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On 1-connected 8-manifolds with the Same Homology as $S^3 \times S^5$

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Abstract In this article, we classify 1-connected 8-dimensional Poincaré complexes, topological manifolds and smooth manifolds whose integral homology groups are isomorphic to those of $S^3 \times S^5$. A topic related to a paper of Escher and Ziller is also discussed.

Keywords 8-manifolds, Poincaré complexes, diffeomorphism groups, S^1-bundles

MR(2010) Subject Classification 57R19, 57P10, 57R22

1 Introduction

The classification of manifolds is an interesting and challenging task in topology. In this article, we mainly concern the following:

Problem 1.1 Classify all 1-connected 8-manifolds with the same homology as $S^3 \times S^5$.

We call such 8-manifolds of type $(*)$. These manifolds are related to a class of 1-connected 7-manifolds satisfying either

$$H^2(M) \cong \mathbb{Z}\{u\}, \quad H^3(M) = 0, \quad H^4(M) \cong \mathbb{Z}_r\{u^2\} \quad (r \geq 1),$$

which will be called of type (r); or

$$H^2(M) \cong \mathbb{Z}\{u\}, \quad H^3(M) = \mathbb{Z}\{v\}, \quad H^4(M) \cong \mathbb{Z}\{u^2\}, \quad H^5(M) \cong \mathbb{Z}\{uv\},$$

i.e., with integral cohomology ring isomorphic to that of $\mathbb{C}P^2 \times S^3$, which will be called of type (0). Here we use the symbol $\mathbb{Z}\{x\}$ or $\mathbb{Z}_r\{x\}$ to represent an infinite cyclic group or a finite cyclic group of order r with a generator x. If we consider the S^1-bundle over a manifold of type (r) $(r \geq 0)$ with Euler class u, easy calculation shows that the total space is of type $(*)$.

There are many well-known examples for manifolds of type (r), such as Aloff–Wallach manifolds, Eschenburg spaces, etc. These manifolds contribute many valuable examples in the study of positively curved manifolds (cf. [28] for a survey). As for manifolds of type (0), the most natural examples may be the total spaces of S^3-bundles over $\mathbb{C}P^2$ which have cross sections. The classification of type (0) manifolds has been considered in [25]. For manifolds of type (r), Kreck and Stolz have constructed invariants which can determine whether two such manifolds can be homotopic, homeomorphic or diffeomorphic [16]. However, the realization of these invariants is unknown, which means we still need more research on the structure of such manifolds. Considering the connection between manifolds of type $(*)$ and (r), knowing more about manifolds of type $(*)$ may help us get more information on manifolds of type (r).

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To state our main result, we take the following notations. Let $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{S}$ be the homotopy equivalence, homeomorphism and diffeomorphism classes of Poincaré complexes, topological manifolds and smooth manifolds of type $(*)$, respectively. Let $X_{r,s} = (S^3 \vee S^5) \cup_{[\iota_3,\iota_5]+ra_3\eta_6+s\eta_5\eta_6} D^8$, where ι_k corresponds to the identity map of S^k, while $a_3\eta_6$ and $\eta_5\eta_6$ are certain elements in $\pi_7(S^3)$ and $\pi_7(S^5)$, respectively (cf. Lemma 2.3). The composition with the inclusion into $S^3 \vee S^5$ is always omitted for short.

Our main result is the following:

Theorem 1.2 (1) $\mathcal{P} = \{S^3 \times S^5, X_{1,0}, X_{0,1}, X_{1,1}, SU(3)\}$.

(2) $\mathcal{T}=\{S^3\times S^5, SU(3), M_{1,0}\}$. *Here* $M_{1,0}$ *is the total space of the only nontrivial* S^3*-bundle over* S^5 *which has a cross section and* $M_{1,0} \simeq X_{1,0}$.

(3) $\mathcal{S} = \{S^3 \times S^5, S^3 \times S^5\#\Sigma^8, SU(3), M_{1,0}\}$. *Here* Σ^8 *is the only exotic* 8*-sphere.*

Remark 1.3 When classifying a certain class of manifolds, a complete set of invariants is often provided. However, it seems not easy to provide complete invariants which can be quickly calculated in our situation. In the topological case, the invariants (μ,ϕ) in [5] can be used. Obviously $\mu_{S^3\times S^5} = 0$, $\mu_{M_{1,0}} = 0$, while $\mu_{SU(3)} \neq 0$. [5, Theorem 1.9] indicates that we must have $\phi_{S^3\times S^5} = 0$ and $\phi_{M_{1,0}} \neq 0$. However, we do not know how to calculate them directly. We guess that $\phi_{X_{0,1}} = 0$ and $\phi_{X_{1,1}} \neq 0$. If it is the case, then (μ,ϕ) together with the first exotic class e defined in [7] can provide complete invariants for $\mathcal{P}$. In the smooth case, Kosiński's invariant $p'(M)$ [15] can distinguish $S^3 \times S^5$ and $S^3 \times S^5\#\Sigma^8$, although it may be hard to calculate it in practice.

Remark 1.4 Since S^3-bundles over S^5 and exotic spheres are stably parallelizable, we know from Theorem 1.2 that all smooth 8-manifolds of type $(*)$ are stably parallelizable. This fact can also be seen directly by obstruction theory and Hirzebruch's Signature Theorem. Note that in [5], Fang and Pan have obtained a complete classification of $(n-2)$-connected $2n$-dimensional stably parallelizable manifolds up to homeomorphism. The topological classification has also been obtained by Ishimoto [10] for $(n-2)$-connected $2n$-dimensional stably parallelizable manifolds with torsion free homology groups.

As a consequence of Theorem 1.2, we have the following rigidity theorem:

Corollary 1.5 (1) *Two topological* 8*-manifolds of type* $(*)$ *are homeomorphic if and only if they are homotopy equivalent. All topological* 8*-manifolds of type* $(*)$ *are smoothable.*

(2) *A closed smooth* 8*-manifold* M *is diffeomorphic to* $M_{1,0}$ *or* $SU(3)$ *if and only if they are homotopy equivalent.*

Recall that for a smooth n-dimensional manifold M, its inertia group $I(M) = \{\Sigma \in \Theta_n | M\#\Sigma \cong M\}$. Here Θ_n is the group of homotopy n-spheres. Theorem 1.2 also implies:

Corollary 1.6 $I(S^3 \times S^5) = 0$, $I(M_{1,0}) = I(SU(3)) = \Theta_8$.

Remark 1.7 The fact $I(S^3 \times S^5) = 0$ has already been obtained by De Sapio [2] and Schultz [21].

For $n \geq 3$, the n-th homotopy group of the total space of a circle bundle coincides with that of the base space. Together with the obvious fact $\pi_i(M_{1,0}) \cong \pi_i(S^3 \times S^5)$, we also have:

Corollary 1.8 *Let N be a 7-dimensional manifold of type (r). Then either*

$$\pi_i(N) \cong \begin{cases} \mathbb{Z}, & i = 2 \\ \pi_i(S^3) \oplus \pi_i(S^5), & i \geq 3 \end{cases}$$

or

$$\pi_i(N) \cong \begin{cases} \mathbb{Z}, & i = 2 \\ \pi_i(SU(3)), & i \geq 3. \end{cases}$$

In [4], Escher and Ziller studied the topology of certain 7-manifolds of type (r), i.e., the total spaces of S^3-bundles over $\mathbb{C}P^2$ and S^1-bundles over S^2-bundles over $\mathbb{C}P^2$. By classifying the total spaces of S^1-bundles over these manifolds, we obtain the following:

Theorem 1.9 *There are infinitely many nonequivalent free smooth S^1-actions on $S^3 \times S^5$, $M_{1,0}$ and $SU(3)$.*

The article is organized as follows. In Section 2, we give some notations and elementary results which will be used later. We deal with the classification problem in Sections 3 and 5. Section 4 is a preparation for Section 5. Finally, in Section 6 we discuss a topic related to a paper of Escher and Ziller [4].

2 Preliminaries

In this section we list some notations and basic results which will be used later. Sometimes we use the same symbol to represent a map and its homotopy class if there is no confusion.

Let $x_0 = (1, 0, \ldots, 0)^T$ be the base point of S^n. There is a principal fiber bundle

$$SO(n) \xrightarrow{i} SO(n+1) \xrightarrow{p} S^n,$$

defined by

$$i : A \mapsto \begin{pmatrix} 1 & \\ & A \end{pmatrix}, \quad p : B \mapsto Bx_0.$$

The fiber bundle $SO(3) \xrightarrow{i} SO(4) \xrightarrow{p} S^3$ has a cross section $\sigma : S^3 \to SO(4)$ given by the canonical identification of S^3 with $Sp(1) \subset SO(4)$. Explicitly σ is defined by

$$\sigma : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}.$$

Therefore we have a canonical diffeomorphism between $SO(4)$ and $SO(3) \times S^3$, which induces the isomorphism $\pi_4(SO(4)) \cong \pi_4(SO(3)) \oplus \pi_4(S^3)$.

Identify $\mathbb{R}^4$ with $\mathbb{H}$, i.e., the set of quaternions, and identify S^3 with the set of unit quaternions. Then for $z \in S^3$ and $x = (0, x_2, x_3, x_4) \in \mathbb{R}^4 \cong \mathbb{H}$, it is easily checked that the first coordinate of zxz^{-1} is also 0. Therefore we can define a map $\rho : S^3 \to SO(3)$ by $\rho(z)x = zxz^{-1}$. Note that ρ is a two-fold covering map of $SO(3)$.

Let η_2 be the Hopf map and $\eta_n = \Sigma^{n-2}\eta_2$.

Lemma 2.1 *Notations as above.*

(1) $\pi_3(SO(3)) \cong \mathbb{Z}\{\rho\}$, $\pi_3(SO(4)) \cong \mathbb{Z}\{i\rho\} \oplus \mathbb{Z}\{\sigma\}$, $\pi_3(SO(5)) \cong \mathbb{Z}\{i\sigma\}$;

(2) $\pi_4(SO(3)) \cong \mathbb{Z}_2\{\rho\eta_3\}$, $\pi_4(SO(4)) \cong \mathbb{Z}_2\{i\rho\eta_3\} \oplus \mathbb{Z}_2\{\sigma\eta_3\}$.

Let $G_n = \{f : S^{n-1} \to S^{n-1} | \deg f = 1\}$ and $F_n = \{f \in G_n | f(x_0) = x_0\}$. We have the following ladder of fibration sequences:

$$\begin{array}{ccccc} SO(n) & \xrightarrow{i} & SO(n+1) & \xrightarrow{p} & S^n \\ \downarrow & & \downarrow & & \| \\ F_{n+1} & \xrightarrow{\bar{i}} & G_{n+1} & \xrightarrow{\bar{p}} & S^n \end{array}$$

Here the vertical arrows are natural inclusions. We denote their induced maps between homotopy groups by

$$\bar{\mu}' : \pi_k(SO(n)) \to \pi_k(F_{n+1}), \quad \bar{\mu} : \pi_k(SO(n+1)) \to \pi_k(G_{n+1}).$$

We define the map $I : \pi_k(F_{n+1}) \to \pi_{k+n}(S^n)$ as follows. Write S^{k+n} as $(S^k \times D^n) \cup (D^{k+1} \times S^{n-1})$, and let $q : D^n \to D^n/\partial D^n \cong S^n$ be the quotient map. For any $\beta = [f] \in \pi_k(F_{n+1})$, its image $I(\beta)$ is defined to be represented by

$$I(f)(x, y) = \begin{cases} f(x)(q(y)), & \text{if } (x, y) \in S^k \times D^n, \\ x_0, & \text{otherwise.} \end{cases}$$

By definition, there is a commutative diagram

$$\begin{array}{ccc} \pi_k(SO(n)) & & \\ \downarrow{\scriptstyle \bar{\mu}'} & \searrow^{J} & \\ \pi_k(F_{n+1}) & \xrightarrow{I} & \pi_{k+n}(S^n) \end{array} \tag{†}$$

Theorem 2.2 ([26]) *I is a group isomorphism.*

For homotopy groups of spheres and J-homomorphisms, we need the following results:

Lemma 2.3 ([22, 23]) (1) $\pi_{n+1}(S^n) \cong \begin{cases} \mathbb{Z}\{\eta_2\}, & \text{if } n = 2, \\ \mathbb{Z}_2\{\eta_n\}, & \text{if } n \geq 3. \end{cases}$

(2) $\pi_{n+2}(S^n) \cong \mathbb{Z}_2\{\eta_n\eta_{n+1}\}$ *for* $n \geq 2$.

(3) $\pi_6(S^3) \cong \mathbb{Z}_{12}\{a_3\}$ *with* $a_3 = J\rho$.

$\pi_7(S^4) \cong \mathbb{Z}\{\nu_4\} \oplus \mathbb{Z}_{12}\{a_4\}$ *with* ν_4 *the Hopf map and* $a_4 = \Sigma a_3$. *Note that* $\nu_4 = J\sigma$.

(4) $\pi_7(S^3) \cong \mathbb{Z}_2\{a_3\eta_6\}$. *Note that* $\eta_3\nu_4 = a_3\eta_6$ *and* $\eta_3 a_4 = 0$.

$\pi_8(S^4) \cong \mathbb{Z}_2\{\nu_4\eta_7\} \oplus \mathbb{Z}_2\{a_4\eta_7\}$,

$\pi_{n+4}(S^n) = 0$, $n \geq 6$.

Lemma 2.4 (1) $\Sigma J = -Ji_*$;

(2) $J(\alpha \circ \beta) = J\alpha \circ \Sigma^n \beta$ *for* $\alpha \in \pi_k(SO(n))$ *and* $\beta \in \pi_l(S^k)$.

Proof (1) See [27].

(2) It can be checked directly by definition. □

Lemma 2.5 (1) $J : \pi_4(SO(3)) \xrightarrow{\cong} \pi_7(S^3)$;

(2) $J : \pi_4(SO(4)) \xrightarrow{\cong} \pi_8(S^4)$ *with* $J(\sigma\eta_3) = \nu_4\eta_7$ *and* $J(i\rho\eta_3) = a_4\eta_7$.

Proof Using Lemma 2.3 and 2.4.

(1) $J(\rho\eta_3) = (J\rho)\eta_6 = a_3\eta_6$.

(2) $J(\sigma\eta_3) = J(\sigma)\eta_7 = \nu_4\eta_7$.

$J(i\rho\eta_3) = J(i\rho)\eta_7 = (-\Sigma J\rho)\eta_7 = a_4\eta_7$. □

Finally we have the following:

Lemma 2.6 $\bar{\mu}' : \pi_4(SO(3)) \xrightarrow{\cong} \pi_4(F_4)$, $\bar{\mu} : \pi_4(SO(4)) \xrightarrow{\cong} \pi_4(G_4)$.

Proof Combining Lemma 2.2, Lemma 2.5 and (†), it is easy to see that $\bar{\mu}'$ is an isomorphism.

For $\bar{\mu}$, we consider the following exact ladder:

$$\begin{array}{ccccccccc}
\pi_5(S^3) & \longrightarrow & \pi_4(SO(3)) & \longrightarrow & \pi_4(SO(4)) & \underset{\sigma_*}{\longrightarrow} & \pi_4(S^3) & \xrightarrow{\partial} & \pi_3(SO(3)) \\
\Vert & & \downarrow{\scriptstyle \bar{\mu}'} & & \downarrow{\scriptstyle \bar{\mu}} & & \Vert & & \downarrow{\scriptstyle \bar{\mu}'} \\
\pi_5(S^3) & \longrightarrow & \pi_4(F_4) & \longrightarrow & \pi_4(G_4) & \longrightarrow & \pi_4(S^3) & \xrightarrow{\bar{\partial}} & \pi_3(F_4)
\end{array}$$

The top row is split, so $\partial = 0$. Then $\bar{\partial} = \bar{\mu}'\partial = 0$. Therefore $\bar{\mu}$ is an isomorphism by the 5-Lemma. □

3 The Classification of Poincaré Complexes and Topological Manifolds

Now we prove the first two statements in Theorem 1.2. First, we deal with the classification of Poincaré complexes.

Let X be a Poincaré complex of type $(*)$. Then

$$X \simeq S^3 \cup_{\phi_1} D^5 \cup_{\phi_2} D^8.$$

Since $\phi_1 \in \pi_4(S^3) \cong \mathbb{Z}_2\{\eta_3\}$, it has two choices:

(1) $\phi_1 = \eta_3$.

In this case, we have $\phi_2 \in \pi_7(S^3 \cup_{\eta_3} D^5) = \pi_7(\Sigma\mathbb{C}P^2) \cong \mathbb{Z}\{\beta\}$ (see [20]). If $\phi_2 = \pm\beta$, then $X \simeq SU(3)$. The cup product structure of X implies that it is the only possibility.

(2) $\phi_1 = 0$.

In this case, we have $\phi_2 \in \pi_7(S^3 \vee S^5)$. By Hilton's theorem [9], we have $\pi_7(S^3 \vee S^5) \cong \mathbb{Z}_2\{a_3\eta_6\} \oplus \mathbb{Z}_2\{\eta_5\eta_6\} \oplus \mathbb{Z}\{[\iota_3, \iota_5]\}$. Therefore we can suppose $\phi_2 = ra_3\eta_6 + s\eta_5\eta_6 + t[\iota_3, \iota_5]$. The cup product structure is essentially determined by the term $t[\iota_3, \iota_5]$, which forces $t = \pm 1$, and we may assume $t = 1$. Let $X_{r,s} = (S^3 \vee S^5) \cup_{ra_3\eta_6 + s\eta_5\eta_6 + [\iota_3, \iota_5]} D^8$, $r, s \in \{0, 1\}$. To see that these $X_{r,s}$ are different from each other, we use the following:

Lemma 3.1 *$X_{r,s} \simeq X_{r',s'}$ if and only if there exists $f \in \mathcal{E}(S^3 \vee S^5)$ such that $f_*(ra_3\eta_6 + s\eta_5\eta_6 + [\iota_3, \iota_5]) = \pm(r'a_3\eta_6 + s'\eta_5\eta_6 + [\iota_3, \iota_5])$, where $\mathcal{E}(X)$ denotes the group of self homotopy equivalences of X.*

Proof "⇒" Let $g : X_{r,s} \to X_{r',s'}$ be a homotopy equivalence. We may assume g is a cellular map. Let f be the restriction of g on the 5-skeleton. Obviously $f \in \mathcal{E}(S^3 \vee S^5)$. Then the only if part follows by the commutative diagram below together with the fact that ∂ sends 1 to the attaching maps.

$$\begin{array}{ccc} \mathbb{Z} \cong \pi_8(X_{r,s}, S^3 \vee S^5) & \xrightarrow{\partial} & \pi_7(S^3 \vee S^5) \\ \cong \downarrow g_* & & \cong \downarrow f_* \\ \mathbb{Z} \cong \pi_8(X_{r',s'}, S^3 \vee S^5) & \xrightarrow{\partial} & \pi_7(S^3 \vee S^5) \end{array}$$

"$\Leftarrow$" The condition implies that f can be extended to a map $g : X_{r,s} \to X_{r',s'}$. It can be easily checked that g induces an isomorphism between the cohomology rings. Therefore g is a homotopy equivalence. □

By Lemma 3.1, we only need to show that

$$f_*(ra_3\eta_6 + s\eta_5\eta_6 + [\iota_3, \iota_5]) = \pm(ra_3\eta_6 + s\eta_5\eta_6 + [\iota_3, \iota_5])$$

for any $f \in \mathcal{E}(S^3 \vee S^5)$. Clearly,

$$\mathcal{E}(S^3 \vee S^5) \cong \left\{ \begin{pmatrix} \pm\iota_3 & \epsilon\eta_3\eta_4 \\ 0 & \pm\iota_5 \end{pmatrix} \middle| \epsilon = 0, 1 \right\}.$$

We only verify the case

$$f = \begin{pmatrix} \iota_3 & \eta_3\eta_4 \\ 0 & \iota_5 \end{pmatrix}$$

below. The rest is similar.

$$\begin{aligned} &f_*(a_3\eta_6) = a_3\eta_6, \\ &f_*(\eta_5\eta_6) = (\eta_3\eta_4 + \iota_5)\eta_5\eta_6 = \eta_5\eta_6, \\ &f_*([\iota_3, \iota_5]) = [\iota_3, \eta_3\eta_4 + \iota_5] = [\iota_3, \eta_3\eta_4] + [\iota_3, \iota_5] = [\iota_3, \iota_5]. \end{aligned}$$

Next we turn to the classification of topological manifolds. By definition $M_{1,0}$ is the total space of the S^3 bundle with clutching map $i\rho\eta_3 \in \pi_4(SO(4))$. Therefore together with [11, (3.7)] and Lemma 2.5, we have

$$M_{1,0} \simeq (S^3 \vee S^5) \cup_{[\iota_3,\iota_5]+J\rho\eta_3} D^8 = (S^3 \vee S^5) \cup_{[\iota_3,\iota_5]+a_3\eta_6} D^8 = X_{1,0}.$$

Obviously $S^3 \times S^5 \simeq X_{0,0}$. To see that there are no other topological manifolds homotopy equivalent to $S^3 \times S^5$, $M_{1,0}$ and $SU(3)$, recall that we have the following surgery exact sequence for a simply connected n-dimensional manifold M for $n \geq 5$ (cf. [24]):

$$L_{n+1}(\mathbb{Z}) \to \mathscr{S}^{\mathrm{TOP}}(M) \to [M, G/\mathrm{TOP}] \xrightarrow{\theta} L_n(\mathbb{Z}).$$

Note that

$$\pi_n(G/\mathrm{TOP}) \cong L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n \equiv 0 \mod 4, \\ \mathbb{Z}_2, & n \equiv 2 \mod 4, \\ 0, & n \equiv 1 \mod 2 \end{cases}$$

(cf. [14]). If M is a topological 8-manifold of type $(*)$, then we have the following:

$$\begin{array}{ccccccc} & & & [S^4\cup D^6, G/\mathrm{TOP}] & & \\ & & & \downarrow{\scriptstyle (\Sigma\phi)^*=0} & & \\ & & & [S^8, G/\mathrm{TOP}] & \searrow{\scriptstyle\cong} & \\ & & & \downarrow & & \\ 0=L_9(\mathbb{Z}) & \longrightarrow & \mathscr{S}^{\mathrm{TOP}}(M) & \longrightarrow [M, G/\mathrm{TOP}] & \xrightarrow{\theta} & L_8(\mathbb{Z})\cong\mathbb{Z} \\ & & & \downarrow & & \\ & & & [S^3\cup D^5, G/\mathrm{TOP}]=0 & & \end{array}$$

Here ϕ denotes the attaching map of the top cell of M. Since $\Sigma\phi$ is a torsion element (cf. [20] for $SU(3)$), we must have $(\Sigma\phi)^*=0$. The fact $[S^3\cup D^5, G/\mathrm{TOP}]=0$ is due to $\pi_3(G/\mathrm{TOP})=\pi_5(G/\mathrm{TOP})=0$. Therefore θ is an isomorphism, which makes the topological structure set $\mathscr{S}^{\mathrm{TOP}}(M)$ only have one element.

The remaining work is to show that $X_{0,1}$ and $X_{1,1}$ are not homotopy equivalent to any topological manifolds. The situation is similar to the well-known 5-dimensional case (cf. [18, pp. 32-33]). We have

$$\Sigma^n(X_{r,1})_+\simeq S^n\vee S^{n+3}\vee(S^{n+5}\cup_{\eta^2}D^{n+8})$$

for n sufficiently large. Let $\nu_{r,1}$ be the Spivak normal bundle of $X_{r,1}$ and $T_{r,1}$ be its Thom space. Then $T_{r,1}$ is the Spanier–Whitehead duality of $\Sigma^n(X_{r,1})_+$ [1]. Therefore

$$T_{r,1}\simeq(S^l\cup_{\eta^2}D^{l+3})\vee S^{l+5}\vee S^{l+8}.$$

This implies that the Thom space of $\nu_{r,1}|_{S^3}$ is homotopy equivalent to $S^l\cup_{\eta^2}D^{l+3}$. Since $\pi_3(\mathrm{BTOP})=0$, we see that $\nu_{r,1}|_{S^3}$ can not be given a topological bundle structure. Therefore $\nu_{r,1}$ also does not have a topological bundle structure, which means $X_{r,1}$ is not homotopy equivalent to a topological manifold.

4 Self Equivalences of $S^k\times S^n$

The key to the classification of smooth 8-manifolds of type $(*)$ is the following observation:

If M^8 is of type $(*)$, then $M\cong S^3\times D^5\cup_f S^3\times D^5$, where $f\in\mathrm{Diff}(S^3\times S^4)$.

Then we need to analyze $\mathrm{Diff}(S^3\times S^4)$. Our progress relies heavily on results in [17]. We will briefly review them and add some easy observations. For simplicity we assume $k<n$, although the case $k=n$ has also been considered in [17].

We will deal with the following categories:

$\mathscr{H}$: Topological spaces and homotopy classes of maps;

$\mathscr{D}$: Smooth manifolds and smooth maps.

Let $\bar{D}=\bar{D}^{k,n}$ ($\bar{H}=\bar{H}^{k,n}$) be the group of concordance (homotopy) classes of self-diffeomorphisms (self-homotopy equivalences) of $S^k\times S^n$. The symbol $\bar{A}$ may refer to any of them. By saying a self-equivalence, we mean a self-diffeomorphism or a self-homotopy equivalence in its suited category. There is a natural homomorphism $\mu:\bar{D}^{k,n}\to\bar{H}^{k,n}$ defined by considering a diffeomorphism merely as a homotopy equivalence.

Let $B = \mathrm{Aut}\, H^*(S^k \times S^n)$ be the group of graded ring automorphisms of $H^*(S^k \times S^n)$. It is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\Phi : \bar{A}^{k,n} \to B$ be the obvious homomorphism. Then Φ is onto, and its kernel $A = A^{k,n}$ is the subgroup of $\bar{A}^{k,n}$ which contains those orientation-preserving self-equivalences restricting to some $S^k \times x_0$ homotopic to the inclusion. Namely we have the following exact sequence:

$$1 \to A \to \bar{A} \xrightarrow{\Phi} B \to 1.$$

Notice that B can be realized by reflections on $S^k \times S^n$, which makes the short exact sequence split. Therefore $\bar{A} \cong A \rtimes B$.

Define subgroups A_1, A_2 and α of A to consist of those elements represented by $f : S^k \times S^n \to S^k \times S^n$ satisfying:

(A_1) f extends to a self-equivalence of $D^{k+1} \times S^n$;

(A_2) f extends to a self-equivalence of $S^k \times D^{n+1}$;

(A_3) There is some $(k+n)$-disk $D \subset S^k \times S^n$ such that $f(D) \subset D$ and $f|_{S^k \times S^n - D}$ is the inclusion.

Subgroups $\bar{A}_1$, $\bar{A}_2$ and $\bar{\alpha}$ of $\bar{A}$ are similarly defined.

Proposition 4.1 ([17, Proposition 2.3]) *$\alpha = 0$ in $\mathscr{H}$, while $\alpha \cong \Theta_{k+n+1}$, the group of exotic $(k+n+1)$-spheres, in $\mathscr{D}$.*

Theorem 4.2 ([17, Theorem 2.4]) *For $n \geq 3$, $A^{k,n} = (A_1 \oplus \alpha) \rtimes_{\hat{\phi}} A_2$, with the action $\hat{\phi}$ trivial on α. Besides, the groups A_1, A_2 and α are abelian.*

Corollary 4.3 *For $n \geq 3$, $\bar{A}^{k,n} \cong (A_1 \oplus \alpha) \rtimes \bar{A}_2$ and $\bar{A}_2 \cong A_2 \rtimes B$.*

Proof We already have

$$\bar{A} \cong ((A_1 \oplus \alpha) \rtimes A_2) \rtimes B.$$

Actually $A_1 \oplus \alpha$ is a normal subgroup of $\bar{A}$ and $A_2 B$ is a subgroup of $\bar{A}$, which can be easily deduced from the observation that A_1, A_2 and α are invariant under the conjugation action of B. Therefore

$$\bar{A} \cong (A_1 \oplus \alpha) \rtimes (A_2 \rtimes B).$$

The elements in B are represented by reflections, which obviously extend to $S^k \times D^{n+1}$. Hence $A_2 \rtimes B \subset \bar{A}_2$. Since $(A_1 \oplus \alpha) \cap \bar{A}_2 = (A_1 \oplus \alpha) \cap A_2 = \{1\}$, we must have $\bar{A}_2 \cong A_2 \rtimes B$. □

Now we turn to the determination of A_1, A_2 and the semi-direct product structure $\hat{\phi}$.

Let FC_m^p be the group of concordance classes of framed imbeddings $S^m \hookrightarrow S^{m+p}$. We construct homomorphisms

$$\lambda_1 : D_1^{k,n} \to FC_n^{k+1}, \quad \lambda_2 : D_2^{k,n} \to FC_k^{n+1}$$

as follows. Let $f : S^k \times S^n \to S^k \times S^n$ represent an element ξ in $D_1^{k,n}$ and $F : D^{k+1} \times S^n \to D^{k+1} \times S^n$ be an extension of f. Then the composition of the following maps defines a framed embedding, which represents the element $\lambda_1(\xi) \in FC_k^{n+1}$:

$$S^n \times D^{k+1} \underset{\cong}{\xrightarrow{F}} S^n \times D^{k+1} \subset S^n \times \mathbb{R}^{k+1} \subset \mathbb{R}^{k+n+1}.$$

The homomorphism λ_2 is defined similarly. Note that we have natural homomorphisms

$$e_1 : \pi_n(SO(k+1)) \to D_1^{k,n}, \quad e_2 : \pi_k(SO(n+1)) \to D_2^{k,n}$$

defined by associating a smooth map $f : S^n \to SO(k+1)$ (or $g : S^k \to SO(n+1)$) to the diffeomorphism $(x,y) \mapsto (f(y)x, y)$ (or $(x,y) \mapsto (x, g(x)y)$), which fit into commutative diagrams

$$\begin{array}{ccc} \pi_n(SO(k+1)) & & \\ \downarrow e_1 & \searrow & \\ D_1^{k,n} & \xrightarrow{\lambda_1} & FC_n^{k+1} \end{array} \qquad \begin{array}{ccc} \pi_k(SO(n+1)) & & \\ \downarrow e_2 & \searrow & \\ D_2^{k,n} & \xrightarrow{\lambda_2} & FC_k^{n+1} \end{array} \qquad (*)$$

where $\pi_p(SO(m)) \to FC_p^m$ is the same as in [8, (5.10)].

Define similar homomorphisms

$$\lambda_1 : H_1^{k,n} \to \pi_n(G_{k+1}), \quad \lambda_2 : H_2^{k,n} \to \pi_k(G_{n+1})$$

in $\mathscr{H}$ as follows. Any element of H_1 can be represented by a map of the form $(x,y) \mapsto (g(x,y), y)$, where $g : S^k \times S^n \to S^k$ corresponds to an element of $\pi_n(G_{k+1})$. This induces the homomorphism λ_1. In a similar fashion, the homomorphism λ_2 is defined. It is clear that the following diagrams commute:

$$\begin{array}{ccc} \pi_n(SO(k+1)) & \xrightarrow{\bar{\mu}} & \pi_n(G_{k+1}) \\ \downarrow e_1 & & \uparrow \lambda_1 \\ D_1^{k,n} & \xrightarrow{\mu} & H_1^{k,n} \end{array} \qquad \begin{array}{ccc} \pi_k(SO(n+1)) & \xrightarrow{\bar{\mu}} & \pi_k(G_{n+1}) \\ \downarrow e_2 & & \uparrow \lambda_2 \\ D_2^{k,n} & \xrightarrow{\mu} & H_2^{k,n} \end{array} \qquad (**)$$

Theorem 4.4 ([17, Theorem 3.3]) λ_1, λ_2 *are isomorphisms, assuming* $n \geq 3$ *and* $k \geq 2$.

Corollary 4.5 $\pi_4(SO(4)) \underset{\cong}{\xrightarrow{e_1}} D_1^{3,4} \underset{\cong}{\xrightarrow{\mu}} H_1^{3,4}$, $\pi_3(SO(5)) \underset{\cong}{\xrightarrow{e_2}} D_2^{3,4}$.

Proof Recall the long exact sequence in [8]:

$$\cdots \to \pi_n(SO(q)) \to FC_n^q \to C_n^q \to \pi_{n-1}(SO(q)) \to \cdots .$$

Here C_n^q is the group of concordance classes of embeddings of S^n in S^{n+q}.

For the case FC_3^5, we have

$$\cdots \to C_4^5 \to \pi_3(SO(5)) \to FC_3^5 \to C_3^5 \to \cdots .$$

Using the fact that $C_n^q = 0$ for $n < 2q - 3$ (cf. [8, (6.6)]), the fact $\pi_3(SO(5)) \xrightarrow{\cong} FC_3^5$ follows.

For FC_4^4, we have

$$\cdots \to FC_5^4 \to C_5^4 \to \pi_4(SO(4)) \to FC_4^4 \to C_4^4 \to \cdots .$$

Note that $C_4^4 = 0$ while $C_5^4 \neq 0$. But $FC_5^4 \to C_5^4$ is surjective, since all 5-spheres in S^9 have trivial normal bundles ([13, Theorem 8.2]). Therefore $\pi_4(SO(4)) \xrightarrow{\cong} FC_4^4$. Then the corollary follows from Theorem 4.4, Lemma 2.6 and commutative diagrams $(*)$ $(**)$. □

The action $\hat{\phi}$ can be divided into two parts, i.e.,

$$\hat{\phi}(g_2).g_1 = \phi(g_2).g_1 + \tau(g_2).g_1 \quad \text{for } g_i \in D_i,$$

where $\phi : D_2 \to \operatorname{Aut} D_1$ is a homomorphism and $\tau : D_2 \to \operatorname{Hom}(D_1, \alpha)$ satisfies

$$\tau(gg') = \tau(g)\phi(g') + \tau(g') \quad \text{for } g, g' \in D_2.$$

In $\mathscr{H}$, the action ϕ can be completely determined, which will be presented below. Note that the homomorphism $\mu : \bar{D} \to \bar{H}$ preserves the action ϕ. Therefore a large amount of information about ϕ in $\mathscr{D}$ can also be known.

Let $\theta = \bar{i}_* I^{-1} : \pi_{n+m}(S^m) \xleftarrow[\cong]{I} \pi_n(F_{m+1}) \xrightarrow{\bar{i}_*} \pi_n(G_{m+1})$. By Theorem 4.4 and the surjectivity of $\bar{i}_* : \pi_k(F_{n+1}) \to \pi_k(G_{n+1})$ when $k < n$, we identify $H_1^{k,n}$ with $\pi_n(G_{k+1})$ and $H_2^{k,n}$ with $\pi_k(G_{n+1}) = \operatorname{im}\theta$.

Proposition 4.6 ([17, Proposition 4.2]) *$\phi(\theta(\xi)).\beta = \beta - \theta(\bar{p}_*(\beta) \circ \xi)$ for $\xi \in \pi_{n+k}(S^n)$ and $\beta \in \pi_n(G_{k+1})$.*

The function $\tau : D_2 \to \operatorname{Hom}(D_1, \alpha)$, or put it another way, the pairing $\tau : D_1^{k,n} \otimes D_2^{k,n} \to \Theta_{k+n+1}$, can be viewed as a special case of the pairing

$$T : \pi_n(SO(k+1)) \otimes \pi_k(SO(n+1)) \to \Theta_{k+n+1}$$

studied by Milnor in [19]. Namely, we have the following commutative diagram:

$$\begin{array}{ccc}
\pi_n(SO(k+1)) \otimes \pi_k(SO(n+1)) & & \\
\downarrow{\scriptstyle e_1 \otimes e_2} & \searrow^{T} & \\
D_1^{k,n} \otimes D_2^{k,n} & \xrightarrow{\tau} & \Theta_{k+n+1}
\end{array}$$

Lemma 4.7 *The pairing τ coincides with T for $k = 3$ and $n = 4$.*

Proof By Corollary 4.5. □

5 The Classification of Smooth Manifolds

Lemma 5.1 *A smooth 8-manifold M is of type $(*)$ if and only if $M \cong S^3 \times D^5 \cup_f S^3 \times D^5$ for some $f \in \operatorname{Diff}(S^3 \times S^4)$.*

Proof The "if" part can be easily seen using van Kampen theorem and Mayer–Vietories sequence.

For the "only if" part, first note that there exists a self-indexed minimal Morse fuction $h : M \to \mathbb{R}$. Namely, the function h has only 4 critical points, with values $0, 3, 5, 8$, respectively. Let $W_1 = h^{-1}(-\infty, 4]$ and $W_2 = h^{-1}[4, +\infty)$. Then $M = W_1 \cup W_2$, and both W_1 and W_2 can be obtained by pasting a 3-handle $D^3 \times D^5$ on D^8. Since orientation preserving framed imbeddings of S^2 in S^7 are all isotopic, we must have $W_1 \cong W_2 \cong S^3 \times D^5$. □

Let $M(f) = S^3 \times D^5 \cup_f S^3 \times D^5$ for $f \in \operatorname{Diff}(S^3 \times S^4)$. Note that if $f_0, f_1 \in \operatorname{Diff}(S^3 \times S^4)$ are concordant, then $M(f_0) \cong M(f_1)$. Therefore, each $\alpha = [f] \in \bar{D} = \bar{D}^{3,4}$ defines an element $[M(f)] \in \mathcal{S}$, which is denoted by $M(\alpha)$.

Proposition 5.2 *$M(\alpha_1) = M(\alpha_2)$ if and only if there exist $\beta_1, \beta_2 \in \bar{D}_2$ such that $\beta_1\alpha_1 = \alpha_2\beta_2$.*

In other words, $\mathcal{S} \cong \bar{D}/\sim$, where $\alpha_1 \sim \alpha_2$ if there exist $\beta_1, \beta_2 \in \bar{D}_2$ such that $\beta_1\alpha_1 = \alpha_2\beta_2$.

Corollary 5.3 *Elements in $\mathcal{S}$ are in one-to-one correspondence with orbits of the conjugation action of $\bar{D}_2$ on $D_1 \oplus \alpha$.*

Proposition 5.2 and Corollary 5.3 are actually special cases of [17, Lemma 5.4 and Proposition 5.7].

Now we analyze the action of $\bar{D}_2$. We only analyze the action of D_2, i.e., $\hat{\phi}$, as we will see that it is already enough for our final result.

Proposition 5.4 *Identify D_1 an D_2 with $\pi_4(SO(4))$ and $\pi_3(SO(5))$ respectively. Notations are as in Lemma* 2.1. *Then*

$$\begin{aligned}\phi(i\sigma)(\sigma\eta_3) &= \sigma\eta_3 + i\rho\eta_3,\\ \phi(i\sigma)(i\rho\eta_3) &= i\rho\eta_3.\end{aligned}$$

Proof By Corollary 4.5, we can pass to $\mathscr{H}$ to do calculations.

$$\bar{\mu}(\phi(i\sigma)) = \phi(\bar{\mu}(i\sigma)) = \phi(\bar{i}_*\bar{\mu}'\sigma) = \phi(\bar{i}_* I^{-1} J\sigma) = \phi(\theta(J\sigma)) = \phi(\theta(\nu_4)).$$

Then using the formula of Proposition 4.6,

$$\begin{aligned}\bar{\mu}(\phi(i\sigma)(\sigma\eta_3)) &= \phi(\bar{\mu}(i\sigma))(\bar{\mu}(\sigma\eta_3))\\ &= \phi(\theta(\nu_4))(\bar{\mu}(\sigma\eta_3))\\ &= \bar{\mu}(\sigma\eta_3) - \theta(\bar{p}_*(\bar{\mu}(\sigma\eta_3)) \circ \nu_4)\\ &= \bar{\mu}(\sigma\eta_3) - \theta(p_*\sigma\eta_3 \circ \nu_4)\\ &= \bar{\mu}(\sigma\eta_3) - \theta(\eta_3\nu_4)\\ &= \bar{\mu}(\sigma\eta_3) - \bar{i}_* I^{-1}(J(\rho\eta_3))\\ &= \bar{\mu}(\sigma\eta_3) - \bar{\mu} i_*(\rho\eta_3)\\ &= \bar{\mu}(\sigma\eta_3 - i\rho\eta_3)\\ &= \bar{\mu}(\sigma\eta_3 + i\rho\eta_3)\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{\mu}(\phi(i\sigma)(i\rho\eta_3)) &= \phi(\theta(\nu_4))(\bar{\mu}(i\rho\eta_3))\\ &= \bar{\mu}(i\rho\eta_3) - \theta(\bar{p}_*(\bar{\mu}(i\rho\eta_3)) \circ \nu_4)\\ &= \bar{\mu}(i\rho\eta_3) - \theta(p_*(i\rho\eta_3) \circ \nu_4)\\ &= \bar{\mu}(i\rho\eta_3)\end{aligned}$$

Since $\bar{\mu}$ is an isomorphism, it completes the proof. □

For $\tau : D_2 \to \mathrm{Hom}(D_1, \alpha)$, by Lemma 4.7, it is equivalent to analyze the Milnor pairing $T : \pi_4(SO(4)) \otimes \pi_3(SO(5)) \to \Theta_8$. It was almost done in [6].

Proposition 5.5

$$\begin{aligned} T : \pi_4(SO(4)) \otimes \pi_3(SO(5)) &\to \Theta_8\\ (0, i\sigma) &\mapsto S^8\\ (\sigma\eta_3, i\sigma) &\mapsto S^8\\ (i\rho\eta_3, i\sigma) &\mapsto \Sigma^8\\ (\sigma\eta_3 + i\rho\eta_3, i\sigma) &\mapsto \Sigma^8\end{aligned}$$

Proof Combine [6, Lemma 1, Theorem 3] and Lemma 2.5 (2). □

Corollary 5.6 *The orbits of $D_1 \oplus \alpha$ under the action $\hat{\phi}$ are*

$$\{0\}, \{\Sigma^8\}, \{i\rho\eta_3, i\rho\eta_3 + \Sigma^8\}, \{\sigma\eta_3, \sigma\eta_3 + i\rho\eta_3, \sigma\eta_3 + \Sigma^8, \sigma\eta_3 + i\rho\eta_3 + \Sigma^8\}.$$

Proof of Theorem 1.2 (3) By Corollary 5.6, there are at most 4 elements in $\mathcal{S}$: $S^3 \times S^5$, $S^3 \times S^5 \# \Sigma^8$, $SU(3)$ and $M_{1,0}$. Since $\alpha \cong \Theta_8$ is invariant under the action of B, we see that 0 and Σ^8 must lie in different orbits. Thus $S^3 \times S^5$ is not diffeomorphic to $S^3 \times S^5 \# \Sigma^8$, and the theorem follows. □

6 Examples

As we have introduced in Section 1, for a 7-manifold of type (r), the total space of the S^1-bundle over it with Euler class u is an 8-manifold of type $(*)$. Conversely, if an 8-manifold of type $(*)$ admits a smooth free S^1-action, the orbit space must be a 7-manifold of type (r).

In [4], Escher and Ziller studied the topology of certain 7-manifolds of type (r), i.e., the total spaces of S^3-bundles over $\mathbb{C}P^2$ and S^1-bundles over S^2-bundles over $\mathbb{C}P^2$. Some of these manifolds are known to admit non-negatively curved metrics and Einstein metrics. However, there are some inaccurate statements in their paper, which are related to the following questions. What are the homotopy groups of these manifolds? What are the total spaces of S^1-bundles over them? We will take a closer look at these manifolds and clarify the answers to these questions.

6.1 S^3-bundles over $\mathbb{C}P^2$

It is well known that a 4-dimensional oriented vector bundle over $\mathbb{C}P^2$ is classified by its second Stiefel–Whitney class, first Pontryagin class and Euler class (cf. [3]). Up to isomorphism, they can be constructed as follows. Let $\mathrm{Vect}_{\mathbb{R}}^{n,+}(X)$ be the set of isomorphism classes of n-dimensional oriented real vector bundles over a space X. Then we have $\mathrm{Vect}_{\mathbb{R}}^{4,+}(S^4) \cong \pi_3(SO(4)) \cong \mathbb{Z}\{\alpha\} \oplus \mathbb{Z}\{\beta\}$ with

	α	β
e	0	ω_{S^4}
p_1	$4\omega_{S^4}$	$-2\omega_{S^4}$

where ω_{S^4} is the orientation cohomology class of S^4. Choose an embedded disk $D \cong D^4$ in $\mathbb{C}P^2$. Let $q : \mathbb{C}P^2 \to S^4$ be the map collapsing the outer of D to a point, and $v : \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ be the map shrinking the boundary of D to a point. Then all 4-dimensional vector bundles over $\mathbb{C}P^2$ can be constructed by the following two pullback squares:

Spin case: Nonspin case:

$$\begin{array}{ccc} q^*(\xi_{k,l}) & \longrightarrow & \xi_{k,l} \\ \downarrow & & \downarrow \\ \mathbb{C}P^2 & \xrightarrow{q} & S^4 \end{array} \qquad \begin{array}{ccc} v^*(\xi'_{k,l}) & \longrightarrow & \xi'_{k,l} \\ \downarrow & & \downarrow \\ \mathbb{C}P^2 & \xrightarrow{v} & \mathbb{C}P^2 \vee S^4 \end{array}$$

where $\xi_{k,l} = k\alpha + l\beta$, $\xi'_{k,l}|_{\mathbb{C}P^2} = \gamma \oplus \epsilon^2$, $\xi'_{k,l}|_{S^4} = \xi_{k,l}$, with γ the tautological line bundle over $\mathbb{C}P^2$.

Let $N_{k,l} = S(q^*(\xi_{k,l}))$ and $N'_{k,l} = S(v^*(\xi'_{k,l}))$, where $S(\xi)$ is the associated sphere bundle of the vector bundle ξ. Namely $N_{k,l}$ and $N'_{k,l}$ are linear S^3-bundles over $\mathbb{C}P^2$. Then $N_{k,l}$ and $N'_{k,l}$ are simply connected, and of type (0) if $l = 0$, or of type (r) $(r \geq 1)$ if $l \neq 0$. Moreover, we have $H^4(N_{k,l}) \cong H^4(N'_{k,l}) \cong \mathbb{Z}_l$.

We are interested in the topology of S^1-bundles over $N_{k,l}$ and $N'_{k,l}$. Let $E_{k,l}$ and $E'_{k,l}$ be the total space of the S^1-bundle over $N_{k,l}$ and $N'_{k,l}$, respectively, with Euler class a chosen generator of their second cohomology groups. Then we have the following pullback squares of fiber bundles:

$$
\begin{array}{ccccccccc}
 & & S^3 & & S^3 & & S^3 & & S^3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & E_{k,l} & \longrightarrow & N_{k,l} & \longrightarrow & S(\xi_{k,l}) & \longrightarrow & S(\gamma^4) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & S^5 & \xrightarrow{\pi} & \mathbb{C}P^2 & \xrightarrow{q} & S^4 & \xrightarrow{f_{k,l}} & BSO(4)
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & S^3 & & S^3 & & S^3 & & S^3 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & E'_{k,l} & \longrightarrow & N'_{k,l} & \longrightarrow & S(\xi'_{k,l}) & \longrightarrow & S(\gamma^4) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & S^5 & \xrightarrow{\pi} & \mathbb{C}P^2 & \xrightarrow{v} & \mathbb{C}P^2 \vee S^4 & \xrightarrow{\widehat{\gamma} \vee f_{k,l}} & BSO(4)
\end{array}
$$

Here γ^4 is the universal 4-dimensional vector bundle, and the map $\widehat{\gamma}$ and $f_{k,l}$ are classifying maps of $\gamma \oplus \epsilon^2$ and $\xi_{k,l}$, respectively. Thus $E_{k,l}$ and $E'_{k,l}$ can be regarded as total spaces of S^3-bundles over S^5. To determine them, we only need to determine the classifying maps.

We will identify $\pi_n(BSO(m))$ with $\pi_{n-1}(SO(m))$. The corresponding element of $\alpha \in \pi_{n-1}(SO(m))$ in $\pi_n(BSO(m))$ will be denoted by $\widetilde{\alpha}$. Under this convention and our notations in Lemma 2.1, we have $f_{k,l} = k\widetilde{i\rho} + l\widetilde{\sigma}$.

Proposition 6.1 (1) $f_{k,l}q\pi = 0$;

(2) $(\widehat{\gamma} \vee f_{k,l})v\pi = k\widetilde{i\rho\eta_3} + l\widetilde{\sigma\eta_3}$.

Proof (1) The fact $\mathbb{C}P^3/\mathbb{C}P^1 \simeq S^4 \cup_{q\pi} D^6$ together with $Sq^2 = 0 : H^4(\mathbb{C}P^3; \mathbb{Z}_2) \to H^6(\mathbb{C}P^3; \mathbb{Z}_2)$ leads to $q\pi = 0$.

(2) We have $\pi_5(\mathbb{C}P^2 \vee S^4) \cong \mathbb{Z}\{\pi\} \oplus \mathbb{Z}_2\{\eta_4\} \oplus \mathbb{Z}\{[\iota_2, \iota_4]\}$. It is easy to see that $v\pi = \pi \pm [\iota_2, \iota_4]$. Then

$$(\widehat{\gamma} \vee f_{k,l})v\pi = (\widehat{\gamma} \vee f_{k,l})(\pi \pm [\iota_2, \iota_4]) = \widehat{\gamma}\pi \pm [\widetilde{\epsilon_1}, k\widetilde{i\rho} + l\widetilde{\sigma\iota_3}],$$

where ϵ_1 generates $\pi_1(SO(4)) \cong \mathbb{Z}_2$. Note that $\widehat{\gamma}\pi = 0$, as γ can be extended to a bundle over $\mathbb{C}P^3$, i.e., the tautological line bundle over $\mathbb{C}P^3$. Thus

$$(\widehat{\gamma} \vee f_{k,l})v\pi = k[\widetilde{\epsilon_1}, \widetilde{i\rho}] + l[\widetilde{\epsilon_1}, \widetilde{\sigma}] = k\widetilde{\langle \epsilon_1, i\rho \rangle} + l\widetilde{\langle \epsilon_1, \sigma \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the Samelson product. Similar arguments as in the proof of [12, Theorem 1.1] show that $\langle \epsilon_1, \sigma \rangle = \sigma\eta_3$ and $\langle \epsilon_1, i\rho \rangle = i\rho\eta_3$. Hence the result follows. $\square$

Immediately, we have

Theorem 6.2 (1) $E_{k,l} \cong S^3 \times S^5$;

$$(2)\ E'_{k,l} \cong \begin{cases} S^3 \times S^5, & \textit{if } k, l \textit{ are even,} \\ M_{1,0}, & \textit{if } k \textit{ is odd, } l \textit{ is even,} \\ SU(3), & \textit{if } l \textit{ is odd.} \end{cases}$$

Consequently, all topological 8-manifolds of type (∗) admit infinitely many nonequivalent free S^1-actions.

Corollary 6.3 (1) $\pi_4(N_{k,l}) \cong \mathbb{Z}_2$;

$$(2)\ \pi_4(N'_{k,l}) \cong \begin{cases} 0, & \textit{if } l \textit{ is odd,} \\ \mathbb{Z}_2, & \textit{if } l \textit{ is even.} \end{cases}$$

Remark 6.4 We do not know whether $S^3 \times S^5 \# \Sigma^8$ admits a smooth free S^1-action.

6.2 S^1-bundles over S^2-bundles over $\mathbb{C}P^2$

Consider S^1-bundles over S^2-bundles over $\mathbb{C}P^2$. It is clear that S^1-bundles over these manifolds are just T^2-bundles over S^2-bundles over $\mathbb{C}P^2$. Recall that a T^2-bundle is determined by two elements in the second cohomology group of the base space. Different pairs of elements which are in the same orbit under $SL(2,\mathbb{Z})$-actions give isomorphic T^2-bundles. Since the second cohomology group of an S^2-bundle over $\mathbb{C}P^2$ is all isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, up to isomorphism, there is a unique T^2-bundle over each S^2-bundle over $\mathbb{C}P^2$ whose total space is simply connected. Thus the total space of an S^1-bundle over such a 7-manifold depends only on the based S^2-bundle over $\mathbb{C}P^2$ when the simply connected condition is required.

A 3-dimensional oriented vector bundle over $\mathbb{C}P^2$ is determined by its second Stiefel–Whitney class and first Pontryagin class, and can be constructed as follows:

Spin case: $\qquad\qquad$ Nonspin case:

$$\begin{array}{ccc} q^*(\xi_k) & \longrightarrow & \xi_k \\ \downarrow & & \downarrow \\ \mathbb{C}P^2 & \xrightarrow{q} & S^4 \end{array} \qquad\qquad \begin{array}{ccc} v^*(\xi'_k) & \longrightarrow & \xi'_k \\ \downarrow & & \downarrow \\ \mathbb{C}P^2 & \xrightarrow{v} & \mathbb{C}P^2 \vee S^4 \end{array}$$

where $\xi_k \oplus \epsilon \cong \xi_{k,0}$ and $\xi'_k \oplus \epsilon \cong \xi'_{k,0}$. Let $L_k = S(q^*(\xi_k))$ and $L'_k = S(v^*(\xi'_k))$. Gysin sequences show that $H^2(L_k) \cong \mathbb{Z}\{x_k\} \oplus \mathbb{Z}\{y_k\}$ and $H^2(L'_k) \cong \mathbb{Z}\{x'_k\} \oplus \mathbb{Z}\{y'_k\}$ with x_k and x'_k pulled back from a generator of $H^2(\mathbb{C}P^2)$.

Consider the S^1-bundles over L_k and L'_k with Euler class x_k and x'_k, and denote their total spaces as P_k and P'_k, respectively. Then we have the following pullback squares of fiber bundles:

$$\begin{array}{ccccccccc} & & S^2 & & S^2 & & S^2 & & S^2 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & P_k & \longrightarrow & L_k & \longrightarrow & S(\xi_k) & \longrightarrow & S(\gamma^3) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^5 & \xrightarrow{\pi} & \mathbb{C}P^2 & \xrightarrow{q} & S^4 & \xrightarrow{f_k} & BSO(3) \end{array}$$

$$\begin{array}{ccccccccc}
 & & S^2 & & S^2 & & S^2 & & S^2 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & P'_k & \longrightarrow & L'_k & \longrightarrow & S(\xi'_k) & \longrightarrow & S(\gamma^3) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^1 & \longrightarrow & S^5 & \xrightarrow{\pi} & \mathbb{C}P^2 & \xrightarrow{v} & \mathbb{C}P^2 \vee S^4 & \xrightarrow{\widehat{\gamma}\vee f_k} & BSO(3)
\end{array}$$

Similar arguments as the first example show that

$$f_k q\pi = 0, \quad (\widehat{\gamma} \vee f_k)v\pi = k\widetilde{\rho\eta_3}.$$

Thus

$$P_k \cong S^5 \times S^2, \quad P'_k \cong \begin{cases} S^5 \times S^2, & \text{if } k \text{ is even} \\ S^5 \widetilde{\times} S^2, & \text{otherwise.} \end{cases}$$

Since $S^5 \widetilde{\times} S^2 \simeq S^2 \cup_{\eta_2\eta_3} e^5 \cup e^7$, we have $\pi_4(S^5 \widetilde{\times} S^2) \cong \pi_4(S^2 \cup_{\eta_2\eta_3} e^5) = 0$. It follows that the total space of the S^1-bundle over $S^5 \widetilde{\times} S^2$ with Euler class a generator of $H^2(S^5 \widetilde{\times} S^2)$ is diffeomorphic to $SU(3)$. For $S^5 \times S^2$, the total space is $S^5 \times S^3$.

In general, let $P_{k,a,b}$ be the total space of the S^1-bundle over L_k with Euler class $ax_k + by_k$, where a, b are relatively prime integers. Similarly, let $P'_{k,a,b}$ be the total space of the S^1-bundle over L'_k with Euler class $ax'_k + by'_k$. Then $P_{k,a,b}$ and $P'_{k,a,b}$ are 7-manifolds of type (r). Let $R_{k,a,b}$ and $R'_{k,a,b}$ be the total space of the S^1-bundle over $P_{k,a,b}$ and $P'_{k,a,b}$, respectively, with Euler class a chosen generator of their second cohomology groups. Therefore, $R_{k,a,b}$ and $R'_{k,a,b}$ are T^2-bundles over L_k and L'_k, respectively. Summarizing the above discussion, we have the following:

Theorem 6.5 (1) $R_{k,a,b} \cong S^3 \times S^5$;

(2) $R'_{k,a,b} \cong \begin{cases} S^3 \times S^5, & \textit{if } k \textit{ is even,} \\ SU(3), & \textit{if } k \textit{ is odd.} \end{cases}$

Consequently, there are infinitely many nonequivalent free smooth T^2-actions on $S^3 \times S^5$ and $SU(3)$.

Corollary 6.6 (1) $\pi_4(P_{k,a,b}) \cong \mathbb{Z}_2$;

(2) $\pi_4(P'_{k,a,b}) \cong \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ \mathbb{Z}_2, & \textit{if } k \textit{ is even.} \end{cases}$

Remark 6.7 Finally, we focus on some statements in [4] related to our discussion:

(1) In [4, p. 27], they stated that $S^3 \times S^5$ and $SU(3)$ are the only homotopy types of an 8-manifold of type $(*)$ which admits free S^1-actions. It turns out that $M_{1,0}$ also admits free S^1-actions and is not homotopy equivalent to any of them.

(2) The remark in [4, p. 34] and the second remark in [4, p. 38] are true. Note that their notations are different from ours.

(3) The suspicion of the remark in [4, p. 45] is not the case, since π_4 is $\mathbb{Z}_2$ in all cases.

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