Acta Mathematica Sinica, English Series
Sep., 2021, Vol. 37, No. 9, pp. 1367–1383
Published online: September 15, 2021
https://doi.org/10.1007/s10114-021-0267-z
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Asymptotic Distributions for Power Variation of the Solution to a Stochastic Heat Equation

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Abstract Let $u = \{u(t, x), t \in [0, T], x \in \mathbb{R}\}$ be a solution to a stochastic heat equation driven by a space-time white noise. We study that the realized power variation of the process u with respect to the time, properly normalized, has Gaussian asymptotic distributions. In particular, we study the realized power variation of the process u with respect to the time converges weakly to Brownian motion.

Keywords Quadratic variation, power variation, stochastic heat equation, weak convergence

MR(2010) Subject Classification 60F05, 60F17, 60G17

1 Introduction

The study of single path behavior of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, for $p > 0$, the p-power variation of a process X, with respect to a subdivision $\pi_n = \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = 1\}$ of $[0, 1]$, is defined to be the sum

$$\sum_{k=1}^{n} |X(t_{n,k}) - X(t_{n,k-1})|^p.$$

For simplicity, consider from now on the case where $t_{n,k} = k/n$, for $n \in \mathbb{N}$ and $k \in \{1, ..., n\}$. In the present paper, we wish to point out some interesting phenomena when X is a solution to a stochastic heat equation. In fact, we will also drop the absolute value (when p is odd). More precisely, we will consider

$$\sum_{k=1}^{n} \Delta X_k^p, \tag{1.1}$$

where $\Delta X_k = \Delta X(k/n)$ denotes the increment $X(k/n) - X((k-1)/n)$.

The analysis of the asymptotic behavior of quantities of type (1.1) is motivated, for instance, by the study of the exact rates of convergence of some approximation schemes of scalar stochastic differential equations driven by a Brownian motion B (see, e.g., Corcuera et al. [2], Neuenkirch and Nourdin [8] and Nourdin [9]), besides, of course, the traditional applications of quadratic variations to parameter estimation problems.

Now, let us recall some known results concerning the p-power variations (for $p \in \mathbb{N}_+$), which are today more or less classical. First, assume that B is the standard Brownian motion. Let μ_p

Received May 29, 2020, revised August 23, 2020, accepted December 22, 2020
Supported by ZJNSF (Grant No. LY20A010020) and NSFC (Grant No. 11671115)

denote the p-moment of a standard Gaussian random variable following an $\mathcal{N}(0,1)$ law, that is, $\mu_{2p-1}=0$ and $\mu_{2p}=(2p-1)!!=(2p)!/(p!2^p)$ for all $p\in\mathbb{N}_+$. By the scaling property of the Brownian motion and using the central limit theorem, it is immediate that (see, e.g., Nourdin [9]), as $n\to\infty$:

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(n^{p/2}\Delta B_k^p-\mu_p)\xrightarrow{\mathcal{L}}\mathcal{N}(0,\mu_{2p}-\mu_p^2). \tag{1.2}$$

Assume that $H\neq 1/2$, that is, the case where the fractional Brownian motion B has no independent increments anymore. Then (1.2) has been extended by Dobrushin and Major [4], Taqqu [13], Breuer and Major [1], Giraitis and Surgailis [7], Corcuera et al. [2] and Nourdin [9]. Swanson [12] extended (1.2) to modifications of the quadratic variation of the solution of the stochastic heat equation driven by a space-time white noise. Motivated by (1.2), in the present paper, we show that (1.2) with different mean and variance also holds for the solution to a stochastic heat equation driven by a space-time white noise.

Consider a centered Gaussian field $W=\{W(t,A);t\in[0,T],A\in\mathcal{B}_b(\mathbb{R}^d)\}$ with covariance

$$\mathbb{E}[W(t,A)W(s,B)]=(s\wedge t)\lambda(A\cap B),\quad s,t\in[0,T],\quad A,B\in\mathcal{B}_b(\mathbb{R}^d), \tag{1.3}$$

where λ denotes the Lebesque measure and $\mathcal{B}_b(\mathbb{R}^d)$ is the collection of all bounded Borel subsets of $\mathbb{R}^d$. Also consider the stochastic partial differential equation

$$\begin{aligned}&\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+\dot{W},\quad t\in[0,T],\quad x\in\mathbb{R}^d,\\ &u(0,x)=0,\quad x\in\mathbb{R}^d,\end{aligned} \tag{1.4}$$

where $T>0$ is a constant and the noise W is defined by (1.3). The noise W is usually referred to as a *space-time white noise* because it behaves as a Brownian motion with respect to both the time and the space variables. It is well known (see for example the now classical paper Dalang [3]) that (1.4) admits a unique mild solution if and only if $d=1$ and this mild solution is defined as

$$u(t,x)=\int_0^t\int_{\mathbb{R}}G(t-s,x-y)W(ds,dy),\quad t\in[0,T],\quad x\in\mathbb{R}, \tag{1.5}$$

where the above integral is a Wiener integral with respect to the Gaussian process W (see, e.g., Dalang [3] or Walsh [15] for details) and G is the Green kernel of the heat equation given by

$$G(t,x)=\begin{cases}(2\pi t)^{-1/2}\mathrm{e}^{-x^2/(2t)} & \text{if } t>0,\ x\in\mathbb{R},\\ 0 & \text{if } t\le 0,\ x\in\mathbb{R}.\end{cases} \tag{1.6}$$

Swanson [12] showed that the covariance function of the solution (1.5) satisfies the following: for every $x\in\mathbb{R}$ we have

$$\mathbb{E}[u(t,x)u(s,x)]=\frac{1}{\sqrt{2\pi}}(\sqrt{t+s}-\sqrt{|t-s|}),\quad \text{for every } s,t\in[0,T]. \tag{1.7}$$

Swanson [12] showed that the process u with respect to the time has infinite quadratic variation and is not a semimartingale, and also investigated central limit theorems for modifications of the quadratic variation of the process u with respect to the time. Tudor and Xiao [14] investigated the exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm of the process u with respect to the time. In fact, they investigated these path properties for a more wide class, namely, the solution to the linear stochastic heat equation driven by a fractional noise in time with correlated spatial structure. Wang and Xiao [16] investigated the exact moduli of non-differentiability of the process u with respect to the time by using general methods for Gaussian random fields. In this paper we show that the fluctuations of the realized power variation of the process u with respect to the time, properly normalized, have Gaussian asymptotic distributions. Our proof is based on the approach method of Swanson [12]. The new ingredients of our present paper are to make use of the product-moments of various orders of the normal correlation surface of two variates in Pearson and Young [10] and to establish exact convergence rates of variances of the realized power variation of the process u with respect to the time.

In order to establish this result we first introduce some notation. Let $F(t) = u(t, x)$, where $x \in \mathbb{R}$ is fixed. We consider discrete Riemann sums over a uniformly spaced time partition $t_k = k\Delta t$, where $\Delta t = n^{-1}$. Let $\Delta F_k = F(t_k) - F(t_{k-1})$ and $\sigma_k^2 = \mathbb{E}[\Delta F_k^2]$. For any $p \in \mathbb{N}_+$ and $n \in \mathbb{N}_+$, we define

$$U_p^n(F)_t = \sum_{k=1}^{\lfloor nt \rfloor} \Delta F_k^{2p}$$

and

$$V_p^n(F)_t = \sum_{k=1}^{\lfloor nt \rfloor} \Delta F_k^{2p-1}.$$

For $k \in \mathbb{N}_+$, let $\gamma_k = 2\sqrt{k} - \sqrt{k-1} - \sqrt{k+1}$. For real number $r \geq 1$, define $J_r = J(r) = \sum_{k=1}^{\infty} \gamma_k^r$. It follows from (2.13) below that J_r is a positive and finite constant depending only on r. For any $p \in \mathbb{N}_+$, we put

$$\kappa_p = \left(\mu_{4p} - \mu_{2p}^2 + \frac{(2p)!(2p)!}{2^{2p-1}} \sum_{u=1}^{p} \frac{J_{2u}}{(p-u)!(p-u)!(2u)!} \right) \left(\frac{2}{\pi} \right)^p \tag{1.8}$$

and

$$\chi_p = \left(\mu_{4p-2} - \frac{(2p-1)!(2p-1)!}{2^{2p-2}} \sum_{u=0}^{p-1} \frac{J_{2u+1}}{(p-1-u)!(p-1-u)!(2u+1)!} \right) \left(\frac{2}{\pi} \right)^{p-1/2}. \tag{1.9}$$

We will first show the exact convergence rate of variance for the realized power variation of the process F.

Theorem 1.1 *Fix $p \in \mathbb{N}_+$. Then*

$$n^{-1+p} \operatorname{Var}(U_p^n(F)_t) \to \kappa_p \, t, \tag{1.10}$$

and

$$n^{-3/2+p} \operatorname{Var}(V_p^n(F)_t) \to \chi_p \, t \tag{1.11}$$

for each fixed $t > 0$ as n tends to infinity.

By (1.10), we have the following convergence in probability for the realized power variation of the process F.

Corollary 1.2 *Fix $p \in \mathbb{N}_+$. Then*

$$n^{-1+p/2} U_p^n(F)_t \ \rightarrow \ K_p\, t \tag{1.12}$$

in L^2 and in probability for each fixed $t > 0$ as n tends to infinity, where $K_p = \mu_{2p}(\frac{2}{\pi})^{p/2}$.

Remark 1.3 Since $U_p^n(F)_t$ is monotone, (1.12) implies that $n^{-1+p/2}U_p^n(F)_t \rightarrow K_p\, t$ uniform convergence in probability in the time interval $[0, T]$ with some $T > 0$.

Remark 1.4 The 4-th variation, namely, $p = 2$ in (1.12), has also be explicitly obtained by Pospíšil and Tribe [11]. The constant is $3/\pi$, see Proposition 3.2 of Pospíšil and Tribe [11]. In this paper, the corresponding constant is equal to $K_2 = 6/\pi$. The difference comes from the factor $1/2$ in front of the Laplace operator in (1.4) because the factor of the covariance function in (1.7) becomes $1/(2\sqrt{\pi})$ in the case of Pospíšil and Tribe [11].

The central limit theorems for the realized power variation of the process F are as follows.

Theorem 1.5 *Fix $p \in \mathbb{N}_+$. Then*

$$\left(F(t), \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (n^{p/2} \Delta F_k^{2p} - K_p) \right) \xrightarrow{\mathcal{L}} (F(t), \kappa_p^{1/2} W(t)), \tag{1.13}$$

as n tends to infinity, where K_p is given in (1.12) and $W = \{W(t), t \in [0, T]\}$ is a Brownian motion independent of the process F, and the convergence is in the space $D([0, T])^2$ equipped with the Skorohod topology.

Theorem 1.6 *Fix $p \in \mathbb{N}_+$. Then*

$$\left(F(t), \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (n^{-1/4+p/2} \Delta F_k^{2p-1}) \right) \xrightarrow{\mathcal{L}} (F(t), \chi_p^{1/2} W(t)), \tag{1.14}$$

as n tends to infinity, where $W = \{W(t), t \in [0, T]\}$ is a Brownian motion independent of the process F, and the convergence is in the space $D([0, T])^2$ equipped with the Skorohod topology.

Throughout this paper, we use C to denote unspecified positive and finite constants whose values may change in each appearance.

2 Proof of Theorem 1.1

We start with the following facts. See, e.g., Corcuera et al. [2] for the first one, and Eqs. (viii) and (ix) in Pearson and Young [10] for the second one. The last one is cited from Lemma 2.1 of Swanson [12].

- Let ξ be a random variable following an $\mathcal{N}(0, \sigma^2)$ law. Then, for any $r \in \mathbb{N}_+$,

$$\mathbb{E}[\xi^r] = \mu_r \sigma^r. \tag{2.1}$$

- Suppose that $(U, V) \sim \mathcal{N}\big(0, \left(\begin{smallmatrix} \sigma_u^2 & \rho \\ \rho & \sigma_v^2 \end{smallmatrix}\right)\big)$, where $\rho = (\sigma_u \sigma_v)^{-1} \mathbb{E}[UV]$. Then, for any $p \in \mathbb{N}_+$,

$$\mathbb{E}[U^{2p} V^{2p}] = \frac{(2p)!(2p)!}{2^{2p}} \sigma_u^{2p} \sigma_v^{2p} \sum_{k=0}^{p} \frac{(2\rho)^{2k}}{(p-k)!(p-k)!(2k)!}, \tag{2.2}$$

and

$$\mathbb{E}[U^{2p-1} V^{2p-1}] = \frac{\rho(2p-1)!(2p-1)!}{2^{2p-2}} \sigma_u^{2p-1} \sigma_v^{2p-1} \sum_{k=0}^{p-1} \frac{(2\rho)^{2k}}{(p-1-k)!(p-1-k)!(2k+1)!}. \tag{2.3}$$

- If $0 \le s < t$, then

$$\left| \mathbb{E}|F(t) - F(s)|^2 - \sqrt{\frac{2(t-s)}{\pi}} \right| \le \frac{1}{t^{3/2}}|t-s|^2. \tag{2.4}$$

Proof of Theorem 1.1 We first prove (1.10). For $1 \le i < j \le \lfloor nt \rfloor$, define $\rho_{ij} = (\sigma_i\sigma_j)^{-1} \cdot \mathbb{E}[\Delta F_i \Delta F_j]$. By (2.1) and (2.2), we have

$$\begin{aligned}
\mathrm{Var}(U_p^n(F)_t) &= \mathbb{E}\left[\left| \sum_{k=1}^{\lfloor nt \rfloor} (\Delta F_k^{2p} - \mu_{2p}\sigma_k^{2p}) \right|^2 \right] \\
&= \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}[(\Delta F_k^{2p} - \mu_{2p}\sigma_k^{2p})^2] + 2\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor} \mathbb{E}[(\Delta F_i^{2p} - \mu_{2p}\sigma_i^{2p})(\Delta F_j^{2p} - \mu_{2p}\sigma_j^{2p})] \\
&= \sum_{k=1}^{\lfloor nt \rfloor} (\mathbb{E}[\Delta F_k^{4p}] - \mu_{2p}^2\sigma_k^{4p}) + 2\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor} (\mathbb{E}[\Delta F_i^{2p}\Delta F_j^{2p}] - \mu_{2p}^2\sigma_i^{2p}\sigma_j^{2p}) \\
&= (\mu_{4p} - \mu_{2p}^2)\sum_{k=1}^{\lfloor nt \rfloor} \sigma_k^{4p} \\
&\quad + \frac{(2p)!(2p)!}{2^{2p-1}} \sum_{u=1}^{p} \frac{2^{2u}}{(p-u)!(p-u)!(2u)!} \sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor} \sigma_i^{2p}\sigma_j^{2p}\rho_{ij}^{2u}.
\end{aligned} \tag{2.5}$$

It follows from (2.4) that $\sigma_k^2 \le Cn^{-1/2}$ for all $1 \le k \le \lfloor nt \rfloor$. By (2.4) and Lagrange mean value theorem, it holds that for any real number $r \ge 2$ and $1 \le k \le \lfloor nt \rfloor$,

$$\begin{aligned}
\left| \sigma_k^r - \left(\frac{2}{n\pi}\right)^{r/4} \right| &\le \frac{r}{2}\left(\sigma_k^{r-2} + \left(\frac{2}{n\pi}\right)^{(r-2)/4} \right) \left| \sigma_k^2 - \sqrt{\frac{2}{n\pi}} \right| \\
&\le \frac{C}{t_k^{3/2}} n^{-3/2-r/4} \le \frac{C}{t_k^{3/4}} n^{-3/4-r/4}.
\end{aligned} \tag{2.6}$$

Note that

$$\frac{1}{n}\sum_{k=1}^{\lfloor nt \rfloor} \frac{1}{t_k^{3/4}} \to \int_0^t x^{-3/4}dx = 4t^{1/4}. \tag{2.7}$$

It follows from (2.6) (with $r = 4p$) and (2.7) that

$$n^{-1+p}\sum_{k=1}^{\lfloor nt \rfloor} \left| \sigma_k^{4p} - \left(\frac{2}{n\pi}\right)^p \right| \to 0. \tag{2.8}$$

Hence

$$n^{-1+p}\sum_{k=1}^{\lfloor nt \rfloor} \sigma_k^{4p} = n^{-1+p}\sum_{k=1}^{\lfloor nt \rfloor}\left(\sigma_k^{4p} - \left(\frac{2}{n\pi}\right)^p \right) + n^{-1+p}\left(\frac{2}{n\pi}\right)^p \lfloor nt \rfloor \to \left(\frac{2}{\pi}\right)^p t. \tag{2.9}$$

It follows from (1.7) that

$$\begin{aligned}
\mathbb{E}[\Delta F_i \Delta F_j] = \sqrt{\frac{1}{2n\pi}}\big(&\sqrt{j+i} - \sqrt{j-i} - \sqrt{j+i-1} + \sqrt{j-i+1} \\
&- \sqrt{j+i-1} + \sqrt{j-i-1} + \sqrt{j+i-2} - \sqrt{j-i}\big),
\end{aligned}$$

which simplifies to

$$\mathbb{E}[\Delta F_i \Delta F_j] = -\sqrt{\frac{1}{2n\pi}}(\gamma_{j+i-1} + \gamma_{j-i}). \tag{2.10}$$

Thus, for $1 \le u \le p$ and $1 \le i < j \le \lfloor nt \rfloor$, by binomial expansion,

$$\begin{aligned}\sigma_i^{2p}\sigma_j^{2p}\rho_{ij}^{2u} &= \sigma_i^{2p-2u}\sigma_j^{2p-2u}(\mathbb{E}[\Delta F_i \Delta F_j])^{2u} \\ &= (2\pi)^{-u}\sigma_i^{2p-2u}\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j+i-1} + n^{-1/2}\gamma_{j-i})^{2u} \\ &= (2\pi)^{-u}\sum_{\nu=0}^{2u}\sigma_i^{2p-2u}\sigma_j^{2p-2u}\binom{2u}{\nu}(n^{-1/2}\gamma_{j+i-1})^{\nu}(n^{-1/2}\gamma_{j-i})^{2u-\nu}. \end{aligned}\tag{2.11}$$

If we write $\gamma_k = f(k-1) - f(k)$, where $f(x) = \sqrt{x+1} - \sqrt{x}$, then for each $k \ge 2$, the mean value theorem gives $\gamma_k = |f'(k-\theta_1)| = \frac{1}{4}(k - \theta_1 + \theta_2)^{-3/2}$ for some $\theta_1, \theta_2 \in [0,1]$. This yields that for all $k \in \mathbb{N}$,

$$0 < \gamma_k \le \frac{1}{\sqrt{2}k^{3/2}}, \tag{2.12}$$

and hence, for any $r \ge 1$,

$$\sum_{k=1}^{M}\gamma_k^r \to J_r \tag{2.13}$$

with some $J_r = J(r) > 0$ as $M \to \infty$.

Note that since $j + i - 1 \ge (j+i)/2$, we have

$$n^{-1/2}\gamma_{j+i-1} \le \frac{n^{-1/2}}{\sqrt{2}((j+i)/2)^{3/2}} = \frac{2}{n^2}\frac{1}{(t_i + t_j)^{3/2}}. \tag{2.14}$$

Note that (2.4) gives that $\sigma_k^2 \le Cn^{-1/2}$ for all $1 \le k \le \lfloor nt \rfloor$, and that (2.12) gives $n^{-1/2}\gamma_{j-i} \le Cn^{-1/2}$ and $n^{-1/2}\gamma_{j+i-1} \le Cn^{-1/2}$ for all $1 \le i < j \le \lfloor nt \rfloor$. Hence, by (2.14), for all $1 \le \nu \le 2u$,

$$\begin{aligned} & n^{-1+p}\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor}\sigma_i^{2p-2u}\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j+i-1})^{\nu}(n^{-1/2}\gamma_{j-i})^{2u-\nu} \\ &\le Cn^{-1/2}\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor}(n^{-1/2}\gamma_{j+i-1}) \\ &\le Cn^{-5/2}\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=i+1}^{\lfloor nt \rfloor}\frac{1}{(t_i + t_j)^{3/2}}, \end{aligned}\tag{2.15}$$

which tends to zero as $n \to \infty$ since $\int_0^t\int_0^t (x+y)^{-3/2}dxdy < \infty$.

Let $B^H = \{B^H(t), t \in \mathbb{R}_+\}$ be a fractional Brownian motion with index $H \in (0,1)$, which is a centered Gaussian process with $\mathbb{E}[(B^H(t) - B^H(s))^2] = |s-t|^{2H}$ for $s, t \in \mathbb{R}_+$. Then, for $H_0 = 1/4$,

$$\begin{aligned} & \mathbb{E}\left[\left(B^{H_0}\left(\frac{j+1}{n}\right) - B^{H_0}\left(\frac{j}{n}\right)\right)\left(B^{H_0}\left(\frac{i+1}{n}\right) - B^{H_0}\left(\frac{i}{n}\right)\right)\right] \\ &= -\frac{1}{2}\left[2\left(\frac{j-i}{n}\right)^{1/2} - \left(\frac{j-i-1}{n}\right)^{1/2} - \left(\frac{j-i+1}{n}\right)^{1/2}\right] \\ &= -\frac{1}{2}n^{-1/2}\gamma_{j-i}. \end{aligned}\tag{2.16}$$

Thus,

$$
\begin{aligned}
n^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\gamma_{j-i} &= n^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor}\gamma_{j-i}\\
&= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}\bigg[\bigg(B^{H_0}\Big(\frac{j+1}{n}\Big)-B^{H_0}\Big(\frac{j}{n}\Big)\bigg)\bigg(B^{H_0}\Big(\frac{i+1}{n}\Big)-B^{H_0}\Big(\frac{i}{n}\Big)\bigg)\bigg]\\
&= -2\sum_{i=1}^{\lfloor nt\rfloor-1}\mathbb{E}\bigg[\bigg(B^{H_0}\Big(\frac{\lfloor nt\rfloor+1}{n}\Big)-B^{H_0}\Big(\frac{i+1}{n}\Big)\bigg)\bigg(B^{H_0}\Big(\frac{i+1}{n}\Big)-B^{H_0}\Big(\frac{i}{n}\Big)\bigg)\bigg]\\
&= -\sum_{i=1}^{\lfloor nt\rfloor-1}\bigg[-\Big(\frac{\lfloor nt\rfloor-i}{n}\Big)^{1/2}+\Big(\frac{\lfloor nt\rfloor+1-i}{n}\Big)^{1/2}-\Big(\frac{1}{n}\Big)^{1/2}\bigg]\\
&= -\Big(\frac{\lfloor nt\rfloor}{n}\Big)^{1/2}+\lfloor nt\rfloor n^{-1/2}. \qquad (2.17)
\end{aligned}
$$

This yields

$$
n^{-1}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\gamma_{j-i}\to t. \qquad (2.18)
$$

Note that for $M>0$, by (2.12), we have

$$
\begin{aligned}
&n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_i^{2p-2u}\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j-i})^{2u}\\
&\quad\le CM^{-3(2u-1)/2}n^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}(n^{-1/2}\gamma_{j-i})\\
&\quad\le CM^{-3(2u-1)/2}n^{-1}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\gamma_{j-i}. \qquad (2.19)
\end{aligned}
$$

This, together with (2.18), yields

$$
n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+M+1}^{\lfloor nt\rfloor}\sigma_i^{2p-2u}\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j-i})^{2u}\le CM^{-3(2u-1)/2}t \qquad (2.20)
$$

which tends to zero by letting $M\to\infty$.

By (2.6) (with $r=2p-2u$), (2.7) and (2.16), we have

$$
\begin{aligned}
&n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\bigg|\sigma_i^{2p-2u}-\Big(\frac{2}{n\pi}\Big)^{p/2-u/2}\bigg|\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j-i})^{2u}\\
&\le Cn^{-5/4}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\frac{1}{t_i^{3/4}}(n^{-1/2}\gamma_{j-i})\\
&\le -Cn^{-5/4}\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{t_i^{3/4}}\bigg[-\Big(\frac{\lfloor nt\rfloor-i}{n}\Big)^{1/2}+\Big(\frac{\lfloor nt\rfloor+1-i}{n}\Big)^{1/2}-\Big(\frac{1}{n}\Big)^{1/2}\bigg]\\
&\le -Cn^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor}\bigg[-\Big(\frac{\lfloor nt\rfloor-i}{n}\Big)^{1/2}+\Big(\frac{\lfloor nt\rfloor+1-i}{n}\Big)^{1/2}\bigg]+n^{-7/4}\sum_{i=1}^{\lfloor nt\rfloor}\frac{C}{t_i^{3/4}}
\end{aligned}
$$

$$= -Cn^{-1/2}\left[\left(\frac{1}{n}\right)^{1/2} + \left(\frac{\lfloor nt\rfloor}{n}\right)^{1/2}\right] + n^{-7/4}\sum_{i=1}^{\lfloor nt\rfloor}\frac{C}{t_i^{3/4}} \tag{2.21}$$

which tends to zero as $n \to \infty$ since $\int_0^t x^{-3/4}dx < \infty$. Hence, we have

$$n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(\sigma_i^{2p-2u} - \left(\frac{2}{n\pi}\right)^{p/2-u/2}\right)\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j-i})^{2u} \to 0. \tag{2.22}$$

Similarly, we have

$$n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\left(\frac{2}{n\pi}\right)^{p/2-u/2}\left(\sigma_j^{2p-2u} - \left(\frac{2}{n\pi}\right)^{p/2-u/2}\right)(n^{-1/2}\gamma_{j-i})^{2u} \to 0. \tag{2.23}$$

For any $M > 0$,

$$n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{i+M}\left(\frac{2}{n\pi}\right)^{p-u}(n^{-1/2}\gamma_{j-i})^{2u} = \left(\frac{2}{\pi}\right)^{p-u}\frac{\lfloor nt\rfloor}{n}\sum_{k=1}^{M}\gamma_k^{2u} \to \left(\frac{2}{\pi}\right)^{p-u}J_{2u}t \tag{2.24}$$

as $n \to \infty$ and $M \to \infty$.

Note that for $1 \le u \le p$ and $1 \le i < j \le \lfloor nt\rfloor$,

$$\begin{aligned}\sigma_i^{2p-2u}\sigma_j^{2p-2u} &= \left(\sigma_i^{2p-2u} - \left(\frac{2}{n\pi}\right)^{p/2-u/2}\right)\sigma_j^{2p-2u} \\ &\quad + \left(\frac{2}{n\pi}\right)^{p/2-u/2}\left(\sigma_j^{2p-2u} - \left(\frac{2}{n\pi}\right)^{p/2-u/2}\right) + \left(\frac{2}{n\pi}\right)^{p-u}.\end{aligned} \tag{2.25}$$

Hence, by (2.22)–(2.24), we have

$$n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{i+M}\sigma_i^{2p-2u}\sigma_j^{2p-2u}(n^{-1/2}\gamma_{j-i})^{2u} \to \left(\frac{2}{\pi}\right)^{p-u}J_{2u}t \tag{2.26}$$

as $n \to \infty$ and $M \to \infty$. Thus, combining (2.11), (2.15), (2.20) and (2.26), we have for any $1 \le u \le p$,

$$n^{-1+p}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\sigma_i^{2p}\sigma_j^{2p}\rho_{ij}^{2u} \to (2\pi)^{-u}\left(\frac{2}{\pi}\right)^{p-u}J_{2u}t \tag{2.27}$$

Therefore, by (2.5), (2.9) and (2.27), we have

$$\begin{aligned}&n^{-1+p}\operatorname{Var}(U_p^n(F)_t) \\ &\quad\to \left(\mu_{4p} - \mu_{2p}^2 + \frac{(2p)!(2p)!}{2^{2p-1}}\sum_{u=1}^{p}\frac{J_{2u}}{(p-u)!(p-u)!(2u)!}\right)\left(\frac{2}{\pi}\right)^p t = \kappa_p\, t.\end{aligned} \tag{2.28}$$

This proves (1.10).

We now prove (1.11). If follows from (2.1) and (2.3) that

$$\begin{aligned}&\operatorname{Var}(V_p^n(F)_t) \\ &= \sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}[\Delta F_i^{2p-1}\Delta F_j^{2p-1}] \\ &= \sum_{k=1}^{\lfloor nt\rfloor}\mathbb{E}[\Delta F_k^{4p-2}] + 2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=i+1}^{\lfloor nt\rfloor}\mathbb{E}[\Delta F_i^{2p-1}\Delta F_j^{2p-1}]\end{aligned}$$

$$
\begin{aligned}
&= \mu_{4p-2} \sum_{k=1}^{\lfloor nt \rfloor} \sigma_k^{4p-2} \\
&\quad + \frac{(2p-1)!(2p-1)!}{2^{2p-3}} \sum_{u=0}^{p-1} \frac{2^{2u}}{(p-1-u)!(p-1-u)!(2u+1)!} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} \sigma_i^{2p-1} \sigma_j^{2p-1} \rho_{ij}^{2u+1}. \quad (2.29)
\end{aligned}
$$

It follows from (2.6) (with $r = 4p-2$) and (2.7) that

$$
\begin{aligned}
& n^{-3/2+p} \sum_{k=1}^{\lfloor nt \rfloor} \sigma_k^{4p-2} \\
&\quad = n^{-3/2+p} \sum_{k=1}^{\lfloor nt \rfloor} \left(\sigma_k^{4p-2} - \left(\frac{2}{n\pi}\right)^{p-1/2} \right) + n^{-3/2+p} \left(\frac{2}{n\pi}\right)^{p-1/2} \lfloor nt \rfloor \\
&\quad \to \left(\frac{2}{\pi}\right)^{p-1/2} t. \quad (2.30)
\end{aligned}
$$

Similarly to the argument of (2.27), we have for any $0 \le u \le p-1$,

$$
n^{-3/2+p} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=i+1}^{\lfloor nt \rfloor} \sigma_i^{2p-1} \sigma_j^{2p-1} \rho_{ij}^{2u+1} \to -(2\pi)^{-u-1/2} \left(\frac{2}{\pi}\right)^{p-u-1} J_{2u+1} t. \quad (2.31)
$$

Therefore, by (2.29)–(2.31), we have

$$
\begin{aligned}
& n^{-3/2+p} \operatorname{Var}(V_p^n(F)_t) \\
&\quad \to \left(\mu_{4p-2} - \frac{(2p-1)!(2p-1)!}{2^{2p-2}} \sum_{u=0}^{p-1} \frac{J_{2u+1}}{(p-1-u)!(p-1-u)!(2u+1)!} \right) \\
&\quad \cdot \left(\frac{2}{\pi}\right)^{p-1/2} t = \chi_p \, t. \quad (2.32)
\end{aligned}
$$

This proves (1.11). The proof of Theorem 1.1 is completed. □

Proof of Corollary 1.2 Write

$$
\begin{aligned}
& n^{-1+p/2} U_p^n(F)_t - K_p t \\
&\quad = n^{-1+p/2} (U_p^n(F)_t - \mathbb{E}[U_p^n(F)_t]) + \mu_{2p} n^{-1+p/2} \sum_{k=1}^{\lfloor nt \rfloor} \left(\sigma_k^{2p} - \left(\frac{2}{n\pi}\right)^{p/2} \right) \\
&\quad + \mu_{2p} \left(\frac{2}{\pi}\right)^{p/2} \left(\frac{\lfloor nt \rfloor}{n} - t \right). \quad (2.33)
\end{aligned}
$$

Obviously, the third term of (2.33) tends to zero as $n \to \infty$. It follows from (2.6) (with $r = 2p$) and (2.7) that the second term of (2.33) tends to zero as $n \to \infty$. Hence, by (1.10), we have

$$
\mathbb{E}[|n^{-1+p/2} U_p^n(F)_t - K_p t|^2] \to 0.
$$

This proves (1.12). □

3 Proofs of Theorems 1.5 and 1.6

The following lemma is needed to prove Theorems 1.5 and 1.6.

Lemma 3.1 *Let $X_1, \dots, X_4$ be mean zero, jointly normal random variables, such that $\mathbb{E}[X_k^2] = 1$ and $\rho_{ij} = \mathbb{E}[X_i X_j]$. Put $\xi_k = X_k^p - E[X_k^p]$. Then, for any $p \in \mathbb{N}_+$,*

$$\left|\mathbb{E}\left[\prod_{k=1}^{4} \xi_k\right]\right| \le C\left(|\rho_{12}\ \rho_{34}| + \frac{1}{\sqrt{1-\rho_{12}^2}} \max_{i \le 2 < j} |\rho_{ij}|\right) \tag{3.1}$$

whenever $|\rho_{12}| < 1$. Moreover,

$$\left|\mathbb{E}\left[\prod_{k=1}^{4} \xi_k\right]\right| \le C \max_{2 \le k \le 4} |\rho_{1k}|. \tag{3.2}$$

Furthermore, there exists $\varepsilon > 0$ such that

$$\left|\mathbb{E}\left[\prod_{k=1}^{4} \xi_k\right]\right| \le C \max_{1 \le i \ne j \le 4} \rho_{ij}^2 \tag{3.3}$$

whenever $|\rho_{ij}| < \varepsilon$ for all $1 \le i \ne j \le 4$.

Proof Following the same lines as the proof of Lemma 3.3 in Swanson [12] with $h_k(X_k) = \xi_k$, $1 \le k \le 4$, we get Lemma 3.1 immediately. □

Proposition 3.2 *Fix $r \in \mathbb{N}_+$. Put*

$$c_r = \begin{cases} \mu_r, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd,} \end{cases}$$

and

$$W_r^n(F)_t = n^{-1/2+r/4} \sum_{k=1}^{\lfloor nt \rfloor} (\Delta F_k^r - c_r \sigma_k^r).$$

Then, there exists a constant C such that

$$\mathbb{E}[|W_r^n(F)_t - W_r^n(F)_s|^4] \le C\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^2 \tag{3.4}$$

for all $0 \le s < t$ and all $n \in \mathbb{N}$. The sequence $\{W_r^n(F)\}$ is therefore relatively compact in the Skorohod space $D_{\mathbb{R}}[0, \infty)$.

Proof We follow the method of Proposition 3.5 in Swanson [12] to prove (3.4). Let $\mathcal{S} = \{k \in \mathbb{N}_+^4 : \lfloor ns \rfloor + 1 \le k_1 \le \cdots \le k_4 \le \lfloor nt \rfloor\}$. For $k \in \mathcal{S}$ and $i \in \{1, 2, 3\}$, define $h_i = k_{i+1} - k_i$ and let $\mathcal{S}_i = \{k \in \mathcal{S} : h_i = \max\{h_1, h_2, h_3\}\}$. Define $N = \lfloor nt \rfloor - (\lfloor ns \rfloor + 1)$ and for $j \in \{0, 1, \dots, N\}$, let $\mathcal{S}_i^j = \{k \in \mathcal{S}_i : \max\{h_1, h_2, h_3\} = j\}$. Further define $\mathcal{T}_i^\ell = \mathcal{T}_i^{j,\ell} = \{k \in \mathcal{S}_i^j : \min\{h_1, h_2, h_3\} = \ell\}$ and $\mathcal{V}_i^v = \mathcal{V}_i^{j,\ell,v} = \{k \in \mathcal{T}_i^\ell : \operatorname{med}\{h_1, h_2, h_3\} = v\}$, where "med" denotes the median function. For $k \in \mathcal{S}$, define

$$U_k = \prod_{j=1}^{4} (\Delta F_{k_j}^r - c_r \sigma_{k_j}^r).$$

Observe that

$$\begin{aligned}\mathbb{E}[|W_r^n(F)_t - W_r^n(F)_s|^4] &= n^{-2+r} \mathbb{E}\left[\left|\sum_{k=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} (\Delta F_k^r - c_r \sigma_k^r)\right|^4\right] \\ &\le 4! n^{-2+r} \sum_{k \in \mathcal{S}} |\mathbb{E}[U_k]|\end{aligned}$$

$$\le 4! n^{-2+r} \sum_{i=1}^{3} \sum_{k \in \mathcal{S}_i} |\mathbb{E}[U_k]|, \tag{3.5}$$

and that

$$\begin{aligned}\sum_{k \in \mathcal{S}_i} |\mathbb{E}[U_k]| &= \sum_{j=0}^{N} \sum_{k \in \mathcal{S}_i^j} |\mathbb{E}[U_k]| \\ &= \sum_{j=0}^{N} \sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor} \sum_{k \in \mathcal{T}_i^\ell} |\mathbb{E}[U_k]| + \sum_{j=0}^{N} \sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j} \sum_{k \in \mathcal{T}_i^\ell} |\mathbb{E}[U_k]| \\ &= \sum_{j=0}^{N} \sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor} \sum_{v=\ell}^{j} \sum_{k \in \mathcal{V}_i^v} |\mathbb{E}[U_k]| + \sum_{j=0}^{N} \sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j} \sum_{v=\ell}^{j} \sum_{k \in \mathcal{V}_i^v} |\mathbb{E}[U_k]|. \end{aligned} \tag{3.6}$$

Let $X_j = \sigma_{k_j}^{-1} \Delta F_{k_j}$ and

$$\xi_j = X_j^r - \mathbb{E}[X_j^r] = \sigma_{k_j}^{-r}(\Delta F_{k_j}^r - c_r \sigma_{k_j}^r).$$

Then

$$|\mathbb{E}[U_k]| = \left(\prod_{j=1}^{4} \sigma_{k_j}^r\right) \left|\mathbb{E}\left[\prod_{j=1}^{4} \xi_j\right]\right|. \tag{3.7}$$

By (2.10) and (2.12), we have for all $i \neq j \in \mathbb{N}$,

$$|\mathbb{E}[\Delta F_i \Delta F_j]| \le \frac{2n^{-1/2}}{|i-j|^{3/2}}. \tag{3.8}$$

It follows from (2.4) that for all $1 \le k \le \lfloor nt \rfloor$,

$$\pi^{-1/2} n^{-1/2} \le |\mathbb{E}[\Delta F_k^2]| \le 2n^{-1/2}. \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$|\rho_{ij}| = |\mathbb{E}[X_i X_j]| = \sigma_{k_i}^{-1} \sigma_{k_j}^{-1} |\mathbb{E}[\Delta F_{k_i} \Delta F_{k_j}]| \le \frac{2\sqrt{\pi}}{|k_i - k_j|^{3/2}}. \tag{3.10}$$

Suppose $0 \le \ell \le \lfloor\sqrt{j}\rfloor$. Fix v and let $k \in \mathcal{V}_i^v$ be arbitrary. If $i = 1$, then $j = \max\{h_1, h_2, h_3\} = h_1 = k_2 - k_1$. If $i = 3$, then $j = \max\{h_1, h_2, h_3\} = h_3 = k_4 - k_3$. In either case, by (3.2) and (3.10), we have

$$|\mathbb{E}[U_k]| \le \frac{Cn^{-r}}{j^{3/2}} \le C\left(\frac{1}{(\ell v)^{3/2}} + \frac{1}{j^{3/2}}\right) n^{-r}.$$

If $i = 2$, then $j = \max\{h_1, h_2, h_3\} = h_2 = k_3 - k_2$ and $\ell v = h_3 h_1 = (k_4 - k_3)(k_2 - k_1)$. Hence, by (3.1) and (3.10),

$$|\mathbb{E}[U_k]| \le C\left(\frac{1}{(\ell v)^{3/2}} + \frac{1}{j^{3/2}}\right) n^{-r}.$$

Now choose $i' \neq i$ such that $h_{i'} = \ell$. With i' given, k is determined by k_i. Since there are two possibilities for i' and $N+1$ possibilities for k_i, $|\mathcal{V}_i^v| \le 2(N+1)$. Therefore,

$$\sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor} \sum_{v=\ell}^{j} \sum_{k \in \mathcal{V}_i^v} |\mathbb{E}[U_k]| \le C(N+1) \sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor} \sum_{v=\ell}^{j} \left(\frac{1}{(\ell v)^{3/2}} + \frac{1}{j^{3/2}}\right) n^{-r}$$

$$
\begin{aligned}
&\le C(N+1)\sum_{\ell=0}^{\lfloor\sqrt{j}\rfloor}\left(\frac{1}{\ell^{3/2}}+\frac{1}{j^{1/2}}\right)n^{-r} \\
&\le C(N+1)n^{-r}. \qquad (3.11)
\end{aligned}
$$

For the second summation, suppose $\lfloor\sqrt{j}\rfloor+1\le\ell\le j$. (In particular, $j\ge1$.) In this case, if $k\in\mathcal{T}_i^\ell$, then $\ell=\min\{h_1,h_2,h_3\}$, so that by (3.3) and (3.10),

$$
|\mathbb{E}[U_k]|\le\frac{Cn^{-r}}{\ell^3}.
$$

Since $\sum_{v=\ell}^{j}|\mathcal{V}_i^v|\le2(N+1)j$, we have

$$
\begin{aligned}
\sum_{\ell=\lfloor\sqrt{j}\rfloor+1}^{j}\sum_{v=\ell}^{j}\sum_{k\in\mathcal{V}_i^v}|\mathbb{E}[U_k]| &\le C(N+1)j\sum_{\lfloor\sqrt{j}\rfloor+1}^{j}\frac{n^{-r}}{\ell^{\sim3}} \\
&\le C(N+1)j\left(\int_{\lfloor\sqrt{j}\rfloor}^{\infty}\frac{1}{x^3}dx\right)n^{-r} \\
&\le C(N+1)n^{-r}. \qquad (3.12)
\end{aligned}
$$

Thus, using (3.5), (3.6), (3.11) and (3.12), we have

$$
n^{-2+r}\mathbb{E}\left[\left|\sum_{j=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\Delta F_j^r-c_r\sigma_j^r)\right|^4\right]\le C\sum_{j=0}^{N}(N+1)n^{-2}=C\left(\frac{\lfloor nt\rfloor-\lfloor ns\rfloor}{n}\right)^2,
$$

which is (3.4).

To show that a sequence of cadlag processes $\{X_n\}$ is relatively compact, it suffices to show that for each $T>1$, there exist constants $\beta>0$, $C>0$, and $\alpha>1$ such that

$$
R_{X_n}(t,h)=\mathbb{E}[|X_n(t+h)-X_n(t)|^\beta|X_n(t)-X_n(t-h)|^\beta]\le Ch^\alpha
$$

for all $n\in\mathbb{N}$, all $t\in[0,T]$ and all $h\in[0,t]$. (See, e.g., Theorem 3.8.8 in Ethier and Kurtz [6].) Taking $\beta=2$ and using (3.4) together with Hölder inequality gives

$$
R_{W_r^n(F)}(t,h)\le C\left(\frac{\lfloor nt+nh\rfloor-\lfloor nt\rfloor}{n}\right)\left(\frac{\lfloor nt\rfloor-\lfloor nt-nh\rfloor}{n}\right).
$$

If $nh<1/2$, then the right-hand side of this inequality is zero. Assume $nh\ge1/2$. Then

$$
\frac{\lfloor nt+nh\rfloor-\lfloor nt\rfloor}{n}\le\frac{nh+1}{n}\le3h.
$$

The other factor is similarly bounded, so that $R_{W_r^n(F)}(t,h)\le Ch^2$. □

Proposition 3.3 *Fix $0\le s<t$ and $r\in\mathbb{N}_+$. Then*

$$
W_r^n(F)_t-W_r^n(F)_s\ \xrightarrow{\mathcal{L}}\ \sigma_r(s,t)\mathcal{N}
$$

as $n\to\infty$, where $\mathcal{N}$ is a standard normal random variable and

$$
\sigma_r(s,t)=\begin{cases}\kappa_p^{1/2}|t-s|^{1/2}, & \text{if } r=2p,\\ \chi_p^{1/2}|t-s|^{1/2}, & \text{if } r=2p-1.\end{cases}
$$

Here κ_p and χ_p are defined in (1.8) and (1.9), respectively.

Proof Let $\{n(j)\}_{j=1}^{\infty}$ be any sequence of natural numbers. We will prove that there exists a subsequence $\{n(j_m)\}$ such that $W_r^{n(j_m)}(F)_t - W_r^{n(j_m)}(F)_s$ converges in law to the given random variable.

For each $m \in \mathbb{N}_+$, choose $n(j_m) \in \{n(j)\}$ such that $n(j_m) > n(j_{m-1})$ and $n(j_m) \geq m^4(t-s)^{-1}$. Let $b = b(m) = n(j_m)(t-s)/m$. For $0 \leq k \leq m$, define $u_k = n(j_m)s + kb$, so that

$$\begin{aligned} W_r^{n(j_m)}(F)_t - W_r^{n(j_m)}(F)_s &= n(j_m)^{-1/2+r/4} \sum_{i=\lfloor n(j_m)s \rfloor+1}^{\lfloor n(j_m)t \rfloor} (\Delta F_i^r - c_r \sigma_i^r) \\ &= n(j_m)^{-1/2+r/4} \sum_{k=1}^{m} \sum_{i=u_{k-1}+1}^{u_k} (\Delta F_i^r - c_r \sigma_i^r). \end{aligned} \tag{3.13}$$

Let us now introduce the filtration

$$\mathcal{F}_t = \sigma\{W(A) : A \subset \mathbb{R} \times [0,t], \lambda(A) < \infty\},$$

where λ denotes Lebesgue measure on $\mathbb{R}^2$. Let $\tau_k = n(j_m)^{-1}u_{k-1}$. For each pair (i,k) such that $u_{k-1} < i \leq u_k$,

$$\xi_{i,k} = \Delta F_i - \mathbb{E}[\Delta F_i | \mathcal{F}_{\tau_k}].$$

Note that $\xi_{i,k}$ is $\mathcal{F}_{\tau_{k+1}}$-measurable and independent of $\mathcal{F}_{\tau_k}$. Recall that

$$F(t) = \int_0^t \int_{\mathbb{R}} G(t-s, x-y) W(ds, dy). \tag{3.14}$$

Also, given constants $0 \leq \tau \leq s \leq t$, we have

$$\mathbb{E}[F(t)|\mathcal{F}_\tau] = \int_0^\tau \int_{\mathbb{R}} G(t-s, x-y) W(ds, dy). \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$F(t+\tau_k) - \mathbb{E}[F(t+\tau_k)|\mathcal{F}_{\tau_k}] = \int_{\tau_k}^{t+\tau_k} \int_{\mathbb{R}} G(t+\tau_k - s, x-y) W(ds, dy).$$

This yields that $\{\xi_{i,k}\}$ has the same law as $\{\Delta F_{i-u_{k-1}}\}$.

Now define $\sigma_{i,k}^2 = \mathbb{E}[\xi_{i,k}^2] = \sigma_{i-u_{k-1}}^2$ and

$$\zeta_{m,k} = \sum_{i=u_{k-1}+1}^{u_k} (\xi_{i,k}^r - c_r \sigma_{i,k}^r),$$

so that $\zeta_{m,k}$, $1 \leq k \leq m$, are independent and

$$W_r^{n(j_m)}(F)_t - W_r^{n(j_m)}(F)_s = n(j_m)^{-1/2+r/4} \sum_{k=1}^{m} \zeta_{m,k} + \varepsilon_m, \tag{3.16}$$

where

$$\varepsilon_m = n(j_m)^{-1/2+r/4} \sum_{k=1}^{m} \sum_{i=u_{k-1}+1}^{u_k} ((\Delta F_i^r - c_r \sigma_i^r) - (\xi_{i,k}^r - c_r \sigma_{i,k}^r)).$$

Since $\xi_{i,k}$ and $\Delta F_i - \xi_{i,k} = \mathbb{E}[\Delta F_i | \mathcal{F}_{\tau_k}]$ are independent, we have

$$\begin{aligned} \sigma_i^2 = \mathbb{E}[\Delta F_i^2] &= \mathbb{E}[\xi_{i,k}^2] + \mathbb{E}[|\Delta F_i - \xi_{i,k}|^2] \\ &= \sigma_{i,k}^2 + \mathbb{E}[|\Delta F_i - \xi_{i,k}|^2]. \end{aligned} \tag{3.17}$$

This, together with (2.4), gives

$$\mathbb{E}[|\Delta F_i - \xi_{i,k}|^2] = \sigma_i^2 - \sigma_{i,k}^2 \le \frac{2n(j_m)^{-2}}{(t_i-\tau_k)^{3/2}} = \frac{2n(j_m)^{-1/2}}{(i-u_{k-1})^{3/2}}. \tag{3.18}$$

Thus, since $\Delta F_i - \xi_{i,k}$ is Gaussian, by (2.1), we have

$$\mathbb{E}[|\Delta F_i - \xi_{i,k}|^4] \le \frac{Cn(j_m)^{-1}}{(i-u_{k-1})^3}. \tag{3.19}$$

Note that (2.1) implies $\mathbb{E}[|\Delta F_i|^{4r-4}] \le C\sigma_i^{4r-4}$ and $\mathbb{E}[|\xi_{i,k}|^{4r-4}] \le C\mathbb{E}[|\Delta F_i|^{4r-4}] \le C\sigma_i^{4r-4}$. By Lagrange mean value theorem,

$$|\Delta F_i^r - \xi_{i,k}^r| \le C(|\Delta F_i|^{r-1} + |\xi_{i,k}|^{r-1})|\Delta F_i - \xi_{i,k}|.$$

Thus, by Hölder inequality, (3.19) and the fact $\sigma_i^2 \le Cn(j_m)^{-1/2}$ for all $i \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[|\Delta F_i^r - \xi_{i,k}^r|^2] &\le C(\mathbb{E}[|\Delta F_i|^{4r-4}] + \mathbb{E}[|\xi_{i,k}|^{4r-4}])^{1/2}(\mathbb{E}[|\Delta F_i - \xi_{i,k}|^4])^{1/2} \\ &\le \frac{Cn(j_m)^{-r/2}}{(i-u_{k-1})^{3/2}}.\end{aligned} \tag{3.20}$$

It follows (3.18) and Lagrange mean value theorem that

$$|\sigma_i^r - \sigma_{i,k}^r| \le C(|\sigma_i|^{r-2} + |\sigma_{i,k}|^{r-2})|\sigma_i^2 - \sigma_{i,k}^2| \le \frac{Cn(j_m)^{-r/4}}{(i-u_{k-1})^{3/4}}. \tag{3.21}$$

Therefore, by Hölder inequality, (3.20) and (3.21),

$$\begin{aligned}\mathbb{E}[|\varepsilon_m|] &\le n(j_m)^{-1/2+r/4}\sum_{k=1}^m \sum_{j=u_{k-1}+1}^{u_k} ((\mathbb{E}[|\Delta F_i^r - \xi_{i,k}^r|^2])^{1/2} + c_r|\sigma_j^r - \bar{\sigma}_{j,k}^r|) \\ &\le Cn(j_m)^{-1/2}\sum_{k=1}^m \sum_{i=u_{k-1}+1}^{u_k} \frac{1}{(i-u_{k-1})^{3/4}} \\ &= Cn(j_m)^{-1/2}\sum_{k=1}^m \sum_{i=1}^{u_k-u_{k-1}} i^{-3/4}.\end{aligned}$$

Since $u_k - u_{k-1} \le b$, this gives

$$\mathbb{E}[|\varepsilon_m|] \le Cn(j_m)^{-1/2} m b^{1/4} = Cm^{3/4} n(j_m)^{-1/4}(t-s)^{1/4}.$$

But since $n(j_m)$ was chosen so that $n(j_m) \ge m^4(t-s)^{-1}$, we have $E[|\varepsilon_m|] \le Cm^{-1/4}|t-s|^{1/2}$ and $\varepsilon_m \to 0$ in L^1 and in probability. Therefore, by (3.16), we need only to show that

$$n(j_m)^{-1/2+r/4}\sum_{k=1}^m \zeta_{m,k} \xrightarrow{\mathcal{L}} \sigma_r(s,t)\mathcal{N}$$

in order to complete the proof.

For this, we will use the Lindeberg–Feller theorem (see, e.g., Theorem 2.4.5 in Durrett [5]), which states the following: for each m, let $\zeta_{m,k}, 1 \le k \le m$, be independent random variables with $\mathbb{E}[\zeta_{m,k}] = 0$. Suppose:

(a) $n(j_m)^{-1+r/2}\sum_{k=1}^m \mathbb{E}[\zeta_{m,k}^2] \to \nu^2$, and

(b) for all $\varepsilon > 0$, $\lim_{m\to\infty} n(j_m)^{-1+r/2}\sum_{k=1}^m \mathbb{E}[|\zeta_{m,k}|^2 I_{\{n(j_m)^{-1/2+r/4}|\zeta_{m,k}|>\varepsilon\}}] \to 0$.

Then $n(j_m)^{-1/2+r/4}\sum_{k=1}^m \zeta_{m,k} \xrightarrow{\mathcal{L}} \nu\mathcal{N}$ as $n \to \infty$.

To verify these conditions, recall that $\{\xi_{i,k}\}$ and $\{\Delta F_{i-u_{k-1}}\}$ have the same law, so that

$$\mathbb{E}[|\zeta_{m,k}|^4] = \mathbb{E}\left[\left|\sum_{i=1}^{u_k-u_{k-1}}(\Delta F_i^r - c_r\sigma_i^r)\right|^4\right].$$

Hence, by (3.4),

$$n(j_m)^{-2+r}\mathbb{E}[|\zeta_{m,k}|^4] \le C(u_k - u_{k-1})^2 n(j_m)^{-2}.$$

Jensen inequality now gives $m^{-1+r/2}\sum_{k=1}^m \mathbb{E}[|\zeta_{m,k}|^2] \le Cmbn(j_m)^{-1} = C(t-s)$, so that by passing to a subsequence, we may assume that (a) holds for some $\nu \ge 0$.

For (b), let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned} n(j_m)^{-1+r/2}\sum_{k=1}^m \mathbb{E}[|\zeta_{m,k}|^2 I_{\{n(j_m)^{-1/2+r/4}|\zeta_{m,k}|>\varepsilon\}}] &\le \varepsilon^{-2}n(j_m)^{-2+r}\sum_{k=1}^m \mathbb{E}[|\zeta_{m,k}|^4] \\ &\le C\varepsilon^{-2}mb^2 n(j_m)^{-2} \\ &= C\varepsilon^{-2}m^{-1}(t-s)^2, \end{aligned}$$

which tends to zero as $m \to \infty$.

It therefore follows that $n(j_m)^{-1/2+r/4}\sum_{k=1}^m \zeta_{m,k} \xrightarrow{\mathcal{L}} \nu\mathcal{N}$ as $n \to \infty$ and it remains only to show that $\nu = \sigma_r(s,t)$. For this, observe that the continuous mapping theorem implies that $|W_r^m(F)_t - W_r^m(F)_s|^2 \xrightarrow{\mathcal{L}} \nu^2\mathcal{N}^2$. By the Skorohod representation theorem, we may assume that the convergence is a.s. By Proposition 3.2, the family $|W_r^m(F)_t - W_r^m(F)_s|^2$ is uniformly integrable. Hence, $|W_r^m(F)_t - W_r^m(F)_s|^2 \to \nu^2\mathcal{N}^2$ in L^1, which implies $\mathbb{E}[|W_r^m(F)_t - W_r^m(F)_s|^2] \to \nu^2$. But by Theorem 1.1, $\mathbb{E}[|W_r^m(F)_t - W_r^m(F)_s|^2] \to \sigma_r(s,t)^2$, so $\nu = \sigma_r(s,t)$ and the proof is complete. □

We now prove Theorems 1.5 and 1.6. It is sufficient to prove Theorem 1.5 since the proof of Theorem 1.6 is similar to that of Theorem 1.5.

Proof of Theorem 1.5 Let $\{n(j)\}_{j=1}^\infty$ be any sequence of natural numbers. By Proposition 3.2, the sequence $\{(F, W_{2p}^{n(j)}(F))\}$ is relatively compact. Therefore, there exists a subsequence $\{n(j_k)\}$ and a cadlag process X such that $(F, W_{2p}^{n(j_k)}(F)) \xrightarrow{\mathcal{L}} (F, X)$. Fix $0 < s_1 < s_2 < \cdots < s_d < s < t$. With notation as in Proposition 3.3, let

$$\zeta_n = n^{-1/2+p/2}\sum_{i=\lfloor ns\rfloor+2}^{\lfloor nt\rfloor}(\xi_{i,k}^r - c_r\sigma_{i,k}^r),$$

and define

$$\eta_n = W_{2p}^n(F)_t - W_{2p}^n(F)_s - \zeta_n.$$

As in the proof of Proposition 3.3, $\eta_n \to 0$ in probability. It therefore follows that

$$(W_{2p}^{n(j_k)}(F)_{s_1}, \dots, W_{2p}^{n(j_k)}(F)_{s_d}, \zeta_{n(j_k)}) \xrightarrow{\mathcal{L}} (X(s_1), \dots, X(s_d), X(t) - X(s)).$$

Note that $\mathcal{F}_{(\lfloor ns\rfloor+1)\Delta t}$ and ζ_n are independent. Hence, $(W_{2p}^n(F)_{s_1}, \dots, W_{2p}^n(F)_{s_d})$ and ζ_n are independent, which implies $X(t) - X(s)$ and $(X(s_1), \dots, X(s_d))$ are independent. This yields that the process X has independent increments.

By Proposition 3.3, the increment $X(t) - X(s)$ is normally distributed with mean zero and variance $\kappa_p|t-s|$. Also, $X(0) = 0$ since $W_{2p}^n(F)_0 = 0$ for all n. Hence, X is equal in law to

$\kappa_p^{1/2}W$, where W is a standard Brownian motion. It remains only to show that F and W are independent.

Fix $0 < s_1 < s_2 < \cdots < s_d$. Let $Z = (F(s_1), \ldots, F(s_d))^T$ and $\Sigma = \mathbb{E}[ZZ^T]$, and define the vectors $b_j \in \mathbb{R}^d$ by $b_j = \mathbb{E}[Z\Delta F_j]$, and $w_j = \Sigma^{-1}b_j$. Let $\xi_j = \Delta F_j - w_j^T Z$, so that ξ_j and Z are independent.

Define

$$\widetilde{W}_{2p}^n(F)_t = n^{-1/2+p/2}\sum_{j=1}^{\lfloor nt\rfloor}(\xi_j^{2p} - c_{2p}\sigma_j^{2p}).$$

Then

$$|W_{2p}^n(F)_t - \widetilde{W}_{2p}^n(F)_t| \le n^{-1/2+p/2}\bigg|\sum_{j=1}^{\lfloor nt\rfloor}(\Delta F_j^{2p} - \xi_j^{2p})\bigg|.$$

By binomial expansion, (2.1) and Hölder inequality,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|W_{2p}^n(F)_t - \widetilde{W}_{2p}^n(F)_t|\Big] &\le Cn^{-1/2+p/2}\sum_{\nu=1}^{2p}\sum_{j=1}^{\lfloor nT\rfloor}(\mathbb{E}[\Delta F_j^{4p-2\nu}])^{1/2}(\mathbb{E}[(w_j^T Z)^{2\nu}])^{1/2}\\ &\le C\sum_{\nu=1}^{2p} n^{-1/2+\nu/4}\sum_{j=1}^{\lfloor nT\rfloor}(\mathbb{E}[(w_j^T Z)^{2\nu}])^{1/2}\\ &\le C\max_{1\le i\le d}\sum_{\nu=1}^{2p} n^{-1/2+\nu/4}\sum_{j=1}^{\lfloor nT\rfloor}|\mathbb{E}[F(s_i)\Delta F_j]|^{\nu}.\end{aligned}$$

Note that by Hölder inequality, we have $|\mathbb{E}[F(s_i)\Delta F_j]| \le C\sigma_j \le Cn^{-1/4}$ for all $1\le i\le d$ and $1\le j\le \lfloor nt\rfloor$, and that by (1.7) and Lagrange mean value theorem,

$$\begin{aligned}\mathbb{E}[F(s_i)\Delta F_j] &= \frac{1}{\sqrt{2\pi}}(\sqrt{s_i+t_j} - \sqrt{s_i+t_{j-1}} - \sqrt{s_i-t_j} + \sqrt{s_i-t_{j-1}})\\ &= \frac{1}{2n\sqrt{2\pi}}\bigg(\Big(s_i + \frac{j-\theta_1}{n}\Big)^{-1/2} + \Big(s_i - \frac{j-\theta_2}{n}\Big)^{-1/2}\bigg),\end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$. Then,

$$\begin{aligned}&n^{-1/2+\nu/4}\sum_{j=1}^{\lfloor nT\rfloor}|\mathbb{E}[F(s_i)\Delta F_j]|^{\nu}\\ &\quad\le Cn^{-5/4}\sum_{j=1}^{\lfloor nT\rfloor}\bigg(\Big(s_i + \frac{j-\theta_1}{n}\Big)^{-1/2} + \Big(s_i - \frac{j-\theta_2}{n}\Big)^{-1/2}\bigg),\end{aligned}$$

which tends to zero as $n\to\infty$ since $\int_0^t((s_i+x)^{-1/2} + (s_i-x)^{-1/2})dx < \infty$. Thus, $(Z, \widetilde{W}_{2p}^n(F)_{s_1}, \ldots, \widetilde{W}_{2p}^n(F)_{s_d}) \xrightarrow{\mathcal{L}} (Z, \kappa_p^{1/2}W(s_1), \ldots, \kappa_p^{1/2}W(s_d))$. Since Z and $\widetilde{W}_{2p}^n(F)$ are independent, this finishes the proof. □

Acknowledgements The author wishes to express their deep gratitude to a referee for his/her valuable comments on an earlier version which improve the quality of this paper.

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