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Quasi-normal Family of Meromorphic Functions Whose Certain Type of Differential Polynomials Have No Zeros

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Abstract Define the differential operators ϕ_n for $n \in \mathbb{N}$ inductively by $\phi_1[f](z) = f(z)$ and $\phi_{n+1}[f](z) = f(z)\phi_n[f](z) + \frac{d}{dz}\phi_n[f](z)$. For a positive integer $k \geq 2$ and a positive number δ, let $\mathcal{F}$ be the family of functions f meromorphic on domain $D \subset \mathbb{C}$ such that $\phi_k[f](z) \neq 0$ and $|\mathrm{Res}(f, a) - j| \geq \delta$ for all $j \in \{0, 1, \ldots, k-1\}$ and all simple poles a of f in D. Then $\mathcal{F}$ is quasi-normal on D of order 1.

Keywords Normal families, quasi-normal families, differential polynomials, meromorphic functions

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1 Introduction

The following theorem was conjectured by Hayman [9, p. 23] and proved by Frank [7] for $k \geq 3$ and by Langley [10] for $k = 2$.

Theorem 1.1 *Let $k \in \mathbb{N}$ with $k \geq 2$ and let f be a function meromorphic on whole $\mathbb{C}$ such that $f \neq 0$ and $f^{(k)} \neq 0$. Then either $f(z) = \mathrm{e}^{a(z+b)}$ or $f(z) = a/(z+b)^n$ for some constants $a \neq 0$, b and $n \in \mathbb{N}$.*

It follows from Theorem 1.1 that either $f'/f = a$ or $f'/f = -n/(z+b)$ for meromorphic functions f on $\mathbb{C}$ with the property $ff^{(k)} \neq 0$. Note that the family $\{-n/(z+b) : n \in \mathbb{N}, b \in \mathbb{C}\}$ is a normal family on $\mathbb{C}$.

A heuristic principle attributed to Bloch says that if the functions meromorphic and possessing a given property on $\mathbb{C}$ must be constants (or weakly, form a family normal on $\mathbb{C}$), then the functions meromorphic and possessing the same property on a domain $D \subset \mathbb{C}$ form a family normal on D. See [2, 13, 16], where the Bloch principle is thoroughly discussed.

The normality criteria corresponding to Theorem 1.1 have been obtained by Schwick [14] for holomorphic case, and by Bergweiler [1] and Bergweiler and Langley [3] for general meromorphic case.

Theorem 1.2 *Let $k \in \mathbb{N}$ with $k \geq 2$ and let $\mathcal{F}$ be a family of functions f meromorphic on $D \subset \mathbb{C}$ such that $f \neq 0$ and $f^{(k)} \neq 0$. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D.*

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In fact, Bergweiler and Langley [3] proved a more general result. They introduced a type of differential operators ϕ_n defined inductively by $\phi_1[f](z) = f(z)$ and

$$\phi_{n+1}[f](z) = f(z)\phi_n[f](z) + \frac{d}{dz}\phi_n[f](z). \tag{1.1}$$

Next we denote by $\mathrm{Res}_1(f, D)$ (or $\mathrm{Res}_1(f)$ simply) the set of residues of f at its simple poles in D; denote by $\mathbb{N}_k = \{0, 1, \ldots, k-1\}$ for $k \in \mathbb{N}$. Denote by

$$\mathrm{dis}(\mathrm{Res}_1(f, D), \mathbb{N}_k) = \inf\{|r - j| : r \in \mathrm{Res}_1(f, D), j \in \mathbb{N}_k\}$$

the distance between $\mathrm{Res}_1(f, D)$ and $\mathbb{N}_k$. We set $\mathrm{dis}(\mathrm{Res}_1(f, D), \mathbb{N}_k) = +\infty$, if f has no simple poles.

Theorem 1.3 ([3, Theorem 1.3]) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\delta \in \mathbb{R}^+$ with $0 < \delta \leq 1$. Let $\mathcal{F}$ be a family of functions f meromorphic on $D \subset \mathbb{C}$ such that*

(i) *$\phi_k[f](z) \neq 0$ for $z \in D$;*

(ii) *$\mathrm{dis}(\mathrm{Res}_1(f, D), \mathbb{N}_k) \geq \delta$; and*

(iii) *if $c \in D$ and $R > 0$ with $\Delta(c, R) \subset D$, if $\Delta(c, \delta R)$ contains two poles of f counting multiplicity, and if $\Delta(c, R) \setminus \Delta(c, \delta R)$ contains no poles of f, then*

$$\left| \sum_{a \in \Delta(c, \delta R)} \mathrm{Res}(f, a) - (k-1) \right| \geq \delta.$$

Then $\mathcal{F}$ is a normal family.

As pointed out in [3], the assumption (iii) in Theorem 1.3 is necessary to obtain normality. We consider here the following question: What can be said under the hypotheses (i) and (ii) of Theorem 1.3? Our answer is that there is quasi-normality.

Recall that a family of functions meromorphic in $D \subset \mathbb{C}$ is said to be normal (quasi-normal) in D in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally uniformly in D (minus a set that has no accumulation point in D). The subtracted set may depend on the subsequence. If there exists an integer ν such that the subtracted sets always can be chosen at most ν points, then $\mathcal{F}$ is said to be quasi-normal of order ν. So, a normal family can be regarded as a quasi-normal family of order 0. See [6, 13, 15]. Now our main result can be stated as follows.

Theorem 1.4 *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\delta > 0$. Let $\mathcal{F}$ be a family of functions f meromorphic on $D \subset \mathbb{C}$ such that*

(i) *$\phi_k[f](z) \neq 0$ for $z \in D$;*

(ii) *$\mathrm{dis}(\mathrm{Res}_1(f, D), \mathbb{N}_k) \geq \delta$.*

Then $\mathcal{F}$ is quasi-normal of order 1. Moreover, each sequence in $\mathcal{F}$ which is not normal at a point $z_0 \in D$ contains a subsequence which converges spherically locally uniformly to the function $(k-1)/(z - z_0)$ on $D \setminus \{z_0\}$.

In Section 2, we state and prove some lemmas and in Section 3, we prove our result Theorem 1.4. We remark that the idea somewhat comes from the papers [4, 12].

2 Preliminary Results

We write $f_n \xrightarrow{\chi} f$ on D to indicate that the sequence $\{f_n\}$ converges spherically locally uniformly to the function f on D; and write $f_n \to f$ on D if the convergence is in Euclidean metric, where the limit function f is allowed to be ∞ identically.

Lemma 2.1 ([3, Lemma 4.2]; [11, Lemma 2]) *Let $\mathcal{F}$ be a family of meromorphic functions on D. Suppose that there exists $\delta > 0$ such that $\mathrm{dis}(\mathrm{Res}_1(f, D), \{0\}) \geq \delta$. Then if $\mathcal{F}$ is not normal at some point $z_0 \in D$, there exist a sequence $\{f_n\} \subset \mathcal{F}$, a sequence of points $\{z_n\} \subset D$ with $z_n \to z_0$, and a sequence of positive numbers $\{\rho_n\}$ with $\rho_n \to 0$, such that the sequence $\{g_n\}$ defined by $g_n(\zeta) = \rho_n f_n(z_n + \rho_n \zeta)$ converges spherically locally uniformly on $\mathbb{C}$ to a nonconstant meromorphic function g of finite order and g satisfies $g^{\#}(\zeta) \leq g^{\#}(0) = 1 + 1/\delta$. Moreover, $\mathrm{dis}(\mathrm{Res}_1(g, \mathbb{C}), \{0\}) \geq \delta$.*

Lemma 2.2 *Let $\{f_n\}$ be a sequence of functions meromorphic on $D \subset \mathbb{C}$, and $z_0 \in D$ be a point such that $f_n \overset{\chi}{\to} f$ on $D \setminus \{z_0\}$. Then the following statements are true*:

(a) *If f_n are holomorphic on D and $f \not\equiv \infty$, then f is holomorphic on whole D and $f_n \to f$ on D*;

(b) *If $f_n \neq 0$ on D and $f \not\equiv 0$, then f is meromorphic on D and $f \neq 0$ or $f \equiv \infty$, and $f_n \overset{\chi}{\to} f$ on D*;

(c) *If f_n are holomorphic on D and $f_n \neq 0$, then either f is holomorphic on D and $f \neq 0$ or $f \equiv c \in \{0, \infty\}$, and $f_n \to f$ on D.*

Proof (a) is a direct corollary to the maximum modulus principle. And (b), (c) follow from (a). □

Lemma 2.3 ([3, Lemma 1.1]) *The operators ϕ_n defined in* (1.1) *have the following properties*:

(a) *For a meromorphic function $f \not\equiv 0$,*

$$\phi_n\left[\frac{f'}{f}\right] = \frac{f^{(n)}}{f}; \tag{2.1}$$

(b) *For meromorphic functions f and $g(z) = af(az + b)$ with constants a and b,*

$$\phi_n[g](z) = a^n \phi_n[f](az + b). \tag{2.2}$$

Lemma 2.4 ([3, Lemma 2.1]) *Let f be meromorphic on D. Then*

(a) *the poles of f with multiplicity $m \geq 2$ are poles of $\phi_n[f]$ multiplicity nm, and*

(b) *the simple poles a of f with $\mathrm{Res}(f, a) \notin \mathbb{N}_n$ are poles of $\phi_n[f]$ with multiplicity n, and*

(c) *the simple poles a of f with $\mathrm{Res}(f, a) \in \mathbb{N}_n$ are at most poles of $\phi_n[f]$ with multiplicity less than n.*

Lemma 2.5 ([3, Theorem 1.1 and Theorem 1.2]) *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\delta > 0$. Let f be a nonconstant meromorphic functions on $\mathbb{C}$ such that $\phi_k[f] \neq 0$ and $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \geq \delta$. Then either*

$$f(z) = \frac{(k-1)(z-\alpha)}{(z-\beta_1)(z-\beta_2)} \tag{2.3}$$

or

$$f(z) = \frac{a}{z-b}, \tag{2.4}$$

where α, β_1, β_2, a, b are constants with $\alpha \neq \beta_1$, $\alpha \neq \beta_2$ and $|a| \geq \delta$.

Lemma 2.6 *The rational function* (2.4) *with $|a| \geq \delta (> 0)$ satisfies $f^{\#}(z) \leq 1/\delta$.*

Proof We have

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2} = \frac{|a|}{|z-b|^2+|a|^2} \le \frac{1}{|a|} \le \frac{1}{\delta}. \qquad \square$$

Lemma 2.7 *If the rational function* (2.3) *has two poles* $\pm\frac{1}{2}$ *with* $\mathrm{dis}(\mathrm{Res}_1(f), \{0\}) \ge \delta (>0)$, *then there exists a positive constant* $K = K(k,\delta)$ *such that*

$$\sup_{z\in\overline{\Delta}(0,1)} f^{\#}(z) \le K. \tag{2.5}$$

Proof By the assumption,

$$f(z) = f_\alpha(z) = \frac{(k-1)(z-\alpha)}{z^2-\frac{1}{4}},$$

so that $\mathrm{Res}_1(f) = \{(k-1)(\frac{1}{2}\pm\alpha)\}$. Hence $|\frac{1}{2}\pm\alpha| \ge \frac{\delta}{k-1}$, since $\mathrm{dis}(\mathrm{Res}_1(f), \{0\}) \ge \delta$.

It is not difficult to see that the family

$$\mathcal{F} = \left\{ f_\alpha(z) = \frac{(k-1)(z-\alpha)}{z^2-\frac{1}{4}} : \left|\frac{1}{2}\pm\alpha\right| \ge \frac{\delta}{k-1} \right\}$$

is normal on $\mathbb{C}$. And hence the conclusion follows from Marty's theorem. $\square$

Lemma 2.8 *Let* $k \in \mathbb{N}$ *with* $k \ge 2$ *and* $\delta > 0$. *Let* $\mathcal{F}$ *be a family of functions meromorphic on* $D \subset \mathbb{C}$ *such that for every* $f \in \mathcal{F}$, $f \ne 0$, $\phi_k[f] \ne 0$ *and* $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \ge \delta$. *Then the family* $\mathcal{F}$ *is normal on* D.

Proof Suppose that $\mathcal{F}$ is not normal at some point $z_0 \in D$. Since $0 \in \mathbb{N}_k$, the assumption $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \ge \delta$ implies $\mathrm{dis}(\mathrm{Res}_1(f), \{0\}) \ge \delta$. Hence by Lemma 2.1, there exist functions f_n in $\mathcal{F}$, points $z_n \to z_0$ in D and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \rho_n f_n(z_n + \rho_n\zeta) \xrightarrow{\chi} g(\zeta) \quad \text{on } \mathbb{C}, \tag{2.6}$$

where g is a nonconstant meromorphic function satisfying $g^{\#}(\zeta) \le g^{\#}(0) = 1 + \frac{1}{\delta}$. And as $f_n \ne 0$, we get $g \ne 0$ on $\mathbb{C}$ by Hurwitz's theorem.

We claim that $\mathrm{dis}(\mathrm{Res}_1(g), \mathbb{N}_k) \ge \delta$. To prove this, let ζ_0 be a simple pole of g. Then by (2.6) and Hurwitz's theorem, g_n has a simple pole $\zeta_n \to \zeta_0$. It is obvious that $\mathrm{Res}(g_n, \zeta_n) = \mathrm{Res}(f_n, z_n + \rho_n\zeta_n)$. Hence by the condition $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \ge \delta$, we get $|\mathrm{Res}(g_n, \zeta_n) - j| \ge \delta$ for all $j \in \mathbb{N}_k$. It follows that $|\mathrm{Res}(g, \zeta_0) - j| \ge \delta$ for all $j \in \mathbb{N}_k$. This proves the claim.

This claim with Lemma 2.4 shows that each pole of g must be a pole of $\phi_k[g]$.

We claim that $\phi_k[g] \not\equiv 0$. In fact, if $\phi_k[g] \equiv 0$, then as just showed, g has no pole; g is a nonconstant entire function. Let $h(\zeta) = \exp(\int_0^\zeta g(t)dt)$. Then h is entire and $g = h'/h$, and hence by Lemma 2.3, $h^{(k)}(\zeta) = h(\zeta)\phi_k[g](\zeta) \equiv 0$. It follows that h is a polynomial. Since $h \ne 0$, h is a constant, and hence $g \equiv 0$. This is a contradiction.

We claim further that $\phi_k[g] \ne 0$. Suppose that $\phi_k[g](\zeta_0) = 0$. Then $g(\zeta_0) \ne \infty$, so that g is holomorphic on some neighbourhood $\Delta(\zeta_0, \eta)$ of ζ_0. It follows that g_n for sufficiently large n are holomorphic on $\Delta(\zeta_0, \eta)$ and $g_n \to g$ on $\Delta(\zeta_0, \eta)$. Hence $\phi_k[g]$ and $\phi_k[g_n]$ are holomorphic on $\Delta(\zeta_0, \eta)$ and $\phi_k[g_n] \to \phi_k[g]$ on $\Delta(\zeta_0, \eta)$. Since $\phi_k[g](\zeta_0) = 0$ and $\phi_k[g] \not\equiv 0$, it follows from Hurwitz's theorem that $\phi_k[g_n](\zeta_n) = 0$ for some $\zeta_n \to \zeta_0$. Direct calculation shows $\phi_k[g_n](\zeta) = \rho_n^k \phi_k[f_n](z_n + \rho_n\zeta)$. Hence $\phi_k[f_n](z_n + \rho_n\zeta_n) = 0$. This contradicts the assumption that $\phi_k[f] \ne 0$ for $f \in \mathcal{F}$.

Thus by Lemma 2.5 with noting that $g \neq 0$, g has the form (2.4), and hence $g^{\#}(\zeta) \leq \frac{1}{\delta}$ by Lemma 2.6. This contradicts the restriction $g^{\#}(0) = 1 + \frac{1}{\delta}$.

Hence the family $\mathcal{F}$ is normal on D. □

Lemma 2.9 *Let $k \in \mathbb{N}$ with $k \geq 2$. Let $\mathcal{F}$ be a family of functions holomorphic on $D \subset \mathbb{C}$ such that $\phi_k[f] \neq 0$ for every $f \in \mathcal{F}$. Then the family $\mathcal{F}$ is normal on D.*

Proof It is similar to the proof of Lemma 2.8 with noting that the limit function g here is an entire function. □

Lemma 2.10 *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\delta > 0$. Let $\mathcal{F}$ be a family of functions meromorphic on $D \subset \mathbb{C}$ such that for every $f \in \mathcal{F}$, $\phi_k[f] \neq 0$ and $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \geq \delta$.*

Let $\{f_n\} \subset \mathcal{F}$ be a sequence and $z_0 \in D$ a point such that

(a) *$f_n \xrightarrow{\chi} f$ on $D \setminus \{z_0\}$, where the limit function f may be ∞ identically;*

(b) *no subsequence of $\{f_n\}$ is normal at z_0; and*

(c) *there exists a neighbourhood $\Delta(z_0, \eta)$ of z_0 in which every f_n has at most one single zero.*

Then the limit function $f(z) = \frac{k-1}{z-z_0}$.

Proof Say $z_0 = 0$. Since $\{f_n\}$ is not normal at 0, by a similar argument showed in the proof of Lemma 2.8, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, points $z_n \to z_0 = 0$ and positive numbers $\rho_n \to 0$ such that

$$g_n(\zeta) = \rho_n f_n(z_n + \rho_n \zeta) \xrightarrow{\chi} g(\zeta) = \frac{(k-1)(\zeta - \alpha)}{(\zeta - \beta_1)(\zeta - \beta_2)} \quad \text{on } \mathbb{C}, \tag{2.7}$$

where α, β_1, β_2 are constants with $\alpha \neq \beta_i$.

It follows from Hurwitz's theorem that g_n has a zero α_n and two poles $\beta_{n,i}$ $(i = 1, 2)$ with

$$\alpha_n \to \alpha, \quad \beta_{n,i} \to \beta_i \quad \text{as } n \to \infty, \tag{2.8}$$

and hence f_n has a zero $z_{n,0} = z_n + \rho_n \alpha_n$ and two poles $z_{n,\infty}^{(i)} = z_n + \rho_n \beta_{n,i}$.

Set

$$R_n(z) = \frac{z - z_{n,0}}{(z - z_{n,\infty}^{(1)})(z - z_{n,\infty}^{(2)})}. \tag{2.9}$$

Then we have

$$L_n(\zeta) := \rho_n R(z_n + \rho_n \zeta) = \frac{\zeta - \alpha_n}{(\zeta - \beta_{n,1})(\zeta - \beta_{n,2})} \xrightarrow{\chi} \frac{g(\zeta)}{k-1} \quad \text{on } \mathbb{C}. \tag{2.10}$$

Define

$$f_n^*(z) := \frac{f_n(z)}{R_n(z)}. \tag{2.11}$$

Let $g_n^*(\zeta) := f_n^*(z_n + \rho_n \zeta)$. Since $g_n^*(\zeta) L_n(\zeta) = g_n(\zeta)$, we see from (2.7) and (2.10) that $g_n^*(\zeta) \to k-1$ on $\mathbb{C} \setminus \{\alpha, \beta_1, \beta_2\}$. We also see that $g_n^* \neq 0$ on $\mathbb{C}$ locally uniformly. Thus by applying Lemma 2.2 (b),

$$g_n^*(\zeta) = f_n^*(z_n + \rho_n \zeta) \to k-1 \quad \text{on } \mathbb{C}. \tag{2.12}$$

Next, we claim that there exists a neighbourhood of $z_0 = 0$ in which all f_n^* for sufficiently large n have no pole.

Suppose this is not the case. Then there exists a subsequence of $\{f_n^*\}$, say itself w.l.g., such that every f_n^* has at least one pole w_n with $w_n \to z_0 = 0$ as $n \to \infty$. We may assume that w_n is the nearest pole to z_n. Then by (2.12), $\frac{w_n - z_n}{\rho_n} \to \infty$. Thus $w_n \neq z_n$, and

$$\rho_n^* := \frac{\rho_n}{w_n - z_n} \to 0 \tag{2.13}$$

with $\rho_n^* \neq 0$. Now set

$$\widehat{R}_n(z) := (w_n - z_n)R(z_n + (w_n - z_n)z) = \frac{z - \rho_n^* \alpha_n}{(z - \rho_n^* \beta_{n,1})(z - \rho_n^* \beta_{n,2})}. \tag{2.14}$$

Then we have

$$\widehat{R}_n(z) \xrightarrow{\chi} \frac{1}{z} \quad \text{on } \mathbb{C}^* = \mathbb{C} \setminus \{0\}. \tag{2.15}$$

Let $\widehat{f}_n^*(z) := f_n^*(z_n + (w_n - z_n)z)$. Since by (c), $f_n^* \neq 0$ on $\Delta(0, \eta)$ and hence $\widehat{f}_n^*(z) \neq 0$ on $\mathbb{C}$ locally uniformly, and since w_n is the nearest pole to z_n, $\widehat{f}_n^*(z) \neq \infty$ on $\Delta(0,1)$ with $\widehat{f}_n^*(1) = \infty$.

Now consider the sequence $\{\widehat{f}_n\}$ defined by

$$\widehat{f}_n(z) = \widehat{R}_n(z)\widehat{f}_n^*(z) = (w_n - z_n)f_n(z_n + (w_n - z_n)z). \tag{2.16}$$

It follows from Lemma 2.3 (b) and the assumption $\phi_k[f_n] \neq 0$ that

$$\phi_k[\widehat{f}_n](z) = (w_n - z_n)^k \phi_k[f_n](z_n + (w_n - z_n)z) \neq 0 \tag{2.17}$$

on $\mathbb{C}$ locally uniformly. Also, we have

$$\mathrm{dis}(\mathrm{Res}_1(\widehat{f}_n), \mathbb{N}_k) \geq \delta. \tag{2.18}$$

Note that $\widehat{f}_n$ either has two simple poles $\rho_n^* \beta_{n,1}$ and $\rho_n^* \beta_{n,2}$ when $\beta_{n,1} \neq \beta_{n,2}$, or has a double poles $\rho_n^* \beta_{n,1}$ when $\beta_{n,2} = \beta_{n,1}$. Hence by Lemma 2.4, $\phi_k[\widehat{f}_n](z)$ has at least $2k$ poles, counting multiplicities, that tend to 0.

Since $\widehat{f}_n^*(z) \neq 0$ on $\mathbb{C}$ locally uniformly, we get $\widehat{f}_n(z) \neq 0$ on $\mathbb{C}^*$ locally uniformly. Applying Lemma 2.8 yields that $\{\widehat{f}_n\}$ and hence $\{\widehat{f}_n^*\}$ is normal on $\mathbb{C}^*$. Since $\widehat{f}_n^* \neq 0, \infty$ on $\Delta(0,1)$, by Lemma 2.2 (c), $\{\widehat{f}_n^*\}$ is normal on $\Delta(0,1)$ and hence on whole $\mathbb{C}$. By taking a subsequence and renumbering, we may say that

$$\widehat{f}_n^* \xrightarrow{\chi} \widehat{f}^* \quad \text{on } \mathbb{C}. \tag{2.19}$$

Since $\widehat{f}_n^*(1) = \infty$, we get $\widehat{f}^*(1) = \infty$; since $\widehat{f}_n^*(0) = f_n^*(z_n) = g_n^*(0) \to k-1$, we get $\widehat{f}^*(0) = k-1$. It follows that $\widehat{f}^*$ is a nonconstant meromorphic function on $\mathbb{C}$. Further, $\widehat{f}^* \neq 0$ on $\mathbb{C}$, since $\widehat{f}_n^* \neq 0$ on $\mathbb{C}$.

Now by (2.19), (2.16) and (2.15), we have $\widehat{f}_n \xrightarrow{\chi} \widehat{f} := \widehat{f}^*/z$ on $\mathbb{C}^*$. Note that $\widehat{f}$ is a nonconstant meromorphic function on $\mathbb{C}$ such that $\widehat{f} \neq 0$ and $\widehat{f}(0) = \widehat{f}(1) = \infty$. In particular, 0 is a simple pole with $\mathrm{Res}(\widehat{f}, 0) = k-1$. Also, it follows from (2.18) that $\mathrm{dis}(\mathrm{Res}_1(\widehat{f}, \mathbb{C}^*), \mathbb{N}_k) \geq \delta$, and in particular, $\mathrm{Res}(\widehat{f}, 1) \notin \mathbb{N}_k$. Thus by Lemma 2.4, 1 is a pole of $\phi_k[\widehat{f}]$, and hence $\phi_k[\widehat{f}] \not\equiv 0$.

Since $\widehat{f}_n \xrightarrow{\chi} \widehat{f}$ on $\mathbb{C}^*$, we have $\widehat{f}_n \to \widehat{f}$ on $\mathbb{C} \setminus A$, where A is the set of poles of $\widehat{f}$. It follows that $\phi_k[\widehat{f}_n] \to \phi_k[\widehat{f}]$ on $\mathbb{C} \setminus A$. Since A has no accumulate points on $\mathbb{C}$ and $\phi_k[\widehat{f}] \not\equiv 0$, by (2.17) and Lemma 2.2 (b), we get $\phi_k[\widehat{f}_n] \xrightarrow{\chi} \phi_k[\widehat{f}]$ on $\mathbb{C}$. Since $\phi_k[\widehat{f}_n](z)$ has at least $2k$ poles tending to 0, 0 is a pole of $\phi_k[\widehat{f}]$ with multiplicity at least $2k$. However, as 0 is a simple pole of $\widehat{f}$, this contradicts Lemma 2.4.

The above claim is thus proved. By removing finitely many functions and renumbering, we may assume that $f_n^* \neq 0, \infty$ on $\Delta(0, \eta)$.

Since $f_n \xrightarrow{\chi} f$ on $D \setminus \{0\}$ and $R_n(z) \to 1/z$ on $\mathbb{C}^*$, it follows from (2.11) that $f_n^* \xrightarrow{\chi} f^*$ on $D \setminus \{0\}$, where $f^*(z) = zf(z)$. Since $f_n^* \neq 0$ on $\Delta(0, \eta)$, it follows from Lemma 2.2 (c) that $f_n^* \xrightarrow{\chi} f^*$ on $\Delta(0, \eta)$ and hence on whole D. Since $f_n^*(z_n) = g_n^*(0) \to k-1$ and $z_n \to 0$, we get $f^*(0) = k - 1$. Thus the function

$$f(z) = \frac{f^*(z)}{z} \not\equiv \infty \tag{2.20}$$

is meromorphic on D.

Next we show that $\phi_k[f] \equiv 0$ on D. Since $f_n \xrightarrow{\chi} f$ on $D\setminus\{0\}$, it follows that $\phi_k[f_n] \to \phi_k[f]$ on $D \setminus A$, where A is the set of poles of f on D. If $\phi_k[f] \not\equiv 0$, then by $\phi_k[f_n] \neq 0$ on D and Lemma 2.2 (b) we get $\phi_k[f_n] \xrightarrow{\chi} \phi_k[f]$ on whole D. Note that f_n either has two simple poles $z_{n,\infty}^{(1)}$ and $z_{n,\infty}^{(2)}$ when $z_{n,\infty}^{(1)} \neq z_{n,\infty}^{(2)}$, or has a double poles $z_{n,\infty}^{(1)}$ when $z_{n,\infty}^{(1)} = z_{n,\infty}^{(2)}$. Hence by Lemma 2.4, $\phi_k[f_n](z)$ has at least $2k$ poles, counting multiplicities, that tend to 0. Thus by $\phi_k[f_n] \xrightarrow{\chi} \phi_k[f]$ on D, 0 is a pole of $\phi_k[f]$ with multiplicity at least $2k$. On the other hand, since $f(z) = f^*(z)/z$ with $f^*(0) = k - 1$, this contradicts Lemma 2.4.

Now we show that f^* is a constant with $f^* = k-1$ to complete the proof. Since $f^*(0) = k-1$, the function

$$h(z) = \frac{f^*(z) - (k-1)}{z} \tag{2.21}$$

is holomorphic at 0, say on $\Delta(0, \eta)$. Let

$$H(z) = z^{k-1} \exp\left(\int_0^z h(t)dt\right), \quad z \in \Delta(0, \eta). \tag{2.22}$$

Then we have

$$\frac{H'(z)}{H(z)} = \frac{k-1}{z} + h(z) = \frac{f^*(z)}{z} = f(z). \tag{2.23}$$

Now applying Lemma 2.3 (a) yields that $H^{(k)}(z) = H(z)\phi_k[f] \equiv 0$, so that H is a polynomial with degree less than k. This occurs only when $h \equiv 0$. Thus f^* is a constant with $f^* = k - 1$.□

Lemma 2.11 *Let $k \in \mathbb{N}$ with $k \geq 2$ and $\delta > 0$. Let $\mathcal{F}$ be a family of functions meromorphic on $D \subset \mathbb{C}$ such that for every $f \in \mathcal{F}$, $\phi_k[f] \neq 0$ and* $\mathrm{dis}(\mathrm{Res}_1(f), \mathbb{N}_k) \geq \delta$.

Let $\{f_n\} \subset \mathcal{F}$ be a sequence and $z_0 \in D$ a point such that

(a) *no subsequence of $\{f_n\}$ is normal at z_0; and*

(b) *every f_n has at least two distinct zeros tending to z_0.*

Then there exists a subsequence of $\{f_n\}$ which we continue to call $\{f_n\}$ such that every f_n has at least two distinct poles a_n and b_n tending to z_0 such that

$$\sup_{z\in\overline{\Delta}(0,1)} h_n^{\#}(z) \to \infty, \tag{2.24}$$

where

$$h_n(z) = (a_n - b_n)f_n\left(\frac{a_n + b_n}{2} + (a_n - b_n)z\right). \tag{2.25}$$

Proof Say $z_0 = 0$. The beginning part is the same as the proof of Lemma 2.10 up to (2.12).

Next, we claim that each f_n^* has at least one pole tending to $z_0 = 0$.

Suppose not, then all f_n^* are holomorphic on some neighbourhood $\Delta(0, \eta) \subset D$ of 0. By the assumption (b), each f_n^* has at least one zero tending to 0. Say the one which is nearest to z_n is $w_n \to 0$. Then $(w_n - z_n)/\rho_n \to \infty$ by (2.12), and hence

$$\rho_n^* = \frac{\rho_n}{w_n - z_n} \to 0 \tag{2.26}$$

with $\rho_n^* \neq 0$. Thus the rational functions

$$\widehat{R}_n(z) := (w_n - z_n)R(z_n + (w_n - z_n)z) = \frac{z - \rho_n^* \alpha_n}{(z - \rho_n^* \beta_{n,1})(z - \rho_n^* \beta_{n,2})} \tag{2.27}$$

satisfy

$$\widehat{R}_n(z) \xrightarrow{\chi} \frac{1}{z} \quad \text{on } \mathbb{C}^* = \mathbb{C} \setminus \{0\}. \tag{2.28}$$

Let $\widehat{f}_n^*(z) := f_n^*(z_n + (w_n - z_n)z)$ and

$$\widehat{f}_n(z) := \widehat{R}_n(z)\widehat{f}_n^*(z) = (w_n - z_n)f_n(z_n + (w_n - z_n)z). \tag{2.29}$$

Since f_n^* are holomorphic on $\Delta(0, \eta)$, the functions $\widehat{f}_n^*$ are holomorphic on $\mathbb{C}$ locally uniformly. Since w_n is the zero of f_n nearest to z_n, $\widehat{f}_n^* \neq 0$ on $\Delta(0, 1)$ with $\widehat{f}_n^*(1) = 0$. By the assumption $\phi_k[f_n] \neq 0$ and Lemma 2.3(b),

$$\phi_k[\widehat{f}_n](z) = (w_n - z_n)^k \phi_k[f_n](z_n + (w_n - z_n)z) \neq 0. \tag{2.30}$$

The assumption $\mathrm{dis}(\mathrm{Res}_1(f_n), \mathbb{N}_k) \geq \delta$ gives

$$\mathrm{dis}(\mathrm{Res}_1(\widehat{f}_n), \mathbb{N}_k) \geq \delta. \tag{2.31}$$

It then follows from Lemma 2.4 that $\phi_k[\widehat{f}_n](z)$ has at least $2k$ poles tending to 0, counting multiplicities.

Since $\widehat{f}_n^*$ are holomorphic on $\mathbb{C}$ locally uniformly, we see from (2.29) that the functions $\widehat{f}_n$ are holomorphic on $\mathbb{C}^*$ locally uniformly. Hence by Lemma 2.9, the sequence $\{\widehat{f}_n\}$ is normal on $\mathbb{C}^*$, and hence so is $\{\widehat{f}_n^*\}$. Since $\widehat{f}_n^* \neq 0, \infty$ on $\Delta(0, 1)$, it follows that $\{\widehat{f}_n^*\}$ is normal on $\Delta(0, 1)$ and hence on whole $\mathbb{C}$. Taking a subsequence and renumbering, we may say that $\widehat{f}_n^* \to \widehat{f}^*$ on $\mathbb{C}$. Since $\widehat{f}_n^*(1) = 0$ and $\widehat{f}_n^*(0) = f_n^*(z_n) = g_n^*(0) \to k - 1$, we get $\widehat{f}^*(1) = 0$ and $\widehat{f}^*(0) = k - 1$. This shows that $\widehat{f}^*$ is a nonconstant entire function.

Let $\widehat{f} = \widehat{f}^*/z$. Then $\phi_k[\widehat{f}] \not\equiv 0$. For otherwise, the same argument used in the final part of the proof of Lemma 2.10 shows that $\widehat{f}^*$ is a constant $\widehat{f}^* \equiv k - 1$, which is a contradiction.

By (2.28) and (2.29), we also have $\widehat{f}_n \to \widehat{f}$ on $\mathbb{C}^*$, and hence $\phi_k[\widehat{f}_n] \to \phi_k[\widehat{f}]$ on $\mathbb{C}^*$. Since $\phi_k[\widehat{f}_n] \neq 0$ and $\phi_k[\widehat{f}] \not\equiv 0$, by Lemma 2.2(b), we have $\phi_k[\widehat{f}_n] \xrightarrow{\chi} \phi_k[\widehat{f}]$ on whole $\mathbb{C}$. Since $\phi_k[\widehat{f}_n](z)$ has at least $2k$ poles tending to 0, it follows that 0 is a pole of $\phi_k[\widehat{f}]$ with multiplicity $2k$ at least. But by Lemma 2.4, 0 is a pole of $\phi_k[\widehat{f}]$ with multiplicity at most k. A contradiction.

Hence, we have proved that each f_n^* has at least one pole $z_n^* \to 0$. Note that $z_n^* \neq z_{n,\infty}^{(i)}$, and $\zeta^* = (z_n^* - z_n)/\rho_n \to \infty$ by (2.12). Now let

$$h_n(z) = (z_{n,\infty}^{(1)} - z_n^*)f_n\left(\frac{z_{n,\infty}^{(1)} + z_n^*}{2} + (z_{n,\infty}^{(1)} - z_n^*)z\right). \tag{2.32}$$

Then we have

$$h_n\left(\frac{1}{2}\right)=\infty,\quad h_n\left(\frac{z_{n,0}-\frac{z_{n,\infty}^{(1)}+z_n^*}{2}}{z_{n,\infty}^{(1)}-z_n^*}\right)=0. \tag{2.33}$$

Since

$$\frac{z_{n,0}-\frac{z_{n,\infty}^{(1)}+z_n^*}{2}}{z_{n,\infty}^{(1)}-z_n^*}=\frac{2\zeta_{n,0}-\zeta_{n,\infty}^{(1)}-\zeta_n^*}{2(\zeta_{n,\infty}^{(1)}-\zeta_n^*)}\to\frac{1}{2},$$

we see from (2.33) that every subsequence of $\{h_n\}$ fails to be equicontinuous in any neighbourhood of $z=1/2$, and hence fails to be normal at $1/2$. Now (2.24) follows from Marty's theorem. The proof is complete. □

3 Proof of Theorem 1.4

Let $\{f_n\}\subset\mathcal{F}$ be a sequence, and let $E\subset D$ be the set of points at which $\{f_n\}$ is not normal.

Claim For each $z_0\in E$, there exists a neighbourhood $\Delta(z_0)$ of z_0 in which every f_n has at most one single zero.

Suppose that this claim is not true. Then for some $z_0\in E$, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that each f_n has at least two distinct zeros tending to z_0 as $n\to\infty$. By Lemma 2.11, there exists a subsequence of $\{f_n\}$, which we continue to call $\{f_n\}$, such that each f_n has at least two distinct poles a_n and b_n tending to z_0 such that

$$\sup_{z\in\overline{\Delta}(0,1)} h_n^{\#}(z)\geq K+1, \tag{3.1}$$

where h_n is defined by (2.25) and K is the constant defined in Lemma 2.7. We may assume that $K>\frac{1}{\delta}$.

Fix $\eta>0$. We may assume that a_n and b_n are two distinct poles of f_n in $\Delta(z_0,\eta)\subset D$ satisfying (3.1) such that

$$\tau_n=\tau(a_n,b_n):=\frac{|a_n-b_n|}{\eta-|\frac{a_n+b_n}{2}-z_0|}\ \text{ is minimal.} \tag{3.2}$$

Obviously, we have $\tau_n=\tau(a_n,b_n)\to 0$.

Now we claim that no subsequence of $\{h_n\}$ is normal on $\mathbb{C}$. Suppose not. By taking a subsequence and renumbering, we may assume that $h_n\stackrel{\chi}{\to}h$ on $\mathbb{C}$, where the limit function h may be ∞ identically.

By (3.1), we have $\sup_{z\in\overline{\Delta}(0,1)}h^{\#}(z)\geq K+1$, so that $h\not\equiv\infty$, and hence it is a nonconstant meromorphic function on $\mathbb{C}$. The assumption $\text{dis}(\text{Res}_1(f_n,D),\mathbb{N}_k)\geq\delta$ gives $\text{dis}(\text{Res}_1(h_n),\mathbb{N}_k)$ $\geq\delta$ and hence $\text{dis}(\text{Res}_1(h),\mathbb{N}_k)\geq\delta$. This guarantees that $\phi_k[h]\not\equiv 0$. In fact, Suppose $\phi_k[h]\equiv 0$. Take a simply connected domain Ω on which h is holomorphic and a point $z_0\in\Omega$. Define $H(z)=\exp(\int_{z_0}^z h(t)dt)$. Then H is holomorphic on Ω with $h=H'/H$, and hence by Lemma 2.3 (a), $H^{(k)}=H\phi_k[h]\equiv 0$. This leads that H is a polynomial with degree less than k. If H is nonconstant, then

$$h=\frac{H'}{H}=\sum_{i=1}^{s}\frac{p_i}{z-z_i},\quad \text{with } p_i\in\mathbb{N} \text{ and } \sum_{i=1}^{s}p_i\leq k-1. \tag{3.3}$$

Obviously, each $1\leq p_i\leq k-1$. This contradicts with $\text{dis}(\text{Res}_1(h),\mathbb{N}_k)\geq\delta$. Thus H is a constant and hence $h\equiv 0$. This is also impossible since h is nonconstant.

By Lemma 2.3 (b) and the assumption $\phi_k[f_n] \neq 0$, we also have

$$\phi_k[h_n](z) = (a_n - b_n)^k \phi_k[f_n]\left(\frac{a_n + b_n}{2} + (a_n - b_n)z\right) \neq 0. \tag{3.4}$$

Since $h_n \to h$ on $\mathbb{C} \setminus A$, where A is the set of poles of h, we get $\phi_k[h_n] \to \phi_k[h]$ on $\mathbb{C} \setminus A$. Now applying Lemma 2.2 (b) then yields that $\phi_k[h_n] \xrightarrow{\chi} \phi_k[h]$ on whole $\mathbb{C}$, and $\phi_k[h] \neq 0$. Thus by Lemma 2.5, h must be a rational function with the form (2.3) or (2.4). Hence by Lemmas 2.6 and 2.7, $\sup_{z\in\overline{\Delta}(0,1)} h^{\#}(z) \leq K$. This contradicts to $\sup_{z\in\overline{\Delta}(0,1)} h^{\#}(z) \geq K + 1$.

Hence, the set $F \subset \mathbb{C}$ of points at which $\{h_n\}$ is not normal is nonempty.

Suppose first that for each $\zeta_0 \in F$, there exists a neighbourhood of ζ_0 in which each h_n has at most one single zero. Then by Lemma 2.8, $\{h_n\}$ is normal on some punctured neighbourhood of ζ_0. It follows that $\{h_n\}$ is quasinormal on $\mathbb{C}$ and the set F has no accumulation point on $\mathbb{C}$. Suppose further that each subsequence of some subsequence of $\{h_n\}$ is not normal at at least two distinct points $\zeta_1, \zeta_2 \in F$. Then by Lemma 2.10, there exists a subsequence of $\{h_n\}$, say itself, such that $h_n(z) \xrightarrow{\chi} (k-1)/(z-\zeta_1)$ and $h_n(z) \xrightarrow{\chi} (k-1)/(z-\zeta_2)$ on $\mathbb{C} \setminus F$. It follows from the uniqueness that $\zeta_1 = \zeta_2$. A contradiction. Hence $\{h_n\}$ is quasinormal on $\mathbb{C}$ of order 1, and the set $F = \{\zeta_0\}$ is a singleton. Applying Lemma 2.10 again, there exists a subsequence of $\{h_n\}$, say itself, such that $h_n(z) \xrightarrow{\chi} (k-1)/(z-\zeta_0)$ on $\mathbb{C} \setminus \{\zeta_0\}$. This is also impossible, since $h_n(\pm 1/2) = \infty$.

So there exists a point $\zeta_0 \in F$ and a subsequence of $\{h_n\}$, which we continue to call $\{h_n\}$, such that each h_n has at least two distinct zeros tending to ζ_0. Then by Lemma 2.11, there exists a subsequence of $\{h_n\}$, which we continue to call $\{h_n\}$, such that each h_n has at least two distinct poles a_n^* and b_n^* tending to ζ_0 and the functions

$$H_n(z) = (a_n^* - b_n^*)h_n\left(\frac{a_n^* + b_n^*}{2} + (a_n^* - b_n^*)z\right) \tag{3.5}$$

satisfy

$$\sup_{z\in\overline{\Delta}(0,1)} H_n^{\#}(z) \geq K + 1, \tag{3.6}$$

where K is the constant defined in Lemma 2.7.

Set

$$A_n = \frac{a_n + b_n}{2} + (a_n - b_n)a_n^*, \quad B_n = \frac{a_n + b_n}{2} + (a_n - b_n)b_n^*. \tag{3.7}$$

Then A_n and B_n are poles of f_n by (2.25). Since $\tau_n = \tau(a_n, b_n) \to 0$ and $a_n^* \to \zeta_0$, we have

$$|A_n - z_0| \leq \left|\frac{a_n + b_n}{2} - z_0\right| + |a_n - b_n||a_n^*| = \eta - \left(\frac{1}{\tau_n} - |a_n^*|\right)|a_n - b_n| < \eta$$

for sufficiently large n. That is, $A_n \in \Delta(z_0, \eta)$. Similarly, $B_n \in \Delta(z_0, \eta)$.

Note that the function $\widehat{H}_n(z) := (A_n - B_n)f_n\left(\frac{A_n+B_n}{2} + (A_n - B_n)z\right) \equiv H_n(z)$, so that by (3.6),

$$\sup_{z\in\overline{\Delta}(0,1)} \widehat{H}_n^{\#}(z) = \sup_{z\in\overline{\Delta}(0,1)} H_n^{\#}(z) \geq K + 1. \tag{3.8}$$

However, we have

$$\frac{\tau(A_n, B_n)}{\tau(a_n, b_n)} = \frac{\eta - |\frac{a_n+b_n}{2} - z_0|}{\eta - |\frac{a_n+b_n}{2} - z_0 + \frac{a_n^*+b_n^*}{2}(a_n - b_n)|}|a_n^* - b_n^*|$$

$$\leq \frac{\eta - |\frac{a_n+b_n}{2} - z_0|}{\eta - |\frac{a_n+b_n}{2} - z_0| - |\frac{a_n^*+b_n^*}{2}(a_n - b_n)|}|a_n^* - b_n^*|$$
$$= \frac{|a_n^* - b_n^*|}{1 - |\frac{a_n^*+b_n^*}{2}|\tau_n} \to 0. \tag{3.9}$$

It follows that $\tau(A_n, B_n) < \tau(a_n, b_n)$ for sufficiently large n, which contradicts that $\tau(a_n, b_n)$ is minimal.

Up to now, we have proved the **Claim** mentioned in the beginning of the proof. Now applying Lemma 2.8 yields that $\{f_n\}$ is normal on some punctured neighbourhood of each $z_0 \in E$. This shows that E has no accumulation points in D and the family $\mathcal{F}$ is quasi-normal on D.

Suppose now that each subsequence of some subsequence of $\{f_n\}$ is not normal at at least two distinct points $z_1, z_2 \in E$. Then by Lemma 2.10, there exists a subsequence of $\{f_n\}$, say itself, such that $f_n(z) \xrightarrow{\chi} (k-1)/(z-z_1)$ and $f_n(z) \xrightarrow{\chi} (k-1)/(z-z_2)$ on $\mathbb{D} \setminus E$. It follows from the uniqueness that $z_1 = z_2$. A contradiction. Hence the set $E = \{z_0\}$ is a singleton, so that the family $\mathcal{F}$ is quasinormal on D of order 1.

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