} - x_{k_{j+1}}| = 0; \quad \min\{|x_{i_j} - u_0|, |x_{i_j} - u_1|\} \geq \delta, \quad \forall i_j.$$

After passing k_j to a subsequence, we may assume $|x_{k_{j+1}} - x_{k_j}| < \frac{\phi(\delta)}{2C}$, $\forall k_j$. By Lemma 2.8,

$$\sum_{i=k_j}^{k_{j+1}-1} a_i(x) \geq \phi(\delta) - C|x_{k_{j+1}} - x_{k_j}| > \frac{\phi(\delta)}{2}.$$

As a result, for any $n \in \mathbb{Z}^+$,

$$\sum_{i=-k_n}^{k_n-1} a_i(x) \geq B + \sum_{i=k_0}^{k_n-1} a_i(x) \geq B + \sum_{j=1}^{n-1} \sum_{i=k_j}^{k_{j+1}-1} a_i(x) \geq B + \frac{n\phi(\delta)}{2}.$$

Hence $\limsup_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x) = +\infty$. Then by Lemma 6.1, $J(x) = +\infty$.

Next we will show $J(x)$ is well defined. Because of what we have just shown above, it is enough to prove that when $x(\pm\infty) \in U$, $\lim_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x)$ either exists and is finite number, or diverges.

We only give details for the case with $x(-\infty) = u_0$ and $x(+\infty) = u_1$ (the other cases are similar). Set

$$A_1 = \liminf_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x), \quad A_2 = \limsup_{n\to+\infty} \sum_{i=-n}^{n-1} a_i(x).$$

First by Lemma 6.1, if $A_2 = +\infty$, $J(x)$ diverges to $+\infty$. Now let's assume A_2 is a finite number. Then it is enough to show $A_1 = A_2$.

By a contradiction argument, let's assume $A_1 \neq A_2$. Since $A_1 \leq A_2$, this implies $A_1 < A_2$. We can find two sequences of positive integers $\{l_j \nearrow +\infty\}_{j\in\mathbb{N}}$ and $\{n_j \nearrow +\infty\}_{j\in\mathbb{N}}$ satisfying $l_j + 1 < n_j < l_{j+1} - 1$, $\forall j \in \mathbb{Z}^+$, and

$$\lim_{j\to+\infty} \sum_{i=-l_j}^{l_j-1} a_i(x) = A_2 > \lim_{j\to+\infty} \sum_{i=-n_j}^{n_j-1} a_i(x) = A_1.$$

Hence for j large enough,

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) + \sum_{i=l_j}^{n_j-1} a_i(x) = \sum_{i=-n_j}^{n_j-1} a_i(x) - \sum_{i=-l_j}^{l_j-1} a_i(x) < \frac{A_1 - A_2}{2} < 0. \tag{6.1}$$

Meanwhile $x(-\infty) = u_0$ implies $|x_{-n_j} - x_{-l_j}| < \frac{|A_1 - A_2|}{4C}$ for j large enough. Then by Lemma 2.8,

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) \geq -C|x_{-n_j} - x_{-l_j}| > \frac{A_1 - A_2}{4}.$$

Similarly,

$$\sum_{i=l_j}^{n_j-1} a_i(x) \geq -C|x_{n_j} - x_{l_j}| > \frac{A_1 - A_2}{4}.$$

Therefore for j large enough,

$$\sum_{i=-n_j}^{-l_j-1} a_i(x) + \sum_{i=l_j}^{n_j-1} a_i(x) > \frac{A_1 - A_2}{2},$$

which contradicts (6.1). □

7 Proof of Lemmas 2.11 and 2.12

Lemma 7.1 *Let $x = \{x_i\}_{i=n_0}^{n_1}$ and $y = \{y_i\}_{i=n_0}^{n_1}$ be two minimal configurations. Then the intersection of their Aubry graphs $Au(x) \cap Au(y)$ contains at most two points. In particular if it indeed contains two points, they must be the endpoints of the graphs, i.e., $x_{n_0} = y_{n_0}$ and $x_{n_1} = y_{n_1}$.*

Proof A proof can be found in [5] or [17]. □

Lemma 7.2 *If $(\xi, \eta, \zeta) \neq (\xi', \eta, \zeta')$ are two locally minimal configurations of a variational principle h, then*

$$(\xi - \xi')(\zeta - \zeta') < 0.$$

Proof It follows from the conditions H_4 and H_5. For a proof see Mather [14]. □

Proof of Lemma 2.11 We claim $h(x_0, x_1, x_2) \leq h(x_0, \xi, x_2)$, $\forall \xi \in (u_1, +\infty)$. Assume this is not true. There is a $\eta \in (u_1, +\infty)$, such that

$$h(x_0, \eta, x_2) < h(x_0, x_1, x_2) \leq h(x_0, u_1, x_2).$$

By Condition H_3,

$$\begin{aligned} &h(x_0, u_1) + h(u_1, \eta) + h(\eta, u_1) + h(u_1, x_2) \\ &\quad < h(x_0, \eta) + h(\eta, x_2) + h(u_1, u_1) + h(u_1, u_1). \end{aligned}$$

As a result,

$$h(u_1, \eta) + h(\eta, u_1) < h(u_1, u_1) + h(u_1, u_1),$$

which is absurd.

Now both (x_0, x_1, x_2) and (u_1, u_1, u_1) are locally minimal configurations satisfying $(x_0 - u_1)(x_2 - u_1) \geq 0$. If $x_1 = u_1$, it will contradict Lemma 7.2. Hence $x_1 \neq u_1$. □

Proof of Lemma 2.12 First it is easy to see there must be a minimal configuration $z = \{z_i\}_{i=n_0}^{n_1}$ satisfying $z_{n_0} = x_{n_0}$ and $z_{n_1} = x_{n_1}$. If $z_i \in [u_0, u_1]$, $\forall i \in (n_0, n_1) \cap \mathbb{Z}$, by the definition of x,

$$h(x_{n_0}, \ldots, x_{n_1}) = h(z_{n_0}, \ldots, z_{n_1}),$$

so x is a minimal configuration as well.

Assume there is an $n_0 < i_0 < n_1$, such that $z_{i_0} \notin [u_0, u_1]$, then $Au(z)$ has at least two intersections with $Au(u_0)$ or $Au(u_1)$. When the two intersections are the two end points of $Au(z)$, a basic result of Aubry–Mather theory (see [5]) implies $z_i = u_0$ or u_1, $\forall i \in (n_0, n_1) \cap \mathbb{Z}$. This means $z_i \in [u_0, u_1]$, $\forall i \in (n_0, n_1) \cap \mathbb{Z}$.

When the two intersections are not the end points of $Au(z)$, it contradicts Lemma 7.1, because z, u_0 and u_1 are minimal configurations. This proves the first part of the lemma.

For the second part, without loss of generality, we assume $x_{n_0} \notin \{u_0, u_1\}$. By Lemma 2.11, $x_{n_0+1} \notin \{u_0, u_1\}$. By repeating this process, we get $x_i \notin \{u_0, u_1\}$, $\forall i \in (n_0, n_1) \cap \mathbb{Z}$. □

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