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Finite NPDM-groups

Jian Ji CAO
School of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006*, P. R. China*
E-mail: 13994371056@163*.com*

Xiu Yun GUO[1)]
Department of Mathematics, Shanghai University, Shanghai 200444*, P. R. China*
E-mail: *xyguo@staff.shu.edu.cn*

Abstract In this paper the classification is given for finite groups in which the normalizer of every non-normal cyclic subgroup of order divided by the minimal prime of $|G|$ is a maximal subgroup.

Keywords Normalizer, cyclic subgroup, maximal subgroup

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1 Introduction

All groups considered in this paper are finite.

It is quite interesting to investigate the structure of groups by using normalizers of some kind of subgroups. For example, a famous p-nilpotent criterion due to Frobenius [9] is that a group G is p-nilpotent if and only if the normalizers of all p-subgroups are p-nilpotent. Bianchi, Gillio Berta Mauri and Hauck in [3] proved that a group is nilpotent if and only if the normalizer of every Sylow subgroup is nilpotent. Ballester-Bolinches and Shemetkov in [2] established a p-nilpotency criterion of a group by using only normalizers of Sylow p-subgroups: a group is p-nilpotent if and only if the normalizers of Sylow p-subgroups are p-nilpotent.

Inspired by the above research, we are interested in the class of groups in which the normalizer of every non-normal cyclic subgroup of order divided by the smallest prime p of $|G|$ is a maximal subgroup. For convenience, if the order of an element x is divided by p, then we call such an element a pd-element, $\langle x\rangle$ a pd-subgroup. We also call a group an NPDM-group if the normalizer of every non-normal cyclic pd-subgroup is maximal in G.

In [4, 5] we give the classification of groups in which the normalizer of every non-normal cyclic subgroup is a maximal subgroup, which is called NCM-groups. The following examples illustrate that there exists NPDM-groups but they are not NCM-groups. Therefore the class of NPDM-groups is larger than the class of NCM-groups.

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1) Corresponding author

Example 1.1 Let $G = (\langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle \times \langle e\rangle) \rtimes \langle f\rangle$ with $a^2 = b^2 = c^2 = d^2 = e^7 = f^3 = 1, a^f = ab, b^f = a, c^f = cd, d^f = c, e^f = e^2$. Then G is an NPDM-group but not an NCM-group.

In fact, it is easy to find that G is an NPDM-group. On the other hand, $N_G(\langle f\rangle) = \langle f\rangle$ and $\langle a, f\rangle = (\langle a\rangle \times \langle b\rangle) \rtimes \langle f\rangle \neq G$. So $N_G(\langle f\rangle)$ is not maximal in G and G is not an NCM-group.

Example 1.2 Let $G = \mathrm{PSL}(2, 11)$. Then G is an NPDM-group but not an NCM-group.

In fact, it is easy to find that every element of G of even order is contained in a cyclic subgroup $S \simeq C_6$ and $N_G(S) \simeq D_{12}$ is a maximal subgroup of G. So G is an NPDM-group. On the other hand, there exists a cyclic subgroup $U \simeq C_5$ such that $N_G(U) \simeq D_{10}$ is not maximal in G. Therefore G is not an NCM-group.

We will investigate NPDM-groups in this paper and then give a classification of this kind of groups.

2 Preliminaries

In this section, we list some basic properties of NPDM-groups and also list some lemmas which will be useful for the proof of our main results.

Lemma 2.1 *Let p be the smallest prime dividing the order of a group G and N be a normal p'-subgroup of a group G. If G is an* NPDM-*group, then G/N is also an* NPDM-*group.*

Proof Let $\langle x\rangle N/N$ be a non-normal pd-subgroup of G/N. Then $\langle x\rangle \ntrianglelefteq G$ and x is a pd-element in G, and therefore $N_G(\langle x\rangle)$ is a maximal subgroup in G. It follows from $N_G(\langle x\rangle)N/N \leq N_{G/N}(\langle x\rangle N/N)$ that $N_{G/N}(\langle x\rangle N/N)$ is a maximal subgroup in G/N. □

Lemma 2.2 *Let p be the smallest prime dividing the order of a group G and E be a non-normal cyclic p-subgroup of G. If G is an* NPDM-*group, then there is a normal Hall p'-subgroup K of $C_G(E)$ such that every subgroup of K is normal in $N_G(E)$ and $N_G(E) = K \rtimes P$ with P a Sylow p-subgroup of $N_G(E)$. Furthermore, K is an abelian group.*

Proof If $C_G(E)$ is a p-subgroup, then, since $N_G(E)/C_G(E)$ is a p-group, there is nothing need to be proved. Now assume that $C_G(E)$ is not a p-subgroup and that F is a cyclic q-subgroup of $C_G(E)$ with $q \neq p$ a prime. Since E is a characteristic subgroup of EF, we see $EF \ntrianglelefteq G$ and $E \trianglelefteq N_G(EF)$. The maximality of $N_G(EF)$ implies that $N_G(E) = N_G(EF)$. By the same reason, we have $F \trianglelefteq N_G(EF)$ and therefore $N_G(F) \geq N_G(EF) = N_G(E)$. It follows that every q-subgroup of $C_G(E)$ is normal in $C_G(E)$, and therefore the Hall p'-subgroup K in $C_G(E)$ is normal in $C_G(E)$, which also implies that every subgroup in K is normal in $N_G(E)$. Noticing that $N_G(E)/C_G(E)$ is a p-subgroup, we see $N_G(E) = K \rtimes P$ with P a Sylow p-subgroup of $N_G(E)$. We also know that K is a Dedekind group, thus K is an abelian group. The proof of the lemma is complete. □

Lemma 2.3 ([7, Theorem 2.2]) *Let $q = p^f \geq 5$ with p an odd prime. Then the maximal subgroup of* $\mathrm{PSL}(2, q)$ *are*:

(1) $C_p^f \rtimes C_{\frac{q-1}{2}}$;

(2) D_{q-1}*, for* $q \geq 13$;

(3) D_{q+1}*, for* $q \neq 7, 9$;

(4) $\mathrm{PGL}(2, q_0)$*, for* $q = {q_0}^2$;

(5) $\mathrm{PSL}(2, q_0)$*, for* $q = {q_0}^r$ *where r is an odd prime*;

(6) A_5, *for* $q \equiv \pm 1 \pmod{10}$ *where either* $q = p$ *or* $q = p^2$ *and* $p \equiv \pm 3 \pmod{10}$;

(7) A_4, *for* $q = p \equiv \pm 3 \pmod 8$ *and* $q \not\equiv \pm 1 \pmod{10}$;

(8) S_4, *for* $q = p \equiv \pm 1 \pmod 8$.

Lemma 2.4 ([7, Theorem 3.5]) *Let* $G = \mathrm{PGL}(2, q)$ *with* $q = p^f \geq 5$ *and* p *an odd prime. Then the maximal subgroup of* G *not containing* $\mathrm{PSL}(2, q)$ *are*:

(1) $C_p^f \rtimes C_{q-1}$;

(2) $D_{2(q-1)}$, *for* $q \neq 5$;

(3) $D_{2(q+1)}$;

(4) S_4, *for* $q = p \equiv \pm 3 \pmod 8$;

(5) $\mathrm{PGL}(2, q_0)$, *for* $q = {q_0}^r$ *where* r *is an odd prime.*

Lemma 2.5 ([9, Theorems 2.8.2, 2.8.3, 2.8.4, 2.8.5]) *Let* $G \simeq \mathrm{PSL}(2, q)$ *such that* $q = p^f$ *and* p *is a prime. Then there exist subgroups* $P \simeq C_p^f$, $U \simeq C_{\frac{q-1}{2}}$, $S \simeq C_{\frac{q+1}{2}}$ *of* G *such that* $N_G(P) \simeq C_p^f \rtimes C_{\frac{q-1}{2}}$, $N_G(U) \simeq D_{q-1}$ *and* $N_G(S) \simeq D_{q+1}$. *Moreover, every element of* G *must conjugate with one element of* P, U *or* S.

Lemma 2.6 ([5, Lemma 2.15]) *Let* $G \simeq \mathrm{PGL}(2, q)$ *such that* $q = p^f \geq 7$ *and* p *is an odd prime. Then there exist subgroups* $P \simeq C_p^f$, $U \simeq C_{q-1}$, $S \simeq C_{q+1}$ *of* G *such that* $N_G(P) \simeq C_p^f \rtimes C_{q-1}$, $N_G(U) \simeq D_{2(q-1)}$ *and* $N_G(S) \simeq D_{2(q+1)}$. *Moreover, every element of* G *must conjugate with one element of* P, U *or* S.

Recall that a group is called an I-group if the centralizer of every involution has a normal 2-complement. We also use $O(G)$ and $S(G)$ to denote the largest normal subgroup of odd order and the largest solvable normal subgroup of a group G respectively. The structure of non-solvable I-groups was given by Gorenstein (see [8]) as follows:

Lemma 2.7 ([8, Theorem A]) *A non-solvable* I*-group* G *has one of the following structures*:

(i) $G/O(G)$ *contains a normal subgroup of odd index isomorphic to* $\mathrm{PSL}(2, q)$, $\mathrm{PGL}(2, q)$, $\mathrm{PGL}^*(2, q)$, q *odd,* $q > 3$, *or* A_7;

(ii) $G/O(G)$ *is isomorphic to* $\mathrm{PSL}(2, 2^n)$ *or* $Sz(2^n)$, $n \geq 3$, *or to* $\mathrm{PSL}(3, 4)$;

(iii) $S(G) = O_{2'2}(G) \supset O(G)$, *and* $G/S(G)$ *is isomorphic to* $\mathrm{PSL}(2, 2^n)$, $n \geq 2$, *or to* $\mathrm{Sz}(2^n)$, $n \geq 3$.

By the way, we often use the following lemma.

Lemma 2.8 *If* M *is a maximal subgroup of a solvable group* G, *then the index* $|G : M|$ *is a prime power.*

3 Solvable NPDM-groups

In this section, we will give the structure of solvable NPDM-groups. In the following of this section, we always assume that G is a solvable group, p is the smallest prime dividing the order of G and P is a Sylow p-subgroup of G. We begin with the following lemma.

Lemma 3.1 *If* G *is an* NPDM*-group but not an* NCM*-group, then*

(1) *there is no non-trivial cyclic normal* p*-subgroup in* G;

(2) *for any* p*-element* $1 \neq x \in G$, *there is an element* $g \in G$ *such that* $P^g \leq N_G(\langle x \rangle)$.

Proof If there is a non-trivial cyclic normal p-subgroup in G, then there exists a normal p-subgroup $\langle x \rangle$ of order p in G. The minimality of p implies that $\langle x \rangle \leq Z(G)$. Let $\langle y \rangle$ be a non-normal cyclic p'-subgroup of G. Then $N_G(\langle x\rangle\times\langle y\rangle)\le N_G(\langle y\rangle)$. It follows from the maximality of $N_G(\langle x\rangle\times\langle y\rangle)$ that $N_G(\langle y\rangle)$ is maximal in G. Thus G is an NCM-group, a contradiction. So (1) is true.

If there is a non-trivial cyclic p-subgroup $\langle x\rangle$ of G such that $|G:N_G(\langle x\rangle)|=p^s$, then there is a Hall p'-subgroup T of G such that $T\le N_G(\langle x\rangle)$. Let $\langle y\rangle\le T$ be a non-normal cyclic subgroup of G. Then, by Lemma 2.2, $N_G(\langle x\rangle)\le N_G(\langle y\rangle)$. The maximality of $N_G(\langle x\rangle)$ implies that $N_G(\langle y\rangle)$ is maximal in G. Noticing that all Hall p'-subgroups of G are conjugate in G, we see that $N_G(\langle z\rangle)$ is maximal in G for every non-normal cyclic p'-subgroup $\langle z\rangle$ of G. Thus G is an NCM-group, a contradiction. So (2) is true. □

Lemma 3.2 *If G is an* NPDM-*group but G is neither p-closed nor p-nilpotent, then $O_{p'}(G)=Z(G)$.*

Proof Since G is neither p-closed nor p-nilpotent, we see that G is not an NCM-group by [4, Lemma 3.2]. By Lemma 3.1 (1), we have $Z(G)\le O_{p'}(G)$. Now we prove $O_{p'}(G)\le Z(G)$. In fact, let $\langle x\rangle$ be a non-trivial p-subgroup of P. By Lemmas 3.1 and 2.2, $N_G(\langle x\rangle)=T_x\rtimes P_x$ with T_x a Hall p'-subgroup of $N_G(\langle x\rangle)$ and P_x a Sylow p-subgroup of G. If $O_{p'}(G)\nleq T_x$, then $O_{p'}(G)\nleq N_G(\langle x\rangle)$. The maximality of $N_G(\langle x\rangle)$ implies that $G=O_{p'}(G)N_G(\langle x\rangle)$ is p-nilpotent, a contradiction. Thus $O_{p'}(G)\le T_x$, and therefore $x\in C_G(O_{p'}(G))$. It follows that $P\le C_G(O_{p'}(G))$ and therefore $P_x\le C_G(O_{p'}(G))$ by Lemma 3.1 (2). Furthermore, by Lemma 2.2 again, $T_x\le C_G(O_{p'}(G))$ and so $N_G(\langle x\rangle)\le C_G(O_{p'}(G))$. The maximality of $N_G(\langle x\rangle)$ implies that $C_G(O_{p'}(G))=G$ or $C_G(O_{p'}(G))=N_G(\langle x\rangle)$. If $C_G(O_{p'}(G))=N_G(\langle x\rangle)$, then it follows from T_x char $C_G(O_{p'}(G))\trianglelefteq G$ that T_x is normal in G. Thus $T_x=O_{p'}(G)$ and $C_G(O_{p'}(G))=O_{p'}(G)\times P_x$, and therefore P_x is normal in G, in contradiction to that G is not p-closed. Hence $C_G(O_{p'}(G))=G$ and $O_{p'}(G)\le Z(G)$. The proof of the lemma is complete. □

Lemma 3.3 *If G is an* NPDM-*group but G is neither p-closed nor p-nilpotent and $O_{p'}(G)=1$, then P is a maximal subgroup of G and there exists a Sylow q-subgroup Q of order q with $q\ne p$ such that $G=PQ$. Furthermore, $QO_p(G)$ is a Frobenius group with kernel $O_p(G)$ and complement Q, $G/O_p(G)$ is also a Frobenius group with kernel $O_p(G)Q/O_p(G)$ and complement $P/O_p(G)$, and $P/O_p(G)$ is cyclic with $|P/O_p(G)|\mid(q-1)$. For the sake of convenience, we call this kind of groups F_{pq}-groups.*

Proof Let $\langle x\rangle$ be a non-trivial p-subgroup of P. By Lemmas 3.1 and 2.2, $N_G(\langle x\rangle)=T_x\rtimes P_x$ with T_x a Hall p'-subgroup of $N_G(\langle x\rangle)$ and P_x a Sylow p-subgroup of G. Noticing that $O_p(G)\le P_x$, we see that $[T_x,O_p(G)]=1$ and therefore $T_x\le C_G(O_p(G))$. It follows from [9, Theorem 3.4.2] that $T_x=1$ and therefore P is a maximal subgroup of G. Thus there exists a prime $q\ne p$ and a Sylow q-subgroup Q of G such that $G=PQ$. Let $H/O_p(G)$ be a chief factor of G. Since $H\nleq P$ and the maximality of P, we have $G=PH$, and therefore $H=O_p(G)Q$ and Q is elementary abelian. It is easy to see that the action of Q on $O_p(G)$ is fixed-point-free. Thus $H=O_p(G)Q$ is a Frobenius group with kernel $O_p(G)$ and complement Q. By Burnside theorem [9, Theorem 5.8.7], Q is cyclic and therefore Q is a cyclic group of order q. On the other hand, it is also easy to see that the action of $P/O_p(G)$ on $QO_p(G)/O_p(G)$ is fixed-point-free. It follows from the structure of Frobenius groups that $P/O_p(G)$ is cyclic or generalized quaternion if $p=2$. Noticing that $\mathrm{Aut}(Q)$ is cyclic, we see that $P/O_p(G)$ is cyclic and $|P/O_p(G)|\mid(q-1)$.

The proof of the lemma is complete. □

Corollary 3.4 *If G is an* NPDM-*group but G is neither p-closed nor p-nilpotent, then there exists a prime q such that the quotient group $G/Z(G)$ is a F_{pq}-group.*

Proof By Lemmas 2.1 and 3.2, we see that $G/Z(G)$ is also an NPDM-group. If $G/Z(G)$ is p-closed or p-nilpotent, then it is easy to know that G is p-closed or p-nilpotent. Thus $G/Z(G)$ is neither p-closed nor p-nilpotent. Now the results follows from Lemmas 3.2 and 3.3. □

Theorem 3.5 *If G is an* NPDM-*group, then G is isomorphic to one of the following groups:*

(I) *G is an* NCM-*group;*

(II) *$G = P \rtimes T$ with P an abelian subgroup and T a Hall p'-subgroup of G. Furthermore $C_T(P)$ is an abelian subgroup and $P \times C_T(P) = C_G(P)$ is maximal in G;*

(III) *$G = T \rtimes P$ with T a Hall p'-subgroup of G. Furthermore, for any p-element $x \in G$ with $\langle x \rangle \ntrianglelefteq G$, there is an element g_x in G such that $N_G(\langle x \rangle) = C_T(x) \rtimes P^{g_x}$ is maximal in G;*

(IV) *there exists a prime q such that G is a F_{pq}-group;*

(V) *there exists a prime q such that the quotient group $G/Z(G)$ is a F_{pq}-group.*

Proof By the above discussion (Lemma 3.3 and Corollary 3.4), we may only investigate the case that G is an NPDM-group but not an NCM-group and G is either p-closed or p-nilpotent. If G is both p-closed and p-nilpotent, then it is easy to see that there exists a non-trivial cyclic normal p-subgroup in G, in contradiction to Lemma 3.1. So we may only investigate the following two cases.

Case 1 $P \trianglelefteq G$. By Lemma 3.1 (2), P is a Dedekind group. If P is a non-abelian group, then $P \simeq Q_8 \times C_2 \times \cdots \times C_2$. Noticing that $C_2 \simeq \mho_1(P)$ char $P \trianglelefteq G$, we see that $\mho_1(P)$ is a normal cyclic 2-subgroup of G, a contradiction. So P is an abelian group. Let $G = P \rtimes T$ with T a Hall p'-subgroup of G. By Lemma 2.2, $N_G(\langle x \rangle) = C_T(x) \times P$ is maximal in G for any $x \in P$ with $\langle x \rangle \ntrianglelefteq G$ and $C_T(x)$ an abelian group. It is easy to find that $N_G(\langle x \rangle) \leq C_G(P)$. By maximality of $N_G(\langle x \rangle)$, we see that $C_G(P) = G$ or $C_G(P) = N_G(\langle x \rangle)$. If $C_G(P) = G$, then every cyclic subgroup of P is normal in G, in contradiction to Lemma 3.1 (1). So $C_G(P) = N_G(\langle x \rangle)$, and furthermore $C_T(x) = C_T(P)$. So G is a group of type (II).

Case 2 G is a p-nilpotent group. Let $G = T \rtimes P$ with T a Hall p'-subgroup of G. Then by Lemma 3.1, for any p-element $x \in G$ with $\langle x \rangle \ntrianglelefteq G$, there is an element g_x in G such that $\langle x \rangle \trianglelefteq P^{g_x}$. Therefore by Lemma 2.2 $N_G(\langle x \rangle) = C_T(x) \rtimes P^{g_x}$ is maximal in G. So G is a group of type (III). The proof is complete. □

4 Non-solvable NPDM-groups

Recall that a group is called a semisimple group if it has no non-trivial solvable normal subgroup. First, we give the structure of semisimple NPDM-groups. We begin with the following lemmas.

Lemma 4.1 *Let G be a semisimple* NPDM-*group. Then G has a unique minimal normal subgroup N such that $N \simeq \mathrm{PSL}(2, q)$ with $q = p^f \geq 5$ and p an odd prime or $N \simeq \mathrm{PSL}(3, 4)$ and $G \lesssim \mathrm{Aut}(N)$.*

Proof Let g be an involution of G. Then, by Lemma 2.2, $C_G(g)$ is a solvable subgroup of G, and therefore $\mathrm{core}_G(C_G(g)) = 1$. The maximality of $C_G(g)$ implies that G is a primitive group. If there are two minimal normal subgroups N and N^* in G, then $C_G(N) = N^*$ by [6, Theorem A.15.2]. It follows that $N^* = C_G(N) \leq C_G(y)$ for any involution $y \in N$, in contradiction to the solvability of $C_G(y)$. So N is the unique minimal normal subgroup of G. Let $N = T_1 \times T_2 \times \cdots \times T_n$ with $T_i \simeq T$ a non-abelian simple group. If $n > 1$, then $T_2 \leq C_G(t)$ for any involution $t \in T_1$, in contradiction to the solvability of $C_G(t)$. So N is a non-abelian simple group. $N \cap C_G(N) = 1$, thus $C_G(N) = 1$. Therefore $G \lesssim \mathrm{Aut}(N)$.

If the centralizer of every involution of N is a 2-group, then by Suzuki's main result in [10] and [11], N is isomorphic to one of the following groups: $\mathrm{PSL}(2,p)$ with p a Fermat or Mersenne prime and $p > 5$, $\mathrm{PSL}(2,9)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSL}(2,2^n)$ with $n \geq 2$ and $\mathrm{Sz}(2^n)$ with $n \geq 3$. For any involution $x \in Z(Q)$ with Q a Sylow 2-subgroup of N, we see that $C_N(x) = Q$. The maximality and the solvability of $C_G(x) = N_G(\langle x \rangle)$ implies that $G = NC_G(x)$. If N is isomorphic to $\mathrm{PSL}(2,2^n)$ with $n \geq 2$ or $\mathrm{Sz}(2^n)$ with $n \geq 3$, then $C_N(x) = Q < N_N(Q)$, in addition, $Q = C_N(x) \trianglelefteq C_G(x)$, we have $C_G(x) < N_G(Q)$. Therefore $Q \trianglelefteq G$, a contradiction. So N is isomorphic to $\mathrm{PSL}(2,p)$ with p a Fermat or Mersenne prime and $p > 5$, $\mathrm{PSL}(2,9)$ or $\mathrm{PSL}(3,4)$.

If there is an involution $z \in N$ such that $C_N(z)$ is not a 2-group. By Lemma 2.2, $C_G(z)$ is a solvable group and every $2'$-subgroup of $C_G(z)$ is normal in $C_G(z)$. So $C_N(z)$ is also a solvable group and $O(C_N(z)) \neq 1$. By WUGT Theorem of [1], N is isomorphic to one of the following groups: $\mathrm{PSL}(2,q)$ with q an odd number, A_7 or A_{11}. If $N \simeq A_7$, then there exists a subgroup $F \times T$ of N with $F \simeq A_4$ and $T \simeq C_3$. Let C_2 be any cyclic subgroup of F. Then $H_1 = C_2 \times T$ is a cyclic subgroup of even order. By maximality of $N_G(H_1)$, we see that $N_G(C_2) = N_G(T) = N_G(H_1)$. $F \leq N_G(T) = N_G(C_2)$ implies that every subgroup of F with order two is normal in F, this is a contradiction. If $N \simeq A_{11}$, then we may choose a subgroup $E \times U$ with $E \simeq A_8$ and $U \simeq C_3$. Similarly, we can get a contradiction. Since $\mathrm{PSL}(2,3)$ is solvable, we see $N \simeq \mathrm{PSL}(2,q)$ with $q \geq 5$. The proof is now complete. □

Lemma 4.2 *Let $G = \mathrm{PSL}(2,q) \rtimes C_f$ with $q = p^f$ and p an odd prime. If $C_f = \langle g \rangle$ and g induces a field automorphism on $\mathrm{PSL}(2,q)$, then g^i $(1 \leq i < f)$ is not contained in $C_G(C_{\frac{q-1}{2}})$ and $C_G(C_{\frac{q+1}{2}})$.*

Proof Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$ be a matrix of $\mathrm{SL}(2,q)$ represent element of cyclic subgroup $C_{\frac{q-1}{2}}$ in $\mathrm{PSL}(2,q)$ of order $\frac{q-1}{2}$. g induces a field automorphism on $\mathrm{PSL}(2,q)$, so $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{g^i} = \begin{pmatrix} a^{p^i} & b^{p^i} \\ c^{p^i} & d^{p^i} \end{pmatrix}$ with $1 \leq i < f$. If g^i is contained in $C_G(C_{\frac{q-1}{2}})$, then $a^{p^i-1} = b^{p^i-1} = c^{p^i-1} = d^{p^i-1} = 1$. It follows that there is a subfield $\mathrm{GF}(p^j) \subset \mathrm{GF}(q)$ with $j = (i,f)$ such that $a,b,c,d \in \mathrm{GF}(p^j)$, furthermore $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,p^j)$. $p^j < q$ implies that the order of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is not equal to $q-1$, a contradiction. Similarly, we can see that g^i $(1 \leq i < f)$ is not contained in $C_G(C_{\frac{q+1}{2}})$. The proof is complete. □

Theorem 4.3 *A group G is a semisimple* NPDM-*group if and only if G is one of the following groups*:

(I) $G \simeq \mathrm{PSL}(2,q)$ *with $q = p^f$, p an odd prime and $q \geq 11$;*

(II) $G \simeq \mathrm{PGL}(2,q)$ *with $q = p^f$, p an odd prime and $q \geq 7$.*

Proof By Lemma 4.1, G has a unique minimal normal subgroup N such that $N \simeq \mathrm{PSL}(2,q)$, where $q = p^f \geq 5$ and p is an odd prime or $N \simeq \mathrm{PSL}(3,4)$. We can also find that the centralizer of every involution of G has a normal 2-complement by Lemma 2.2. If $N \simeq \mathrm{PSL}(3,4)$, then by Lemma 2.7, $G \simeq \mathrm{PSL}(3,4)$. Also by Suzuki's main result in [10] and [11], we see that the centralizer of every involution of G is a 2-group. So every Sylow 2-subgroup of G is maximal in G. On the other hand, there is a subgroup of $G \simeq \mathrm{PSL}(3,4)$ with order 960 that contains a Sylow 2-subgroup of G, a contradiction. If $N \simeq \mathrm{PSL}(2,q)$ with $q = p^f \geq 5$ and p an odd prime, then by Lemma 2.7, G contains a normal subgroup of odd index isomorphic to $\mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$ or $\mathrm{PGL}^*(2,q)$. By Lemma 4.1 again, $G \lesssim \mathrm{Aut}\,(N) = \mathrm{PGL}(2,q) \rtimes C_f$. So, there is a cyclic subgroup C_n with odd order in C_f which normalizes $\mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$ or $\mathrm{PGL}^*(2,q)$. We confirm that $C_n = 1$. Otherwise, let $C_f = \langle g \rangle$ and $C_n = \langle g^i \rangle \neq 1$ with $1 \leq i < f$. Then g^i induces a field automorphism on $\mathrm{PSL}(2,q)$. We consider the following two cases.

Case 1 $q \equiv 1 \,(\mathrm{mod}\ 4)$. In this case, we can find that $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ with $a^2 = -1$ represents an involution τ in $\mathrm{PSL}(2,q)$. If f is an odd, then $p^{f-1}+p^{f-2}+\cdots+1$ is also an odd. Thus $4|(p^f-1)$ implies that $4|(p-1)$ and so $4|(p^i-1)$. Furthermore $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)^{g^i} = \left(\begin{smallmatrix} a^{p^i} & 0 \\ 0 & a^{-p^i} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. So $g^i \in C_G(\tau)$. By Lemmas 2.2 and 2.3, $N_G(\langle g^i \rangle) = C_G(\tau) = \langle D_{q-1}, g^i \rangle = T \rtimes Q$ with T an abelian Hall $2'$-subgroup of $C_G(\tau)$ and Q a Sylow 2-subgroup of $C_G(\tau)$. Let $C_{\frac{q-1}{2}}$ be a cyclic subgroup that contains τ. Then $C_{\frac{q-1}{2}}$ is represented by $\langle A \rangle$ where $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. If $\langle e \rangle$ is a Sylow 2-subgroup of $C_{\frac{q-1}{2}}$, then $\langle \tau \rangle$ char $\langle e \rangle \trianglelefteq N_G(\langle e \rangle)$ implies that $N_G(\langle e \rangle) = N_G(\langle \tau \rangle) = N_G(\langle g^i \rangle)$. So $[g^i, e] \leq \langle g^i \rangle \cap \langle e \rangle = 1$. Noticing that $g^i \in T$ and T is an abelian Hall $2'$-subgroup of $N_G(\langle g^i \rangle)$, we see that $g^i \in C_G(C_{\frac{q-1}{2}})$, in contradiction to Lemma 4.2. If f is an even, then i is also an even since $\langle g^i \rangle$ has odd order. It follows that $p-1$ and $p^{i-1}+p^{i-2}+\cdots+1$ are all even, furthermore $4|(p^i-1)$. So $g^i \in C_G(\tau)$. Similarly, we get another contradiction.

Case 2 $q \equiv -1 \,(\mathrm{mod}\ 4)$. In this case, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ represents an involution τ' in $\mathrm{PSL}(2,q)$. $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{g^i} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1^{p^i} \\ (-1)^{p^i} & 0 \end{smallmatrix}\right)$, since p^i is an odd. So $g^i \in C_G(\tau')$. Similarly, we see that $g^i \in C_G(C_{\frac{q+1}{2}})$, where $C_{\frac{q+1}{2}}$ is represented by $\langle B \rangle$ with $B = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ such that $a^2+b^2 = 1$, in contradiction to Lemma 4.2.

So G is isomorphic to $\mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$ or $\mathrm{PGL}^*(2,q)$. We consider the following three cases separately.

(1) $G \simeq \mathrm{PSL}(2,q)$ with $q = p^f$ and p an odd prime is an NPDM-group if and only if $q \geq 11$.

Let $C = \langle u \rangle$ be a cyclic group of $G \simeq \mathrm{PSL}(2,q)$ of even order. Then, by Lemma 2.5, u is conjugate with one element of the following subgroups: $U \simeq C_{\frac{q-1}{2}}$, $S \simeq C_{\frac{q+1}{2}}$. Without loss of generality, we may assume that C is contained in U or S. It follows that $N_G(C) \simeq D_{q+1}$ or $N_G(C) \simeq D_{q-1}$.

If $q < 13$ and $q \neq 7, 11$, then, by Lemma 2.5, there is a cyclic subgroup F of even order such that $N_G(F) \simeq D_{q-1}$. By Lemma 2.3, if $q < 13$, then $N_G(F) \simeq D_{q-1}$ is not maximal in G, a contradiction. If $q = 7$, then $|U| = 3$ implies that $N_G(C) \simeq D_8$. On the other hand, by Lemma 2.3 again, we see that $N_G(C) \simeq D_8$ is not maximal in G, a contradiction. So $q \geq 11$.

Conversely, if $G \simeq \mathrm{PSL}(2,11)$, then $|U| = 5$ implies that $N_G(C) \simeq D_{12}$. By Lemma 2.3, $N_G(C)$ is maximal in G. If $G \simeq \mathrm{PSL}(2,q)$ with $q \geq 13$, then, by Lemma 2.3 again, we see that $N_G(C) \simeq D_{q+1}$ or $N_G(C) \simeq D_{q-1}$ is also maximal in G. So G is a group of type (I).

(2) $G \simeq \mathrm{PGL}(2,q)$ with $q = p^f$ and p an ddd prime is an NPDM-group if and only if $q \geq 7$.

Assume that $G \simeq \mathrm{PGL}(2,5)$. By [9, Theorem 2.7.2 (a)], $\mathrm{GL}(2,5)$ has an abelian subgroup

$D \simeq C_4 \times C_4$ which contains $Z(\mathrm{GL}(2,5))$. Furthermore, $N_{\mathrm{GL}(2,5)}(D)/D \simeq C_2$. Hence G has a cyclic subgroup $H(\simeq D/Z(\mathrm{GL}(2,5)))$ with order 4 such that $N_G(H)/H \simeq C_2$ by [9, Theorem 2.7.2 (a)] again. Hence $N_G(H) \simeq D_8$ or Q_8, but by Lemma 2.4, $N_G(H)$ is not a maximal subgroup of PGL(2, 5). So $q \geq 7$.

Conversely, let E be a cyclic group of $G \simeq \mathrm{PGL}(2,q)$ of even order with $q \geq 7$. Then, similarly, by Lemma 2.6 we can find that $N_G(E)$ contains $N_G(U)$ or $N_G(S)$. By Lemma 2.4, we see that $N_G(E)$ is maximal in G. So G is a group of type (II).

(3) $G = \mathrm{PGL}^*(2,q)$ with q an odd and $q = r^2$ for some integer r is not an NPDM-group.

If $q < 13$, then $q = 9$ and $G \simeq \mathrm{PGL}^*(2,9)$. Let S be a Sylow 2-subgroup of $\mathrm{PGL}^*(2,9)$. Then $S \simeq C_8 \rtimes C_2 = \langle a \rangle \rtimes \langle b \rangle$ with $\langle a \rangle \simeq C_8$ and $\langle b \rangle \simeq C_2$. It is easy to find that $\langle ab \rangle$ is a cyclic subgroup of order 4 and $N_G(\langle ab \rangle) = \langle a^2, ab \rangle < S$. Therefore $N_G(\langle ab \rangle)$ is not maximal in G and so $\mathrm{PGL}^*(2,9)$ is not an NPDM-group. So $q \geq 13$.

As is well-known, $\mathrm{Aut}(\mathrm{PSL}(2,q)) = P\Gamma L(2,q) = \mathrm{PSL}(2,q) \rtimes F$ with $q = p^n$, F a cyclic subgroup of order n. If $|F|$ is even, then there are three subgroups of $P\Gamma L(2,q)$ containing $\mathrm{PSL}(2,q)$ as a subgroup of index 2. They are $\mathrm{PGL}(2,q)$, $\langle \mathrm{PSL}(2,q), c \rangle$ and $\mathrm{PGL}^*(2,q) = \langle \mathrm{PSL}(2,q), ac \rangle$ where $\langle ac \rangle$ and $\langle a \rangle$ are the maximal cyclic subgroups of a Sylow 2-subgroup of $\mathrm{PGL}^*(2,q)$ and $\mathrm{PGL}(2,q)$ respectively, c induces a field automorphism on $\mathrm{PSL}(2,q)$. By [8, Lemma 2.3], every Sylow 2-subgroup of $\mathrm{PGL}^*(2,q)$ is a semidihedral and every involution of $\mathrm{PGL}^*(2,q)$ lies in $\mathrm{PSL}(2,q)$. If $\mathrm{PGL}^*(2,q)$ is an NPDM-group, then by Lemmas 2.2 and 2.3 $N_G(\langle ac \rangle) = \langle D_{q-1}, ac \rangle$ or $\langle D_{q+1}, ac \rangle$ has a normal 2-complement T and $T \leq C_G(ac)$. So ac is contained in $C_{P\Gamma L(2,q)}(C_{\frac{q-1}{2}})$ or $C_{P\Gamma L(2,q)}(C_{\frac{q+1}{2}})$. By (2), $\mathrm{PGL}(2,q)$ with $q \geq 13$ is an NPDM-group. Similarly we can find that a is contained in $C_{P\Gamma L(2,q)}(C_{\frac{q-1}{2}})$ or $C_{P\Gamma L(2,q)}(C_{\frac{q+1}{2}})$. So c is also abelian with cyclic subgroup $C_{\frac{q-1}{2}}$ or $C_{\frac{q+1}{2}}$ of $\mathrm{PGL}^*(2,q)$, in contradiction to Lemma 4.2. So $\mathrm{PGL}^*(2,q)$ is not an NPDM-group. The proof is complete. □

Corollary 4.4 *Let G be a non-solvable group. If G is a non-semisimple* NPDM-*group. Then $S(G) = O(G)$ and $G/O(G)$ is isomorphic to the groups stated in the above theorem.*

Proof If G is a non-semisimple group, then by Lemma 2.7 and Theorem 4.3 we have that $S(G) = O(G)$. Also by Lemma 2.1, $G/O(G)$ is a semisimple NPDM-group. So $G/O(G)$ is a group stated in the above theorem. □

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