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p-Laplacian Equations on Locally Finite Graphs

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Abstract This paper is mainly concerned with the following nonlinear p-Laplacian equation

$$-\Delta_p u(x) + (\lambda a(x) + 1)|u|^{p-2}(x)u(x) = f(x, u(x)), \quad \text{in } V$$

on a locally finite graph $G = (V, E)$ with more general nonlinear term, where Δ_p is the discrete p-Laplacian on graphs, $p \geq 2$. Under some suitable conditions on f and $a(x)$, we can prove that the equation admits a positive solution by the Mountain Pass theorem and a ground state solution u_λ via the method of Nehari manifold, for any $\lambda > 1$. In addition, as $\lambda \to +\infty$, we prove that the solution u_λ converge to a solution of the following Dirichlet problem

$$\begin{cases} -\Delta_p u(x) + |u|^{p-2}(x)u(x) = f(x, u(x)), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \{x \in V : a(x) = 0\}$ is the potential well and $\partial\Omega$ denotes the the boundary of Ω.

Keywords p-Laplacian equation, locally finite graph, ground state solution

MR(2010) Subject Classification 35J05, 35J20, 35J60

1 Introduction

In Euclidean space, the nonlinear Schrödinger type equation of the form

$$-\Delta u(x) + V(x)u(x) = f(x, u(x)), \quad \text{in } \Omega$$

has been extensively studied during the past several decades, where $\Omega \subset \mathbb{R}^N$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function and $V(x)$ is a given potential. It has attracted great interest because of its importance in applications. The readers can refer to [2–4, 7, 9, 17, 20, 21, 25–27, 32, 33] and the references therein. In particular, Bartsch and Wang [2] showed the existence of a least energy solution $u_\lambda(x)$ of the following nonlinear Schrödinger equation

$$-\Delta u(x) + (\lambda a(x) + 1)u(x) = u^p(x), \quad u > 0,\, x \in \mathbb{R}^N, \tag{1.1}$$

for large λ, where $1 < p < (N+2)/(N-2), N \geq 3$ and $a(x) \geq 0$. As $\lambda \to \infty$ they proved $u_\lambda(x)$ converged strongly in $H^1(\mathbb{R}^N)$ to a least energy solution of the elliptic problem

$$\begin{cases} -\Delta u + u = u^p, \quad u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

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Recently, there were many works about the differential equations on graphs (cf. [5, 6, 10–16, 18, 19, 22–24, 29–31]). For example, Grigor'yan, Lin and Yang studied some nonlinear elliptic equations on graphs by using the variational method. Specifically, in [13], using the calculus of variations and a method of upper and lower solutions, they studied the Kazdan–Warner equation

$$\Delta u = c - h\mathrm{e}^{u}, \quad \text{in } V \tag{1.2}$$

on finite graphs, where Δ is a discrete graph Laplacian, $c \in \mathbb{R}$ is a constant and $h : V \to \mathbb{R}$ is a function. In [14], for any $p \geq 2$, they proved the existence of nontrivial solutions to the Yamabe type equation

$$\begin{cases} -\Delta u - \alpha u = |u|^{p-2}u, & \text{in } \Omega^{\circ}, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

by using the Mountain Pass theorem, where Ω is a bounded domain on locally finite graphs or finite graphs, Ω° and $\partial\Omega$ denote the interior and the boundary of Ω respectively. And they also considered similar problems involving the p-Laplacian and poly-Laplacian by the same method in [14]. In [15], they established existence results for the equation

$$-\Delta u(x) + h(x)u(x) = f(x, u(x)), \quad x \in V. \tag{1.3}$$

In particular, they proved that the equation (1.3) has strictly positive solutions under the assumption for h and f. Specifically, $h : V \to \mathbb{R}$ and $f : V \times \mathbb{R} \to \mathbb{R}$ are two functions satisfying the assumptions:

(H_1) *There exists a constant* $h_0 > 0$ *such that* $h(x) \geq h_0$ *for all* $x \in V$;

(H_2) $\frac{1}{h} \in L^1(V)$;

(F_1) $f(x,s)$ *is continuous in* $s \in \mathbb{R}$, $f(x,0) = 0$, *and for any fixed* $M > 0$ *there exists a constant* A_M *such that* $\max_{s\in[0,M]} f(x,s) \leq A_M$ *for all* $x \in V$;

(F_2) *There exists a constant* $\theta > 2$ *such that if* $s > 0$ *then there holds*

$$0 < \theta F(x,s) := \theta \int_0^s f(x,t)dt \leq sf(x,s), \quad \forall x \in V;$$

(F_3) $\limsup_{s\to 0^+} \frac{2F(x,s)}{s^2} < \lambda_1 := \inf_{\int_V u^2 = 1 d\mu} \int_V (|\nabla u|^2 + hu^2) d\mu$.

Zhang and Zhao [31] studied the convergence of ground state solutions for the following Schrödinger equation

$$-\Delta u + (\lambda a + 1)u = |u|^{p-1}u, \quad \text{in } V$$

on a locally finite graph, where Δ is discrete graph Laplacian. And Keller and Schwarz [19] studied the Kazdan–Warner equation on canonically compact graphs.

In addition, there were many works about the differential equations on infinite graphs. In [10], under the assumption that $h \leq 0$ and some other conditions, Ge and Jiang proved the existence of a solution to the Kazdan–Warner equation (1.2) on an infinite graph by using a heat flow method. In [11], under the assumption that the graph Laplacian Δ is a bounded operator, g is bounded and h is large at infinity, they also proved the existence of a solution to the graph Yamabe equation

$$\Delta u + hu = g|u|^{p-2}u$$

on an infinite graph.

In this paper, motivated by [15, 31], we consider the nonlinear p-Laplacian equation

$$-\Delta_p u(x) + (\lambda a(x) + 1)|u|^{p-2}(x)u(x) = f(x, u(x)), \quad \text{in } V \tag{1.4}$$

on a locally finite graph $G = (V, E)$ with more general nonlinear term, where Δ_p is the discrete p-Laplacian on graphs, $p \geq 2$ and $f(x, u)$ is continuous in $u \in \mathbb{R}$, for any $x \in V$. We prove that the equation admits a positive solution by the Mountain Pass theorem and a ground state solution u_λ via the method of Nehari manifold, for any $\lambda > 1$. Moreover, as $\lambda \to +\infty$, the solution u_λ converges to a solution of the Dirichlet problem.

Before we state our works, let us introduce some concepts and assumptions. Let $G = (V, E)$ be a graph, where V denotes the set of vertices and E denotes the set of edges, $\omega_{xy} : V \times V \to \mathbb{R}^+$ be an edge weight function and $\mu : V \to \mathbb{R}^+$ be a finite positive function on $G = (V, E)$. We say it is a *uniformly positive measure* if there exists a constant $\mu_{\min} > 0$ such that $\mu(x) \geq \mu_{\min}$ for all $x \in V$. We say that a graph is *locally finite* if for any $x \in V$, there holds $\sum_{y \sim x} 1 < \infty$. A graph is *connected* if any two vertices x and y can be connected via finitely many edges. Note that the information of G contains V, E, μ and ω. Throughout this paper, we always assume that G satisfies the following assumptions.

(G_1) *For any* $xy \in E$, $\omega_{xy} = \omega_{yx} > 0$ *and* $M := \sup_{x \in V} \frac{\deg_x}{\mu(x)} < +\infty$, *where* $\deg_x := \sum_{y \in V} \omega_{xy}$.

(G_2) *G is a locally finite and connected graph with uniformly positive measure.*

The *graph distance* $d(x, y)$ of two vertices $x, y \in V$ is defined by the minimal number of edges which connect these two vertices. We call $\Omega \subset V$ a *bounded domain* in V, if the distance $d(x, y)$ is uniformly bounded from above for any $x, y \in \Omega$. Note that a bounded domain of a locally finite graph contains only finitely many vertices. We denote the boundary of Ω by

$$\partial\Omega := \{y \in V, y \notin \Omega : \exists\, x \in \Omega \text{ such that } xy \in E\}$$

and the interior of Ω by Ω°. Note that $\Omega^\circ = \Omega$ which is different from the Euclidean case.

For any function $u : V \to \mathbb{R}$, the μ-Laplacian (or Laplacian for short) of u is defined as

$$\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)). \tag{1.5}$$

The associated gradient form is defined by

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{1.6}$$

Write $\Gamma(u) = \Gamma(u, u)$. Sometimes we use $\nabla u \nabla v$ instead of $\Gamma(u, v)$. We denote the length of its gradient by

$$|\nabla u|(x) := \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}. \tag{1.7}$$

With respect to the vertex weight μ, the integral of u over V is defined by

$$\int_V u\, d\mu = \sum_{x \in V} \mu(x) u(x). \tag{1.8}$$

The p-Laplacian of $u : V \to \mathbb{R}$, namely $\Delta_p u$, is defined by

$$\Delta_p u(x) := \frac{1}{2\mu(x)} \sum_{y\sim x} (|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))\omega_{xy}(u(y) - u(x)). \tag{1.9}$$

Remark 1.1 Ge [12] studied the following p-th Yamabe equation on a connected finite graph G:

$$\Delta_p \varphi + h\varphi^{p-1} = \lambda f \varphi^{\alpha-1}, \tag{1.10}$$

where Δ_p is the discrete p-Laplacian, h and $f > 0$ are fixed real functions defined on all vertices. He proved that the above equation always has a positive solution for some constant $\lambda \in \mathbb{R}$. However, it is remarkable that their Δ_p considered in the equation (1.10) is different from ours when $p \neq 2$.

To state the main results, we introduce the following assumptions on $f : V \times \mathbb{R} \to \mathbb{R}$ and $a : V \to \mathbb{R}$. Here $\lambda > 1$ and $p \geq 2$ are constants.

(f_1) *For any $x \in V$, $f(x, s)$ is continuous in $s \in \mathbb{R}$, $f(x, 0) = 0$, and for any fixed $M > 0$ there exists a constant A_M such that $\max_{|s|<M} f(x, s) \leq A_M$ for all $x \in V$.*

(f_2) *There exists some $\alpha > p$ such that for any $s \in \mathbb{R} \setminus \{0\}$ there holds*

$$0 < \alpha F(x, s) := \alpha \int_0^s f(x, t)dt \leq sf(x, s), \quad \forall x \in V.$$

(f_3) *For any $x \in V$, there holds*

$$\limsup_{|s|\to 0} \frac{|f(x, s)|}{|s|^{p-1}} < \lambda_p := \inf_{u\neq 0} \frac{\int_V (|\nabla u|^p + (\lambda a + 1)|u|^p)d\mu}{\int_V |u|^p d\mu}.$$

(f_4) *There exist some $q > p$ and $C > 0$ such that*

$$|f(x, s)| \leq C(1 + |s|^{q-1}), \quad \text{uniformly in } x \in V.$$

(f_5) *$s \mapsto \frac{f(x,s)}{|s|^{p-1}}$ is strictly increasing on $(-\infty, 0)$ and $(0, +\infty)$ for all $x \in V$.*

Our assumptions on the potential $a(x)$ are:

(A_1) *$a(x) \geq 0$ and the potential well $\Omega = \{x \in V : a(x) = 0\}$ is a non-empty, connected and bounded domain in V.*

(A_2) *$(a(x) + 1)^{-1} \in L^{\frac{1}{p-1}}(V)$.*

Remark 1.2 For brevity, we use $\int_V f(x, u)d\mu$ for $\int_V f(\cdot, u(\cdot))d\mu$ and $\int_V F(x, u)d\mu$ for $\int_V F(\cdot, u(\cdot))d\mu$.

Define

$$W^{1,p}(V) := \{u : V \to \mathbb{R} : \|u\|_{W^{1,p}(V)} < +\infty\}, \tag{1.11}$$

where

$$\|u\|_{W^{1,p}(V)} = \left(\int_V (|\nabla u|^p + |u|^p)d\mu \right)^{\frac{1}{p}}.$$

We can verify that $W^{1,p}(V)$ is reflexive (see, Corollary 5.8 in Appendix). Consider a function space

$$E_\lambda := \left\{ u \in W^{1,p}(V) : \int_V \lambda a|u|^p d\mu < +\infty \right\}$$

with the norm

$$\|u\|_{E_\lambda} = \left(\int_V (|\nabla u|^p + (\lambda a + 1)|u|^p) d\mu \right)^{\frac{1}{p}}.$$

Clearly, E_λ is a Banach space and also a reflexive space.

The functional related to (1.4) is

$$J_\lambda(u) = \frac{1}{p} \int_V (|\nabla u|^p + (\lambda a + 1)|u|^p) d\mu - \int_V F(x, u) d\mu. \tag{1.12}$$

Let

$$H_\lambda(u) = \frac{1}{p} \int_V (|\nabla u|^p + (\lambda a + 1)|u|^p) d\mu, \quad u \in E_\lambda. \tag{1.13}$$

Then

$$J_\lambda(u) = H_\lambda(u) - \int_V F(x, u) d\mu.$$

We can easily verify that $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and

$$J'_\lambda(u)v = H'_\lambda(u)v - \int_V f(x, u) v d\mu, \quad \forall v \in E_\lambda, \tag{1.14}$$

where

$$H'_\lambda(u)v = \int_V (|\nabla u|^{p-2} \Gamma(u, v) + (\lambda a + 1)|u|^{p-2} uv) d\mu.$$

The Nehari manifold related to (1.4) is defined as

$$\mathcal{N}_\lambda := \{u \in E_\lambda \setminus \{0\} : J'_\lambda(u)u = 0\}.$$

Let m_λ be

$$m_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

If m_λ can be achieved by some function $u_\lambda \in \mathcal{N}_\lambda$ and u_λ is a critical point of the functional J_λ, then we call u_λ a *ground state solution* of (1.4). Our main theorems are

Theorem 1.3 *Let $G = (V, E)$ be a graph satisfying* (G_1)–(G_2). *Assume $a(x) : V \to [0, +\infty)$ is a function satisfying* $(A_1), (A_2)$ *and $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies* (f_1)–(f_3). *Then for any positive constants $\lambda > 1$ and $p \geq 2$, the equation* (1.4) *has a positive solution.*

Remark 1.4 If we replace $\lambda a + 1$ by a function h satisfying (H_1) and (H_2), then we have the same conclusion as Grigor'yan–Lin–Yang in [15] and our results generalize their work from $p = 2$ to $p > 2$.

Theorem 1.5 *Let $G = (V, E)$ be a graph satisfying* (G_1)–(G_2). *Assume $a(x) : V \to [0, +\infty)$ is a function satisfying* $(A_1), (A_2)$ *and $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies* (f_1)–(f_5). *Then for any positive constants $\lambda > 1$ and $p \geq 2$, there exists a ground state solution u_λ of the equation* (1.4).

Remark 1.6 We can easily check that the function $|u|^{q-2}u$ is a typical example of f that satisfy the assumption (f_1)–(f_5), where $q > p$.

For the asymptotic behavior of u_λ as $\lambda \to +\infty$, we introduce the limit problem which is defined on the potential well Ω:

$$\begin{cases} -\Delta_p u(x) + |u|^{p-2}(x) u(x) = f(x, u(x)), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

where $\Omega = \{x \in V : a(x) = 0\}$ is the potential well.

Let $W_0^{1,p}(\Omega)$ be the completion of $C_c(\Omega)$ under the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla u|^p d\mu + \int_\Omega |u|^p d\mu \right)^{\frac{1}{p}}.$$

The functional related to (1.15) is

$$J_\Omega(u) = \frac{1}{p}\int_{\Omega\cup\partial\Omega} |\nabla u|^p d\mu + \frac{1}{p}\int_\Omega |u|^p d\mu - \int_\Omega F(x,u)d\mu. \tag{1.16}$$

We can easily verify that $J_\Omega \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ and

$$J'_\Omega(u)\phi = \int_{\Omega\cup\partial\Omega} |\nabla u|^{p-2}\Gamma(u,\phi)d\mu + \int_\Omega |u|^{p-2}u\phi d\mu - \int_\Omega f(x,u)\phi d\mu, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{1.17}$$

The corresponding Nehari manifold is

$$\mathcal{N}_\Omega := \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : \|u\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega f(x,u)ud\mu \right\}.$$

And

$$m_\Omega := \inf_{u\in\mathcal{N}_\Omega} J_\Omega(u).$$

Similar to (1.4), the Dirichlet problem also has a ground state solution.

Theorem 1.7 *Let $G = (V,E)$ be a graph satisfying* (G_1)–(G_2)*, Ω be a non-empty, connected and bounded domain in V. Assume $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies* (f_1)–(f_5)*. Then for any $p \geq 2$, the equation* (1.15) *has a ground state solution $u_0 \in W_0^{1,p}(\Omega)$.*

We can prove that (1.15) is some kind of limit problem for (1.4). More precisely, we have

Theorem 1.8 *Under the same assumptions as in Theorem* 1.5*, we have that, for any sequence $\lambda_k \to \infty$, up to a subsequence, the corresponding ground state solutions u_{λ_k} of* (1.4) *converge in $W^{1,p}(V)$ to a ground state solution of* (1.15)*.*

As far as we know, there is no such results on p-Laplacian equations defined on locally finite graphs. Our works generalize the results in [31] to p-Laplacian equations but the proofs are more complicated than those in [31].

This paper is organized as follows. In Section 2, we mainly prove that the formula of integration by parts and Sobolev embedding theorem hold on the graph. In Section 3, we prove the existence of positive and ground state solutions of (1.4) by using the Mountain Pass theorem and the method of Nehari manifold. In Section 4, we demonstrate the desired convergence behavior, namely as $\lambda \to +\infty$, the ground state solutions u_λ of (1.4) tend to a ground state solution of (1.15) in Ω. In Section 5, some necessary lemmas are given in Appendix.

2 Preliminaries

In this section, we introduce some preliminaries and basic functional settings. In particular, we shall prove formulas of integration by parts and embedding theorems of Sobolev spaces on locally finite graphs.

Lemma 2.1 *Assume that $u \in W^{1,p}(V)$ and its p-Laplacian is defined by* (1.9)*. Then for any $v \in C_c(V)$ we have*

$$\int_V |\nabla u|^{p-2}\nabla u\nabla v d\mu = \int_V |\nabla u|^{p-2}\Gamma(u,v)d\mu = -\int_V (\Delta_p u)v d\mu.$$

Proof By the definition of $\Gamma(u,v)$, we have

$$\begin{aligned}\int_V |\nabla u|^{p-2}\Gamma(u,v)d\mu &= \int_V |\nabla u|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x))d\mu \\ &= \sum_{x\in V}|\nabla u|^{p-2}(x)\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x))\mu(x) \\ &= \sum_{x\in V}\frac{1}{2}|\nabla u|^{p-2}(x)\sum_{y\sim x}\omega_{xy}(u(y)-u(x))v(y) \\ &\quad - \sum_{x\in V}\frac{1}{2}|\nabla u|^{p-2}(x)\sum_{y\sim x}\omega_{xy}(u(y)-u(x))v(x) \\ &= \sum_{y\in V}\frac{1}{2}|\nabla u|^{p-2}(y)\sum_{x\sim y}\omega_{xy}(u(x)-u(y))v(x) \\ &\quad - \sum_{x\in V}\frac{1}{2}|\nabla u|^{p-2}(x)\sum_{y\sim x}\omega_{xy}(u(y)-u(x))v(x) \\ &= I + II,\end{aligned}$$

where $I=\sum_{y\in V}\frac{1}{2}|\nabla u|^{p-2}(y)\sum_{x\sim y}\omega_{xy}(u(x)-u(y))v(x)$ and $II=-\sum_{x\in V}\frac{1}{2}|\nabla u|^{p-2}(x)\sum_{y\sim x}$ $\omega_{xy}(u(y)-u(x))v(x)$. Noting that $v\in C_c(V)$, there are finite vertices in V such that $v(x)\neq 0$. Then we have

$$\begin{aligned}I &= \sum_{y\in V}\frac{1}{2}|\nabla u|^{p-2}(y)\sum_{x\sim y}\omega_{xy}(u(x)-u(y))v(x) \\ &= \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}|\nabla u|^{p-2}(y)\omega_{xy}(u(x)-u(y))v(x).\end{aligned}$$

And then we can obtain

$$\begin{aligned}\int_V |\nabla u|^{p-2}\Gamma(u,v)d\mu &= I + II \\ &= \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}|\nabla u|^{p-2}(y)\omega_{xy}(u(x)-u(y))v(x) + II \\ &= -\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}(|\nabla u|^{p-2}(y)+|\nabla u|^{p-2}(x))\omega_{xy}(u(y)-u(x))v(x)\frac{1}{\mu(x)}\mu(x) \\ &= -\sum_{x\in V}\sum_{y\sim x}\frac{1}{2\mu(x)}(|\nabla u|^{p-2}(y)+|\nabla u|^{p-2}(x))\omega_{xy}(u(y)-u(x))v(x)\mu(x) \\ &= -\int_V(\Delta_p u)v d\mu.\end{aligned}$$

□

Remark 2.2 By Lemma 2.1, we know that the definition of p-Laplacian in (1.9) is reasonable.

Lemma 2.3 *Let $\Omega\subset V$ be a bounded domain. Assume that $u\in W^{1,p}(V)$ and its p-Laplacian $\Delta_p u$ is defined by* (1.9). *Then for any $v\in C_c(\Omega)$, we have*

$$\int_{\Omega\cup\partial\Omega}|\nabla u|^{p-2}\nabla u\nabla v d\mu = \int_{\Omega\cup\partial\Omega}|\nabla u|^{p-2}\Gamma(u,v)d\mu = -\int_\Omega(\Delta_p u)v d\mu.$$

Proof By using Lemma 2.1, we only need to prove that

$$\int_{V\setminus\{\Omega\cup\partial\Omega\}}|\nabla u|^{p-2}\Gamma(u,v)d\mu = 0,$$

i.e., $\Gamma(u,v)(x)=0$, for all $x\in V\setminus\{\Omega\cup\partial\Omega\}$. Since $v\in C_c(\Omega)$, $v=0$ on $V\setminus\Omega$. Thus, for any $x\in V\setminus\{\Omega\cup\partial\Omega\}$, there hold $v(x)=0$ and $v(y)=0$ for all $y\sim x$. Therefore, we get for $x\in V\setminus\{\Omega\cup\partial\Omega\}$,

$$\Gamma(u,v)(x)=\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x))=0,$$

and the lemma is proved. □

The weak solution of (1.4) and (1.15) are defined as

Definition 2.4 *If for any $\phi\in E_\lambda$, there holds*

$$\int_V(|\nabla u|^{p-2}\Gamma(u,\phi)+(\lambda a+1)|u|^{p-2}u\phi)d\mu=\int_V f(x,u)\phi d\mu,\quad u\in E_\lambda,$$

then u is called a weak solution of (1.4).

Definition 2.5 *If for any $\phi\in W_0^{1,p}(\Omega)$, there holds*

$$\int_{\Omega\cup\partial\Omega}|\nabla u|^{p-2}\Gamma(u,\phi)d\mu+\int_\Omega|u|^{p-2}u\phi d\mu=\int_\Omega f(x,u)\phi d\mu,\quad u\in W_0^{1,p}(\Omega),$$

then u is called a weak solution of (1.15).

Finally in this section, we prove the Sobolev embedding theorems on the graphs.

Lemma 2.6 *Assume that $\lambda>1$ and $a(x)$ satisfies* (A_1) *and* (A_2). *Then E_λ is continuously embedded into $L^q(V)$ for any $q\in[1,+\infty]$ and the embedding is independent of λ. Namely there exists a constant ξ depending on $q,p,\mu_{\min}$ and $\|(a+1)^{-1}\|_{\frac{1}{p-1}}$ such that for any $u\in E_\lambda$,*

$$\|u\|_q\le\xi\|u\|_{E_\lambda}.\tag{2.1}$$

Moreover, for any bounded sequence $\{u_k\}\subset E_\lambda$, there exists $u\in E_\lambda$ such that, up to subsequence,

$$\begin{cases}u_k\rightharpoonup u, & \text{in } E_\lambda;\\ u_k(x)\to u(x), & \forall x\in V;\\ u_k\to u, & \text{in } L^q(V),\quad \forall q\in[1,+\infty].\end{cases}$$

Proof Suppose $u\in E_\lambda$. At any vertex $x_0\in V$, we have

$$\begin{aligned}\|u\|_{E_\lambda}^p&=\int_V(|\nabla u|^p+(\lambda a+1)|u|^p)d\mu\\&\ge\int_V|u|^pd\mu\\&=\sum_{x\in V}|u(x)|^p\mu(x)\\&\ge\mu_{\min}|u(x_0)|^p,\end{aligned}$$

which gives

$$u(x_0)\le\left(\frac{1}{\mu_{\min}}\right)^{\frac{1}{p}}\|u\|_{E_\lambda}.\tag{2.2}$$

Therefore, $E_\lambda\hookrightarrow L^\infty(V)$ continuously and the embedding is independent of λ. Thus $E_\lambda\hookrightarrow L^q(V)$ continuously for any $p\le q\le\infty$. In fact, for any $u\in E_\lambda$, we have $u\in L^p(V)$. Then, for any $p \le q$,

$$\int_V |u|^q d\mu = \int_V |u|^p |u|^{q-p} d\mu \le (\mu_{\min})^{\frac{p-q}{p}} \|u\|_{E_\lambda}^{q-p} \int_V |u|^p d\mu \le (\mu_{\min})^{\frac{p-q}{p}} \|u\|_{E_\lambda}^q < +\infty, \quad (2.3)$$

which implies that $u \in L^q(V)$ and

$$\|u\|_q = \left(\int_V |u|^q d\mu \right)^{\frac{1}{q}} \le (\mu_{\min})^{\frac{p-q}{pq}} \|u\|_{E_\lambda} \quad \text{for any } p \le q. \quad (2.4)$$

Next, we prove that $E_\lambda \hookrightarrow L^q(V)$ continuously for any $1 \le q < p$. Indeed, (A_2) implies that

$$(\lambda a + 1)^{-1} \in L^{\frac{1}{p-1}}(V), \quad \forall \lambda > 1. \quad (2.5)$$

Then, for any $u \in E_\lambda$,

$$\begin{aligned} \int_V |u| d\mu &= \int_V (\lambda a+1)^{-\frac{1}{p}} (\lambda a+1)^{\frac{1}{p}} |u| d\mu \\ &\le \left(\int_V (\lambda a+1)^{-\frac{1}{p}\cdot\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} \left(\int_V (\lambda a+1)|u|^p d\mu \right)^{\frac{1}{p}} \\ &= \|(\lambda a+1)^{-1}\|_{\frac{1}{p-1}}^{\frac{1}{p}} \left(\int_V (\lambda a+1)|u|^p d\mu \right)^{\frac{1}{p}} \\ &\le \|(\lambda a+1)^{-1}\|_{\frac{1}{p-1}}^{\frac{1}{p}} \|u\|_{E_\lambda} \\ &\le \|(a+1)^{-1}\|_{\frac{1}{p-1}}^{\frac{1}{p}} \|u\|_{E_\lambda} \\ &< +\infty, \end{aligned} \quad (2.6)$$

which implies that $u \in L^1(V)$. And it follows from

$$\|u\|_{L^\infty(V)} \le \frac{1}{\mu_{\min}} \int_V |u| d\mu$$

that

$$\int_V |u|^q d\mu = \int_V |u|^{q-1} |u| d\mu \le \frac{1}{{\mu_{\min}}^{q-1}} \left(\int_V |u| d\mu \right)^q \le \frac{1}{{\mu_{\min}}^{q-1}} \|(a+1)^{-1}\|_{\frac{1}{p-1}}^{\frac{q}{p}} \|u\|_{E_\lambda}^q. \quad (2.7)$$

Therefore, for any $1 \le q \le p$, $E_\lambda \hookrightarrow L^q(V)$ continuously and

$$\|u\|_q = \left(\int_V |u|^q d\mu \right)^{\frac{1}{q}} \le (\mu_{\min})^{\frac{1-q}{q}} \|(a+1)^{-1}\|_{\frac{1}{p-1}}^{\frac{1}{p}} \|u\|_{E_\lambda}. \quad (2.8)$$

By (2.4) and (2.8), we can obtain that there exists a constant η depending on $q, p, \mu_{\min}$ and $\|(\lambda a+1)^{-1}\|_{\frac{1}{p-1}}$ such that for any $u \in E_\lambda$,

$$\|u\|_q \le \xi \|u\|_{E_\lambda}.$$

Let p^* be the exponent conjugate to p. Each element $v \in L^{p^*}(V)$ defines a linear functional ϕ_v on $L^p(V)$ via

$$\phi_v(u) = \int_V uv d\mu, \quad u \in L^p(V).$$

Noting that E_λ is reflexive, for $\{u_k\}$ bounded in E_λ, we have that, up to a subsequence, $u_k \rightharpoonup u$ in E_λ. On the other hand, $\{u_k\} \subset E_\lambda$ is also bounded in $L^p(V)$ and we have $u_k \rightharpoonup u$ in $L^p(V)$, which tell us that

$$\lim_{n\to\infty} \phi_v(u_n - u) = \lim_{k\to\infty} \int_V (u_k - u) v d\mu$$

$$= \lim_{k\to\infty}\sum_{x\in V}\mu(x)(u_k(x)-u(x))v(x) = 0, \quad \forall v \in L^{p^*}(V). \tag{2.9}$$

Take any $x_0 \in V$ and let

$$v_0(x) = \begin{cases} 1, & x = x_0; \\ 0, & x \neq x_0. \end{cases}$$

Obviously it belongs to $L^{p^*}(V)$. By substituting v_0 into (2.9) we have

$$\lim_{k\to\infty}\mu(x_0)(u_k(x_0)-u(x_0)) = 0, \tag{2.10}$$

which implies that $\lim_{k\to\infty} u_k(x) = u(x)$ for any $x \in V$.

We now prove $u_k \to u$ in $L^q(V)$ for all $1 \le q \le +\infty$. Since $\{u_k\}$ is bounded in E_λ and $u \in E_\lambda$, there exists some constant C_1 such that

$$\int_V (\lambda a+1)|u_k - u|^p d\mu \le C_1.$$

Let $x_0 \in V$ be fixed. For any $\epsilon > 0$, in view of (2.5), there exists some $R > 0$ such that

$$\int_{\mathrm{dist}(x,x_0)>R} (\lambda a+1)^{-\frac{1}{p-1}} d\mu < \epsilon^p.$$

Hence by the Hölder inequality,

$$\begin{aligned} &\int_{\mathrm{dist}(x,x_0)>R} |u_k - u| d\mu \\ &\quad = \int_{\mathrm{dist}(x,x_0)>R} (\lambda a+1)^{-\frac{1}{p}}(\lambda a+1)^{\frac{1}{p}}|u_k-u|d\mu \\ &\quad \le \left(\int_{\mathrm{dist}(x,x_0)>R}(\lambda a+1)^{-\frac{1}{p-1}}d\mu\right)^{\frac{p-1}{p}}\left(\int_{\mathrm{dist}(x,x_0)>R}(\lambda a+1)|u_k-u|^p d\mu\right)^{\frac{1}{p}} \\ &\quad \le C_1^{\frac{1}{p}}\epsilon^{p-1}. \end{aligned} \tag{2.11}$$

Moreover, we have that up to a subsequence,

$$\lim_{k\to+\infty}\int_{\mathrm{dist}(x,x_0)\le R}|u_k-u|d\mu = 0. \tag{2.12}$$

Combining (2.11) and (2.12), we conclude

$$\liminf_{k\to+\infty}\int_V |u_k-u|d\mu = 0.$$

In particular, there holds up to a subsequence, $u_k \to u$ in $L^1(V)$. Since

$$\|u_k-u\|_{L^\infty(V)} \le \frac{1}{\mu_{\min}}\int_V |u_k-u|d\mu,$$

there holds for any $1 < q < +\infty$,

$$\int_V |u_k-u|^q d\mu \le \frac{1}{\mu_{\min}^{q-1}}\left(\int_V |u_k-u|d\mu\right)^q.$$

Therefore, up to a subsequence, $u_k \to u$ in $L^q(V)$ for all $1 \le q \le +\infty$. □

Lemma 2.7 *Assume that Ω is a bounded domain in V. Then $W_0^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for any $q \in [1, +\infty]$. In particular, there exists a constant C depending only on q and Ω such that for any $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_q \le C\|u\|_{W_0^{1,p}(\Omega)}. \tag{2.13}$$

Moreover, $W_0^{1,p}(\Omega)$ is pre-compact, namely, if u_k is bounded in $W_0^{1,p}(\Omega)$, then up to a subsequence, there exists some $u \in W_0^{1,p}(\Omega)$ such that $u_k \to u$ in $W_0^{1,p}(\Omega)$.

Proof Since Ω is a finite set in V, $W_0^{1,p}(\Omega)$ is a finite dimensional space. Hence, $W_0^{1,p}(\Omega)$ is pre-compact. And the proof of (2.13) is similar to [14, Theorem 7], which we omit here. □

3 The Existence of Positive and Ground State Solutions

3.1 Existence of Positive Solution

In this subsection, under the assumptions (f_1)–(f_3) and $(A_1), (A_2)$, we prove that the equation (1.4) admits a positive solution by using the Mountain Pass theorem.

Definition 3.1 ($(PS)_c$ condition [1, Definition 1.16]) *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$. We say the function J satisfies the $(PS)_c$ condition, if any $\{u_k\} \subset X$ such that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$ has a convergent subsequence.*

Lemma 3.2 (Mountain Pass theorem [1, Theorem 1.17]) *Let $(X, \|\cdot\|)$ be a Banach space and $J \in C^1(X, \mathbb{R})$ be a functional satisfying the $(PS)_c$ condition. If there exist $e \in X$ and $r > 0$ satisfying $\|e\| > r$ such that*

$$b := \inf_{\|u\|=r} J(u) > J(0) \ge J(e),$$

then c is a critical value of J, where

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

and

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.3 *If $u \in E_\lambda$ is a weak solution of* (1.4)*, then u is also a point-wise solution of* (1.4)*.*

Proof If $u \in E_\lambda$ is a weak solution of (1.4), then for any $\phi \in E_\lambda$, there holds

$$\int_V (|\nabla u|^{p-2}\Gamma(u, \phi) + (\lambda a + 1)|u|^{p-2}u\phi)d\mu = \int_V f(x, u)\phi d\mu.$$

Then Lemma 2.1 gives

$$\int_V (-\Delta_p u\phi + (\lambda a + 1)|u|^{p-2}u\phi)d\mu = \int_V f(x, u)\phi d\mu, \quad \forall\phi \in C_c(V). \tag{3.1}$$

For any fixed $x_0 \in V$, taking a test function $\phi : V \to \mathbb{R}$ in (3.1) with

$$\phi(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \ne x_0, \end{cases}$$

and $\phi \in E_\lambda$, then we have

$$-\Delta_p u(x_0) + (\lambda a(x_0) + 1)|u(x_0)|^{p-2}u(x_0) - f(x_0, u(x_0)) = 0.$$

Since x_0 is arbitrary, we conclude that u is a point-wise solution of (1.4).

Define

$$\tilde{f}(x,s)=\begin{cases}0, & s<0,\\ f(x,s), & s\geq 0.\end{cases} \tag{3.2}$$

We consider the following equation

$$-\Delta_p u(x)+(\lambda a(x)+1)|u|^{p-2}(x)u(x)=\tilde{f}(x,u(x)),\quad \text{in } V. \tag{3.3}$$

Lemma 3.4 *If f satisfies* (f_2) *and $u\in E_\lambda$ is a nontrivial weak solution of* (3.3)*, then u is a strictly positive solution of* (1.4)*.*

Proof Let $u\in E_\lambda$ be a nontrivial weak solution of (3.3). And let $u^-=\min\{u,0\}$. We claim that

$$\Gamma(u^-,u)(x)\geq |\nabla u^-|^2(x) \tag{3.4}$$

and

$$\Gamma(u)(x)\geq |\nabla u^-|^2(x). \tag{3.5}$$

In fact, by the definition of $\Gamma(u,v)$ we have

$$\begin{aligned}\Gamma(u^+,u^-)(x)&=\Gamma(u^-,u^+)(x)\\ &=\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u^-(y)-u^-(x))(u^+(y)-u^+(x))\\ &=\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}[u^-(y)u^+(y)-u^-(y)u^+(x)-u^-(x)u^+(y)+u^-(x)u^+(x)]\\ &=-\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}[u^-(y)u^+(x)+u^-(x)u^+(y)]\\ &\geq 0.\end{aligned} \tag{3.6}$$

It follows from (3.6) that

$$\Gamma(u^-,u)=\Gamma(u^-,u^++u^-)=\Gamma(u^-,u^-)+\Gamma(u^-,u^+)\geq\Gamma(u^-,u^-)=|\nabla u^-|^2.$$

Thus, we have

$$\begin{aligned}\Gamma(u)=\Gamma(u^-+u^+,u)&=\Gamma(u^-,u)+\Gamma(u^+,u)\\ &=\Gamma(u^-,u)+\Gamma(u^+,u^++u^-)\\ &=\Gamma(u^-,u)+\Gamma(u^+,u^+)+\Gamma(u^+,u^-)\\ &\geq\Gamma(u^-,u^-)=|\nabla u^-|^2,\end{aligned}$$

which implies (3.5) and

$$|\nabla u|^p\geq|\nabla u^-|^p. \tag{3.7}$$

Note that

$$|u^-|^p=|u^-|^{p-2}|u^-|^2=|u-u^+|^{p-2}(u-u^+)u^-=|u-u^+|^{p-2}uu^-=|u|^{p-2}uu^-\leq|u|^p. \tag{3.8}$$

By (3.7) and (3.8), we have $u^-\in E_\lambda$ whenever $u\in E_\lambda$.

Testing the above equation u^-, we have

$$\int_V -u^-\Delta_p u d\mu+\int_V(\lambda a(x)+1)|u|^{p-2}uu^-d\mu=\int_V u^-\tilde{f}(x,u)d\mu.$$

Then, by (3.4), (3.7) and the definition of weak solution of the equation (3.3), we have

$$\begin{aligned}\|u^-\|_{E_\lambda}^p &= \int_V (|\nabla u^-|^p + (\lambda a(x)+1)|u^-|^p)d\mu \\ &\le \int_V (|\nabla u^-|^{p-2}\Gamma(u^-,u) + (\lambda a(x)+1)|u^-|^p)d\mu \\ &= \int_V (|\nabla u^-|^{p-2}\Gamma(u,u^-) + (\lambda a(x)+1)|u^-|^p)d\mu \\ &\le \int_V (|\nabla u|^{p-2}\Gamma(u,u^-) + (\lambda a(x)+1)|u^-|^p)d\mu \\ &= \int_V u^-\tilde{f}(x,u)d\mu \le 0.\end{aligned}$$

We have by the above inequality that $u^- \equiv 0$. We claim that $u(x) > 0$ for all $x \in V$. In fact, if $u(x_0) = 0$ for some x_0, then one can see from (3.3) that

$$-\Delta_p u(x_0) + (\lambda a(x_0)+1)|u(x_0)|^{p-2}u(x_0) = \tilde{f}(x_0, u(x_0)),$$

then we get

$$-\Delta_p u(x_0) = \tilde{f}(x,0) = 0.$$

By the definition of $\Delta_p u$, we have $u(x) = u(x_0) = 0$ for all $x \sim x_0$. Thus, $u \equiv 0$, which is a contradiction. Therefore, u is a strictly positive solution of (3.3). By this together with the hypothesis (f_2), we have $f(x,u) > 0$. Hence $\tilde{f}(x,u) = f(x,u)$ and u is a strictly positive solution of (1.4). □

By Lemma 3.4, we know that we only need to prove that the equation (3.3) has a nontrivial weak solution in order to prove Theorem 1.3. Thus, we assume $f(x,u) \equiv 0$ as $u \le 0$ in the following and f also satisfies (f_1)–(f_3).

Lemma 3.5 *Assume* (f_1)–(f_3) *hold. Then there exist positive constants δ, r such that $J_\lambda(u) \ge \delta$ for all functions u with $\|u\|_{E_\lambda} = r$.*

Proof By (f_3), there exist positive constants τ and σ such that if $|s| \le \sigma$, then

$$|f(x,s)| \le (\lambda_p - \tau)|s|^{p-1} \tag{3.9}$$

and

$$|F(x,s)| \le \frac{\lambda_p - \tau}{p}|s|^p. \tag{3.10}$$

Assume $\|u\|_{E_\lambda} \le 1$. If $|s| \ge \sigma$, then by (f_1) and (f_2)

$$|F(x,s)| \le \frac{1}{\sigma^{p+1}}|s|^{p+1}|F(x,s)| \le C|s|^{p+2},$$

where C depends on σ and A_1. Thus, for all $(x,s) \in V \times \mathbb{R}$, there holds

$$|F(x,s)| \le \frac{\lambda_p - \tau}{p}|s|^p + C|s|^{p+2}. \tag{3.11}$$

Hence,

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\|u\|_{E_\lambda}^p - \int_V F(x,u)d\mu \\ &\geq \frac{1}{p}\|u\|_{E_\lambda}^p - \frac{\lambda_p - \tau}{p}\|u\|_p^p - C\|u\|_{p+2}^{p+2} \\ &\geq \left(\frac{1}{p} - \frac{\lambda_p - \tau}{p\lambda_p}\right)\|u\|_{E_\lambda}^p - C\xi\|u\|_{E_\lambda}^{p+2} \\ &\geq \left(\frac{\tau}{p\lambda_p} - C\xi\|u\|_{E_\lambda}^2\right)\|u\|_{E_\lambda}^p. \end{aligned}$$

Setting $r = \min\{1, (\frac{\tau}{2p\lambda_p C\xi})^{\frac{1}{2}}\}$, we have $J_\lambda(u) \geq \frac{\tau}{2p\lambda_p} r^p := \delta$ for all u with $\|u\|_{E_\lambda} = r$. This completes the proof of the lemma. □

Lemma 3.6 *Assume* (f_2) *holds. Then there exists some non-negative function* $u \in E_\lambda$ *such that* $J_\lambda(tu) \to -\infty$ *as* $t \to +\infty$.

Proof We obtain from (f_1) and (f_2) that there exist positive constants C_1 and C_2 such that

$$F(x,s) \geq C_1 s^\alpha - C_2, \quad \forall (x,s) \in V \times \mathbb{R}. \tag{3.12}$$

Indeed, for some $0 < R \in \mathbb{R}$, if $0 < |s| \leq R$, then by (f_1) we have

$$|F(x,s)| = \left|\int_0^s f(x,t)dt\right| \leq C, \quad \forall |s| \leq R. \tag{3.13}$$

For $|s| \geq R$, if $s \geq R$, then by (f_2) we have

$$0 < \alpha F(x,s) \leq s f(x,s) = s\frac{dF(x,s)}{ds}$$

and

$$\alpha \frac{ds}{s} \leq \frac{dF(x,s)}{F(x,s)} \tag{3.14}$$

and after integrating over the interval $[R, s_0]$, we obtain

$$\alpha(\ln s_0 - \ln R) \leq \ln F(x,s_0) - \ln F(x,R),$$

that is

$$\ln\frac{s_0^\alpha}{R^\alpha} \leq \ln\frac{F(x,s_0)}{F(x,R)},$$

which implies that

$$F(x,s_0) \geq C_3 s_0^\alpha. \tag{3.15}$$

For $s \leq -R$, by (f_2), we have

$$0 < \alpha F(x,s) \leq s f(x,s) = s\frac{dF(x,s)}{ds}$$

and

$$\alpha \frac{ds}{s} \geq \frac{dF(x,s)}{F(x,s)}, \tag{3.16}$$

since $s < 0$ and $F(x,s) > 0$. Then after integrating over the interval $[s_0, -R]$, we obtain

$$\alpha(\ln R - \ln|s_0|) \geq \ln F(x,-R) - \ln F(x,s_0),$$

that is

$$\ln \frac{R_0^\alpha}{|s_0|^\alpha} \geq \ln \frac{F(x,-R)}{F(x,s_0)},$$

which implies that

$$F(x,s_0) \geq C_4 s_0^\alpha. \tag{3.17}$$

Therefore, by (3.13), (3.15) and (3.17), we can obtain (3.12).

Let x_0 be fixed. Take a function

$$u(x) = \begin{cases} 1, & x = x_0; \\ 0, & x \neq x_0. \end{cases}$$

Note that $F(x,0) = 0$, then we have

$$\begin{aligned} J_\lambda(tu) &= \frac{t^p}{p}\int_V (|\nabla u|^p + (\lambda a(x)+1)|u|^p)d\mu - \int_V F(x,tu)d\mu \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}\sum_{x\in V}(\lambda a(x)+1)|u(x)|^p\mu(x) - \sum_{x\in V}\mu(x)F(x,tu(x)) \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}(\lambda a(x_0)+1)|u(x_0)|^p\mu(x_0) - \mu(x_0)F(x_0,tu(x_0)) \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}(\lambda a(x_0)+1)\mu(x_0) - \mu(x_0)F(x_0,t) \\ &\leq \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}(\lambda a(x_0)+1)\mu(x_0) - \mu(x_0)C_1t^\alpha + \mu(x_0)C_2. \end{aligned}$$

By the definition of $u(x)$, the nonzero terms of $\sum_{x\in V}|\nabla u|^p(x)\mu(x)$ are finite, since $G=(V,E)$ is a locally finite graph. Then $\sum_{x\in V}|\nabla u|^p(x)\mu(x)$ is bounded. Therefore,

$$J_\lambda(tu) \leq \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}(\lambda a(x_0)+1)\mu(x_0) - \mu(x_0)C_1t^\alpha + \mu(x_0)C_2 \to -\infty$$

as $t \to +\infty$, since $\alpha > p$.

Next, we prove that J_λ satisfies the $(PS)_c$ condition. And first we need the following two lemmas.

Lemma 3.7 *For any $u, v \in E_\lambda$, it holds that*

$$(H'_\lambda(u) - H'_\lambda(v))(u-v) \geq (\|u\|_{E_\lambda}^{p-1} - \|v\|_{E_\lambda}^{p-1})(\|u\|_{E_\lambda} - \|v\|_{E_\lambda}). \tag{3.18}$$

Proof We follow the idea of the proof of [20, Lemma 3.1]. By direct computations, we have

$$\begin{aligned} &(H'_\lambda(u) - H'_\lambda(v))(u-v) \\ &= H'_\lambda(u)(u-v) - H'_\lambda(v)(u-v) \\ &= \int_V (|\nabla u|^{p-2}\Gamma(u,u-v) + (\lambda a(x)+1)|u|^{p-2}u(u-v))d\mu \\ &\quad - \int_V (|\nabla v|^{p-2}\Gamma(v,u-v) + (\lambda a(x)+1)|v|^{p-2}v(u-v)d)\mu \\ &= \int_V (|\nabla u|^{p-2}\Gamma(u,u) - |\nabla u|^{p-2}\Gamma(u,v) + (\lambda a(x)+1)|u|^{p-2}(u^2-uv))d\mu \end{aligned}$$

$$
\begin{aligned}
&- \int_V (|\nabla v|^{p-2}\Gamma(v,u) - |\nabla v|^{p-2}\Gamma(v,v) + (\lambda a(x)+1)|v|^{p-2}(vu - v^2))d\mu \\
&= \int_V (|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\Gamma(u,v) - |\nabla v|^{p-2}\Gamma(v,u))d\mu \\
&\quad + \int_V (\lambda a(x)+1)(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)d\mu \\
&= \|u\|_{E_\lambda}^p + \|v\|_{E_\lambda}^p - \int_V (|\nabla u|^{p-2}\Gamma(u,v) + (\lambda a(x)+1)|u|^{p-2}uv)d\mu \\
&\quad - \int_V (|\nabla v|^{p-2}\Gamma(v,u) + (\lambda a(x)+1)|v|^{p-2}vu)d\mu.
\end{aligned}
$$

Applying Hölder's inequality,

$$
\begin{aligned}
&\int_V (|\nabla u|^{p-2}\Gamma(u,v) + (\lambda a(x)+1)|u|^{p-2}uv)d\mu \\
&= \int_V \left(|\nabla u|^{p-2} \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y)-u(x))(v(y)-v(x)) + (\lambda a(x)+1)|u|^{p-2}uv \right) d\mu \\
&= \int_V \left(|\nabla u|^{p-2} \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}^{\frac{1}{2}}(u(y)-u(x))\omega_{xy}^{\frac{1}{2}}(v(y)-v(x)) + (\lambda a(x)+1)|u|^{p-2}uv \right) d\mu \\
&\le \int_V (|\nabla u|^{p-2}(\Gamma(u))^{\frac{1}{2}}(\Gamma(v))^{\frac{1}{2}} + (\lambda a(x)+1)|u|^{p-2}uv)d\mu \\
&= \int_V (|\nabla u|^{p-2}|\nabla u||\nabla v| + (\lambda a(x)+1)|u|^{p-2}uv)d\mu \\
&\le \left(\int_V |\nabla u|^p d\mu \right)^{\frac{p-1}{p}} \left(\int_V |\nabla v|^p d\mu \right)^{\frac{1}{p}} \\
&\quad + \left(\int_V (\lambda a(x)+1)|u|^p d\mu \right)^{\frac{p-1}{p}} \left(\int_V (\lambda a(x)+1)|v|^p d\mu \right)^{\frac{1}{p}}.
\end{aligned}
$$

Using the following inequality

$$
(a+b)^\beta (c+d)^{1-\beta} \ge a^\beta c^{1-\beta} + b^\beta d^{1-\beta} \tag{3.19}
$$

which holds for any $\beta \in (0,1)$ and for any $a,b,c,d \ge 0$. (For the proof of (3.19), we refer to Lemma 5.9 in Appendix.) Set $\beta = \frac{p-1}{p}$ and

$$
a = \int_V |\nabla u|^p d\mu, \quad b = \int_V (\lambda a(x)+1)|u|^p d\mu, \quad c = \int_V |\nabla v|^p d\mu, \quad d = \int_V (\lambda a(x)+1)|v|^p d\mu, \tag{3.20}
$$

we get that

$$
\begin{aligned}
&\int_V (|\nabla u|^{p-2}\Gamma(u,v) + (\lambda a(x)+1)|u|^{p-2}uv)d\mu \\
&\le \left(\int_V (|\nabla u|^p + (\lambda a(x)+1)|u|^p)d\mu \right)^{\frac{p-1}{p}} \left(\int_V (|\nabla v|^p + (\lambda a(x)+1)|v|^p)d\mu \right)^{\frac{1}{p}} \\
&= \|u\|_{E_\lambda}^{p-1}\|v\|_{E_\lambda}.
\end{aligned} \tag{3.21}
$$

Similarly, we obtain

$$
\int_V (|\nabla v|^{p-2}\Gamma(v,u) + (\lambda a(x)+1)|v|^{p-2}vu)d\mu \le \|v\|_{E_\lambda}^{p-1}\|u\|_{E_\lambda}.
$$

Therefore, we have

$$
\begin{aligned}
(H'_\lambda(u)-H'_\lambda(v))(u-v) &\geq \|u\|^p_{E_\lambda}+\|v\|^p_{E_\lambda}-\|u\|^{p-1}_{E_\lambda}\|v\|_{E_\lambda}-\|v\|^{p-1}_{E_\lambda}\|u\|_{E_\lambda}\\
&=(\|u\|^{p-1}_{E_\lambda}-\|v\|^{p-1}_{E_\lambda})(\|u\|_{E_\lambda}-\|v\|_{E_\lambda}).
\end{aligned}
$$

□

Lemma 3.8 *If $u_n \rightharpoonup u$ in E_λ and $H'_\lambda(u_n)(u_n-u)\to 0$, then $u_n\to u$ in E_λ.*

Proof Since E_λ is a reflexive Banach space, weak convergence and norm convergence imply strong convergence. Therefore we only need to show that $\|u_n\|_{E_\lambda}\to\|u\|_{E_\lambda}$.

Note that

$$
\lim_{n\to\infty}(H'_\lambda(u_n)-H'_\lambda(u))(u_n-u)=\lim_{n\to\infty}H'_\lambda(u_n)(u_n-u)-H'_\lambda(u)(u_n-u)=0.
$$

By Lemma 3.7 we have

$$
(H'_\lambda(u_n)-H'_\lambda(u))(u_n-u)\geq(\|u_n\|^{p-1}_{E_\lambda}-\|u\|^{p-1}_{E_\lambda})(\|u_n\|_{E_\lambda}-\|u\|_{E_\lambda}).
$$

Hence $\|u_n\|_{E_\lambda}\to\|u\|_{E_\lambda}$ as $n\to\infty$ and the assertion follows. □

Now, we prove that J_λ satisfies the $(PS)_c$ condition.

Lemma 3.9 *Under the assumptions* $(A_1), (A_2)$ *and* (f_1)–(f_3), *J_λ satisfies the $(PS)_c$ condition for any $c\in\mathbb{R}$.*

Proof Note that $J_\lambda(u_k)\to c$ and $J'_\lambda(u_k)\to 0$ as $k\to+\infty$ are equivalent to

$$
\frac{1}{p}\|u_k\|^p_{E_\lambda}-\int_V F(x,u_k)d\mu=c+o_k(1), \tag{3.22}
$$

and

$$
\begin{aligned}
&H'_\lambda(u_k)\varphi-\int_V f(x,u_k)\varphi d\mu\\
&\quad=\int_V(|\nabla u_k|^{p-2}\Gamma(u_k,\varphi)+(\lambda a(x)+1)|u_k|^{p-2}u_k\varphi)d\mu-\int_V f(x,u_k)\varphi d\mu\\
&\quad=o_k(1)\|\varphi\|_{E_\lambda},\quad \forall\varphi\in E_\lambda.
\end{aligned} \tag{3.23}
$$

Here and in the sequel, $o_k(1)\to 0$ as $k\to+\infty$. Taking $\varphi=u_k$ in (3.23), we have

$$
\|u_k\|^p_{E_\lambda}=\int_V f(x,u_k)u_k d\mu+o_k(1)\|u_k\|_{E_\lambda}.
$$

In view of (f_2), we have by combining (3.22) and (3.23) that

$$
\begin{aligned}
\|u_k\|^p_{E_\lambda}&=p\int_V F(x,u_k)d\mu+pc+o_k(1)\\
&\leq\frac{p}{\alpha}\int_V f(x,u_k)u_k d\mu+pc+o_k(1)\\
&=\frac{p}{\alpha}\|u_k\|^p_{E_\lambda}+o_k(1)\|u_k\|_{E_\lambda}+pc+o_k(1).
\end{aligned} \tag{3.24}
$$

Since $\alpha>p$, by (3.24) we get that $\{u_k\}$ is bounded in E_λ. Then Lemma 2.6 implies that up to a subsequence, there exists $u\in E_\lambda$ such that $u_k\rightharpoonup u$ in E_λ, $u_k\to u$ in $L^q(V)$ for any $1\leq q\leq+\infty$. It follows from (f_1) that

$$
\left|\int_V f(x,u_k)(u_k-u)d\mu\right|\leq C\int_V|u_k-u|d\mu=o_k(1).
$$

Replacing φ by $u_k - u$ in (3.23), we have

$$\begin{aligned} H'_\lambda(u_k)(u_k - u) &= \int_V (|\nabla u_k|^{p-2}\Gamma(u_k, u_k - u) + (\lambda a + 1)|u_k|^{p-2}u_k(u_k - u))d\mu \\ &= \int_V f(x, u_k)(u_k - u)d\mu + o_k(1)\|u_k - u\|_{E_\lambda} \\ &= o_k(1), \end{aligned}$$

which implies that $H'_\lambda(u_k)(u_k - u) \to 0$ as $k \to \infty$. Then, it follows from Lemma 3.8 that $u_k \to u$ in E_λ as $k \to \infty$. □

Proof of Theorem 1.3 By Lemma 3.5, Lemma 3.6 and Lemma 3.9, J_λ satisfies all the assumptions of the Mountain Pass theorem. Thus we obtain that $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t))$ is the critical value of J_λ. In particular, there exists some $u \in E_\lambda$ such that $J_\lambda(u) = c$. By Lemma 3.5,

$$J_\lambda(u) = c \geq \delta > 0, \tag{3.25}$$

so, $u \not\equiv 0$. Thus, u is a nontrivial weak solution of (3.3). By Lemma 3.4, we obtain that u is a a positive solution of the equation (1.4). □

3.2 Existence of a Ground State Solution

In this subsection, under the assumptions $(A_1), (A_2)$ and (f_1)–(f_5), we prove the existence of a ground state solution by the method of Nehari manifold.

Lemma 3.10 *Assume* (f_1)–(f_5) *hold. Then for any* $u \in E_\lambda \setminus \{0\}$ *there exists a unique* $t(u) > 0$ *such that* $t(u)u \in \mathcal{N}_\lambda$. *The function*

$$t : E_\lambda \setminus \{0\} \to (0, +\infty) : u \mapsto t(u)$$

is continuous and the map $\psi : u \mapsto t(u)u$ *defines a homeomorphism of the unit sphere of* E_λ *with* $\mathcal{N}_\lambda$.

Proof Let $u \in E_\lambda \setminus \{0\}$ be fixed and define the function $g(t) := J_\lambda(tu)$ on $[0, +\infty)$. Clearly we have

$$\begin{aligned} g'(t) = 0 &\Leftrightarrow tu \in \mathcal{N}_\lambda \\ &\Leftrightarrow \|u\|^p_{E_\lambda} = \frac{1}{t^{p-1}} \int_V f(x, tu)ud\mu. \end{aligned} \tag{3.26}$$

It is easy to verify that $g(0) = 0$. By (f_3), there exist positive constants τ and δ such that

$$|f(x, s)| \leq (\lambda_p - \tau)|s|^{p-1}, \quad \forall |s| < \delta \tag{3.27}$$

and

$$|F(x, s)| \leq \frac{\lambda_p - \tau}{p}|s|^p, \quad \forall |s| < \delta. \tag{3.28}$$

Then, by (3.28) and Lemma 2.6, for $|tu| < \delta$ we have

$$\begin{aligned} g(t) = J_\lambda(tu) &= \frac{t^p}{p}\|u\|^p_{E_\lambda} - \int_V F(x, tu)d\mu \\ &\geq \frac{t^p}{p}\|u\|^p_{E_\lambda} - t^p\frac{\lambda_p - \tau}{p}\|u\|^p_p \\ &\geq \frac{t^p}{p}\|u\|^p_{E_\lambda} - t^p\frac{\lambda_p - \tau}{p\lambda_p}\|u\|^p_{E_\lambda} \end{aligned}$$

$$= \frac{t^p \tau}{p\lambda_p} \|u\|_{E_\lambda}^p > 0.$$

Next, we prove that $g(t) < 0$ for t large. By (3.12) and (3.28), we can obtain that there exist constants $C_3 > 0$ and $C_4 > 0$ such that

$$F(x,u) \geq C_3|u|^\alpha - C_4|u|^p, \quad \forall (x,u) \in V \times \mathbb{R}, \tag{3.29}$$

Then, by (3.29) and (2.1), we have

$$\begin{aligned} g(t) = J_\lambda(tu) &= \frac{t^p}{p}\|u\|_{E_\lambda}^p - \int_V F(x,tu)d\mu \\ &\leq \frac{t^p}{p}\|u\|_{E_\lambda}^p - \int_V (C_3|tu|^\alpha - C_4|tu|^p)d\mu \\ &\leq \frac{t^p}{p}\|u\|_{E_\lambda}^p - C_3 t^\alpha \|u\|_\alpha^\alpha + C_4 \xi t^p \|u\|_{E_\lambda}^p \\ &\leq (1 + C_4\xi)\frac{t^p}{p}\|u\|_{E_\lambda}^p - C_3 t^\alpha \|u\|_\alpha^\alpha. \end{aligned} \tag{3.30}$$

Note that $u \in E_\lambda \setminus \{0\}$ and by (2.2), we have

$$\begin{aligned} \|u\|_\alpha^\alpha &= \int_V |u|^\alpha d\mu = \int_V |u|^{\alpha-p}|u|^p d\mu \\ &\leq (\mu_{\min})^{\frac{p-\alpha}{p}} \|u\|_{E_\lambda}^{\alpha-p} \int_V |u|^p d\mu \leq (\mu_{\min})^{\frac{p-\alpha}{p}} \|u\|_{E_\lambda}^\alpha < +\infty. \end{aligned} \tag{3.31}$$

Therefore, we can obtain

$$g(t) = J_\lambda(tu) \leq (1 + C_4\xi)\frac{t^p}{p}\|u\|_{E_\lambda}^p - C_3 t^\alpha \|u\|_\alpha^\alpha \to -\infty \quad \text{as } t \to +\infty, \tag{3.32}$$

since $\alpha > p$. Thus, $g(t) < 0$ for t large.

Therefore, there exists $t = t(u)$ such that $\max_{[0,+\infty)} g = g(t(u))$. Thus, $g'(t(u)) = 0$ and $t(u)u \in \mathcal{N}_\lambda$.

We claim that $t(u)$ is unique. In fact, for any $t > 0$, we have

$$\frac{f(x,tu)}{t^{p-1}}u = \frac{f(x,tu)}{(tu)^{p-1}}u^p = \begin{cases} \dfrac{f(x,tu)}{|tu|^{p-1}}u^p, & u > 0, \\ -\dfrac{f(x,tu)}{|tu|^{p-1}}|u|^p, & u < 0. \end{cases}$$

It follows from (f$_5$) that $\frac{f(x,tu)}{t^{p-1}}u$ is an increasing function for $t > 0$, hence $\frac{1}{t^{p-1}}\int_V f(x,tu)ud\mu$ is an increasing function for $t > 0$. Then there exists a unique $t(u)$ such that $\|u\|_{E_\lambda}^p = \frac{1}{t^{p-1}}\int_V f(x,tu)ud\mu$, i.e., there exists a unique $t(u)$ such that $t(u)u \in \mathcal{N}_\lambda$.

To prove the continuity of $t(u)$, assume that $u_n \to u$ in $E_\lambda \setminus \{0\}$. We only need to prove that $t(u_n) \to t(u)$ as $u_n \to u$ in $E_\lambda \setminus \{0\}$. We claim that $\{t(u_n)\}$ is bounded. Otherwise, $t(u_n) \to +\infty$ as $n \to \infty$. Note that

$$g(t(u_n)) = \max_{t \geq 0} g(t) > 0. \tag{3.33}$$

However, by (3.29) and (2.1), we have

$$g(t(u_n)) = J_\lambda(t(u_n)u_n) = \frac{t(u_n)^p}{p}\|u_n\|_{E_\lambda}^p - \int_V F(x, t(u_n)u_n)d\mu$$

$$\begin{aligned}
&\le \frac{t(u_n)^p}{p}\|u_n\|_{E_\lambda}^p - \int_V (C_3|t(u_n)u_n|^\alpha - C_4|t(u_n)u_n|^p)d\mu \\
&\le \frac{t(u_n)^p}{p}\|u_n\|_{E_\lambda}^p - C_3 t(u_n)^\alpha \|u_n\|_\alpha^\alpha + C_4\xi t(u_n)^p \|u_n\|_{E_\lambda}^p \\
&\le (1+C_4\xi)\frac{t(u_n)^p}{p}\|u_n\|_{E_\lambda}^p - C_3 t(u_n)^\alpha \|u_n\|_\alpha^\alpha \\
&\to -\infty \quad \text{as } n\to\infty,
\end{aligned} \tag{3.34}$$

which is a contradiction. Hence, using the boundedness of $\{t(u_n)\}$, we have that there exists t_0 such that

$$t(u_n) \to t_0 \quad \text{as } n\to\infty. \tag{3.35}$$

Then, we only need to prove $t_0 = t(u)$, i.e., $J'_\lambda(t_0u)u = 0$. Noting that $J'_\lambda(t_nu_n)u_n = 0$, we only need to prove that

$$J'_\lambda(t_nu_n)u_n \to J'_\lambda(t_0u)u \quad \text{as } n\to\infty. \tag{3.36}$$

Since

$$J'_\lambda(t_nu_n)u_n = \|t(u_n)u_n\|_{E_\lambda}^p - \int_V f(x,t(u_n)u_n)u_n d\mu \tag{3.37}$$

and

$$\|t(u_n)u_n\|_{E_\lambda}^p \to \|t_0u\|_{E_\lambda}^p \quad \text{as } n\to\infty, \tag{3.38}$$

it is enough to prove

$$\int_V f(x,t(u_n)u_n)u_n d\mu \to \int_V f(x,t_0u)u d\mu \quad \text{as } n\to\infty. \tag{3.39}$$

Indeed, by (f_3) and (f_4), there exists a positive constant C_δ such that

$$|f(x,u)| \le (\lambda_p - \tau)|u|^{p-1} + C_\delta |u|^{q-1}, \tag{3.40}$$

where q is given in (f_4). Note that $u_n \to u$ in $E_\lambda \setminus \{0\}$ and $t(u_n) \to t_0$ as $n\to\infty$, then $t(u_n)u_n \to t_0u$ in $L^q(V)$ and

$$t(u_n)u_n \to t_0u \quad \text{in } L^p(V)\cap L^q(V).$$

It follows from Lemma 5.12 in Appendix that

$$f(x,t(u_n)u_n) \to f(x,t_0u), \quad \text{in } L^{\frac{p}{p-1}}(V) + L^{\frac{q}{q-1}}(V). \tag{3.41}$$

Then by (3.41) and (f_1), we have

$$\begin{aligned}
&\left|\int_V (f(x,t(u_n)u_n)u_n - f(x,t_0u)u)d\mu\right| \\
&\quad\le \int_V |f(x,t(u_n)u_n)u_n - f(x,t_0u)u| d\mu \\
&\quad= \int_V |[f(x,t(u_n)u_n) - f(x,t_0u)]u_n + f(x,t_0u)(u_n-u)| d\mu \\
&\quad\le \int_V |[f(x,t(u_n)u_n) - f(x,t_0u)]u_n| d\mu + \int_V |(f(x,t_0u)||(u_n-u)| d\mu \\
&\quad\le \|f(x,t(u_n)u_n) - f(x,t_0u)\|_{L^{\frac{p}{p-1}}(V)+L^{\frac{q}{q-1}}(V)} \|u_n\|_{L^p(V)\cap L^q(V)} + C\int_V |(u_n-u)| d\mu \\
&\quad\to 0 \quad \text{as } n\to\infty,
\end{aligned}$$

which implies (3.39). Thus $t_0 = t(u)$.

Finally, we prove that the map $\psi : u \mapsto t(u)u$ is a homeomorphism of the unit sphere B of E_λ with $\mathcal{N}_\lambda$.

(i) Obviously, ψ is continuous, since $t(u)$ is continuous.

(ii) ψ is injective.

Let $u, v \in B \subset E_\lambda$ and $u \neq v$. We have $\|u\|_{E_\lambda} = \|v\|_{E_\lambda} = 1$. We need to prove that $\psi(u) \neq \psi(v)$, i.e., $t(u)u \neq t(v)v$. Indeed, if $t(u) \neq t(v)$, then

$$\|t(u)u\|_{E_\lambda} = \|u\|_{E_\lambda} t(u) = t(u) \neq t(v) = \|v\|_{E_\lambda} t(v) = \|t(v)v\|_{E_\lambda}, \tag{3.42}$$

where $t(u), t(v) > 0$. Thus $t(u)u \neq t(v)v$.

If $t(u) = t(v)$, we also have $t(u)u \neq t(v)v$, since $u \neq v$.

(iii) ψ is surjective.

For any $u \in \mathcal{N}_\lambda$, let $v = \frac{u}{\|u\|_{E_\lambda}}$, then $v \in B$. Note that $\|u\|_{E_\lambda} v = u$ and t is unique, we have $t(v) = \|u\|_{E_\lambda}$. Thus $\psi(v) = t(v)v = \|u\|_{E_\lambda} \frac{u}{\|u\|_{E_\lambda}} = u \in \mathcal{N}_\lambda$. □

Define

$$c_1 := \inf_{u \in E_\lambda, u \neq 0} \max_{t \geq 0} J_\lambda(tu)$$

and

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\},$$

then we have the following lemma.

Lemma 3.11 *Assume* (f_1)–(f_5) *hold. Then* $c_1 = c = m_\lambda > 0$.

Proof First we prove that $c_1 = m_\lambda$. Let $u \in E_\lambda \setminus \{0\}$ be fixed and define the function $g(t) := J_\lambda(tu)$ on $[0, +\infty)$. Lemma 3.10 implies that for any $u \in E_\lambda \setminus \{0\}$ there exists a unique $t(u) > 0$ such that

$$\max_{t \geq 0} g(t) = \max_{t \geq 0} J_\lambda(tu) = g(t(u)) = J_\lambda(t(u)u). \tag{3.43}$$

Then we have

$$c_1 = \inf_{u \in E_\lambda, u \neq 0} \max_{t \geq 0} J_\lambda(tu) = \inf_{u \in E_\lambda, u \neq 0} J_\lambda(t(u)u) = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u) = m_\lambda. \tag{3.44}$$

Next, we prove that $c_1 \geq c$. Indeed, (3.32) implies that there exists $t_0 > 0$ such that $J_\lambda(t_0 u) < 0$. Define

$$l : t \in [0,1] \to t t_0 u \in E_\lambda,$$

then $l(t) \in \Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\}$, since $l(0) = 0, J_\lambda(l(1)) < 0$. For any $u \in E_\lambda \backslash \{0\}$,

$$\max_{t \geq 0} J_\lambda(tu) \geq \max_{t \in [0,1]} J_\lambda(t t_0 u) = \max_{t \in [0,1]} J_\lambda(l(t)) \geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)),$$

then

$$\inf_{u \in E_\lambda, u \neq 0} \max_{t \geq 0} J_\lambda(tu) \geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

That is $c_1 \geq c$.

Next, we prove that $c_1 = m_\lambda \leq c$. By Lemma 3.10 we know that for any $u \in E_\lambda \setminus \{0\}$ there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\lambda$. Then we can separate E_λ into two components according to $t(u) \geq 1$ or $t(u) < 1$. That means, $E_\lambda = E_\lambda^1 \cup E_\lambda^2$, where $E_\lambda^1 = \{u \in E_\lambda : t(u) \geq 1\}$, $E_\lambda^2 = \{u \in E_\lambda : t(u) < 1\}$.

We claim that every $\gamma \in \Gamma$ has to cross $\mathcal{N}_\lambda$. In fact, it is easy to see that $\gamma(t)$ and 0 are in the same component E_λ^1, if t is small enough. We only need to prove $\gamma(1) \in E_\lambda^2$. Set $g(t) = J(t(\gamma(1))), t \in [0, +\infty)$, then $g(0) = 0$ and $g(1) < 0$. By (f_3), if t small enough, we have

$$|f(x, t\gamma(1))| < \lambda_p t^{p-1} |\gamma(1)|^{p-1}.$$

Then, if t small enough, we have

$$\begin{aligned} g(t) &= \frac{t^p}{p}\|\gamma(1)\|_{E_\lambda}^p - \int_V F(x, t\gamma(1))d\mu \\ &\geq \frac{t^p}{p}\|\gamma(1)\|_{E_\lambda}^p - \frac{t}{\alpha}\int_V f(x, t\gamma(1))\gamma(1)d\mu \\ &\geq \frac{t^p}{p}\|\gamma(1)\|_{E_\lambda}^p - \frac{t^p}{\alpha}\|\gamma(1)\|_p^p \\ &> 0, \end{aligned}$$

since $\alpha > p$. Thus, there exists $t \in (0, 1)$ such that $g(t) = \max_{t\in[0,1]} g(t)$, i.e., $J_\lambda'(t\gamma(1))\gamma(1) = 0$. Therefore, $t(\gamma(1)) < 1$, i.e., $\gamma(1) \in E_\lambda^2$. Thus, by the continuity of the map $\gamma(t) \to t(\gamma(t))$ we know that every $\gamma \in \Gamma$ has to cross $\mathcal{N}_\lambda$.

Thus, $\forall \gamma \in \Gamma$, $\gamma(t) \cap \mathcal{N}_\lambda \neq \emptyset$, then there exists $t_0 \in (0, 1)$ such that $\gamma(t_0) \in \mathcal{N}_\lambda$. Thus we can obtain

$$\inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) \leq J_\lambda(\gamma(t_0)) \leq \max_{t\in[0,1]} J_\lambda(\gamma(t)).$$

Thus

$$m_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) \leq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) = c.$$

Therefore, $c_1 = c = m_\lambda$.

Finally, we prove that $m_\lambda > 0$. If $u \in \mathcal{N}_\lambda$, we have

$$\|u\|_{E_\lambda}^p = \int_V f(x, u)ud\mu.$$

By (3.40), we have

$$\begin{aligned} \|u\|_{E_\lambda}^p &= \int_V f(x, u)ud\mu \\ &\leq \int_V |f(x, u)u|d\mu \\ &\leq \int_V ((\lambda_p - \tau)|u|^p + C_\delta|u|^q)d\mu \\ &= (\lambda_p - \tau)\|u\|_p^p + C_\delta\|u\|_q^q \\ &\leq \frac{\lambda_p - \tau}{\lambda_p}\|u\|_{E_\lambda}^p + C_\delta\xi^q\|u\|_{E_\lambda}^q, \end{aligned}$$

then

$$\frac{\tau}{\lambda_p}\|u\|_{E_\lambda}^p \leq C_\delta\xi^q\|u\|_{E_\lambda}^q.$$

Since $q > p$, we get

$$\|u\|_{E_\lambda} \geq \left(\frac{\tau}{\lambda_p C_\delta \xi^q}\right)^{\frac{1}{q-p}} > 0.$$

This gives that

$$\begin{aligned} m_\lambda &= \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) \\ &= \inf_{u\in\mathcal{N}_\lambda} \left(\frac{1}{p}\|u\|^p_{E_\lambda} - \int_V F(x,u)d\mu\right) \\ &\geq \inf_{u\in\mathcal{N}_\lambda} \left(\frac{1}{p}\|u\|^p_{E_\lambda} - \int_V \frac{1}{\alpha} f(x,u)ud\mu\right) \\ &= \inf_{u\in\mathcal{N}_\lambda} \left(\frac{1}{p} - \frac{1}{\alpha}\right)\|u\|^p_{E_\lambda} \\ &\geq \left(\frac{1}{p} - \frac{1}{\alpha}\right)\left(\frac{\tau}{\lambda_p C_\delta \xi^q}\right)^{\frac{p}{q-p}} > 0, \end{aligned}$$

since $p < \alpha$. □

By Theorem 1.3, we know that J_λ satisfies the $(PS)_c$ condition, and there exists a solution u such that $J_\lambda(u) = c = m_\lambda$, thus we completed the proof of Theorem 1.5.

In the following, we can provide another way to prove Theorem 1.5.

Lemma 3.12 ((Deformation Lemma) [1, Lemma 2.3]) *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X,\mathbb{R})$, $S \subset X, c \in \mathbb{R}$. If there exist $\epsilon, \delta > 0$ such that*

$$\forall u \in J^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\delta} : \|J'(u)\|_{X^*} \geq \frac{8\epsilon}{\delta},$$

where $S_{2\delta} = \{x \in X : d(x,S) < 2\delta\}$, X^* is the dual space of X, then there exists $\eta \in C([0,1]\times X, X)$ such that

(i) $\eta(t,u) = u$, *if $t = 0$ or if $u \notin J^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\delta}$,*

(ii) $\eta(1, J^{-1}((-\infty, c+\epsilon]) \cap S) \subset J^{-1}((-\infty, c-\epsilon])$,

(iii) $J(\eta(\cdot,u))$ *is non-increasing, $\forall u \in X$.*

Lemma 3.13 *If $u_\lambda \in \mathcal{N}_\lambda$ and $J_\lambda(u_\lambda) = m_\lambda$, then u_λ is a critical point of J_λ.*

Proof Assume that $u_\lambda \in \mathcal{N}_\lambda$, $J_\lambda(u_\lambda) = m_\lambda$ and $J'_\lambda(u_\lambda) \neq 0$. Then there exist $\delta > 0$, $\zeta > 0$ such that

$$\|u - u_\lambda\|_{E_\lambda} \leq 3\delta \Rightarrow \|J'_\lambda(u)\|_{(E_\lambda)^*} \geq \zeta,$$

where $(E_\lambda)^*$ is the dual space of E_λ. For $\epsilon := \min\{\frac{m_\lambda}{2}, \frac{\zeta\delta}{8}\}$, $S := B(u_\lambda, \delta)$, Lemma 3.12 yields a deformation η satisfying (i)–(iii). We claim that

$$\max_{t>0} J_\lambda(\eta(1, tu_\lambda)) < m_\lambda. \tag{3.45}$$

In fact, by Lemma 3.10 we know that for any $u \in E_\lambda \setminus \{0\}$, there exists $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\lambda$ and $\max_{t>0} J_\lambda(tu) = J_\lambda(t(u)u)$. Since $u_\lambda \in \mathcal{N}_\lambda$,

$$J_\lambda(tu_\lambda) \leq J_\lambda(u_\lambda) = \max_{t>0} J_\lambda(tu_\lambda) = m_\lambda \leq m_\lambda + \epsilon,$$

that is $tu_\lambda \in J_\lambda^{-1}((-\infty, m_\lambda + \epsilon])$.

1° If for any $t > 0$, $tu_\lambda \in S$, then we have $\eta(1, tu_\lambda) \in J_\lambda^{-1}((-\infty, m_\lambda - \epsilon])$ by using (ii) in Lemma 3.12.

2° If for some $\tilde{t} > 0$, $\tilde{t}u_\lambda \notin S$, then there exist t_1 and t_2 satisfying $t_1 < 1 < t_2$ such that

$$tu_\lambda \in S, \quad \forall t \in (t_1, t_2)$$

and

$$tu_\lambda \notin S, \quad \forall t \in [0, t_1] \text{ or } t \in [t_2, +\infty).$$

Let $g(t) := J_\lambda(tu_\lambda)$. We have

$$\begin{aligned} g'(t) = J'_\lambda(tu_\lambda)u_\lambda &= t^{p-1}\|u_\lambda\|^p_{E_\lambda} - \int_V f(x, tu_\lambda)u_\lambda d\mu \\ &= t^{p-1}\int_V f(x, u_\lambda)u_\lambda d\mu - \int_V f(x, tu_\lambda)u_\lambda d\mu \\ &= t^{p-1}\Big(\int_V f(x, u_\lambda)u_\lambda d\mu - \frac{1}{t^{p-1}}\int_V f(x, tu_\lambda)u_\lambda d\mu\Big), \end{aligned}$$

then we can obtain that $J_\lambda(tu_\lambda)$ is increasing in $[0, 1]$ and decreasing in $[1, +\infty]$ about t. Thus, $J_\lambda(tu_\lambda)$ is increasing in $[0, t_1]$ and decreasing in $[t_2, +\infty]$ about t. And we have

$$J_\lambda(tu_\lambda) \le \max\{J_\lambda(t_1u_\lambda), J_\lambda(t_2u_\lambda)\} \le J_\lambda(u_\lambda) - d = m_\lambda - d,$$

where $d = \min\{J_\lambda(u_\lambda) - J_\lambda(t_1u_\lambda), J_\lambda(u_\lambda) - J_\lambda(t_2u_\lambda)\}$. Thus, by (iii) we have

$$J_\lambda(\eta(1, tu_\lambda)) \le J_\lambda(\eta(0, tu_\lambda)) = J_\lambda(tu_\lambda) \le m_\lambda - d.$$

Combining 1° and 2°, we get

$$J_\lambda(\eta(1, tu_\lambda)) \le \max\{m_\lambda - \epsilon, m_\lambda - d\} < m_\lambda, \quad \forall t \ge 0,$$

which gives (3.45).

Since for any $u \in E_\lambda \setminus \{0\}$, $J_\lambda(tu) \to -\infty$ as $t \to +\infty$, there exists $t_0 > 0$ such that

$$J_\lambda(t_0u_\lambda) < 0 < m_\lambda - 2\epsilon.$$

Then we have $t_0u_\lambda \notin J_\lambda^{-1}([m_\lambda - 2\epsilon, m_\lambda + 2\epsilon]) \cap S_{2\delta}$. Let $\gamma(t) = \eta(1, tt_0u_\lambda)$, $t \in [0, 1]$. It follows from (i) that $\gamma(0) = \eta(1, 0) = 0$ and $J_\lambda(\gamma(1)) = J_\lambda(\eta(1, t_0u_\lambda)) = J_\lambda(t_0u_\lambda) < 0$. Hence, by the definition of Γ we have $\gamma(t) \in \Gamma$.

By the definition of c, we have

$$m_\lambda > \max_{t\ge 0} J_\lambda(\eta(1, tu_\lambda)) \ge \max_{t\in[0,1]} J_\lambda(\eta(1, tt_0u_\lambda)) = \max_{t\in[0,1]} J_\lambda(\gamma(t)) \ge \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)) = c = m_\lambda,$$

which is a contradiction. Thus, $J'_\lambda(u_\lambda) = 0$, i.e., u_λ is a critical point of J_λ. □

4 Convergence of Ground State Solutions

In this section, under the assumptions (f_1)–(f_5), we prove that the ground state solution u_λ of (1.4) converges to a ground state solution of (1.15) as $\lambda \to +\infty$, which also implies Theorem 1.7.

Lemma 4.1 *There exists $\nu > 0$ such that for any critical point $u \in E_\lambda \setminus \{0\}$ of J_λ, we have $\|u\|_{E_\lambda} \ge \nu$, where ν is independent of λ.*

Proof Lemma 2.6 tells us that

$$\|u\|_q \le \xi\|u\|_{E_\lambda},$$

where ξ is independent of λ and q is given in (f_4). Since u is a critical point of J_λ and by (3.40) we have that

$$\begin{aligned}0 = J'_\lambda(u)u &= \|u\|^p_{E_\lambda} - \int_V f(x,u)ud\mu \\ &\geq \|u\|^p_{E_\lambda} - \int_V ((\lambda_p - \tau)|u|^p + C_\delta |u|^q)d\mu \\ &\geq \|u\|^p_{E_\lambda} - \frac{\lambda_p - \tau}{\lambda_p}\|u\|^p_{E_\lambda} - C_\delta \xi^q \|u\|^q_{E_\lambda} \\ &= \|u\|^p_{E_\lambda}\Big(\frac{\tau}{\lambda_p} - C_\delta \xi^q \|u\|^{q-p}_{E_\lambda}\Big).\end{aligned}$$

Then we have

$$\|u\|_{E_\lambda} \geq \Big(\frac{\tau}{\lambda_p C_\delta \xi^q}\Big)^{\frac{1}{q-p}} \tag{4.1}$$

and we can choose $\nu = \big(\frac{\tau}{\lambda_p C_\delta \xi^q}\big)^{\frac{1}{q-p}}$. □

Lemma 4.2 *There exists $C_1 > 0$ which is independent of λ such that if $\{u_k\}$ is a $(PS)_c$ sequence of J_λ, then*

$$\limsup_{k\to+\infty} \|u_k\|^p_{E_\lambda} \leq \frac{\alpha p}{\alpha - p} c \tag{4.2}$$

and either $c \geq C_1$ or $c = 0$, where α is given in (f_2).

Proof Since $J_\lambda(u_k) \to c$ and $J'_\lambda(u_k) \to 0$ as $k \to +\infty$, we have

$$\begin{aligned}c &= \limsup_{k\to+\infty}\Big(J_\lambda(u_k) - \frac{1}{\alpha}J'_\lambda(u_k)u_k\Big) \\ &= \limsup_{k\to+\infty}\Big[\Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|u_k\|^p_{E_\lambda} + \int_V \Big(\frac{1}{\alpha}f(x,u_k)u_k - F(x,u_k)\Big)d\mu\Big] \\ &\geq \limsup_{k\to+\infty}\Big(\frac{1}{p} - \frac{1}{\alpha}\Big)\|u_k\|^p_{E_\lambda} \\ &= \frac{\alpha - p}{\alpha p}\limsup_{k\to+\infty}\|u_k\|^p_{E_\lambda},\end{aligned}$$

which gives (4.2).

For any $u \in E_\lambda$, by (f_3), (3.40) and Lemma 2.6 we have

$$\begin{aligned}J'_\lambda(u)u &= \|u\|^p_{E_\lambda} - \int_V f(x,u)ud\mu \\ &\geq \|u\|^p_{E_\lambda} - \frac{\lambda_p - \tau}{\lambda_p}\|u\|^p_{E_\lambda} - C_\delta \xi^q \|u\|^q_{E_\lambda} \\ &= \|u\|^p_{E_\lambda}\Big(\frac{\tau}{\lambda_p} - C_\delta \xi^q \|u\|^{q-p}_{E_\lambda}\Big),\end{aligned}$$

and there exists $\rho = (\frac{\tau}{2\lambda_p C_\delta \xi^q})^{\frac{1}{q-p}}$ such that

$$J'_\lambda(u)u \geq \frac{\tau}{2\lambda_p}\|u\|^p_{E_\lambda} \quad \text{for } \|u\|_{E_\lambda} \leq \rho.$$

Take $C_1 = \frac{\alpha-p}{\alpha p}\rho^p$ and suppose $c < C_1$. Since $\{u_k\}$ is a $(PS)_c$ sequence, (4.2) gives

$$\limsup_{k\to+\infty}\|u_k\|^p_{E_\lambda} \leq \frac{\alpha p}{\alpha - p}c < \frac{\alpha p}{\alpha - p}C_1 = \rho^p.$$

Hence, for k large, we have

$$\frac{\tau}{2\lambda_p}\|u_k\|_{E_\lambda}^p \le J'_\lambda(u_k)u_k = o_k(1)\|u_k\|_{E_\lambda}.$$

Then we have $\|u_k\|_{E_\lambda} \to 0$ as $k \to +\infty$ which gives $J_\lambda(u_k) \to c = 0$ and the desired results are proved for $C_1 = \frac{\alpha-p}{\alpha p}\rho^p = \frac{\alpha-p}{\alpha p}(\frac{\tau}{2\lambda_p C_\delta \xi^q})^{\frac{p}{q-p}}$. □

Remark 4.3 By Lemma 3.11, we know that for any ground state solutions u_λ, there exists a $(PS)_c$ sequence $\{u_k\}$ which converges weakly to u_λ in E_λ, where $c = m_\lambda$. By weak lower semi-continuity of the norm $\|\cdot\|_{E_\lambda}$, we get that $\|u_\lambda\|_{E_\lambda}$ is bounded by $\frac{\alpha p m_\lambda}{\alpha-p}$.

For the ground states m_λ and m_Ω, we have

Lemma 4.4 $m_\lambda \to m_\Omega$ *as* $\lambda \to \infty$.

Proof Since $\mathcal{N}_\Omega \subset \mathcal{N}_\lambda$, we obviously have that $m_\lambda \le m_\Omega$ for any $\lambda > 0$. Take a sequence $\lambda_k \to \infty$ such that

$$\lim_{k\to\infty} m_{\lambda_k} = M \le m_\Omega, \tag{4.3}$$

where m_{λ_k} is the ground state of the ground state solution $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$ of (1.4). Then Lemma 3.11 tells us that $M > 0$. By Remark 4.3, $\{u_{\lambda_k}\}$ is uniformly bounded in E_λ, up to a subsequence, we assume that there exists some $u_0 \in W^{1,p}(V)$ such that

$$u_{\lambda_k} \rightharpoonup u_0 \quad \text{in } E_\lambda,$$
$$u_{\lambda_k}(x_0) \to u_0(x_0),$$

and

$$u_{\lambda_k} \to u_0 \quad \text{in } L^q(V),$$

where q is given in (f_4).

We claim that $u_0|_{\Omega^c} = 0$. If not, there exists a vertex $x_0 \notin \Omega$ such that $u_0(x_0) \ne 0$. Since $u_{\lambda_k} \in \mathcal{N}_{\lambda_k}$, we have

$$\begin{aligned}
J_\lambda(u_{\lambda_k}) &= J_\lambda(u_{\lambda_k}) - \frac{1}{\alpha}J'_\lambda(u_{\lambda_k})u_{\lambda_k} \\
&= \left(\frac{1}{p} - \frac{1}{\alpha}\right)\|u_{\lambda_k}\|_{E_{\lambda_k}}^p + \int_V \left(\frac{1}{\alpha}f(x,u_{\lambda_k})u_{\lambda_k} - F(x,u_{\lambda_k})\right)d\mu \\
&\ge \left(\frac{1}{p} - \frac{1}{\alpha}\right)\|u_{\lambda_k}\|_{E_{\lambda_k}}^p \\
&= \frac{\alpha-p}{\alpha p}\|u_{\lambda_k}\|_{E_{\lambda_k}}^p \\
&\ge \frac{\alpha-p}{\alpha p}\lambda_k \int_V a|u_{\lambda_k}|^p d\mu \\
&\ge \frac{\alpha-p}{\alpha p}\lambda_k a(x_0)|u_{\lambda_k}(x_0)|^p \mu(x_0).
\end{aligned}$$

Since $x_0 \notin \Omega$, $a(x_0) > 0$ and $\mu(x_0) \ge \mu_{\min} > 0$, $u_{\lambda_k}(x_0) \to u_0(x_0) \ne 0$ as $\lambda_k \to \infty$, then we know that

$$\lim_{k\to\infty} J_{\lambda_k}(u_{\lambda_k}) = \infty,$$

which is a contradiction to the fact that $m_{\lambda_k} \le m_\Omega$.

In view of (f_1), there exists some constant C such that

$$|F(x,u_{\lambda_k})-F(x,u_0)|\le C|u_{\lambda_k}-u_0|,$$

which leads to

$$\left|\int_V (F(x,u_{\lambda_k})-F(x,u_0))d\mu\right|\to 0 \quad \text{as } k\to+\infty \tag{4.4}$$

Similar to the proof of Lemma 3.10, we can prove that there exists $t>0$ such that $tu_0\in\mathcal{N}_\Omega$, i.e.,

$$\int_{\Omega\cup\partial\Omega}(|t\nabla u_0|^p+|tu_0|^p)d\mu=\int_\Omega f(x,tu_0)tu_0d\mu. \tag{4.5}$$

By (4.4), we get that

$$\begin{aligned}
J_\Omega(tu_0)&=\frac{1}{p}\int_{\Omega\cup\partial\Omega}(|t\nabla u_0|^p+|tu_0|^p)d\mu-\int_\Omega F(x,tu_0)d\mu\\
&\le\frac{1}{p}\int_V(|t\nabla u_0|^p+|tu_0|^p)d\mu-\int_V F(x,tu_0)d\mu\\
&\le\liminf_{k\to\infty}\left[\int_V\frac{1}{p}(|t\nabla u_{\lambda_k}|^p+(\lambda_k a+1)|tu_{\lambda_k}|^p)d\mu-\int_V F(x,tu_{\lambda_k})d\mu\right]\\
&=\liminf_{k\to\infty}J_{\lambda_k}(tu_{\lambda_k})\\
&\le\liminf_{k\to\infty}J_{\lambda_k}(u_{\lambda_k})=M.
\end{aligned}$$

Consequently, $M\ge m_\Omega$. Then we get that

$$\lim_{\lambda\to\infty}m_\lambda=m_\Omega. \qquad \square$$

Proofs of Theorem 1.7 *and Theorem* 1.8 We need to prove that for any sequence $\lambda_k\to\infty$, the corresponding $u_{\lambda_k}\in\mathcal{N}_{\lambda_k}$ satisfying $J_{\lambda_k}(u_{\lambda_k})=m_{\lambda_k}$ converges in $W^{1,p}(V)$ to a ground state solution u_0 of (1.15) along subsequence.

Lemma 4.2 gives that u_{λ_k} is bounded in E_{λ_k} and the upper-bound is independent of λ_k. Consequently, we have that $\{u_{\lambda_k}\}$ is also bounded in $W^{1,p}(V)$. Therefore, we can assume that for any $q\in[1,+\infty)$,

$$u_{\lambda_k}\rightharpoonup u_0 \quad \text{in } W^{1,p}(V).$$

Moreover, we get from Lemma 4.1 that $u_0\not\equiv 0$. We have proved in Lemma 4.4 that $u_0|_{\Omega^c}=0$ and (4.4). Now we claim that as $k\to\infty$, we have

$$\lambda_k\int_V a|u_{\lambda_k}|^p d\mu\to 0, \tag{4.6}$$

and

$$\int_V|\nabla u_{\lambda_k}|^p d\mu\to\int_V|\nabla u_0|^p d\mu. \tag{4.7}$$

In fact, similar to the proof of (4.5) in Lemma 4.4, we can also find $t>0$ such that $tu_0\in\mathcal{N}_\Omega$. If

$$\lim_{k\to\infty}\lambda_k\int_V a|u_{\lambda_k}|^p d\mu=\theta>0,$$

or

$$\liminf_{k\to+\infty}\int_V|\nabla u_{\lambda_k}|^p d\mu>\int_V|\nabla u_0|^p d\mu,$$

we have

$$\begin{aligned} J_\Omega(tu_0) &= \frac{1}{p}\int_{\Omega\cup\partial\Omega}(|t\nabla u_0|^p+|tu_0|^p)d\mu-\int_\Omega F(x,tu_0)d\mu \\ &= \frac{1}{p}\int_V(|t\nabla u_0|^p+|tu_0|^p)d\mu-\int_V F(x,tu_0)d\mu \\ &< \liminf_{k\to+\infty}\left[\frac{1}{p}\int_V(|t\nabla u_{\lambda_k}|^p+(\lambda_k a+1)|tu_{\lambda_k}|^p)d\mu-\int_V F(x,tu_{\lambda_k})d\mu\right] \\ &= \liminf_{k\to+\infty} J_{\lambda_k}(tu_{\lambda_k}) \\ &\le \liminf_{k\to+\infty} J_{\lambda_k}(u_{\lambda_k}) \\ &= m_\Omega, \end{aligned}$$

which is contradiction.

Now we prove that u_0 is a ground state solution of (1.15). In fact, since $J'_{\lambda_k}(u_{\lambda_k})=0$, for any $0\neq\phi\in W_0^{1,p}(\Omega)$, we have

$$\int_V(|\nabla u_{\lambda_k}|^{p-2}\nabla u_{\lambda_k}\nabla\phi+(\lambda_k a+1)|u_{\lambda_k}|^{p-2}u_{\lambda_k}\phi)d\mu=\int_V f(x,u_{\lambda_k})\phi d\mu.$$

Since $\Omega=\{x\in V: a(x)=0\}$, $a(x)\phi(x)\equiv 0$, for any $x\in V$. Then

$$\int_{\Omega\cup\partial\Omega}|\nabla u_{\lambda_k}|^{p-2}\nabla u_{\lambda_k}\nabla\phi d\mu+\int_\Omega|u_{\lambda_k}|^{p-2}u_{\lambda_k}\phi d\mu=\int_\Omega f(x,u_{\lambda_k})\phi d\mu.$$

Let $k\to\infty$. The above equality becomes

$$\int_{\Omega\cup\partial\Omega}|\nabla u_0|^{p-2}\nabla u_0\nabla\phi d\mu+\int_\Omega|u_0|^{p-2}u_0\phi d\mu=\int_\Omega f(x,u_0)\phi d\mu.$$

which tells us that $J'_\Omega(u_0)=0$, $u_0\in\mathcal{N}_\Omega$ and u_0 is a solution of (1.15).

On the other hand, by (4.4), (4.6) and (4.7), we have

$$\begin{aligned} J_{\lambda_k}(u_{\lambda_k}) &= \frac{1}{p}\int_V(|\nabla u_{\lambda_k}|^p+(\lambda_k a+1)|u_{\lambda_k}|^p)d\mu-\int_V F(x,u_{\lambda_k})d\mu \\ &= \frac{1}{p}\int_V(|\nabla u_0|^p+|u_0|^p)d\mu-\int_V F(x,u_0)d\mu+o_k(1) \\ &= \frac{1}{p}\int_{\Omega\cup\partial\Omega}|\nabla u_0|^p d\mu+\int_\Omega|u_0|^p d\mu-\int_\Omega F(x,u_0)d\mu+o_k(1) \\ &= J_\Omega(u_0)+o_k(1). \end{aligned}$$

Since $J_{\lambda_k}(u_{\lambda_k})=m_{\lambda_k}$, Lemma 4.4 tells $J_\Omega(u_0)=m_\Omega$. Thus we get that u_0 is a solution of (1.15) which achieves the ground state. Thus Theorem 1.7 and Theorem 1.8 are proved. □

5 Appendix

In the Appendix, we mainly prove that $L^p(V)$ is complete, uniformly convex, reflexive and $W^{1,p}(V)$ is reflexive.

Lemma 5.1 *If $G=(V,E)$ is a connected locally finite graph then the set of vertices V is either finite or countably infinite.*

Define

$$L^p(V):=\left\{u:V\to\mathbb{R}:\int_V|u|^p d\mu<+\infty\right\}$$

with the norm

$$\|u\|_p := \left(\int_V |u|^p d\mu\right)^{\frac{1}{p}}$$

where V is the vertex set of a locally finite graph.

Lemma 5.2 *If $G = (V, E)$ satisfies* (G_1)–(G_2)*, then $L^p(V)$ is a Banach space,* $1 \le p \le \infty$.

The proof of Lemma 5.1 and Lemma 5.2 are standard, so we omit it.

Lemma 5.3 *Let $z, w \in \mathbb{C}$. If $1 < p \le 2$ and $p' = \frac{p}{p-1}$, then*

$$\left|\frac{z+w}{2}\right|^{p'} + \left|\frac{z-w}{2}\right|^{p'} \le \left(\frac{1}{2}|z|^p + \frac{1}{2}|w|^p\right)^{\frac{1}{p-1}}. \tag{5.1}$$

If $2 \le p < \infty$, then

$$\left|\frac{z+w}{2}\right|^{p} + \left|\frac{z-w}{2}\right|^{p} \le \frac{1}{2}|z|^p + \frac{1}{2}|w|^p. \tag{5.2}$$

Proof For the proof of Lemma 5.3 we can refer to [1, Lemma 2.37]. □

Definition 5.4 *The space X is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta(\varepsilon) > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \varepsilon$ then $\left\|\frac{x+y}{2}\right\| \le 1 - \delta(\varepsilon)$.*

Lemma 5.5 *If $G = (V, E)$ satisfies* (G_1)–(G_2) *and $1 < p < \infty$, then $L^p(V)$ is uniformly convex.*

Proof Let $u, v \in L^p(V)$ satisfy $\|u\|_p = \|v\|_p = 1$ and $\|u - v\|_p \ge \varepsilon$ where $\varepsilon \in (0, 2]$. If $2 \le p < \infty$, then (5.2) implies that

$$\left|\frac{u(x)+v(x)}{2}\right|^{p} + \left|\frac{u(x)-v(x)}{2}\right|^{p} \le \frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p, \quad \forall x \in V.$$

Then we have

$$\sum_{x\in V}\left(\left|\frac{u(x)+v(x)}{2}\right|^{p} + \left|\frac{u(x)-v(x)}{2}\right|^{p}\right)\mu(x) \le \sum_{x\in V}\left(\frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p\right)\mu(x).$$

That is

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p,$$

which implies that

$$\left\|\frac{u+v}{2}\right\|_p^p \le 1 - \frac{\varepsilon^p}{2^p}.$$

If $1 < p \le 2$, then (5.1) implies that

$$\left|\frac{u(x)+v(x)}{2}\right|^{p'} + \left|\frac{u(x)-v(x)}{2}\right|^{p'} \le \left(\frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p\right)^{\frac{1}{p-1}}, \quad \forall x \in V,$$

where $p' = \frac{p}{p-1}$. Then we have

$$\left(\left|\frac{u(x)+v(x)}{2}\right|^{p'} + \left|\frac{u(x)-v(x)}{2}\right|^{p'}\right)^{p-1} \le \frac{1}{2}|u(x)|^p + \frac{1}{2}|v(x)|^p, \quad \forall x \in V.$$

Note that

$$\left|\frac{u(x)+v(x)}{2}\right|^{p} + \left|\frac{u(x)-v(x)}{2}\right|^{p} \le \left(\left|\frac{u(x)+v(x)}{2}\right|^{p'} + \left|\frac{u(x)-v(x)}{2}\right|^{p'}\right)^{p-1}, \quad \forall x \in V.$$

Thus, we can also obtain

$$\left\|\frac{u+v}{2}\right\|_p^p + \left\|\frac{u-v}{2}\right\|_p^p \le \frac{1}{2}\|u\|_p^p + \frac{1}{2}\|v\|_p^p,$$

which implies that

$$\left\|\frac{u+v}{2}\right\|_p^p \le 1 - \frac{\varepsilon^p}{2^p}.$$

In either case there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left\|\frac{u+v}{2}\right\|_p \le 1 - \delta.$$ □

Corollary 5.6 *If $G = (V, E)$ satisfies* (G_1)–(G_2) *and $1 < p < \infty$, then $L^p(V)$ is reflexive.*

Proof A uniformly convex Banach space is reflexive ([1, Theorem 1.21]). Thus, Lemma 5.2 and Lemma 5.5 imply that $L^p(V)$ is reflexive. □

Define

$$W^{1,p}(V) := \{u : V \to \mathbb{R} : \|u\|_{W^{1,p}(V)} < +\infty\}, \tag{5.3}$$

where

$$\|u\|_{W^{1,p}(V)} = \left(\int_V (|\nabla u|^p + |u|^p) d\mu\right)^{\frac{1}{p}}.$$

Proposition 5.7 *If $G = (V, E)$ satisfies* (G_1)–(G_2)*, then $W^{1,p}(V)$ is the completion of $C_c(V)$ under the norm $\|u\|_{W^{1,p}(V)}^p = \int_V (|\nabla u|^p + |u|^p) d\mu$.*

Proof The proof of Proposition 5.7 is similar to [16, Proposition 2.1].

Corollary 5.8 *If $G = (V, E)$ satisfies* (G_1)–(G_2) *and $2 < p < \infty$, then $W^{1,p}(V)$ is reflexive.*

Proof Let $\{u_k\} \subset W^{1,p}(V)$ be a sequence and $u_k \to u$ in $L^p(V)$. Next we prove that $u \in W^{1,p}(V)$. Obviously, $u \in L^p(V)$. We only need to prove that $\|\nabla u\|_p^p = \int_V |\nabla u|^p d\mu < +\infty$. Indeed, by Hölder's inequality and (G_1), we have

$$\begin{aligned}
\|\nabla u\|_p^p &= \int_V |\nabla u|^p d\mu \\
&= \sum_{x\in V} \mu(x) \left[\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy} (u(y) - u(x))^2\right]^{\frac{p}{2}} \\
&= 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{1}{\mu(x)}\right)^{\frac{p}{2}} \left[\sum_{y\sim x} \omega_{xy} (u(y) - u(x))^2\right]^{\frac{p}{2}} \\
&= 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{\deg_x}{\mu(x)}\right)^{\frac{p}{2}} \left[\sum_{y\sim x} \frac{\omega_{xy}}{\deg_x} (u(y) - u(x))^2\right]^{\frac{p}{2}} \\
&= 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{\deg_x}{\mu(x)}\right)^{\frac{p}{2}} \left[\sum_{y\sim x} \left(\frac{\omega_{xy}}{\deg_x}\right)^{\frac{p-2}{p}+\frac{2}{p}} (u(y) - u(x))^2\right]^{\frac{p}{2}} \\
&= 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{\deg_x}{\mu(x)}\right)^{\frac{p}{2}} \left[\sum_{y\sim x} \left(\frac{\omega_{xy}}{\deg_x}\right)^{\frac{p-2}{p}} \left(\frac{\omega_{xy}}{\deg_x}\right)^{\frac{2}{p}} (u(y) - u(x))^2\right]^{\frac{p}{2}} \\
&\le 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{\deg_x}{\mu(x)}\right)^{\frac{p}{2}} \left(\sum_{y\sim x} \frac{\omega_{xy}}{\deg_x}\right)^{\frac{p-2}{2}} \sum_{y\sim x} \frac{\omega_{xy}}{\deg_x} |u(y) - u(x)|^p
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-\frac{p}{2}} \sum_{x\in V} \mu(x) \left(\frac{\deg_x}{\mu(x)}\right)^{\frac{p}{2}} \sum_{y\sim x} \frac{\omega_{xy}}{\deg_x} |u(y)-u(x)|^p \\
&\le 2^{-\frac{p}{2}} M^{\frac{p}{2}-1} \sum_{x\in V}\sum_{y\sim x} \omega_{xy} |u(y)-u(x)|^p \\
&\le 2^{-\frac{p}{2}} M^{\frac{p}{2}-1} \sum_{x\in V}\sum_{y\sim x} \omega_{xy} (|u(y)|^p + |u(x)|^p) \\
&\le 2^{1-\frac{p}{2}} M^{\frac{p}{2}-1} \sum_{x,y\in V} \omega_{xy} |u(x)|^p \\
&= 2^{1-\frac{p}{2}} M^{\frac{p}{2}-1} \sum_{x} \deg_x |u(x)|^p \\
&\le 2^{1-\frac{p}{2}} M^{\frac{p}{2}-1} \sum_{x\in V} M\mu(x) |u(x)|^p \\
&= 2^{1-\frac{p}{2}} M^{\frac{p}{2}} \|u\|_p^p \\
&< +\infty,
\end{aligned}
$$

since $u \in L^p(V)$.

Thus, $W^{1,p}(V) \subset L^p(V)$ is closed with respect to the norm topology of $L^p(V)$. The reflexivity of $L^p(V)$ and [1, Theorem 1.22] imply that $W^{1,p}(V)$ is reflexive. □

Lemma 5.9 *The following inequality*

$$(a+b)^\beta (c+d)^{1-\beta} \ge a^\beta c^{1-\beta} + b^\beta d^{1-\beta} \tag{5.4}$$

holds for any $\beta \in (0,1)$ and $a,b,c,d \ge 0$.

Proof If $a = 0$ or $c = 0$, it is obvious. If $a \ne 0, c \ne 0$, then for any $a,b,c,d > 0$, (5.4) is equivalent to

$$\left(1+\frac{b}{a}\right)^\beta \left(1+\frac{d}{c}\right)^{1-\beta} \ge 1 + \left(\frac{b}{a}\right)^\beta \left(\frac{d}{c}\right)^{1-\beta}.$$

Let $u = \frac{b}{a}$, $v = \frac{d}{c}$. We only need to prove

$$Q(u) = (1+u)^\beta (1+v)^{1-\beta} - u^\beta v^{1-\beta} - 1 \ge 0.$$

Without loss of generality, we assume $u \ge v$. Then we have

$$\frac{1+v}{1+u} \ge \frac{v}{u} \quad \text{and} \quad \left(\frac{1+v}{1+u}\right)^{1-\beta} \ge \left(\frac{v}{u}\right)^{1-\beta}.$$

Note that $Q(0) = 0$, and

$$
\begin{aligned}
Q'(u) &= \beta(1+u)^{\beta-1}(1+v)^{1-\beta} - \beta u^{\beta-1} v^{1-\beta} \\
&= \beta\left[\left(\frac{1+v}{1+u}\right)^{1-\beta} - \left(\frac{v}{u}\right)^{1-\beta}\right] \ge 0.
\end{aligned}
$$

Hence, $Q(u)$ is an increasing function about u. And we get $Q(u) \ge Q(0) = 0$. □

Next, we consider the continuity of the operator

$$A : L^p(V) \to L^q(V) : u \mapsto f(x,u).$$

Lemma 5.10 *Let $G = (V, E)$ be a locally finite graph and $1 \le p < \infty$. If $u_n \to u$ in $L^p(V)$, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ and $h \in L^p(V)$ such that for all $x \in V$, $v_n(x) \to u(x)$ and*

$$|u(x)| \le h(x), \quad |v_n(x)| \le h(x).$$

Proof The proof is similar to [28, Theorem A.1]. Going if necessary to a subsequence, we can assume that $u_n(x) \to u(x)$ for all $x \in V$. There exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$\|v_{j+1} - v_j\|_p \le 2^{-j}, \quad \forall j \ge 1.$$

Let us define

$$h(x) := |v_1(x)| + \sum_{j=1}^{\infty} |v_{j+1}(x) - v_j(x)|.$$

It is clear that, for all $x \in V$, $|v_n(x)| \le h(x)$ and so $|u(x)| \le h(x)$. □

Definition 5.11 *On the space $L^p(V) \cap L^q(V)$, we define the norm*

$$\|u\|_{L^p(V)\cap L^q(V)} := \|u\|_p + \|u\|_q.$$

On the space $L^p(V) + L^q(V)$, we define the norm

$$\|u\|_{L^p(V)+L^q(V)} := \inf\{\|v\|_p + \|w\|_q : v \in L^p(V), w \in L^q(V), u = v + w\}.$$

Lemma 5.12 *Let $G = (V, E)$ be a locally finite graph and $1 \le p, q, r, s < \infty$. For any $x \in V$, $f(x, u)$ is continuous in $u \in \mathbb{R}$ and*

$$|f(x, u)| \le c(|u|^{\frac{p}{r}} + |u|^{\frac{q}{s}}).$$

Then, for every $u \in L^p(V) \cap L^q(V), f(\cdot, u) \in L^r(V) + L^s(V)$ and the operator

$$A : L^p(V) \cap L^q(V) \to L^r(V) + L^s(V) : u \mapsto f(x, u)$$

is continuous.

Proof Let $\Phi \in C_0^\infty((-2, 2))$ be such that $\Phi = 1$ on $(-1, 1)$ and define

$$h_1(x, u) := \Phi(u) f(x, u), \quad h_2(x, u) := (1 - \Phi(u)) f(x, u).$$

We can assume that $\frac{p}{r} \le \frac{q}{s}$. Hence we obtain

$$|h_1(x, u)| \le a|u|^{\frac{p}{r}}, \quad |h_2(x, u)| \le b|u|^{\frac{q}{s}}.$$

Assume that $u_n \to u$ in $L^p(V) \cap L^q(V)$. Let $\{v_n\}$ and h be given by the preceding lemma. Since

$$|h_1(x, v_n) - h_1(x, u)|^r \le 2^r a^r |h|^p,$$

it follows from Lebesgue dominated convergence theorem that $h_1 v_n \to h_1 u$ in $L^r(V)$. And then $h_1 u_n \to h_1 u$ in $L^r(V)$. Similarly, we have $h_2(x, u_n) \to h_2(x, u)$ in $L^s(V)$.

Since

$$|f(x, u_n) - f(x, u)|_{r,s} \le |h_1(x, u_n) - h_1(x, u)|_r + |h_2(x, u_n) - h_2(x, u)|_s$$

it follows that $f(x, u_n) \to f(x, u)$ in $L^r(V) + L^s(V)$. □

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