Acta Mathematica Sinica, English Series
Jun., 2022, Vol. 38, No. 6, pp. 1041–1053
Published online: June 15, 2022
https://doi.org/10.1007/s10114-022-0246-z
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Imaginary Modules over the Affine Nappi–Witten Algebra

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Abstract In this paper, we consider the imaginary highest weight modules and the imaginary Whittaker modules for the affine Nappi–Witten algebra. We show that simple singular imaginary Whittaker modules at level (κ, c) $(\kappa \in \mathbb{C}^\star)$ are simple imaginary highest weight modules. The necessary and sufficient conditions for these imaginary modules to be simple are given. All simple imaginary modules are classified.

Keywords Affine Nappi–Witten algebra, imaginary Verma modules, imaginary highest weight modules, imaginary Whittaker modules, simple modules

MR(2010) Subject Classification 17B10, 17B20, 17B65, 17B66, 17B68

1 Introduction

Two-dimensional conformal field theory has many applications both in physics and mathematics. One of the richest classes of models on conformal field theory consists of the Wess–Zumino–Novikov–Witten (WZNW) models, which were studied originally within the framework of semisimple (abelian) groups [16]. However, there has been a great interest in WZNW models based on non-abelian non-semisimple Lie groups (see [11, 13, 14]) while few results on WZNW models for non-reductive groups. In the early 1990s, Nappi and Witten showed in [14] that a WNZW model (NW model) based on a central extension of the two-dimensional Euclidean group describes the homogeneous four-dimensional space-time corresponding to a gravitational wave. The corresponding Lie algebra H_4 is called Nappi–Witten algebra and the affine Nappi–Witten algebra $\widehat{H}_4$ is defined to be the central extension of the loop algebra of H_4.

The study of representation theory of the affine Nappi–Witten algebra $\widehat{H}_4$ was started in [13]. In [2], the authors conducted a systematic study of representations of the affine Nappi–Witten algebra and gave a necessary and sufficient condition for each Verma module to be irreducible. A class of polynomial representations for the affine Nappi–Witten algebra was

Received May 17, 2020, revised December 28, 2021, accepted January 25, 2022
Supported by NSF of China (Grant Nos. 11801117, 11801390), the Natural Science Foundation of Guangdong Province, China (Grant No. 2018A030313268) and the General Finacial Grant from the China Postdoctoral Science Foundation (Grant No. 2016M600140)
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constructed in [4]. The authors classified a class of simple restricted modules for the affine Nappi–Witten algebra in [8]. Verma modules for two classes of twisted affine Nappi–Witten algebras were studied in [6].

In the present paper, we study a new class of modules over $\widehat{H}_4$ which we call the imaginary modules. It consists of two subclasses: the imaginary highest weight modules and the imaginary Whittaker modules. Whittaker modules were first introduced for $\mathfrak{sl}_2(\mathbb{C})$ by Arnal and Pinzcon [1]. In [10], Kostant studied Whittaker modules in the setting of complex semisimiple Lie algebras $\mathfrak{g}$ and showed that irreducible Whittaker modules correspond to maximal ideals of the center of $U(\mathfrak{g})$. The definition of Whittaker modules can be easily generalized to other Lie algebras with a triangular decomposition (see [3, 10, 15]). For imaginary modules, V. Futorny studies imaginary Verma modules for affine Lie algebras [7]. In [3], Christodoulopoulou constructed imaginary Whittaker modules for a non-twisted affine Lie algebra from irreducible Whiitaker modules of its Lie subalgebra $\widetilde{\mathfrak{t}}$, an infinite-dimensional Heisenberg algebra adjoining a degree derivation. Xu studied the simplicity of a family of weighted imaginary Whittaker modules for affine Nappi–Witten algebra (see [17]). In this paper, we will classify all simple imaginary Whittaker modules.

For the affine Nappi–Witten algebra $\widehat{H}_4$, we give two definitions for imaginary modules. In the first definition, imaginary Whittaker (highest weight) modules are realized as generalized Verma modules from irreducible Whittaker (highest weight) modules of an infinite-dimensional Heisenberg algebra as in [3]. In the second definition, imaginary Whittaker (highest weight) modules are realized as Whittaker (highest weight) modules associated with a triangular decomposition, which is different from the triangular decomposition given in [2]. We show the two definitions for imaginary modules coincide to some extent (see Remark 2.5).

Since $\widehat{H}_4$ is a central extension of loop algebra of H_4, the central element K acts as a scalar, called the central charge, on an irreducible module of $\widehat{H}_4$. Also, we call an $\widehat{H}_4$-module of central charge l a module at level l. We study the simplicity of imaginary modules and give a classification of simple imaginary modules.

This paper is organized as follows. In Section 2, we recall the definition of the affine Nappi–Witten algebra $\widehat{H}_4$ and introduce the two definitions of imaginary modules for $\widehat{H}_4$ (Definitions 2.1 and 2.4). We also collect some basic results in this section. In Section 3, we determine the set of imaginary Whittaker vectors (see Proposition 3.2). And we give a classification of simple imaginary modules (see Theorem 4.3, Theorem 4.8 and Theorem 4.10).

Throughout the paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}^\star, \mathbb{N}$, $\mathbb{C}$ and $\mathbb{C}^\star$ the set of all integers, nonnegative integers, nonzero integers, positive integers, complex numbers and nonzero complex numbers, respectively. Denote by V^* the dual space of a vector space V.

2 Preliminaries

We will recall some basic definitions and results in this section.

Verma Modules and Whittaker Modules Let $\mathfrak{g}$ be a Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ where $\mathfrak{g}_0, \mathfrak{g}_\pm$ are Lie subalgebras and $\mathfrak{g}_0$-modules. In this paper, all $\mathfrak{g}_0$'s are finite dimensional. One can define Verma modules and Whittaker modules over $\mathfrak{g}$ as follows.

Definition 2.1 *Let $\varphi : \mathfrak{g}_+ \to \mathbb{C}$ be a Lie algebra homomorphism and let V be a $\mathfrak{g}$-module.*

1. *A nonzero vector $w \in V$ is called a* weight vector *if $xw = \lambda(x)w$ for some $\lambda \in (\mathfrak{g}_0)^*$ and for all $x \in \mathfrak{g}_0$.*
2. *A nonzero vector $w \in V$ is called a* Whittaker vector of type φ *if $xw = \varphi(x)w$ for all $x \in \mathfrak{g}_+$. In particular, a Whittaker vector of type* 0 *is called a* singular vector*.*
3. *V is called a* highest weight module of weight $\lambda \in (\mathfrak{g}_0)^*$ *if V contains a cyclic singular weight vector of weight λ.*
4. *V is called a* Whittaker module of type φ *if V contains a cyclic Whittaker vector of type φ. If $\varphi \neq 0$, we call V a* nonsingular Whittaker module of type φ*, otherwise, we call V a* singular Whittaker module*.*

Let Z be the finite dimensional center of $\mathfrak{g}$. For any $\mathfrak{g}$-module on which any $z \in Z$ acts as a scalar $\mathfrak{z}(z)$ with $\mathfrak{z} \in Z^*$, we call it a $\mathfrak{g}$-module at level $\mathfrak{z}$. Denote by $\mathbb{C}_\varphi$ the 1-dimensional $\mathfrak{g}_+$-module with respect to the Lie algebra homomorphism $\varphi : \mathfrak{g}_+ \to \mathbb{C}$. Then $\mathbb{C}_\varphi$ becomes a $(\mathfrak{g}_+ + Z)$-module $\mathbb{C}_{\mathfrak{z},\varphi}$ by letting the action of Z on $\mathbb{C}_\varphi$ as $\mathfrak{z}(z)$ for some $\mathfrak{z} \in Z^*$ and all $z \in Z$.

Set

$$M(\mathfrak{z}, \varphi)_\mathfrak{g} = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+ + Z)} \mathbb{C}_{\mathfrak{z},\varphi}. \tag{2.1}$$

Define an action of $U(\mathfrak{g})$ on $M(\mathfrak{z}, \varphi)_\mathfrak{g}$ by left multiplication. $M(\mathfrak{z}, \varphi)_\mathfrak{g}$ is a Whittaker $\mathfrak{g}$-module at level $\mathfrak{z}$, called the *universal Whittaker module of type φ at level $\mathfrak{z}$*.

For φ with $\varphi([\mathfrak{g}_0, \mathfrak{g}_+]) = 0$, $\mathbb{C}_{\mathfrak{z},\varphi}$ becomes a $(\mathfrak{g}_+ + \mathfrak{g}_0 + Z)$-module $\mathbb{C}_{\lambda,\mathfrak{z},\varphi}$ by letting the action of $\mathfrak{g}_0$ on $\mathbb{C}_{\mathfrak{z},\varphi}$ as $\lambda(x)$ for some $\lambda \in (\mathfrak{g}_0)^*$ and all $x \in \mathfrak{g}_0$. Set

$$M(\lambda, \mathfrak{z}, \varphi)_\mathfrak{g} = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_+ + \mathfrak{g}_0 + Z)} \mathbb{C}_{\lambda,\mathfrak{z},\varphi}. \tag{2.2}$$

Define an action of $U(\mathfrak{g})$ on $M(\lambda, \mathfrak{z}, \varphi)_\mathfrak{g}$ by left multiplication. $M(\lambda, \mathfrak{z}, 0)_\mathfrak{g}$ is a highest weight $\mathfrak{g}$-module of weight λ at level $\mathfrak{z}$, called the *Verma module of weight λ at level $\mathfrak{z}$*. Clearly, $M(\lambda, \mathfrak{z}, 0)_\mathfrak{g} = M(\mathfrak{z}, 0)/\sum_{x\in\mathfrak{g}_0} U(\mathfrak{g})(x - \lambda(x))(1 \otimes 1)$.

$M(\mathfrak{z}, \varphi)_\mathfrak{g}$ ($M(\lambda, \mathfrak{z}, 0)_\mathfrak{g}$, respectively) is universal in the sense that any simple Whittaker $\mathfrak{g}$-module of type φ (highest weight $\mathfrak{g}$-module of weight λ, respectively) at level $\mathfrak{z}$ is a simple quotient of $M(\mathfrak{z}, \varphi)_\mathfrak{g}$ ($M(\lambda, \mathfrak{z}, 0)_\mathfrak{g}$, respectively). Furthermore, any simple highest weight module is a simple singular Whittaker module. However, the converse is not always true, for example, quasi-Whittaker modules defined in [5] provide an example of simple singular Whittaker modules which are not highest weight modules for the Schrödinger algebra. Denote by $\mathbf{1} = 1 \otimes 1 \in M(\mathfrak{z}, \varphi)_\mathfrak{g}$ ($M(\lambda, \mathfrak{z}, 0)_\mathfrak{g}$, respectively) the cyclic Whittaker (singular weight, respectively) vector.

The Affine Nappi–Witten Algebra The *Nappi–Witten Lie algebra* H_4 is a four dimensional vector space

$$H_4 = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c \oplus \mathbb{C}d$$

equipped with the bracket relations

$$[a, b] = c, \quad [d, a] = a, \quad [d, b] = -b, \quad [c, H_4] = 0.$$

Let $(\ ,\ )$ be the symmetric bilinear form on H_4 defined by

$$\begin{cases}(a,b)=(c,d)=1,\\ (\ ,\ )=0, \quad \text{otherwise.}\end{cases}$$

It is straightforward to check that $(\ ,\ )$ is the unique (up to multiplication of scalars on (a,b) and (c,d) respectively) non-degenerate H_4-invariant symmetric bilinear on H_4.

To the pair $(H_4,(\ ,\ ))$, we associate the *affine Nappi–Witten Lie algebra* $\widehat{H}_4$ with the underlying vector space

$$\widehat{H}_4 = H_4 \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$$

equipped with the bracket relations

$$\begin{aligned}&[h_1\otimes t^m, h_2\otimes t^n] = [h_1,h_2]\otimes t^{m+n} + m(h_1,h_2)\delta_{m+n,0}K, \quad \forall h_1,h_2\in H_4,\ m,n\in\mathbb{Z},\\ &[\widehat{H}_4, K]=0.\end{aligned}$$

Denote any element $x\otimes m\in\widehat{H}_4$ by $x(m)$.

The quotient algebra $\widetilde{H}_4 = \widehat{H}_4/\mathbb{C}K$ is the *loop Nappi–Witten algebra*. Denote by any $\mathfrak{z}\in(\mathbb{C}K\oplus\mathbb{C}c)^*$ by (κ,z). Clearly, the category of $\widehat{H}_4$-modules at level $(0,c)$ is equivalent to the category of $\widetilde{H}_4$-modules at level c.

Triangular Decompositions for the Affine Nappi–Witten Algebra It is clear that the Lie algebra $\widehat{H}_4$ is $\mathbb{Z}$-graded:

$$\widehat{H}_4 = \coprod_{n\in\mathbb{Z}} \widehat{H}_4^{(n)},$$

where $\widehat{H}_4^{(0)} = H_4\oplus\mathbb{C}K$, $\widehat{H}_4^{(n)} = H_4\otimes t^n$, $n\neq 0$.

Then one has the following triangular decomposition for $\widehat{H}_4$ (see [2]):

$$\widehat{H}_4 = \widehat{H}_4^{(+)}\oplus\widehat{H}_4^{(0)}\oplus\widehat{H}_4^{(-)}, \tag{2.3}$$

where $\widehat{H}_4^{(\pm)} = \coprod_{n\in\mathbb{N}}\widehat{H}_4^{(\pm n)}$.

Define

$$\begin{aligned}\widehat{H}_4^{+} &= (\mathbb{C}c\oplus\mathbb{C}d)\otimes t\mathbb{C}[t]\oplus\mathbb{C}a\otimes\mathbb{C}[t,t^{-1}],\\ \widehat{H}_4^{0} &= \mathbb{C}c\oplus\mathbb{C}d\oplus\mathbb{C}K,\\ \widehat{H}_4^{-} &= (\mathbb{C}c\oplus\mathbb{C}d)\otimes t^{-1}\mathbb{C}[t^{-1}]\oplus\mathbb{C}b\otimes\mathbb{C}[t,t^{-1}].\end{aligned}$$

It is easy to check that these are subalgebras of $\widehat{H}_4$ and

$$\widehat{H}_4 = \widehat{H}_4^{+}\oplus\widehat{H}_4^{0}\oplus\widehat{H}_4^{-} \tag{2.4}$$

is another triangular decomposition for $\widehat{H}_4$.

Simple Highest Weight Modules and Whittaker Modules for the Heisenberg Algebra The subalgebra $\mathfrak{h}$ spanned by $\{c(i), d(j), K \mid i,j\in\mathbb{Z}^{\star}\}$ is isomorphic to an *infinite dimensionial Heisenberg Lie algebra*. $\mathfrak{h}$ has a classical triangular decomposition

$$\mathfrak{h} = \mathfrak{h}_+\oplus\mathfrak{h}_0\oplus\mathfrak{h}_-, \tag{2.5}$$

where $\mathfrak{h}_+ = \mathrm{span}_{\mathbb{C}}\{c(i), d(j) \mid i, j \in \mathbb{N}\}$, $\mathfrak{h}_0 = \mathbb{C}K$, $\mathfrak{h}_- = \mathrm{span}_{\mathbb{C}}\{c(i), d(j) \mid i, j \in -\mathbb{N}\}$. With respect to this triangular decomposition, highest weight modules and Whittaker modules for the Heisenberg algebra $\mathfrak{h}$ are defined. Moreover, singular Whittaker modules for $\mathfrak{h}$ are highest weight modules.

Classification of simple highest weight modules of infinite-dimensional Heisenberg Lie algebra is well-known (see [9]).

Theorem 2.2 *The Verma module $M(\kappa, \kappa, 0)_{\mathfrak{h}} = M(\kappa, 0)_{\mathfrak{h}}$ over the infinite-dimensional Heisenberg algebra at level $\kappa \in \mathbb{C}$ is simple if and only if $\kappa \neq 0$. Any simple quotient of $M(0, 0)_{\mathfrak{h}}$ is one-dimensional.*

The following theorem in [3] gives a classification of simple nonsingular Whittaker modules of infinite-dimensional Heisenberg algebra.

Theorem 2.3 *Let $\phi : \mathfrak{h}_+ \to \mathbb{C}$ be a nonzero Lie algebra homomorphism and $\kappa \in \mathbb{C}$.*

1. *For $\kappa \neq 0$, the universal Whittaker module $M(\kappa, \phi)_{\mathfrak{h}}$ is the unique (up to isomorphism) irreducible Whittaker $\mathfrak{h}$-module of type ϕ at level κ.*
2. *The universal Whittaker module $M(0, \phi)_{\mathfrak{h}}$ is reducible, and any simple quotient of $M(0, \phi)_{\mathfrak{h}}$ is one-dimensional.*

Clearly, any simple quotient of $M(0, \phi)_{\mathfrak{h}}$ for some Lie algebra homomorphism $\phi : \mathfrak{h}_+ \to \mathbb{C}$, is isomorphic to

$$L(\tilde{\phi}) = M(0, \phi)_{\mathfrak{h}} / U(\mathfrak{h}) \sum_{i \in \mathbb{N}} ((c(-i) - \tilde{\phi}(c(-i)))\mathbf{1} + (d(-i) - \tilde{\phi}(d(-i)))\mathbf{1}),$$

where $\tilde{\phi} : \mathfrak{h} \to \mathbb{C}$ is a Lie algebra homomorphism with $\tilde{\phi}|_{\mathfrak{h}_+} = \phi$.

Imaginary Modules Now let us give the construction of imaginary Verma modules and imaginary Whittaker modules for $\widehat{H}_4$.

Let $\mathfrak{n}^+ = \mathbb{C}a \otimes \mathbb{C}[t, t^{-1}]$, $\mathfrak{n}^- = \mathbb{C}b \otimes \mathbb{C}[t, t^{-1}]$. Then $\widehat{H}_4$ has the following decomposition:

$$\widehat{H}_4 = \mathfrak{n}^+ \oplus (\mathbb{C}c \oplus \mathbb{C}d \oplus \mathfrak{h}) \oplus \mathfrak{n}^-.$$

The subalgebra $\mathfrak{p} = \mathfrak{n}^+ \oplus (\mathbb{C}c \oplus \mathbb{C}d \oplus \mathfrak{h})$ is a parabolic subalgebra of $\widehat{H}_4$ and $\mathfrak{n}^+$ is an ideal of $\mathfrak{p}$. Following from [3, 7], we can define the imaginary Verma modules and imaginary Whittaker modules over $\widehat{H}_4$ as the following generalized Verma modules.

Definition 2.4 *Let V be a simple $\mathfrak{h}$-module. One can define a $U(\mathfrak{p})$-module structure on V be letting*

$$cw = \lambda(c)w, \quad dw = \lambda(d)w, \quad \mathfrak{n}^+ w = 0, \quad \forall w \in V,$$

where $\lambda \in (\mathbb{C}c \oplus \mathbb{C}d)^*$. Set $\widetilde{V}(\lambda) = U(\widehat{H}_4) \otimes_{U(\mathfrak{p})} V$. Define an action of $U(\widehat{H}_4)$ on $\widetilde{V}(\lambda)$ by left multiplication on $U(\widehat{H}_4)$.

1. *If V is a simple highest weight module for $\mathfrak{h}$, then $\widetilde{V}(\lambda)$ is called an* imaginary highest weight module *for $\widehat{H}_4$.*
2. *If V is a simple Whittaker module for $\mathfrak{h}$, then $\widetilde{V}(\lambda)$ is called an* imaginary Whittaker module *for $\widehat{H}_4$.*

Let $\varphi : \widehat{H}_4^+ \to \mathbb{C}$ be a nonzero Lie algebra homomorphism. Since

$$\varphi(a(i)) = [\varphi(d(1)), \varphi(a(i-1))] = 0, \quad \forall i \in \mathbb{Z},$$

φ is uniquely determined by its restriction on $(\mathbb{C}c\oplus\mathbb{C}d)\otimes t\mathbb{C}[t]$. We also denote the restriction by φ for convenience.

With respect to the triangular decomposition (2.4), we can also define highest weight modules and Whittaker modules in the classical way. We also call them imaginary highest weight modules and imaginary Whittaker modules, respectively. In particular, we call Whittaker vectors, singular vectors and weight vectors with respective to the triangular decomposition (2.4) imaginary Whittaker vectors, imaginary singular vectors and imaginary weight vectors, respectively. For any $\mathfrak{z}\in(\mathbb{C}K+\mathbb{C}c)^*$ with $(\mathfrak{z}(K),\mathfrak{z}(c))=(\kappa,c)\in\mathbb{C}^2$, we write $M(\mathfrak{z},\varphi)_{\widehat{H}_4}$ as $M(\kappa,c,\varphi)_{\widehat{H}_4}$. We call any simple quotient of $M(\kappa,c,\varphi)_{\widehat{H}_4}$ a *simple imaginary module of type* φ *at level* (κ,c). Also, we write $M(\lambda,\kappa,c,\varphi)_{\widehat{H}_4}$ for $M(\lambda,\mathfrak{z},\varphi)_{\widehat{H}_4}$ with $\lambda\in(\mathbb{C}d)^*\cong\mathbb{C}$.

Remark 2.5 Here is the reason why we call modules with respect to the decomposition (2.4) imaginary modules. Suppose V is a simple Whittaker (higheset weight, respectively) module over $\mathfrak{h}$ with cyclic vector w, then w is a cyclic imaginary Whittaker vector (imaginary singular weight vector, respectively) for the imaginary module $\widetilde{V}(\lambda)$ since for $i\in\mathbb{Z}$, and $j,k\in\mathbb{N}$,

$$a(i)w=0,\quad c(j)w=\varphi(c(j))w,\quad d(k)w=\varphi(d(k))w,\quad cw=\lambda(c)w,\quad dw=\lambda(d)w$$

for some φ. Hence $\widetilde{V}(\lambda)$ is a quotient of $M(\kappa,\lambda(c),\varphi)_{\widehat{H}_4}$ $(M(\lambda(d),\kappa,\lambda(c),0)_{\widehat{H}_4}$, respectively) by the $\widehat{H}_4$-homomorphism mapping $\mathbf{1}$ to w.

The goal of this paper is to classify all simple imginary Whittaker modules over $\widehat{H}_4$. To classify these modules, it suffices to classify all simple quotients of $M(\kappa,c,\varphi)_{\widehat{H}_4}$ for any $\kappa,c\in\mathbb{C}$ and Lie algebra homomorphism φ.

3 Imaginary Whittaker Vectors and Singular Weight Vectors

In this section, we will determine the set of all imaginary Whittaker (singular weight, respectively) vectors in $M(\kappa,c,\varphi)_{\widehat{H}_4}(\varphi\neq 0)$ $(M(\lambda,\kappa,c)_{\widehat{H}_4}$, respectively).

Partitions and Basis for Imaginary Modules To prove our main result, we need some concepts on partitions.

A *non-decreasing partition* is a sequence $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)$ of integers in non-decreasing order and we denote $\mathcal{P}$ for the set of all non-decreasing partitions. A *nonnegative* (*positive, respectively*) *non-increasing partition* is a sequence $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)$ of non-negative (positive, respectively) integers in non-increasing order and we denote $\mathcal{T}$ ($\mathcal{T}^+$, respectively) for the set of all nonnegative (positive, respectively) non-increasing partitions. For a partition $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)$, we call r the *length* of α, denoted by $l(\alpha)$.

Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)$ and $\beta=(\beta_1,\beta_2,\dots,\beta_s)$ be two non-decreasing partitions. Define an ordering on $\mathcal{P}$ as follows: $\alpha\succ\beta$ if $r>s$; or if $r=s$ and

$$\alpha_1=\beta_1,\quad \alpha_2=\beta_2,\dots,\alpha_{i-1}=\beta_{i-1},\quad \alpha_i<\beta_i$$

for some i, $1\le i\le r$.

For a sequence $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_r)$ of integers and $x\in H_4$, we denote

$$\begin{aligned}-\alpha&:=(-\alpha_1,-\alpha_2,\dots,-\alpha_r),\\ x(\alpha)&:=x(\alpha_1)x(\alpha_2)\cdots x(\alpha_r),\\ x(-\alpha)&:=x(-\alpha_1)x(-\alpha_2)\cdots x(-\alpha_r).\end{aligned}$$

It is clear that if α is a partition in $\mathcal{T}$, then $-\alpha$ is a partition in $\mathcal{P}$. In particular, we view $\emptyset$ as a non-decreasing partition as well as a non-negative non-increasing partition and set

$$l(\emptyset) = 0, \quad x(\emptyset) = 1, \quad \forall x \in H_4.$$

For an integer i and a non-decreasing (non-increasing) partition α, we denote $m_i(\alpha)$ the times i occurring in α. We construct a new partition called α *minus* i, denoted by $\alpha \setminus i$, as follows: if i occurs in α, we delete one i from α; if i does not occur in α, we keep the partition invariant.

Following from PBW theorem, we know that $M(\kappa, c, \varphi)_{\widehat{H}_4}(\varphi \neq 0)$ has a basis

$$\{b(\alpha)c(-\beta)d(-\gamma)\mathbf{1} \mid \alpha \in \mathcal{P}, \beta \in \mathcal{T}^+, \gamma \in \mathcal{T}\},$$

and $M(\lambda, \kappa, c, \varphi)_{\widehat{H}_4}$ has a basis

$$\{b(\alpha)c(-\beta)d(-\gamma)\mathbf{1} \mid \alpha \in \mathcal{P}, \beta, \gamma \in \mathcal{T}^+\}.$$

Also, $\widetilde{M(\kappa, \phi)_{\mathfrak{h}}}(\lambda)(\kappa \neq 0)$ has a basis

$$\{b(\alpha)c(-\beta)d(-\gamma)\mathbf{1} \mid \alpha \in \mathcal{P}, \beta, \gamma \in \mathcal{T}^+\},$$

and $\widetilde{L(\tilde{\phi})}(\lambda)$ has a basis

$$\{b(\alpha)\mathbf{1} \mid \alpha \in \mathcal{P}\}.$$

Since we have defined an ordering on $\mathcal{P}$, there is a natural lexicographic ordering on these bases.

Imaginary Whittaker Vectors and Singular Weight Vectors Now, let us determine the set of all imaginary Whittaker (singular weight) vectors.

Following from straightforward computation using PBW theorem, we have the following lemma.

Lemma 3.1 *Let v be an imaginary Whiitaker (Whittaker weight, respectively) vector of $M(\kappa, c, \varphi)_{\widehat{H}_4}$ ($M(\lambda, \kappa, c, \varphi)_{\widehat{H}_4}$, respectively). Then following equations hold for any $i \in \mathbb{Z}$, $j \in \mathbb{N}$ ($\mathbb{Z}_+$, respectively), $\beta \in \mathcal{T}^+, \gamma \in \mathcal{T}$ ($\mathcal{T}^+$, respectively),*

$$\begin{aligned}
&a(i)c(-\beta)d(-\gamma)v = 0,\\
&c(j)c(-\beta)d(-\gamma)v = jm_j(\gamma) \cdot \kappa \cdot c(-\beta)d(-\gamma \setminus j)v + \varphi(c(j)) \cdot c(-\beta)d(-\gamma)v,\\
&d(j)c(-\beta)d(-\gamma)v = -jm_j(\beta) \cdot \kappa \cdot c(-\beta \setminus j)d(-\gamma)v + \varphi(d(j)) \cdot c(-\beta)d(-\gamma)v.
\end{aligned}$$

Proposition 3.2 gives the set of all imaginary Whittaker (Whittaker weight, respectively) vectors in $M(\kappa, c, \varphi)_{\widehat{H}_4}$ ($M(\lambda, \kappa, c, \varphi)_{\widehat{H}_4}$, respectively).

Proposition 3.2 1. *Any imaginary Whittaker vector in $M(0, c, \varphi)_{\widehat{H}_4}$ is a linear combination of*

$$\{c(-\beta)d(-\gamma)\mathbf{1} \mid \beta \in \mathcal{T}^+, \gamma \in \mathcal{T}\}.$$

2. *Any imaginary singular weight vector in $M(\lambda, 0, c, 0)_{\widehat{H}_4}$ is a linear combination of*

$$\{c(-\beta)d(-\gamma)\mathbf{1} \mid \beta, \gamma \in \mathcal{T}^+\}.$$

3. *Suppose $\kappa \neq 0$. Then the set of all imaginary Whittaker vectors in $M(\kappa, c, \varphi)_{\widehat{H}_4}$ is*

$$\mathbb{C}[d]\mathbf{1} \setminus \{0\}.$$

4. *Suppose $\kappa \neq 0$. Then the set of all imaginary Whittaker weight vectors in $M(\lambda,\kappa,c,\varphi)_{\widehat{H}_4}$ is $\mathbb{C}^\star\mathbf{1}$.*

Proof We only prove the first statement, similar arguments deduce the other statements. It follows from Lemma 3.1 that any linear combination of $\{c(-\beta)d(-\gamma)\mathbf{1} \mid \beta \in \mathcal{T}^+, \gamma \in \mathcal{T}\}$ is an imaginary Whittaker vector in $M(0,c,\varphi)_{\widehat{H}_4}$. Now we consider a nonzero vector X of the form

$$X = \sum_{i=1}^{n} k_i b(\alpha^i)c(-\beta^i)d(-\gamma^i)\mathbf{1}$$

with $k_i \in \mathbb{C}^*$, $\alpha^1 \neq \emptyset$ and $b(\alpha^i)c(-\beta^i)d(-\gamma^i)$ decreasing in the lexicographic order on the PBW basis. By Lemma 3.1,

$$a(j)X = \sum_{i=1}^{n} k_i [a(j), b(\alpha^i)]c(-\beta^i)d(-\gamma^i)\mathbf{1}.$$

Set $m = l(\alpha^1)$. Denote s the maximal positive integer satisfying the following condition:

$$l(\alpha^i) = m, \quad \alpha^i_t = \alpha^1_t, \quad \forall 1 \leq i \leq s,\ 1 \leq t \leq m-1.$$

Set $r = \max\{l(\beta^1), l(\beta^2), \dots, l(\beta^s)\}$ and take p the minimal positive integer satisfying

$$p \leq s, \quad l(\beta^p) = r.$$

Then for sufficiently small j, the vector $a(j)X$ is nonzero with the leading term

$$b(\alpha^p_1)b(\alpha^p_2)\cdots b(\alpha^p_{m-1})c(j+\alpha^p_m)c(-\beta^p)d(-\gamma^p)\mathbf{1}.$$

Hence X is not an imaginary Whittaker vector. We finish the proof. □

4 Simple Imaginary Modules

In this section, we will classify all simple imaginary Whittaker modules.

4.1 Simple Imaginary Modules at Level (κ, c)

First let us classify all simple imaginary modules at level (κ, c) for $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$. Throughout this subsection, we assume that $\kappa \neq 0$. In the following lemma, we establish a bijection between the set of submodules of $M(\kappa,c,\varphi)_{\widehat{H}_4}$ and the set of ideals of the polynomial ring $\mathbb{C}[d]$.

Lemma 4.1 *Let $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$. Then there is a one-one correspondence between the set of proper submodules of $M(\kappa,c,\varphi)$ and the set of proper ideals of $\mathbb{C}[d]$.*

Proof For any proper submodule W of $M(\kappa,c,\varphi)$, let $S = \{f(d) \mid f(d)\mathbf{1} \in W\} = W \cap \mathbb{C}[d]$. Then the lemma holds if $W = U(\widehat{H}_4)S\mathbf{1}$. It is trivial that $W \supseteq U(\widehat{H}_4)S\mathbf{1}$. The hard part is to prove that $W \subseteq U(\widehat{H}_4)S\mathbf{1}$. By PBW theorem, we write a nonzero vector w in W as $\Sigma_{i=1}^N b(\alpha^i)c(-\beta^i)d(-\gamma^i)g_i(d)\mathbf{1}$, with $(l(\alpha^i), l(\beta^i), \alpha^i, -\beta^i, -\gamma^i)$ decreasing in the natural lexicographic order on the product set $\mathbb{Z}^2 \times \mathcal{P}^3$ and $g_i(d) \in \mathbb{C}[d]$. To prove $w \in U(\widehat{H}_4)S\mathbf{1}$, we need to prove $g_i(d) \in S$. Indeed we only need to prove $g_1(d) \in S$. For sufficient small r, the leading term of $a(r)^p w$ is a nonzero multiple of

$$c(r+\alpha^1_1)c(r+\alpha^1_2)\cdots c(r+\alpha^1_p)c(-\beta^1_q)\cdots c(-\beta^1_2)c(-\beta^1_1)d(-\gamma^1_s)\cdots d(-\gamma^1_2)d(-\gamma^1_1)g_1(d)\mathbf{1},$$

where $p = l(\alpha^1)$, $q = l(\beta^1)$ and $s = l(\gamma^1)$. Then

$$\prod_{\substack{1\le k\le s\\1\le j\le q\\1\le i\le p}} (c(\gamma^1_{s+1-k}) - \varphi(c(\gamma^1_{s+1-k})))$$
$$(d(\beta^1_{q+1-j}) - \varphi(d(\beta^1_{q+1-j})))(d(-r-\alpha^1_i) - \varphi(d(-r-\alpha^1_i)))a(r)^p w$$

is a nonzero multiple of $g_1(d)\mathbf{1}$. Hence $g_1(d) \in S$. We finish the proof. □

To classify all simple imaginary modules at level (κ, c) with $\kappa \neq 0$, we only need the maximal submodules of $M(\kappa, c, \varphi)$.

Lemma 4.2 *Let $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$ and let W be a maximal submodule of $M(\kappa, c, \varphi)$. Then there exists $\lambda \in \mathbb{C}$ such that $W \cong U(\widehat{H}_4)(d-\lambda)\mathbf{1}$.*

Proof Let $S = \{f(d) \mid f(d)v \in W\} = W \cap \mathbb{C}[d]$. By Lemma 4.1, S is a maximal ideal of $\mathbb{C}[d]$ and hence $S = (d-\lambda)\mathbb{C}[d]$ for $\lambda \in \mathbb{C}$. Then we finish the proof. □

Theorem 4.3 gives a classification of simple imaginary Whittaker modules of type φ at level (κ, c) for $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$.

Theorem 4.3 *Let $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$ and let V be a simple imaginary Whittaker modules of type φ for $\widehat{H}_4$ at level (κ, c). Then $V \cong M(\lambda, \kappa, c, \varphi)_{\widehat{H}_4}$ for some $\lambda \in \mathbb{C}$.*

Proof Following from the universal property of $M(\kappa, c, \varphi)_{\widehat{H}_4}$, there exists a surjective module homomorphism $\phi : M(\kappa, c, \varphi) \to V; \mathbf{1} \mapsto w$. So V is isomorphic to some simple quotient of $M(\kappa, c, \varphi)_{\widehat{H}_4}$, and therefore by Lemma 4.2, $V \cong M(\lambda, \kappa, c, \varphi)_{\widehat{H}_4}$ for some $\lambda \in \mathbb{C}$. □

Following from Theorem 4.3, we have

Corollary 4.4 *Let $\kappa \in \mathbb{C}^\star, c \in \mathbb{C}$.*

1. *Any simple imaginary Whittaker module at level (κ, c) is a simple imaginary weight module. In particular, any simple singular imaginary Whittaker module is a simple imaginary highest weight module.*
2. *$\widetilde{M(\kappa, \varphi)}_{\mathfrak{h}}(\lambda)$ $(\kappa \neq 0)$ is simple for any $\lambda \in (\mathbb{C}c \oplus \mathbb{C}d)^*$.*

4.2 Simple Imaginary Modules at Level $(0, c)$

Now let us study simple imaginary modules at level $(0, c)$ for $c \in \mathbb{C}$. We will consider such modules as $\widetilde{H}_4$-modules at level c. Following from Schur's Lemma and the fact $c(i)$'s are central in $\widetilde{H}_4$, we have

Lemma 4.5 *Let $c \in \mathbb{C}$ and V be a simple imaginary Whittaker module over $\widehat{H}_4$ of type φ at level $(0, c)$. Then V is a quotient of $\widehat{L}(\tilde{\varphi})$ for some Lie algebra homomorphism*

$$\tilde{\varphi} : \mathrm{span}_{\mathbb{C}}\{c(i) \mid i \in \mathbb{Z}\} \to \mathbb{C}$$

with

$$\tilde{\varphi}|_{\mathrm{span}_{\mathbb{C}}\{c(i)\,|\,i\in\mathbb{N}\}} = \varphi, \quad \tilde{\varphi}(c) = c,$$

where $\widehat{L}(\tilde{\varphi})$ is defined as follows: $\widehat{L}(\tilde{\varphi}) = M(0, c, \varphi)_{\widehat{H}_4} / \sum_{i\in\mathbb{N}} U(\widehat{H}_4)(c(-i) - \tilde{\varphi}(c(-i)))\mathbf{1}$.

To describe our results, we need the following definition in combinatorics.

Definition 4.6 1. *Let* $s \in \mathbb{N}$. *An* order-s homogeneous linear recurrence with constant coefficients *is an equation of the form* $f(n) = \sum_{j=1}^{s-1} r_j f(n+j)$, *where* $r_1, \ldots, r_{s-2} \in \mathbb{C}$ *and* $r_{s-1} \in \mathbb{C}^*$.

2. *A sequence* $\{f(n) \mid n \in \mathbb{Z}\}$ *is a* constant-recursive sequence of order s *if there exists* $s \in \mathbb{N}$ *and an order-s homogeneous linear recurrence with constant coefficients that it satisfies for all* $n \in \mathbb{Z}$ *and for any* $t < s$, *any order-t homogeneous linear recurrence with constant coefficients does not hold for some* n.

Let $c_i \in \mathbb{C}, i \in \mathbb{Z}$. To classify simple imaginary modules for $\widehat{H}_4$ of type φ at level $(0, c_0)$ on which $c(i)$ acts as c_i for all $i \in \mathbb{Z}$, we need to classify maximal submodules of $\widehat{L}(\tilde{\varphi})$, where $\tilde{\varphi}$ is defined by $\tilde{\varphi}(c(i)) = c_i, i \in \mathbb{Z}$. Suppose that $\{c_i \mid i \in \mathbb{Z}\}$ is not a constant-recursive sequence, we establish a bijection of the set of submodules of $\widehat{L}(\tilde{\varphi})$ and the set of ideals of the polynomial ring $\mathbb{C}[d(-i) \mid i \in \mathbb{Z}_+]$ in the following lemma.

Lemma 4.7 *Suppose that* $\{c_i \mid i \in \mathbb{Z}\}$ *is not a constant-recursive sequence. There is a one-one correspondence between the set of submodules of* $\widehat{L}(\tilde{\varphi})$ *and the set of proper ideals of* $\mathbb{C}[d(-i) \mid i \in \mathbb{Z}_+]$.

Proof For any proper submodule W of $\widehat{L}(\tilde{\varphi})$, let $S = \{f(d) \mid f(d)\mathbf{1} \in W\} = W \cap \mathbb{C}[d(-i) \mid i \in \mathbb{Z}_+]$. We want to prove that $W = U(\widehat{H}_4)S\mathbf{1}$. As in the proof of Lemma 4.1, we only need to prove that $W \subseteq U(\widehat{H}_4)S\mathbf{1}$. A nonzero vector w in W is of the form

$$w = \sum_{k=1}^{n} b(\alpha^k) g_k(d)\mathbf{1},$$

where α^k decreases in $\mathcal{P}$ and $g_k(d) \in \mathbb{C}[d(-i) \mid i \in \mathbb{Z}_+]$. Define the height of w to be the length $l(\alpha^1)$. As in the proof of Lemma 4.1, we only need to prove that $g_1(d) \in S$. We use induction on the height of w. The case of height zero is trivial. Choose p to be the maximal integer satisfying

1. $l(\alpha^1) = l(\alpha^2) = \cdots = l(\alpha^p) = t$;
2. α^i coincides with α^1 in the first $t-1$ entries for every $1 \le i \le p$.

Since $\{c_i \mid i \in \mathbb{Z}\}$ is not a constant-recursive sequence, there exists $j_1 \in \mathbb{Z}$ such that

$$\mathbf{u}_1 = (mc_{\alpha_t^1 + j_1}, c_{\alpha_t^2 + j_1}, \ldots, c_{\alpha_t^p + j_1})$$

is a nonzero vector, where m is the multiplicity of α_t^1 in α^1. Then the dimension of space $W_1 = \{\mathbf{v} \in \mathbb{C}^p \mid (\mathbf{u}_1, \mathbf{v}) = 0\}$ is $p-1$. Choose a nonzero vector $\mathbf{v}_1$ in W_1. Because of the non-constant-recursive condition, there exists $j_2 \in \mathbb{Z}$ such that the inner product of

$$\mathbf{u}_2 = (mc_{\alpha_t^1 + j_2}, c_{\alpha_t^2 + j_2}, \ldots, c_{\alpha_t^p + j_2})$$

and $\mathbf{v}_1$ is nonzero. Applying this procedure several times, we obtain a sequence of nonzero vectors $\mathbf{u}_k = (mc_{\alpha_t^1 + j_k}, c_{\alpha_t^2 + j_k}, \ldots, c_{\alpha_t^p + j_k})$, $1 \le k \le p$, and a sequence of nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$ such that

$$(\mathbf{u}_j, \mathbf{v}_i) = 0, \quad j \le i;$$
$$(\mathbf{u}_{i+1}, \mathbf{v}_i) \ne 0.$$

Therefore we get a sequence of integers $j_1, j_2, \ldots, j_p$ such that the determinant of the following matrix

$$A = \begin{pmatrix} mc_{\alpha_t^1+j_1} & mc_{\alpha_t^1+j_2} & \cdots & mc_{\alpha_t^1+j_p} \\ c_{\alpha_t^2+j_1} & c_{\alpha_t^2+j_2} & \cdots & c_{\alpha_t^2+j_p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\alpha_t^p+j_1} & c_{\alpha_t^p+j_2} & \cdots & c_{\alpha_t^p+j_p} \end{pmatrix}$$

is nonzero. So there exists a column vector

$$\mathbf{k} = (k_1, k_2, \ldots, k_p)^T$$

such that $A\mathbf{k} = (1, 0, 0, \ldots, 0)^T$. Then we know that

$$b(\alpha_1^1)b(\alpha_2^1)\cdots b(\alpha_{t-1}^1)g_1(d)\mathbf{1}$$

is the leading term of the vector

$$(k_1 a(j_1) + k_2 a(j_2) + \cdots + k_p a(j_p))w.$$

By induction on height, $g_1(d)$ lies in S. We finish the proof. □

Theorem 4.8 gives a classification for simple imaginary modules over $\widehat{H}_4$ at zero level under the condition that $\{c_i \,|\, i \in \mathbb{Z}\}$ is not a constant-recursive sequence.

Theorem 4.8 *Suppose $\{c_i \,|\, i \in \mathbb{Z}\} \subseteq \mathbb{C}$ is not a constant-recursive sequence. Let V be a simple imaginary module for $\widehat{H}_4$ of type φ at level $(0, c_0)$ on which $c(i)$ acts as c_i for all $i \in \mathbb{Z}$. Then there exists an extended Lie algebra homomorphism $\hat{\varphi} : \mathfrak{h} \to \mathbb{C}$ such that $V \cong \widetilde{L(\hat{\varphi})}(\lambda)$, where $\lambda = \hat{\varphi}|_{\mathbb{C}c+\mathbb{C}d}$.*

Proof As in the proof of Lemma 4.2, V is a simple quotient of $\widehat{L}(\tilde{\varphi})$ and hence is determined by a maximal ideal S of $\mathbb{C}[d(-i) \,|\, i \in \mathbb{Z}_+]$. By the generalized Hilbert's Nullstellensatz Theorem (see [12]), S is generated by $\{d(-i) - d_{-i} \,|\, i \in \mathbb{Z}_+\}$ for some $d_{-i} \in \mathbb{C}$. Extend $\tilde{\varphi}$ to a Lie algebra homomorphism $\hat{\varphi} : \mathfrak{h} \to \mathbb{C}$ by setting $\hat{\varphi}(d(-i)) = d_{-i}$. Recall the notations in Section 2, we have

$$\widetilde{L(\hat{\varphi})}(\lambda) \cong \widehat{L}(\hat{\varphi}) / \sum_{j \in \mathbb{Z}_+} U(\widehat{H}_4)(d(-j) - \hat{\varphi}(d(-j)))\mathbf{1}.$$

Then we finish the proof. □

Now suppose $\{c_i \,|\, i \in \mathbb{Z}\}$ being a constant-recursive sequence of order s such that

$$c_i = \sum_{j=1}^{s-1} r_j c_{i+j}, \quad \forall i \in \mathbb{Z},$$

where $r_1, r_2, \ldots, r_{s-2} \in \mathbb{C}$ and $r_{s-1} \in \mathbb{C}^*$. We will study the simple imaginary Whittaker modules of type φ at level $(0, c_0)$ on which $c(i)$ acts as c_i for all $i \in \mathbb{Z}$. Let $\mathcal{D} = \sum_{i \in \mathbb{Z}} \mathbb{C}d(i)$ and $\mathcal{B}_\varphi = \sum_{r \in \mathbb{Z}} \mathbb{C}B(r)$ with $B(r) = b(r) - \sum_{j=1}^{s-1} r_j b(r+j)$. Then $\mathcal{D} + \mathcal{B}_\varphi$ is a subalgebra of $\widehat{H}_4$ and any simple $(\mathcal{D}+\mathcal{B}_\varphi)$-module V can be viewed as a simple $\mathcal{F} = (\mathcal{D} + \mathcal{B}_\varphi + \sum_{i \in \mathbb{Z}}(\mathbb{C}c(i) + \mathbb{C}a(i)) + \mathbb{C}K)$-module by letting $c(i)$ acts as c_i and the actions of $a(i)$'s and K are trivial. Denote such a module by $V^{\mathcal{F}}$.

Proposition 4.9 *Let V be a nonzero $(\mathcal{D}+\mathcal{B}_\varphi)$-module. Then $\mathrm{Ind}_{\mathcal{F}}^{\widehat{H}_4}(V^{\mathcal{F}})$ is a simple $\widehat{H}_4$-module at level $(0,c_0)$ if and only if V is a simple $(\mathcal{D}+\mathcal{B}_\varphi)$-module.*

Proof Let V be a nonzero $(\mathcal{D}+\mathcal{B}_\varphi)$-module. Let $\{v_i\,|\,i\in I\}$ be a basis of V, where I is a countable set. Since I is a countable basis, we may define a total order on I. Since $\{B(r),b(i),a(r),c(r),d(r),K\,|\,r\in\mathbb{Z},1\le i\le s-1\}$ is a basis for $\widehat{H}_4$, by PBW theorem, $\{b(1)^{j_1}b(2)^{j_2}\cdots b(s-1)^{j_{s-1}}v_i\,|\,j_1,j_2,\dots,j_{s-1}\in\mathbb{Z}_+,i\in I\}$ is a basis of $\mathrm{Ind}_{\mathcal{F}}^{\widehat{H}_4}(V^{\mathcal{F}})$. A nonzero vector w in $\mathrm{Ind}_{\mathcal{F}}^{\widehat{H}_4}(V^{\mathcal{F}})$ can be written as

$$w=\sum_{p=1}^{N}k_p b(1)^{j_1^p}b(2)^{j_2^p}\cdots b(s-1)^{j_{s-1}^p}v_{i_p}$$

with $(\sum_{h=1}^{s-1}j_h^p,j_1^p,j_2^p,\dots,j_{s-1}^p,i_p)$ decreasing in the natural lexicographic order on $(\mathbb{Z}_+)^s\times I$. Denote t the maximal integer such that j_t^1 is nonzero. Let $\{q_1=1,q_2,\dots,q_M\}$ be the maximal subset of $\{1,2,\dots,N\}$ satisfying

1. $b(1)^{j_1^{q_h}}b(2)^{j_2^{q_h}}\cdots b(s-1)^{j_{s-1}^{q_h}}=b(1)^{j_1^1}b(2)^{j_2^1}\cdots b(t)^{j_t^1-1}b(t_h)$ for $1\le h\le M$, where $t_h\ge t$;
2. $i_1=i_{q_1}=i_{q_2}=\cdots=i_{q_M}$.

It is clear that $M\le s-1$. Then the coefficient of the term $b(1)^{j_1^1}\cdots b(t)^{j_t^1-1}v_{i_1}$ in $a(l)w$ is

$$k_{q_1}j_t^1\xi(c(l+t_1))+k_{q_2}\xi(c(l+t_2))+\cdots+k_{q_M}\xi(c(l+t_M)).$$

Since $\{\xi(c(i))\,|\,i\in\mathbb{Z}\}$ is a constant-recursive sequence of order s, there exists an integer l such that the coefficient is nonzero. We note that the leading term in $a(l)w$ is not necessarily a multiple $b(1)^{j_1^1}\cdots b(t)^{j_t^1-1}v_{i_1}$ but of the form $b(1)^{j_1^1}\cdots b(t)^{j_t^1-1}v_{\tilde{i}}$. Applying this procedure several times, we may reduce w into a nonzero vector in V. □

Theorem 4.10 classifies all simple imaginary Whittaker modules of type φ on which K acts trivially and $c(i)$ acts as c_i for all $i\in\mathbb{Z}$, under the condition that $\{c_i\,|\,i\in\mathbb{Z}\}$ is a constant-recursive sequence.

Theorem 4.10 *Suppose $\{c_i\,|\,i\in\mathbb{Z}\}$ is a constant-recursive sequence of order s. Let V be an imaginary module of type φ for $\widehat{H}_4$ at level $(0,c_0)$ on which $c(i)$ acts as c_i for all $i\in\mathbb{Z}$. Then there exists a simple $(\mathcal{D}+\mathcal{B}_\varphi)$-module W such that*

1. *$d(i)$'s $(i>0)$ have a common eigenvector on W;*
2. *$V\cong\mathrm{Ind}_{\mathcal{F}}^{\widehat{H}_4}(W^{\mathcal{F}})$.*

Proof Let v be the cyclic imaginary Whittaker vector of V and $W=U(\mathcal{D}+\mathcal{B}_\varphi)v$. Then there is a surjective homomorphism from $\mathrm{Ind}_{\mathcal{F}}^{\widehat{H}_4}(W^{\mathcal{F}})$ to V by mapping v to v. Since V is simple, W is a simple $(\mathcal{D}+\mathcal{B}_\varphi)$-module. This completes the proof. □

Acknowledgements The authors would like to thank the referee for valuable suggestions to improve the paper.

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