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Certain Cuntz Semigroup Properties of Certain Crossed Product C*-algebras

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Abstract We show that the following properties of the C*-algebras in a class Ω are inherited by simple unital C*-algebras in the class TAΩ: (1) (m, n)-decomposable, (2) weakly (m, n)-divisible, (3) weak Riesz interpolation. As an application, let A be an infinite dimensional simple unital C*-algebra such that A has one of the above-listed properties. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ also has the property under consideration.

Keywords C*-algebras, tracial approximation, Cuntz semigroup

MR(2010) Subject Classification 46L35, 46L05, 46L80

1 Introduction

The Elliott program for the classification of amenable C*-algebras might be said to have begun with the K-theoretical classification of AF algebras in [6]. Since then, many classes of C*-algebras have been classified by the Elliott invariant. Among them, one important class is the class of simple unital AH algebras without dimension growth (in the real rank zero case see [8], and in the general case see [9]). To axiomatize Elliott–Gong's decomposition theorem for AH algebras of real rank zero (classified by Elliott and Gong in [8]) and Gong's decomposition theorem [15] for simple AH algebras (classified by Elliott et al. in [9]), Lin introduced the concepts of TAF and TAI [21, 22]. Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C*-algebras with this abstract approximation property and certain additional properties are AH algebras without dimension growth. More precisely, Lin introduced the class of tracially approximate interval algebras (also called C*-algebras of tracial topological rank one). This axiomatization has proved to be very important in the classification of simple amenable C*-algebras. For example, it led to the classification of unital simple separable amenable C*-algebras with finite nuclear dimension in the UCT class (see [10, 16, 32]). The isomorphism theorem was established first for those separable amenable C*-algebras with generalized topological tracial rank at most one (see [16]). Simple C*-algebras with generalized tracial topological rank at most one have good regularity properties. There are three regularity properties of particular interest: tensorial absorption of the Jiang–Su algebra $\mathcal{Z}$, also called $\mathcal{Z}$-stability; finite nuclear dimension [36]; and strict

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comparison of positive elements. The last property can be reformulated as an algebraic property of the Cuntz semigroup, called almost unperforation. Toms and Winter have conjectured (see e.g. [12]) that these three fundamental properties are equivalent for all separable, simple, unital, amenable C*-algebras (and this has now almost completely been proved (see [2, 3, 18, 29, 31, 33–35])).

Inspired by Lin's tracial approximation by interval algebras in [22], Elliott and Niu in [11] considered the natural notion of tracial approximation by other classes of C*-algebras. Let Ω be a class of unital C*-algebras. Then the class of unital simple separable C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$, is defined as follows. A simple unital C*-algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there are a projection $p \in A$ and a C*-subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that

(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(2) $pxp \in_\varepsilon B$ for all $x \in F$,

(3) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The question of which properties pass from a class Ω to the class $\mathrm{TA}\Omega$ is interesting and sometimes important. In fact, the property of being of stable rank one, and the property that the strict order on projections is determined by traces, are important in the classification theorem of [16].

In [11], Elliott and Niu showed that the following properties of C*-algebras in a class Ω are inherited by a simple unital C*-algebras in the class $\mathrm{TA}\Omega$: (1) being stable finite, (2) having stable rank one, (3) having at least one tracial state, (4) the strict order on projections determined by traces, (5) any state of the order-unit K_0-group comes from a tracial state of the algebra, (6) if the restriction of a tracial state to the order-unit K_0-group is the average of two distinct states on the K_0-group, then it is the average of two distinct tracial states, (7) the property of being K_1-injective.

In [7], Elliott et al. showed that some regularity properties of C*-algebras in a class Ω are inherited by a simple unital C*-algebras in the class $\mathrm{TA}\Omega$.

In this paper, we show that the following Cuntz semigroup properties of unital C*-algebras in a class Ω are inherited by simple unital C*-algebras in the class $\mathrm{TA}\Omega$:

(1) (m, n)-decomposable,

(2) weakly (m, n)-divisible,

(3) weak Riesz interpolation property.

The Rokhlin property in ergodic theory was adapted to the context of von Neumann algebras by Connes in [4]. It was adapted by Herman and Ocneanu for automorphisms of UHF algebras in [17]. Rørdam [28] and Kishimoto [20] considered the Rokhlin property in a much more general C*-algebra context. More recently, Phillips and Osaka studied actions of a finite group and of the group $\mathbb{Z}$ of integers on certain simple C*-algebras which have a modified Rokhlin property (see Definition 2.7) in [25, 27].

In [14], the following result was obtained: Let Ω be a class of unital C*-algebras such that Ω is closed under passing to unital hereditary C*-subalgebras and tensoring by matrix algebras. Let $A \in \mathrm{TA}\Omega$ be an infinite dimensional simple unital C*-algebra. Suppose that $\alpha : G \to$

Aut(A) is an action of a finite group G on A which has the tracial Rokhlin property (see Definition 2.7). Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ belongs to TAΩ.

Using the results mentioned above, we get the following theorem: Let A be an infinite dimensional separable simple unital C*-algebra such that A has (m, n)-decomposable (respectively, weakly (m, n)-divisible, or weak Riesz interpolation). Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G, A, \alpha)$ has (m, n)-decomposable (respectively, weakly (m, n)-divisible, or weak Riesz interpolation property).

2 Preliminaries and Definitions

Let a and b be positive elements of a C*-algebra A. We write $[a] \leq [b]$ if there is a partial isometry $v \in A^{**}$ with $vv^* = P_a$ such that, for every $0 \leq c \in \mathrm{Her}(a)$, $cv \in A$ and $v^*cv \in \mathrm{Her}(b)$. ($[a] \leq [b]$ implies that a is Cuntz subequivalent to b, i.e. $a \lesssim b$. If A has stable rank one then, by [5], $[a] \leq [b]$ if $a \lesssim b$ but even in this case the preorder relation $[a] \leq [b]$ is not necessarily an order relation.) We write $[a] = [b]$ if, for some v as above, $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leq [b]$ if in addition there are n mutually orthogonal positive elements $b_1, b_2, \ldots, b_n \in \mathrm{Her}(b)$ such that $[a] \leq [b_i], i = 1, 2, \ldots, n$ (see [26, Definition 1.1], [24, Definition 3.2], or [23, Definition 3.5.2].)

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the C*-algebra of $n \times n$ matrices with entries elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic inductive limit of the sequence $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is the canonical embedding as the upper left-hand corner block. Let $\mathrm{M}_\infty(A)_+$ (respectively, $\mathrm{M}_n(A)_+$) denote the positive elements of $\mathrm{M}_\infty(A)$ (respectively, $\mathrm{M}_n(A)$). Given $a, b \in \mathrm{M}_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \lesssim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\lim_{n\to\infty} \|v_n b v_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \lesssim b$ and $b \lesssim a$. We write $\langle a \rangle$ for the equivalence class of a.

The object $\mathrm{W}(A) := \mathrm{M}_\infty(A)_+/\sim$ will be called the Cuntz semigroup of A (See [5]). Observe that any $a, b \in \mathrm{M}_\infty(A)_+$ are Cuntz equivalent to orthogonal elements $a', b' \in \mathrm{M}_\infty(A)_+$ (i.e., $a'b' = 0$), and so $\mathrm{W}(A)$ becomes an ordered semigroup when equipped with the addition operation

$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle$$

whenever $ab = 0$, and the order relation

$$\langle a \rangle \leq \langle b \rangle \Leftrightarrow a \lesssim b.$$

Let A be a stably finite unital C*-algebra. Recall that a positive element $a \in A$ is called purely positive if a is not Cuntz equivalent to a projection. This is equivalent to saying that 0 is an accumulation point of $\sigma(a)$ (recall that $\sigma(a)$ denotes the spectrum of a).

Given a in $\mathrm{M}_\infty(A)_+$ and $\varepsilon > 0$, we denote by $(a - \varepsilon)_+$ the element of $\mathrm{C}^*(a)$ corresponding (via the functional calculus) to the function $f(t) = \max(0, t - \varepsilon),\ t \in \sigma(a)$. By the functional calculus, it follows in a straightforward manner that $((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+$.

Theorem 2.1 ([1, 30]) *Let A be a stably finite* C*-*algebra.*

(1) *Let $a,\ b \in A_+$ and $\varepsilon > 0$ be such that $\|a - b\| < \varepsilon$. Then there is a contraction d in A with $(a - \varepsilon)_+ = dbd^*$.*

(2) *Let $a,\ p$ be positive elements in $\mathrm{M}_\infty(A)$ with p a projection. If $p \lesssim a$, then there is b in $\mathrm{M}_\infty(A)_+$ such that $bp = 0$ and $b + p \sim a$.*

(3) *The following conditions are equivalent*: *$(1)'a \lesssim b$, $(2)'$ for any $\varepsilon > 0$, $(a - \varepsilon)_+ \lesssim b$, $(3)'$ for any $\varepsilon > 0$, there is $\delta > 0$, such that $(a - \varepsilon)_+ \lesssim (b - \delta)_+$.*

(4) *Let a be a purely positive element of A (i.e., a is not Cuntz equivalent to a projection). Let $\delta > 0$, and let $f \in C_0(0, 1]$ be a non-negative function with $f = 0$ on $(\delta, 1)$, $f > 0$ on $(0, \delta)$, and $\|f\| = 1$. We have $f(a) \neq 0$ and $(a - \delta)_+ + f(a) \lesssim a$.*

Let Ω be a class of unital C*-algebras. In this paper, we shall study the class of simple unital C*-algebras which can be tracially approximated by C*-algebras in Ω, denoted by $\mathrm{TA}\Omega$.

Definition 2.2 ([11, 13, 22]) *A simple unital* C*-*algebra A is said to belong to the class $\mathrm{TA}\Omega$ if, for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any non-zero element $a \geq 0$, there exist a non-zero projection $p \in A$ and a* C*-*subalgebra B of A with $1_B = p$ and $B \in \Omega$ such that*

(1) *$\|xp - px\| < \varepsilon$ for all $x \in F$,*

(2) *$pxp \in_\varepsilon B$ for all $x \in F$,*

(3) *$[1 - p] \leq [a]$.*

Let $\mathfrak{I}^k$ denote the class of all unital C*-algebras which are unital hereditary subalgebras of C*-algebras of the form $C(X) \otimes F$, where X is a k-dimensional finite CW complex and F is a finite dimensional C*-algebra. A is said to have tracial topological rank at most k if $A \in \mathrm{TA}\mathfrak{I}^k$.

Lemma 2.3 ([11]) *If the class Ω is closed under tensoring with matrix algebras, or under passing to unital hereditary* C*-*subalgebras, then the class $\mathrm{TA}\Omega$ is closed under passing to matrix algebras, or unital hereditary* C*-*subalgebras.*

Definition 2.4 ([19]) *Let A be unital* C*-*algebra. Let $m, n \geq 1$ be integers. A is said to have (m, n)-decomposable if, for every u in $\mathrm{W}(A)$, any $\varepsilon > 0$, there exist elements $x_1, x_2, \ldots, x_m \in \mathrm{W}(A)$, such that $x_1 + x_2 + \cdots + x_m \leq u$ and $(u - \varepsilon)_+ \leq nx_j$, for all $j = 1, 2, \ldots, m$.*

Definition 2.5 ([19]) *Let A be unital* C*-*algebra. Let $m, n \geq 1$ be integers. A is said to have weakly (m, n)-divisible if, for every u in $\mathrm{W}(A)$, any $\varepsilon > 0$, there exist elements $x_1, x_2, \ldots, x_n \in \mathrm{W}(A)$, such that $mx_j \leq u$ for all $j = 1, 2, \ldots, n$ and $(u - \varepsilon)_+ \leq x_1 + x_2 + \cdots + x_n$.*

Definition 2.6 *Let A be unital* C*-*algebra. A is said to have weak Riesz interpolation provided $\mathrm{W}(A)$ satisfies the conditions that given any $x_1,\ x_2,\ y_1,\ y_2$ in $\mathrm{W}(A)$, any $\varepsilon > 0$, such that $x_i \leq y_j$ for $1 \leq i \leq 2$, $1 \leq j \leq 2$, there exists z such that $(x_i - \varepsilon)_+ \leq z \leq y_j$ for $1 \leq i \leq 2$, $1 \leq j \leq 2$.*

Definition 2.7 ([27]) *Let A be a simple infinite dimensional unital* C*-*algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the tracial Rokhlin property if, for any finite set $F \subseteq A$, any $\varepsilon > 0$, and any non-zero positive element $b \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that*

(1) *$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,*

(2) *$\|e_g d - de_g\| < \varepsilon$ for all $g \in G$ and all $d \in F$,*

(3) *with* $e = \sum_{g\in G} e_g$, *the projection* $1-e$ *is Murray–von Neumann equivalent to a projection in the hereditary* C*-*subalgebra of* A *generated by* b,

(4) $\|ebe\| \geq \|b\| - \varepsilon$.

The following lemma is obvious, and we omit the proof.

Lemma 2.8 *The property of* (m,n)-*decomposable* (*respectively, weakly* (m,n)-*divisible, or weak Riesz interpolation*) *is preserved under tensoring with matrix algebras and under passing to unital hereditary* C*-*subalgebras.*

3 The Main Results

Theorem 3.1 *Let* Ω *be a class of stably finite unital* C*-*algebras such that for any* $B \in \Omega$, B *is weakly* (m,n)-*divisible* ($m \neq n$). *Then* A *is weakly* (m,n)-*divisible for any simple unital* C*-*algebra* $A \in \mathrm{TA}\Omega$.

Proof By Lemma 2.3, enlarging the class Ω, we may suppose it is closed under passing to matrix algebras and unital hereditary C*-subalgebras (i.e., Morita equivalent C*-algebras).

Let $u \in \mathrm{M}_\infty(A)_+$. For any $\varepsilon > 0$, we need to show that there exist $x_1, x_2, \ldots, x_n \in \mathrm{M}_\infty(A)_+$ such that $x_j \oplus x_j \oplus \cdots \oplus x_j \precsim u$ for all $1 \leq j \leq n$, where x_j repeats m times, and $(u - 20\varepsilon)_+ \precsim \bigoplus_{i=1}^n x_i$.

(Here, and elsewhere, by $x \oplus y$ for $x, y \in \mathrm{M}_\infty(A)_+$ we mean $x + y'$ where $y' \in \mathrm{M}_\infty(A)_+$, $y' \sim y$, $y'x = 0$.)

We may assume that $u \in A_+$ and we divide the proof into two cases.

Case (I) We assume that u is Cuntz equivalent to a projection. There exists projection $r \in A_+$ such that $u \sim r$, we may assume $u = r$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{r\}$, any $\varepsilon' > 0$, there exist a sub-C*-subalgebra B of A and a nonzero projection $p \in A$ with $B \in \Omega$ and $1_B = p$, such that

(1) $\|rp - pr\| < \varepsilon'$,

(2) $prp \in_{\varepsilon'} B$.

By (1) and (2), there exist projections $r_1 \in B$ and $r_2 \in (1-p)A(1-p)$ such that $\|r - r_1 - r_2\| < 4\varepsilon'$.

Since $r_1 \in B$ and $B \in \Omega$, there exist $x_1', x_2', \ldots, x_n' \in \mathrm{M}_\infty(B)_+$ such that $x_j' \oplus x_j' \oplus \cdots \oplus x_j' \precsim r_1$ where x_j' repeat m times, and $r_1 = (r_1 - \varepsilon)_+ \precsim \bigoplus_{i=1}^n x_i'$.

We prove this part by three steps.

First, If $x_1', x_2', \ldots, x_n' \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_1 \sim \bigoplus_{i=1}^n x_i'$, then there exist some j and a nonzero projection q such that $(x_j' \oplus q) \oplus (x_j' \oplus q) \oplus \cdots \oplus (x_j' \oplus q) \precsim r_1$ where $x_j' \oplus q$ repeat m times, otherwise, this contradicts the stable finiteness of A (since $m \neq n$ and C*-algebras in Ω are stably finite).

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

(1)$'$ $\|r_2 t - t r_2\| < \varepsilon'$,

(2)$'$ $t r_2 t \in_{\varepsilon'} D$,

(3)$'$ $[1 - p - t] \leq [q]$.

By $(1)'$ and $(2)'$, there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x_1'', x_2'', \dots, x_n'' \in \mathrm{M}_\infty(D)_+$ such that $x_j'' \oplus x_j'' \oplus \cdots \oplus x_j'' \lesssim r_3$ where x_j'' repeat m times and $r_3 = (r_3 - \varepsilon)_+ \lesssim \bigoplus_{i=1}^n x_i''$.

Therefore we have

$$\begin{aligned}
&((x_j' \oplus q) + x_j'') \oplus ((x_j' \oplus q) + x_j'') \oplus \cdots \oplus ((x_j' \oplus q) + x_j'') \\
&\lesssim r_1 + r_4 + r_3 = r \\
&\lesssim r_1 + r_3 \oplus q \\
&\lesssim (x_j' \oplus q) \bigoplus_{i=1, i\neq j}^{n} x_i' \bigoplus_{i=1}^{n} x_i'' \\
&\lesssim (x_j' \oplus q \oplus x_j'') \bigoplus_{i=1, i\neq j}^{n} (x_i' \oplus x_i''),
\end{aligned}$$

and

$$\begin{aligned}
&(x_i' \oplus x_i'') \oplus (x_i' \oplus x_i'') \oplus \cdots \oplus (x_i' \oplus x_i'') \\
&\lesssim r_1 + r_4 + r_3 = r \\
&\lesssim r_1 + r_3 + q \\
&\lesssim (x_j' \oplus q) \bigoplus_{i=1, i\neq j}^{n} (x_i') \bigoplus_{i=1}^{n} x_i'' \\
&\lesssim (x_j' \oplus q \oplus x_j'') \bigoplus_{i=1, i\neq j}^{n} (x_i' + x_i''),
\end{aligned}$$

for all $i \neq j$ and $1 \leq i \leq n$ where $(x_i' \oplus x_i'')$ repeats m times.

Second, if $x_1', x_2', \dots, x_k' \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_1 < \bigoplus_{i=1}^k x_i'$. Then there exists a nonzero projection s such that $r_1 \oplus s \lesssim \bigoplus_{i=1}^k x_i'$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

$(1)''$ $\|r_2 t - t r_2\| < \varepsilon'$,

$(2)''$ $t r_2 t \in_{\varepsilon'} D$,

$(3)''$ $[1-p-t] \leq [s]$.

By $(1)''$ and $(2)''$, there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x_1'', x_2'', \dots, x_n'' \in \mathrm{M}_\infty(D)_+$ such that $x_j'' \oplus x_j'' \oplus \cdots \oplus x_j'' \lesssim r_3$ where x_j'' repeat m times and $r_3 = (r_3 - \varepsilon)_+ \lesssim \bigoplus_{i=1}^n x_i''$. Therefore we have

$$\begin{aligned}
&(x_i' \oplus x_i'') \oplus (x_i' \oplus x_i'') \oplus \cdots \oplus (x_i' \oplus x_i'') \\
&\lesssim r_1 + r_4 + r_3 = r \\
&\lesssim r_1 + r_3 \oplus s \\
&\lesssim \bigoplus_{i=1}^{n} x_j' \bigoplus_{i=1}^{n} x_i''
\end{aligned}$$

$$\lesssim \bigoplus_{i=1}^{n}(x'_j \oplus x''_j),$$

for all $1 \leq j \leq n$, where $(x'_i \oplus x''_i)$ repeat m times.

Third, we assume that there is a purely positive element x'_1. Since $r_1 \lesssim \bigoplus_{i=1}^n x'_i$, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $r_1 = (r_1 - \varepsilon)_+ \lesssim (x'_1 - \delta)_+ \bigoplus_{i=2}^n x'_i$,

By Theorem 2.1, there exists a nonzero positive element d such that $(x'_1 - \delta)_+ + d \lesssim x'_1$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

(1)‴ $\|r_2 t - t r_2\| < \varepsilon'$,

(2)‴ $t r_2 t \in_{\varepsilon'} D$,

(3)‴ $[1-p-t] \leq [d]$.

By (1)‴ and (2)‴, there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x''_1, x''_2, \ldots, x''_n \in \mathrm{M}_\infty(D)_+$ such that $x''_j \oplus x''_j \oplus \cdots \oplus x''_j \lesssim r_3$ where x''_j repeat m times and $r_3 \lesssim \bigoplus_{i=1}^n x''_i$.

Therefore we have

$$\begin{aligned}
&(x'_i \oplus x''_i) \oplus (x'_i \oplus x''_i) \oplus \cdots \oplus (x'_i \oplus x''_i)\\
&\quad \lesssim r_1 + r_4 + r_3 = r\\
&\quad \lesssim r_1 \oplus r_3 \oplus d\\
&\quad \lesssim (x'_1 - \delta)_+ \oplus d \bigoplus_{i=1}^{n} x'_j \bigoplus_{i=1}^{n} x''_i\\
&\quad \lesssim \bigoplus_{i=1}^{n}(x'_j \oplus x''_j),
\end{aligned}$$

for all $1 \leq j \leq n$, where $(x'_i \oplus x''_i)$ repeat m times.

Case (II) We suppose that u is not Cuntz equivalent to a projection.

We need to show that for any $u \in \mathrm{M}_\infty(A)_+$, any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in \mathrm{M}_\infty(A)_+$ such that $x_j \oplus x_j \oplus \cdots \oplus x_j \lesssim u$ for all $1 \leq j \leq n$, where x_j repeat m times, and $(u - 20\varepsilon)_+ \lesssim \bigoplus_{i=1}^n x_i$.

By Theorem 2.1, for $\varepsilon > 0$, there is a non-zero positive element d such that $(u-\varepsilon)_+ + d \lesssim u$.

For $\varepsilon > 0$, there exists $\delta' > 0$ with $\delta' < \varepsilon$ such that $(a-\varepsilon)_+ + (b-\varepsilon)_+ \lesssim (a+b-\delta')_+$ for any $a, b \in A_+$.

Since $A \in \mathrm{TA}\Omega$, with $G = \{(u-\varepsilon)_+,\ d,\ u\}$, for $\varepsilon' > 0$, with $\varepsilon' < \delta'$, there exist a C*-subalgebra C of A and a non-zero projection $r \in A$ with $C \in \Omega$ and $1_C = r$ such that

(1) $\|xr - rx\| < \varepsilon'/3$ for any $x \in G$,

(2) $rxr \in_{\varepsilon'/3} C$ for any $x \in G$.

By (1) and (2), there are $u_1, d_1 \in C$ and $u_2 \in (1-r)A(1-r)$ such that

$$\|u - u_1 - u_2\| < 3\varepsilon',$$

and

$$\|(u-\varepsilon)_+ - (u_1 - \varepsilon)_+ - (u_2 - \varepsilon)_+\| < 3\varepsilon'.$$

Therefore we have $(u_1-2\varepsilon)_+ + (u_2-2\varepsilon)_+ \lesssim (u-\varepsilon)_+$.

Since $(u_1-3\varepsilon)_+ + (d_1-\varepsilon)_+ \in C$ and $C\in\Omega$, for $\delta'>0$, there exist $x'_1, x'_2, \ldots, x'_n \in \mathrm{M}_\infty(B)_+$ such that $x'_j\oplus x'_j\oplus\cdots\oplus x'_j \lesssim (u_1-3\varepsilon)_+ + (d_1-\varepsilon)_+$ where x'_j repeat m times, and $(u_1-4\varepsilon)_+ + (d-2\varepsilon)_+ \lesssim (((u_1-3\varepsilon)_+ + (d_1-\varepsilon)_+)-\delta')_+ \lesssim \bigoplus_{i=1}^n x'_i$.

Since $(1-r)A(1-r)\in \mathrm{TA}\Omega$, with $F=\{u_2\}$ and $\varepsilon'>0$, with $\varepsilon'<\delta'$, there exist a C*-subalgebra E of $(1-r)A(1-r)$ and a non-zero projection $t\in(1-r)A(1-r)$ with $E\in\Omega$ and $1_E=t$ such that

$(1)'$ $\|tu_2-u_2t\|<\varepsilon'$,

$(2)'$ $tu_2t\in_{\varepsilon'} E$,

$(3)'$ $[1-r-t]\le[(d_1-2\varepsilon)_+]$.

By $(1)'$ and $(2)'$, there are $u_3\in E$ and $u_4\in(1-r-t)A(1-r-t)$ such that

$$\|u_2-u_3-u_4\|<3\varepsilon'.$$

Then $(u_3-3\varepsilon)_+ + (u_4-3\varepsilon)_+ \lesssim (u_2-2\varepsilon)_+$.

Since $(u_3-3\varepsilon)_+\in E$ and $E\in\Omega$, there exist there exist $x''_1, x''_2, \ldots, x''_n\in \mathrm{M}_\infty(E)_+$ such that $x''_j\oplus x''_j\oplus\cdots\oplus x''_j\lesssim (u_3-3\varepsilon)_+$ where x''_j repeat m times and $(u_3-4\varepsilon)_+\lesssim\bigoplus_{i=1}^n x''_i$.

Therefore, we have

$$\begin{aligned}
&x'_j\oplus x'_j\oplus\cdots\oplus x'_j + x''_j\oplus x''_j\oplus\cdots\oplus x''_j\\
&\quad\lesssim (u_1-3\varepsilon)_+ + (d_1-\varepsilon)_+ \oplus ((u_3-3\varepsilon)_+ + (u_4-3\varepsilon)_+)\\
&\quad\lesssim (u_1-3\varepsilon)_+ + (d_1-\varepsilon)_+ \oplus (u_2-2\varepsilon)_+\\
&\quad\lesssim (u-\varepsilon)_+ + (d_1-\varepsilon)_+ \lesssim (u-\varepsilon)_+ + d\\
&\quad\lesssim u,
\end{aligned}$$

for all $1\le j\le n$, and $x'_j,\ x''_j$ repeat m times, and

$$\begin{aligned}
(u-20\varepsilon)_+ &\lesssim (u_1-10\varepsilon)_+ + (u_2-10\varepsilon)_+\\
&\lesssim (u_1-10\varepsilon)_+ + (u_3-4\varepsilon)_+ + (u_4-4\varepsilon)_+\\
&\lesssim (u_1-4\varepsilon')_+ + (d_1-2\varepsilon)_+ + (u_3-4\varepsilon)_+\\
&\lesssim \bigoplus_{i=1}^n x'_i \oplus \bigoplus_{i=1}^n x''_i.
\end{aligned}$$

□

Theorem 3.2 ([14]) *Let Ω be a class of unital C*-algebras which is closed under passing to unital hereditary C*-subalgebras and closed under passing to tensoring with matrix algebras. Let $A\in\mathrm{TA}\Omega$ be an infinite dimensional simple unital C*-algebra. Suppose that $\alpha: G\to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product algebra $\mathrm{C}^*(G,A,\alpha)$ belongs to $\mathrm{TA}\Omega$.*

Corollary 3.3 *Let A be a unital simple C*-algebra such that A is weakly (m,n)-divisible. Suppose that $\alpha: G\to\mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra $\mathrm{C}^*(G,A,\alpha)$ is weakly (m,n)-divisible.*

Proof This follows from Theorems 3.1 and 3.2. □

Theorem 3.4 *Let Ω be a class of stably finite unital C^*-algebras such that for any $B \in \Omega$, B is (m,n)-decomposable $(m \neq n)$. Then A is (m,n)-decomposable for any simple unital C^*-algebra $A \in \mathrm{TA}\Omega$.*

Proof By Lemma 2.3, enlarging the class Ω, we may suppose it is closed under passing to matrix algebras and unital hereditary C*-subalgebras (i.e., Morita equivalent C*-algebras).

We need to show that for any $u \in \mathrm{M}_\infty(A)_+$, there exists $x_1, x_2, \ldots, x_m \in \mathrm{M}_\infty(A)_+$ such that $\bigoplus_{i=1}^m x_i \precsim u$ and and $(u - 20\varepsilon)_+ \precsim x_j \oplus x_j \oplus \cdots \oplus x_j$ for some $\varepsilon > 0$ and for all $1 \leq j \leq m$, where x_j repeat n times.

We may assume $u \in A_+$ and we divide the proof into two cases.

Case (I) We assume that u is Cuntz equivalent to a projection. There exists a projection $r \in A_+$ such that $u \sim r$. We may assume $u = r$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{r\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-subalgebra B of A and a nonzero projection $p \in A$ with $B \in \Omega$ and $1_B = p$, such that

(1) $\|rp - pr\| < \varepsilon'$,

(2) $prp \in_{\varepsilon'} B$.

By (1) and (2), there exist projections $r_1 \in B$ and $r_2 \in (1-p)A(1-p)$ such that $\|r - r_1 - r_2\| < 4\varepsilon$.

Since $r_1 \in B$ and $B \in \Omega$, there exist $x_1', x_2', \ldots, x_n' \in M_\infty(B)_+$ such that such that $\bigoplus_{i=1}^m x_i' \precsim r_1$ and and $r_1 = (r_1 - \varepsilon)_+ \precsim x_j' \oplus x_j' \oplus \cdots \oplus x_j'$ for some $\varepsilon > 0$ and for all $1 \leq j \leq m$, where x_j' repeat n times.

We prove this part by three steps.

(I) If $x_1', x_2', \ldots, x_m' \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_1 \sim x_j' \oplus x_j' \oplus \cdots \oplus x_j'$ for all $1 \leq j \leq m$ where x_j' repeat n times. Then there exists a nonzero projection q such that $\bigoplus_{j=1}^m (x_j' \oplus q) \precsim r_1$, otherwise, this contradicts the stable finiteness of A (since $m \neq n$ and C*-algebras in Ω are stably finite).

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

(1) $\|r_2 t - t r_2\| < \varepsilon'$,

(2) $t r_2 t \in_{\varepsilon'} D$,

(3) $[1 - p - t] \leq [q]$.

By (1) and (2), there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x_1'', x_2'', \ldots, x_k'' \in \mathrm{M}_\infty(D)_+$ such that $\bigoplus_{j=1}^m x_i'' \precsim r_3$ and $r_3 = (r_3 - \varepsilon)_+ \precsim x_j'' \oplus x_j'' \oplus \cdots \oplus x_j''$ for all $1 \leq j \leq m$, where x_j'' repeat n times.

Therefore we have

$$\begin{aligned} \bigoplus_{i=1}^m ((x_j' \oplus q) \oplus x_j'') &\precsim r_1 + r_4 + r_3 = r \\ &\precsim r_1 + r_3 \oplus q \\ &\precsim ((x_j' \oplus q) \oplus x_j'') \oplus \cdots \oplus ((x_j' \oplus q) \oplus x_j''), \end{aligned}$$

for all $1 \leq j \leq m$, where $(x_j' \oplus q) \oplus x_j''$ repeat n times.

(II.I) If $x'_1, x'_2, \ldots, x'_m \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_1 < x'_j \oplus x'_j \oplus \cdots \oplus x'_j$, for all $1 \le j \le m$ where x'_j repeat n times. Then there exists a nonzero projection s such that $r_1 \oplus s \precsim x'_j \oplus x'_j \oplus \cdots \oplus x'_j$ for all $1 \le j \le m$ where x'_j repeat n times.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

$(1)'$ $\|r_2 t - t r_2\| < \varepsilon'$,

$(2)'$ $t r_2 t \in_{\varepsilon'} D$,

$(3)'$ $[1-p-t] \le [s]$.

By $(1)'$ and $(2)'$, there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x''_1, x''_2, \ldots, x''_k \in \mathrm{M}_\infty(D)_+$ such that $\bigoplus_{i=1}^m x''_i \precsim r_3$, $r_3 = (r_3 - \varepsilon)_+ \precsim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $1 \le j \le m$, where x''_j repeat n times.

Therefore we have

$$\begin{aligned}\bigoplus_{i=1}^{k}(x'_j \oplus x''_j) &\precsim r_1 + r_4 + r_3 = r\\ &\precsim r_1 + r_3 + s\\ &\precsim (x'_j \oplus x''_j) \oplus (x'_j \oplus x''_j) \oplus \cdots \oplus (x'_j \oplus x''_j),\end{aligned}$$

for all $1 \le j \le m$, where $x'_j \oplus x''_j$ repeat n times.

(II.II) If $x'_1, x'_2, \ldots, x'_m \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_1 < x'_1 \oplus x'_1 \oplus \cdots \oplus x'_1$, and $r_1 \sim x'_j \oplus x'_j \oplus \cdots \oplus x'_j$ for all $2 \le j \le m$ where x'_j repeat n times, then there exists a nonzero projection s such that $r_1 \oplus s \precsim x'_1 \oplus x'_1 \oplus \cdots \oplus x'_1$ where x'_1 repeats n times.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

$(1)'$ $\|r_2 t - t r_2\| < \varepsilon'$,

$(2)'$ $t r_2 t \in_{\varepsilon'} D$,

$(3)'$ $[1-p-t] \le [s]$.

By $(1)'$ and $(2)'$, there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x''_1, x''_2, \ldots, x''_k \in \mathrm{M}_\infty(D)_+$ such that $\bigoplus_{i=1}^m x''_i \precsim r_3$, $r_3 = (r_3 - \varepsilon)_+ \precsim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $1 \le j \le m$, where x''_j repeat n times.

(II.II.I) If $x''_1, x''_2, \ldots, x''_m \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_3 \sim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $1 \le j \le m$ where x''_j repeat n times. Then there exists a nonzero projection q such that $\bigoplus_{j=1}^m (x''_j \oplus q) \precsim r_1$, otherwise, this contradicts the stable finiteness of A (since $m \ne n$ and C*-algebras in Ω are stably finite).

Therefore we have

$$\begin{aligned}\bigoplus_{i=1}^{m}((x'_j \oplus q) \oplus x''_j) &\precsim r_1 + r_4 + r_3 = r\\ &\precsim r_1 + r_3 \oplus q\end{aligned}$$

$$\precsim ((x'_j \oplus q) \oplus x''_j) \oplus \cdots \oplus ((x'_j \oplus q) \oplus x''_j),$$

for all $1 \leq j \leq m$, where $(x'_j \oplus q) + x''_j$ repeat n times.

(II.II.II) If $x''_1, x''_2, \ldots, x''_m \in \mathrm{M}_\infty(B)_+$ are all projections, and $r_3 \sim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $1 \leq j \leq m$ where x''_j repeat n times. Then there exists a nonzero projection s such that $r_3 \oplus s \precsim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $1 \leq j \leq m$, where x''_j repeat n times.

Since $(1-p-t)A(1-p-t) \in \mathrm{TA}\Omega$, for $F = \{r_4\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra E of $(1-p-t)A(1-p-t)$ and a nonzero projection $t \in (1-p-t)A(1-p-t)$ with $E \in \Omega$ and $1_E = t'$, such that

(1)′ $\|r_2t' - t'r_2\| < \varepsilon'$,

(2)′ $t'r_2t' \in_{\varepsilon'} E$,

(3)′ $[1 - p - t'] \leq [s]$.

By (1)′ and (2)′, there exist projections $r_5 \in E$ and $r_6 \in (1 - p - t)E(1 - p - t)$ such that $\|r_4 - r_5 - r_6\| < 4\varepsilon'$.

Since $r_5 \in E$ and $E \in \Omega$, there exist $x'''_1, x'''_2, \ldots, x'''_m \in \mathrm{M}_\infty(E)_+$ such that $\bigoplus_{i=1}^m x'''_i \precsim r_5$, $r_5 = (r_5 - \varepsilon)_+ \precsim x'''_j \oplus x'''_j \oplus \cdots \oplus x'''_j$ for all $1 \leq j \leq m$, where x'''_j repeat n times.

Therefore we have

$$\begin{aligned}\bigoplus_{i=1}^{m}(x'_j \oplus x''_j \oplus x'''_j) &\precsim r_1 + r_3 + r_5 + r_6 = r\\ &\precsim r_1 + r_3 + r_5 \oplus s\\ &\precsim (x'_j \oplus x''_j \oplus x'''_j) \oplus \cdots \oplus (x'_j \oplus x''_j \oplus x'''_j),\end{aligned}$$

for all $1 \leq j \leq m$, where $x'_j \oplus x''_j \oplus x'''_j$ repeat n times.

(II.II.III) If $x''_1, x''_2, \ldots, x''_m \in \mathrm{M}_\infty(B)_+$ are Cuntz equivalent to projections, and $r_3 < x''_1 \oplus x''_1 \oplus \cdots \oplus x''_1$, $r_3 \sim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ for all $2 \leq j \leq m$ where x''_j repeat n times. Then there exists a nonzero projection s such that $r_3 \oplus s' \precsim x''_1 \oplus x''_1 \oplus \cdots \oplus x''_1$.

Since $(1-p-t)A(1-p-t) \in \mathrm{TA}\Omega$, for $F = \{r_4\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p-t)A(1-p-t)$ and a nonzero projection $t' \in (1-p-t)A(1-p-t)$ with $E \in \Omega$ and $1_E = t'$, such that

(1)′ $\|r_2t' - t'r_2\| < \varepsilon'$,

(2)′ $t'r_2t' \in_{\varepsilon'} D$,

(3)′ $[1 - p - t'] \leq [s']$.

By (1)′ and (2)′, there exist projections $r_5 \in E$ and $r_6 \in (1 - p - t)A(1 - p - t)$ such that $\|r_4 - r_5 - r_6\| < 4\varepsilon'$.

Since $r_5 \in E$ and $E \in \Omega$, there exist $x'''_1, x'''_2, \ldots, x'''_m \in \mathrm{M}_\infty(E)_+$ such that $\bigoplus_{i=1}^m x'''_i \precsim r_5$, $r_5 = (r_5 - \varepsilon)_+ \precsim x'''_j \oplus x'''_j \oplus \cdots \oplus x'''_j$ for all $1 \leq j \leq m$, where x'''_j repeat n times.

We repeat the steps (II.II.I), (II.II.II), (II.II.III) for $x'''_1, x'''_2, \ldots, x'''_m \in \mathrm{M}_\infty(E)_+$, and the steps (II.II.I), (II.II.II), (II.II.III), there exist $x'_1, x'_2, \ldots, x'_m$, $x''_1, x''_2, \ldots, x''_m$, $\ldots, {x_1}^m, {x_2}^m, \ldots, {x_m}^m$ and $s, s', \ldots, s^m$, take g such that $g \precsim s$, $g \precsim s', \ldots, g \precsim s^m$, and we have

$$\begin{aligned}&\bigoplus_{i=1}^{m} x'_i \bigoplus_{i=1}^{m} x''_i \cdots \bigoplus_{i=1}^{m} {x_i}^m\\ &\quad\precsim r_1 + r_3 + r_5 + \cdots + r_{2m} = r\end{aligned}$$

$$\begin{aligned}&\lesssim r_1 + r_3 + r_5 + \cdots + r_{2m-1} \oplus g\\&\lesssim (x'_j \oplus x''_j \cdots \oplus x_j{}^m) \oplus (x'_j \oplus x''_j \cdots \oplus x_j{}^m) \oplus \cdots \oplus (x'_j \oplus x''_j \cdots \oplus x_j{}^m),\end{aligned}$$

for all $1 \le j \le m$, where $(x'_j \oplus x''_j \cdots \oplus x_j{}^m)$ repeat n times.

(III) We assume that at least one of $x'_1, x'_2, \ldots, x'_m$ is not Cuntz equivalent to a projection.

(III.I) We may assume that all $x'_1, x'_2, \ldots, x'_m$ are not Cuntz equivalent to projections.

By Theorem 2.1, there exists a nonzero positive element d such that $(x'_j - \varepsilon)_+ + d \lesssim x'_j$ for all $1 \le j \le m$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

(1) $\|r_2 t - t r_2\| < \varepsilon'$,

(2) $t r_2 t \in_{\varepsilon'} D$,

(3) $[1 - p - t] \le [s]$.

By (1) and (2), there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x''_1, x''_2, \ldots, x''_k \in \mathrm{M}_\infty(D)_+$ such that $\bigoplus_{i=1}^m x''_i \lesssim r_3$ and and $r_3 = (r_3 - \varepsilon)_+ \lesssim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ and for all $1 \le j \le n$, where x''_j repeat n times.

Therefore we have

$$\begin{aligned}&(x''_j \oplus x'_j) \oplus (x''_j \oplus x'_j) \oplus \cdots \oplus (x''_j \oplus x'_j)\\&\quad\lesssim r_1 + r_3 + r_4 = r\\&\quad\lesssim r_1 \oplus d \oplus r_3\\&\quad\lesssim \bigoplus_{j=1}^m (x'_j - \varepsilon)_+ \oplus d \bigoplus_{j=1}^m x''_j\\&\quad\lesssim \bigoplus_{j=1}^m (x''_j \oplus x'_j).\end{aligned}$$

(III.II) We may assume that there exists x'_j for some j such that $r_1 \sim x'_j \oplus x'_j \oplus \cdots \oplus x'_j$, where x'_j repeat n times.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $F = \{r_2\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra D of $(1-p)A(1-p)$ and a nonzero projection $t \in (1-p)A(1-p)$ with $D \in \Omega$ and $1_D = t$, such that

(1) $\|r_2 t - t r_2\| < \varepsilon'$,

(2) $r_2 t r_2 \in_{\varepsilon'} D$.

By (1) and (2), there exist projections $r_3 \in D$ and $r_4 \in (1-p-t)A(1-p-t)$ such that $\|r_2 - r_3 - r_4\| < 4\varepsilon'$.

Since $r_3 \in D$ and $D \in \Omega$, there exist $x''_1, x''_2, \ldots, x''_k \in \mathrm{M}_\infty(D)_+$ such that $\bigoplus_{i=1}^m x''_i \lesssim r_3$ and and $r_3 = (r_3 - \varepsilon)_+ \lesssim x''_j \oplus x''_j \oplus \cdots \oplus x''_j$ and for all $1 \le j \le n$, where x''_j repeat n times.

If all $x''_1, x''_2, \ldots, x''_k \in \mathrm{M}_\infty(D)_+$ are not purely positive elements. We repeat Step (II), and get the result.

If there exists x'''_1 which is not Cuntz equivalent to a projection. By Theorem 2.1, there exists a nonzero positive element d such that $(x'''_1 - \varepsilon)_+ + d \lesssim x'''_1$.

Since $(1-p-t)A(1-p-t) \in \mathrm{TA}\Omega$, for $F = \{r_4\}$, any $\varepsilon' > 0$, with ε' sufficiently small, there exist a sub-C*-algebra E of $(1-p-t)A(1-p-t)$ and a nonzero projection $t' \in (1-p-t)A(1-p-t)$ with $E \in \Omega$ and $1_E = t'$, such that

$(1)'$ $\|r_2t' - t'r_2\| < \varepsilon'$,

$(2)'$ $t'r_2t' \in_{\varepsilon'} E$,

$(3)'$ $[1-p-t] \leq [s]$.

By $(1)'$ and $(2)'$, there exist projections $r_5 \in E$ and $r_6 \in (1-p-t-t')A(1-p-t-t')$ such that $\|r_4 - r_5 - r_6\| < 4\varepsilon'$.

Since $r_5 \in E$ and $E \in \Omega$, there exist $x_1''', x_2''', \ldots, x_m''' \in \mathrm{M}_\infty(E)_+$ such that $\bigoplus_{i=1}^m x_i''' \lesssim r_5$, $r_5 = (r_5 - \varepsilon)_+ \lesssim x_j''' \oplus x_j''' \oplus \cdots \oplus x_j'''$ for all $1 \leq j \leq m$, where x_j''' repeat n times.

If $x_1''', x_2''', \ldots, x_m''' \in \mathrm{M}_\infty(E)_+$ are not purely positive elements, and we repeat Step (II), and one of $x_1''', x_2''', \ldots, x_m''' \in \mathrm{M}_\infty(E)_+$ is not Cuntz equivalent to a projection, we repeat Step (III.I). Inductively, we repeat Step (II) and Step (III.I), there exist $x_1', x_2', \ldots, x_m'$, $x_1'', x_2'', \ldots, x_m'', \ldots$, ${x_1}^m, {x_2}^m, \ldots, {x_m}^m$ and $d, d', \ldots, d^m$, take g such that $g \lesssim d, g \lesssim d' \ldots, g \lesssim d^m$, and we have

$$\begin{aligned}
&\bigoplus_{i=1}^m x_i' \bigoplus_{i=1}^m x_i'' \cdots \bigoplus_{i=1}^m {x_i}^m \\
&\quad \lesssim r_1 + r_3 + r_5 + \cdots + r_{2m} = r \\
&\quad \lesssim r_1 + r_3 + r_5 + \cdots + r_{2m-1} \oplus g \\
&\quad \lesssim (x_j' \oplus ((x_j'' - \varepsilon)_+ + g) \cdots \oplus {x_j}^m) \oplus (((x_j' - \varepsilon)_+ + g) \oplus x_j'' \cdots \oplus {x_j}^m) \\
&\qquad \oplus \cdots \oplus (x_j' \oplus x_j'' \cdots \oplus (({x_j}^m - \varepsilon)_+ + g)) \\
&\quad \lesssim (x_i' \oplus x_j'' \oplus \cdots \oplus {x_j}^m) \oplus (x_i' \oplus x_j'' \oplus \cdots \oplus {x_j}^m) \cdots \oplus (x_i' \oplus x_j'' \oplus \cdots \oplus {x_j}^m),
\end{aligned}$$

for all $1 \leq j \leq m$, where $(x_j' \oplus x_j'' \cdots \oplus {x_j}^m)$ repeat n times.

Case (II) We suppose that u is not Cuntz equivalent to a projection.

We need to show that for any $u \in \mathrm{M}_\infty(A)_+$, any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in M_\infty(A)_+$ such that $\bigoplus_{i=1}^m x_i \lesssim u$ and $(u - 20\varepsilon)_+ \lesssim x_j \oplus x_j \oplus \cdots \oplus x_j$ for all $1 \leq j \leq m$, where x_j repeat m times.

By Theorem 2.1, for $\varepsilon > 0$, there is a non-zero positive element d such that $(u-\varepsilon)_+ + d \lesssim u$.

For $\varepsilon > 0$, there exists $\delta' > 0$ with $\delta' < \varepsilon$ such that $(a-\varepsilon)_+ + (b-\varepsilon)_+ \lesssim (a+b-\delta')_+$ for any $a, b \in A_+$.

Since $A \in \mathrm{TA}\Omega$, with $G = \{(u-\varepsilon)_+,\ d,\ u\}$, for $\varepsilon' > 0$, with $\varepsilon' < \delta'$, there are a C*-subalgebra C of A and a non-zero projection $r \in A$ with $C \in \Omega$ and $1_C = r$ such that

(1) $\|xr - rx\| < \varepsilon'/3$ for any $x \in G$,

(2) $rxr \in_{\varepsilon'/3} C$ for any $x \in G$.

By (1) and (2), there are $u_1, d_1 \in C$ and $u_2 \in (1-r)A(1-r)$ such that

$$\|u - u_1 - u_2\| < 4\varepsilon',$$

and

$$\|(u-\varepsilon)_+ - (u_1-\varepsilon)_+ - (u_2-\varepsilon)_+\| < 4\varepsilon'.$$

Therefore we have $(u_1 - 2\varepsilon)_+ + (u_2 - 2\varepsilon)_+ \lesssim (u-\varepsilon)_+$.

Since $(u_1-3\varepsilon)_++(d_1-\varepsilon)_+\in C$ and $C\in\Omega$, for $\delta'>0$, there exist $x_1',x_2',\ldots,x_n'\in \mathrm{M}_\infty(B)_+$ such that $\bigoplus_{i=1}^m x_i'\lesssim(u_1-3\varepsilon)_++(d_1-\varepsilon)_+$, and $(u_1-4\varepsilon)_++(d-2\varepsilon)_+\lesssim(((u_1-3\varepsilon)_++(d_1-\varepsilon)_+)-\delta')_+\lesssim x_j'\oplus x_j'\oplus\cdots\oplus x_j'$ for all $1\le j\le m$, where x_j' repeat m times.

Since $(1-r)A(1-r)\in \mathrm{TA}\Omega$, with $F=\{u_2\}$ and $\varepsilon'>0$, with $\varepsilon'<\delta'$, there are a C*-subalgebra E of $(1-r)A(1-r)$ and a non-zero projection $t\in(1-r)A(1-r)$ with $E\in\Omega$ and $1_E=t$ such that

(1)′ $\|tu_2-u_2t\|<\varepsilon'$,

(2)′ $tu_2t\in_{\varepsilon'}E$,

(3)′ $[1-r-t]\le[(d_1-2\varepsilon)_+]$.

By (1)′ and (2)′, there is $u_3\in E$ and $u_4\in(1-r-t)A(1-r-t)$ such that

$$\|u_2-u_3-u_4\|<3\varepsilon'.$$

Then $(u_3-3\varepsilon)_++(u_4-3\varepsilon)_+\lesssim(u_2-2\varepsilon)_+$.

Since $(u_3-3\varepsilon)_+\in E$ and $E\in\Omega$, there exist $x_1'',x_2'',\ldots,x_n''\in\mathrm{M}_\infty(E)_+$ such that $x_j''\oplus x_j''\oplus\cdots\oplus x_j''\lesssim(u_3-3\varepsilon)_+$ where x_j'' repeat m times and $(u_3-4\varepsilon)_+\lesssim\bigoplus_{i=1}^n x_i''$.

Therefore,

$$\begin{aligned}\bigoplus_{i=1}^k x_i'\bigoplus_{i=1}^k x_i''&\lesssim((u_1-3\varepsilon)_++(d_1-\varepsilon)_+)\oplus((u_3-3\varepsilon)_++(u_4-3\varepsilon)_+)\\&\lesssim(u_1-3\varepsilon)_++(d_1-\varepsilon)_+\oplus(u_2-2\varepsilon)_+\\&\lesssim(u-\varepsilon)_++(d_1-\varepsilon)_+\\&\lesssim(u-\varepsilon)_++d\\&\lesssim u,\end{aligned}$$

and

$$\begin{aligned}(u-20\varepsilon)_+&\lesssim(u_1-10\varepsilon)_++(u_2-10\varepsilon)_+\\&\lesssim(u_1-10\varepsilon)_++(u_3-4\varepsilon)_++(u_4-4\varepsilon)_+\\&\lesssim(u_1-4\varepsilon')_++(d_1-2\varepsilon)_++(u_3-4\varepsilon)_+\\&\lesssim x_j'\oplus x_j'\oplus\cdots\oplus x_j'\oplus x_j''\oplus x_j''\oplus\cdots\oplus x_j''.\end{aligned}$$ □

Corollary 3.5 *Let A be a unital simple* C*-*algebra such that A is (m,n)- decomposable. Suppose that $\alpha:G\to\mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product C*-algebra* $\mathrm{C}^*(G,A,\alpha)$ *is (m,n)-decomposable.*

Proof This follows from Theorems 3.2 and 3.4. □

Theorem 3.6 *Let Ω be a class of unital* C*-*algebras such that B has the weak Riesz interpolation for any unital* C*-*algebra $B\in\Omega$. Then A has the weak Riesz interpolation for any simple unital* C*-*algebra $A\in\mathrm{TA}\Omega$.*

Proof By Lemma 2.3, enlarging the class Ω, we may suppose it is closed under passing to matrix algebras and unital hereditary C*-subalgebras (i.e., Morita equivalent C*-algebras).

We need to show that there exists $z\in\mathrm{M}_\infty(A)_+$ such that $(x_i-200\varepsilon)_+\le z\le y_j$ for any $x_1,\ x_2,\ y_1,\ y_2$ in $\mathrm{M}_\infty(A)_+$ with $x_i\le y_j$ for all $1\le i\le 2$, $1\le j\le 2$ and for any $\varepsilon>0$.

We may assume that x_1,x_2,y_1,y_2 are all in A_+.

Since $x_i \leq y_j$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$, there exist $v_{i,j}$ such that $\|v_{i,j}^* y_j v_{i,j} - x_i\| < \varepsilon$.

Since $A \in \mathrm{TA}\Omega$, for $F = \{x_i, y_j, v_{i,j}\}$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $p \in A$ and a C*-subalgebra $B \subseteq A$ with $B \in \Omega$, $1_B = p$ such that

(1) $\|xp - px\| < \varepsilon'$ for all $x \in F$,

(2) $pxp \in_{\varepsilon}' B$ for all $x \in F$.

By (1) and (2) there exist positive elements x_1', x_2', y_1', $y_2' \in B$ and x_1'', x_2'', y_1'', $y_1'' \in (1-p)A(1-p)$ such that

$$\|x_1 - x_1' - x_1''\| < \varepsilon, \quad \|y_1 - y_1' - y_1''\| < \varepsilon,$$
$$\|x_2 - x_2' - x_2''\| < \varepsilon, \quad \|y_2 - y_2' - y_2''\| < \varepsilon.$$

Since $x_i \leq y_j$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$, therefore we have

$$(x_i' - 6\varepsilon)_+ \precsim (y_j' - 2\varepsilon)_+, \quad (x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+,$$

for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $B \in \Omega$ and B has the weak Riesz interpolation, we may assume that there exists $z' \in B_+$ such that $(x_i' - 8\varepsilon)_+ \precsim z' \precsim (y_j' - 2\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

We divide the proof into two cases.

(1) We assume that z' is not Cuntz equivalent to a projection. Since $(x_i' - 8\varepsilon)_+ \precsim z'$, By Theorem 2.1, there exists $\delta > 0$ such that $(x_i' - 10\varepsilon)_+ \precsim (z' - \delta)_+$.

Also by Theorem 2.1, there is a non-zero positive element d such that $(z' - \delta)_+ + d \precsim z'$.

Since $(x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$, there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j'' - 2\varepsilon)_+ w_{i,j} - (x_i'' - 6\varepsilon)_+\| < \varepsilon$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

(1)$'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

(2)$'$ $sxs \in_{\varepsilon}' D$ for all $x \in G$,

(3)$'$ $[1 - p - s] \leq [d]$.

By (1)$'$ and (2)$'$ there exist positive elements x_1''', x_2''', y_1''', $y_2''' \in D$ and x_1^4, x_2^4, y_1^4, $y_2^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$, we have $(x_i''' - 12\varepsilon)_+ \precsim (y_j''' - 4\varepsilon)_+$ and $(x_i^4 - 12\varepsilon)_+ \precsim (y_j^4 - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i''' - 14\varepsilon)_+ \precsim z''' \precsim (y_j''' - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Therefore we have

$$\begin{aligned}(x_i - 200\varepsilon)_+ &\precsim (x_i' - 10\varepsilon)_+ \oplus (x_i''' - 14\varepsilon)_+ \oplus (x_i^4 - 14\varepsilon)_+ \\ &\precsim (x_i' - 10\varepsilon)_+ \oplus (x_i''' - 14\varepsilon)_+ \oplus d \\ &\precsim (z' - \delta)_+ \oplus d \oplus z'''\end{aligned}$$

$$\begin{aligned} &\precsim z' \oplus z''' \\ &\precsim (y'_j - 2\varepsilon)_+ \oplus (y'''_j - 4\varepsilon)_+ \oplus (y^4_j - 12\varepsilon)_+ \\ &\precsim y_j, \end{aligned}$$

for all $1 \le i \le 2$, $1 \le j \le 2$.

(2) We assume that z' is Cuntz equivalent to a projection.

(2.1) z' is not equivalent to $(y'_j - 2\varepsilon)_+$ for $1 \le j \le 2$.

There is a non-zero positive element d such that $z' + d \precsim (y'_j - 2\varepsilon)_+$.

Since $(x''_i - 6\varepsilon)_+ \precsim (y''_j - 2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w^*_{i,j}(y''_j - 2\varepsilon)_+ w_{i,j} - (x''_i - 6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x''_i, y''_j, w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

(1)′ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

(2)′ $sxs \in'_{\varepsilon} D$ for all $x \in G$,

(3)′ $[1-p-s] \le [d]$.

By (1)′ and (2)′ there exist positive elements x'''_1, x'''_2, y'''_1, $y'''_2 \in D$ and x^4_1, x^4_2, y^4_1, $y^4_2 \in (1-p-s)A(1-p-s)$ such that

$$\|x''_1 - x'''_1 - x^4_1\| < \varepsilon, \quad \|y''_1 - y'''_1 - y^4_1\| < \varepsilon,$$
$$\|x''_2 - x'''_2 - x^4_2\| < \varepsilon, \quad \|y''_2 - y'''_2 - y^4_2\| < \varepsilon.$$

Since $(x''_i - 6\varepsilon)_+ \precsim (y''_j - 2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x'''_i - 12\varepsilon)_+ \precsim (y'''_j - 4\varepsilon)_+$ and $(x^4_i - 12\varepsilon)_+ \precsim (y^4_j - 4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $D \in \Omega$ and D has the weakly Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x'''_i - 14\varepsilon)_+ \precsim z''' \precsim (y'''_j - 4\varepsilon)_+$ for all i, j.

Therefore we have

$$\begin{aligned} (x_i - 200\varepsilon)_+ &\precsim (x'_i - 8\varepsilon)_+ \oplus (x'''_i - 14\varepsilon)_+ \oplus (x^4_i - 12\varepsilon)_+ \\ &\precsim (x'_i - 8\varepsilon)_+ \oplus (x'''_i - 14\varepsilon)_+ \oplus d \\ &\precsim z' \oplus z''' \oplus d \\ &\precsim (y'_j - 2\varepsilon)_+ \oplus (y'''_j - 4\varepsilon)_+ \oplus (y^4_j - 12\varepsilon)_+ \\ &\precsim y_j, \end{aligned}$$

for all $1 \le i \le 2$, $1 \le j \le 2$.

(2.2) z' is not equivalent to $(y'_1 - 2\varepsilon)_+$ and z' is equivalent to $(y'_2 - 2\varepsilon)_+$.

There is a non-zero positive element d such that $z' + d \precsim (y'_1 - 2\varepsilon)_+$.

Since $(x''_i - 6\varepsilon)_+ \precsim (y''_j - 2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w^*_{i,j}(y''_j - 2\varepsilon)_+ w_{i,j} - (x''_i - 6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x''_i, y''_j, w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

(1)′ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

(2)′ $sxs \in'_{\varepsilon} D$ for all $x \in G$, (3)′ $[1-p-s] \leq [d]$.

By (1)′ and (2)′ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$, we have $(x_i''' - 12\varepsilon)_+ \precsim (y_j''' - 4\varepsilon)_+$ and $(x_i^4 - 12\varepsilon)_+ \precsim (y_j^4 - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i''' - 14\varepsilon)_+ \precsim z''' \precsim (y_j''' - 4\varepsilon)_+$ for all $i,\ j$.

Therefore we have

$$\begin{aligned}(x_j - 200\varepsilon)_+ &\precsim (x_j' - 8\varepsilon)_+ \oplus (x_j''' - 14\varepsilon)_+ \oplus (x_j^4 - 12\varepsilon)_+ \\ &\precsim (x_j' - 8\varepsilon)_+ \oplus (x_j''' - 14\varepsilon)_+ \oplus (y_2^4 - 4\varepsilon)_+ \\ &\precsim z' \oplus (y_2^4 - 12\varepsilon)_+ \oplus z''' \\ &\precsim (y_2' - 2\varepsilon)_+ \oplus (y_2''' - 4\varepsilon)_+ \oplus (y_2^4 - 12\varepsilon)_+ \\ &\precsim y_2,\end{aligned}$$

for all $1 \leq j \leq 2$.

We also have

$$\begin{aligned}(x_j - 200\varepsilon)_+ &\precsim (x_j' - 8\varepsilon)_+ \oplus (x_j''' - 14\varepsilon)_+ \oplus (x_j^4 - 12\varepsilon)_+ \\ &\precsim (x_j' - 8\varepsilon)_+ \oplus (x_j''' - 14\varepsilon)_+ \oplus (y_2^4 - 4\varepsilon)_+ \\ &\precsim z' \oplus (y_2^4 - 12\varepsilon)_+ \oplus z''' \\ &\precsim z' \oplus d \oplus z''' \\ &\precsim (y_1' - 2\varepsilon)_+ \oplus (y_1''' - 4\varepsilon)_+ \oplus (y_1^4 - 12\varepsilon)_+ \\ &\precsim y_1,\end{aligned}$$

for all $1 \leq j \leq 2$.

(2.3) z' is equivalent to $(y_j' - 2\varepsilon)_+$.

(2.3.1) $(x_i' - 8\varepsilon)_+$ are not Cuntz equivalent to projections.

By Theorem 2.1, there is a non-zero positive element d such that $(x_i' - 10\varepsilon)_+ + d \precsim (x_i' - 8\varepsilon)_+$.

Since $(x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j'' - 2\varepsilon)_+ w_{i,j} - (x_i'' - 6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

(1)′ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

(2)′ $sxs \in_{\varepsilon}' D$ for all $x \in G$,

(3)′ $[1-p-s] \leq [d]$.

By (1)′ and (2)′ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$

$$\|x_2''-x_2'''-x_2^4\|<\varepsilon,\quad \|y_2''-y_2'''-y_2^4\|<\varepsilon.$$

Since $(x_i''-6\varepsilon)_+\lesssim(y_j''-2\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$. We have $(x_i'''-12\varepsilon)_+\lesssim(y_j'''-4\varepsilon)_+$ and $(x_i^4-12\varepsilon)_+\lesssim(y_j^4-4\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$.

Since $D\in\Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z'''\in D_+$ such that $(x_i'''-14\varepsilon)_+\lesssim z'''\lesssim(y_j'''-4\varepsilon)_+$ for all $i,\ j$.

Therefore we have

$$\begin{aligned}(x_i-200\varepsilon)_+&\lesssim(x_i'-10\varepsilon)_+\oplus(x_i'''-14\varepsilon)_+\oplus(x_i^4-12\varepsilon)_+\\&\lesssim(x_i'-10\varepsilon)_+\oplus(x_i'''-14\varepsilon)\oplus d\\&\lesssim(x_i'-8\varepsilon)_+\oplus z'''\\&\lesssim z'\oplus z'''\\&\lesssim(y_j'-2\varepsilon)_+\oplus(y_j'''-4\varepsilon)_+\oplus(y_j^4-12\varepsilon)_+\\&\lesssim y_j,\end{aligned}$$

for all $1\le i\le 2$, $1\le j\le 2$.

(2.3.2) $(x_i'-8\varepsilon)_+$ are Cuntz equivalent to projections, and $(x_i'-8\varepsilon)_+$ are not equivalent to z'. By Theorem 2.1, there is a non-zero positive element d such that $(x_i'-8\varepsilon)_++d\lesssim z'$.

Since $(x_i''-6\varepsilon)_+\lesssim(y_j''-2\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j''-2\varepsilon)_+w_{i,j}-(x_i''-6\varepsilon)_+\|<\varepsilon$.

Since $(1-p)A(1-p)\in \mathrm{TA}\Omega$, for $G=\{x_i'',y_j'',w_{i,j}\}$, $1\le i\le 2$, $1\le j\le 2$, any $\varepsilon'>0$ with ε' sufficiently small, there exist a projection $s\in(1-p)A(1-p)$ and a C*-subalgebra $D\subseteq(1-p)A(1-p)$ with $D\in\Omega,\ 1_D=s$ such that

(1)′ $\|xs-sx\|<\varepsilon'$ for all $x\in G$,

(2)′ $sxs\in{}'_{\varepsilon}D$ for all $x\in G$,

(3)′ $[1-p-s]\le[d]$.

By (1)′ and (2)′ there exist positive elements x_1''', x_2''', y_1''', $y_2'''\in D$ and x_1^4, x_2^4, y_1^4, $y_2^4\in(1-p-s)A(1-p-s)$ such that

$$\begin{aligned}&\|x_1''-x_1'''-x_1^4\|<\varepsilon,\quad \|y_1''-y_1'''-y_1^4\|<\varepsilon,\\&\|x_2''-x_2'''-x_2^4\|<\varepsilon,\quad \|y_2''-y_2'''-y_2^4\|<\varepsilon.\end{aligned}$$

Since $(x_i''-6\varepsilon)_+\lesssim(y_j''-2\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$, we have $(x_i'''-12\varepsilon)'\lesssim(y_j'''-4\varepsilon)_+$ and $(x_i^4-12\varepsilon)_+\lesssim(y_j^4-4\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$.

Since $D\in\Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z'''\in D_+$ such that $(x_i'''-14\varepsilon)_+\lesssim z'''\lesssim(y_j'''-4\varepsilon)_+$ for all $1\le i\le 2$, $1\le j\le 2$.

Therefore we have

$$\begin{aligned}(x_i-200\varepsilon)_+&\lesssim(x_i'-8\varepsilon)_+\oplus(x_i'''-14\varepsilon)_+\oplus(x_i^4-12\varepsilon)_+\\&\lesssim(x_i'-8\varepsilon)_+\oplus(x_i'''-14\varepsilon)\oplus d\\&\lesssim z'\oplus z'''\\&\lesssim(y_j'-2\varepsilon)_+\oplus(y_j'''-4\varepsilon)_+\oplus(y_j^4-12\varepsilon)_+\\&\lesssim y_j,\end{aligned}$$

for all $1\le i\le 2$, $1\le j\le 2$.

(2.3.3) $(x_1'-8\varepsilon)_+$ is not Cuntz equivalent to a projection, and $(x_2'-8\varepsilon)_+$ is Cuntz equivalent to a projection and is not equivalent to z'.

By Theorem 2.1, there is a non-zero positive element d such that $(x_1'-10\varepsilon)_+ + d \precsim (x_1' - 8\varepsilon)_+ \precsim z'$, and $(x_2'-10\varepsilon)_+ + d \precsim z'$.

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j''-2\varepsilon)_+ w_{i,j} - (x_i''-6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

$(1)'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $sxs \in_{\varepsilon}' D$ for all $x \in G$,

$(3)'$ $[1-p-s] \le [d]$.

By $(1)'$ and $(2)'$ there exist positive elements x_1''', x_2''', y_1''', $y_2''' \in D$ and x_1^4, x_2^4, y_1^4, $y_2^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1''-x_1'''-x_1^4\| < \varepsilon, \quad \|y_1''-y_1'''-y_1^4\| < \varepsilon,$$
$$\|x_2''-x_2'''-x_2^4\| < \varepsilon, \quad \|y_2''-y_2'''-y_2^4\| < \varepsilon.$$

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_i'''-12\varepsilon)_+ \precsim (y_j'''-4\varepsilon)_+$ and $(x_i^4-12\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i'''-14\varepsilon)_+ \precsim z''' \precsim (y_j'''-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Therefore we have

$$\begin{aligned}(x_i-200\varepsilon)_+ &\precsim (x_i'-10\varepsilon)_+ \oplus (x_i'''-14\varepsilon)_+ \oplus (x_i^4-12\varepsilon)_+ \\ &\precsim (x_i''-10\varepsilon)_+ \oplus (x_i'''-14\varepsilon)_+ \oplus d \\ &\precsim (x_i''-8\varepsilon)_+ \oplus (x_i'''-14\varepsilon)_+ \\ &\precsim z' \oplus z''' \\ &\precsim (y_j'-2\varepsilon)_+ \oplus (y_j'''-4\varepsilon)_+ \oplus (y_j^4-12\varepsilon)_+ \\ &\precsim y_j,\end{aligned}$$

for all $1 \le i \le 2$, $1 \le j \le 2$.

(2.3.4) $(x_1'-8\varepsilon)_+$ is not Cuntz equivalent to a projection, and $(x_2'-8\varepsilon)_+$ is Cuntz equivalent to a projection and is equivalent to z'.

By Theorem 2.1, there is a non-zero positive element d such that $(x_1'-10\varepsilon)_+ + d \precsim (x_1'-8\varepsilon)_+$.

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j''-2\varepsilon)_+ w_{i,j} - (x_i''-6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

$(1)'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $sxs \in_{\varepsilon}' D$ for all $x \in G$,

$(3)'$ $[1-p-s] \le [d]$.

By $(1)'$ and $(2)'$ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$, we have $(x_i'''-12\varepsilon)' \precsim (y_j'''-4\varepsilon)_+$ and $(x_i^4-12\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i'''-14\varepsilon)_+ \precsim z''' \precsim (y_j'''-4\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$.

Therefore we have

$$\begin{aligned}(x_1-200\varepsilon)_+ &\precsim (x_1'-10\varepsilon)_+ \oplus (x_1'''-14\varepsilon)_+ \oplus (x_1^4-12\varepsilon)_+ \oplus (x_2^4-12\varepsilon)_+\\ &\precsim (x_1'-10\varepsilon)_+ \oplus (x_1'''-14\varepsilon)_+ \oplus d \oplus (x_2^4-12\varepsilon)_+\\ &\precsim (x_1'-8\varepsilon)_+ \oplus (x_1'''-14\varepsilon)_+ \oplus (x_2^4-12\varepsilon)_+\\ &\precsim z' \oplus z''' \oplus (x_2^4-12\varepsilon)_+\\ &\precsim (y_j'-2\varepsilon)_+ \oplus (y_j'''-4\varepsilon)_+ \oplus (y_j^4-4\varepsilon)_+\\ &\precsim y_j,\end{aligned}$$

for all $1 \le j \le 2$.

We also have

$$\begin{aligned}(x_2-200\varepsilon)_+ &\precsim (x_2'-8\varepsilon)_+ \oplus (x_2'''-14\varepsilon)_+ \oplus (x_2^4-12\varepsilon)_+\\ &\precsim z' \oplus z''' \oplus (x_2^4-12\varepsilon)_+\\ &\precsim (y_j'-2\varepsilon)_+ \oplus (y_j'''-4\varepsilon)_+ \oplus (y_j^4-4\varepsilon)_+\\ &\precsim y_j,\end{aligned}$$

for all $1 \le j \le 2$.

(2.3.5) $(x_i'-8\varepsilon)_+$ are Cuntz equivalent to projections and are equivalent to z'.

(2.3.5.1) $(x_j''-6\varepsilon)_+$ are not Cuntz equivalent to projections. By Theorem 2.1, there is a non-zero positive element d such that $(x_j''-10\varepsilon)_+ + d \precsim (x_j''-6\varepsilon)_+$.

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j''-2\varepsilon)_+ w_{i,j} - (x_i''-6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

$(1)'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $sxs \in_\varepsilon' D$ for all $x \in G$.

By $(1)'$ and $(2)'$ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''', d''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4, d^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i''-10\varepsilon)_+ + d \precsim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$, we have $(x_i'''-20\varepsilon)_+ + (d'''-\varepsilon)_+ \precsim (y_j'''-4\varepsilon)_+$ and $(x_i^4-20\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2,\ 1 \le j \le 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i''' - 24\varepsilon)_+ + (d''' - 4\varepsilon)_+ \precsim z''' \precsim (y_j''' - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $(x_i^4 - 20\varepsilon)_+ \precsim (y_j^4 - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$ there exist $z_{i,j}$ such that $\|z_{i,j}^*(y_j^4 - 4\varepsilon)_+ z_{i,j} - (x_i^4 - 20\varepsilon)_+\| < \varepsilon$.

Since $(1-p-s)A(1-p-s) \in \mathrm{TA}\Omega$, for $H = \{x_i^4, y_j^4, z_{i,j}\}$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $t \in (1-p-s)A(1-p-s)$ and a C*-subalgebra $E \subseteq (1-p-s)A(1-p-s)$ with $E \in \Omega$, $1_E = t$ such that

(1)′ $\|xt - tx\| < \varepsilon'$ for all $x \in H$,

(2)′ $txt \in_{\varepsilon}' E$ for all $x \in H$,

(3)′ $[1-p-s-t] \leq [(d''' - 4\varepsilon)_+]$.

By (1)′ and (2)′ there exist positive elements $x_1^5,\ x_2^5,\ y_1^5,\ y_2^5 \in E$ and $x_1^6,\ x_2^6,\ y_1^6,\ y_2^6 \in (1-p-s-t)A(1-p-s-t)$ such that

$$\|x_1'^4 - x_1^5 - x_1^6\| < \varepsilon, \quad \|y_1^4 - y_1^5 - y_1^6\| < \varepsilon,$$
$$\|x_2'^4 - x_2^5 - x_2^6\| < \varepsilon, \quad \|y_2^4 - y_2^5 - y_2^6\| < \varepsilon.$$

Since $(x_i^4 - 20\varepsilon)_+ \precsim (y_j^4 - 4\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$, we have $(x_i^5 - 24\varepsilon)' \precsim (y_j^5 - 8\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

Since $E \in \Omega$ and E has the weak Riesz interpolation, we may assume that there exists a projection $z^4 \in E_+$ such that $(x_i^5 - 26\varepsilon) \precsim z^4 \precsim (y_j^5 - 8\varepsilon)_+$ for all $i,\ j$.

Therefore we have

$$\begin{aligned}(x_i - 200\varepsilon)_+ &\precsim (x_i' - 8\varepsilon)_+ \oplus (x_j''' - 24\varepsilon)_+ \oplus (x_i^5 - 24\varepsilon)_+ \oplus (x_i^6 - 24\varepsilon)_+\\ &\precsim (x_i' - 8\varepsilon)_+ \oplus (x_j''' - 24\varepsilon)_+ \oplus (x_i^5 - 24\varepsilon)_+ \oplus (d''' - 4\varepsilon)_+\\ &\precsim z' \oplus z''' \oplus z^4\\ &\precsim (y_i' - 2\varepsilon)_+ \oplus (y_j''' - 4\varepsilon)_+ \oplus (y_j^5 - 8\varepsilon)_+\\ &\precsim y_j,\end{aligned}$$

for all $1 \leq i \leq 2$, $1 \leq j \leq 2$.

(2.3.5.2) $(x_1'' - 6\varepsilon)_+$ is not Cuntz equivalent to a projection, $(x_2'' - 6\varepsilon)_+$ is Cuntz equivalent to a projection and $(x_2'' - 6\varepsilon)_+$ is not equivalent to $(y_j'' - 2\varepsilon)_+$.

By Theorem 2.1, there is a non-zero positive element d such that $(x_1'' - 10\varepsilon)_+ + d \precsim (x_1'' - 8\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$ and $(x_2'' - 10\varepsilon)_+ + d \precsim (y_j'' - 2\varepsilon)_+$.

Since $(x_i'' - 6\varepsilon)_+ \precsim (y_j'' - 2\varepsilon)_+$ for all $1 \leq i \leq 2$, $1 \leq j \leq 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j'' - 2\varepsilon)_+ w_{i,j} - (x_i'' - 6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

(1)′ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

(2)′ $sxs \in_{\varepsilon}' D$ for all $x \in G$.

By (1)′ and (2)′ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''', d''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4, d^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$

$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i'' - 10\varepsilon)_+ + d \lesssim (y_j'' - 2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_i''' - 20\varepsilon)_+ + (d''' - 2\varepsilon)_+ \lesssim (y_j''' - 4\varepsilon)_+$ and $(x_i^4 - 20\varepsilon)_+ \lesssim (y_j^4 - 4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $D \in \Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_i''' - 24\varepsilon)_+ + (d''' - 4\varepsilon)_+ \lesssim z''' \lesssim (y_j''' - 4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $(x_i^4 - 20\varepsilon)_+ \lesssim (y_j^4 - 4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $z_{i,j}$ such that $\|z_{i,j}^*(y_j^4 - 4\varepsilon)_+ z_{i,j} - (x_i''^4 - 20\varepsilon)_+\| < \varepsilon$.

Since $(1-p-s)A(1-p-s) \in \mathrm{TA}\Omega$, for $H = \{x_i^4, y_j^4, z_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $t \in (1-p-s)A(1-p-s)$ and a C*-subalgebra $E \subseteq (1-p-s)A(1-p-s)$ with $E \in \Omega$, $1_E = t$ such that

$(1)'$ $\|xt - tx\| < \varepsilon'$ for all $x \in H$,

$(2)'$ $txt \in_{\varepsilon}' E$ for all $x \in H$,

$(3)'$ $[1-p-s-t] \le [(d''' - 4\varepsilon)_+]$.

By $(1)'$ and $(2)'$ there exist positive elements x_1^5, x_2^5, y_1^5, $y_2^5 \in E$ and x_1^6, x_2^6, y_1^6, $y_2^6 \in (1-p-s-t)A(1-p-s-t)$ such that

$$\|x_1^4 - x_1^5 - x_1^6\| < \varepsilon, \quad \|y_1^4 - y_1^5 - y_1^6\| < \varepsilon,$$
$$\|x_2^4 - x_2^5 - x_2^6\| < \varepsilon, \quad \|y_2^4 - y_2^5 - y_2^6\| < \varepsilon.$$

Since $(x_i^4 - 20\varepsilon)_+ \lesssim (y_j^4 - 4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_i^5 - 24\varepsilon)_+ \lesssim (y_j^5 - 8\varepsilon)_+$ and $(x_i^6 - 24\varepsilon)_+ \lesssim (y_j^6 - 8\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $E \in \Omega$ and E has the weak Riesz interpolation, we may assume that there exists a projection $z^4 \in E_+$ such that $(x_i^5 - 26\varepsilon) \lesssim z^4 \lesssim (y_j^5 - 8\varepsilon)_+$ for all i, j.

Therefore we have

$$\begin{aligned}
(x_i - 200\varepsilon)_+ &\lesssim (x_i' - 8\varepsilon)_+ \oplus (x_i''' - 26\varepsilon)_+ \oplus (x_i^5 - 24\varepsilon)_+ \oplus (x_i^6 - 24\varepsilon)_+ \\
&\lesssim (x_i' - 8\varepsilon)_+ \oplus (x_i''' - 26\varepsilon)_+ \oplus (x_i^5 - 24\varepsilon)_+ \oplus (d''' - 4\varepsilon)_+ \\
&\lesssim z' \oplus z''' \oplus z^4 \\
&\lesssim (y_i' - 2\varepsilon)_+ \oplus (y_j''' - 4\varepsilon)_+ \oplus (y_j^5 - 8\varepsilon)_+ \\
&\lesssim y_j,
\end{aligned}$$

for all $1 \le i \le 2$, $1 \le j \le 2$.

(2.3.5.3) $(x_j'' - 6\varepsilon)_+$ are Cuntz equivalent to projections and $(x_j'' - 6\varepsilon)_+$ is not equivalent to $(y_j'' - 2\varepsilon)_+$.

By Theorem 2.1, there is a non-zero positive element d such that $(x_i'' - 6\varepsilon)_+ + d \lesssim (y_j'' - 2\varepsilon)_+$.

Since $(x_i'' - 6\varepsilon)_+ \lesssim (y_j'' - 2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j'' - 2\varepsilon)_+ w_{i,j} - (x_i'' - 6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p) \in \mathrm{TA}\Omega$, for $G = \{x_i'', y_j'', w_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $s \in (1-p)A(1-p)$ and a C*-subalgebra $D \subseteq (1-p)A(1-p)$ with $D \in \Omega$, $1_D = s$ such that

$(1)'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $sxs \in_{\varepsilon}' D$ for all $x \in G$.

By $(1)'$ and $(2)'$ there exist positive elements $x_1''',\ x_2''',\ y_1''',\ y_2''', d''' \in D$ and $x_1^4,\ x_2^4,\ y_1^4,\ y_2^4, d^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_i''-6\varepsilon)_+ + d \precsim (y_j''-2\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$, we have $(x_i'''-12\varepsilon)_+ + (d'''-2\varepsilon)_+ \precsim (y_j'''-4\varepsilon)_+$ and $(x_i^4-12\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$.

Since $D\in\Omega$ and D has the weak Riesz interpolation, we may assume that there exists a projection $z'''\in D_+$ such that $(x_i'''-14\varepsilon)_+ + (d'''-4\varepsilon)_+ \precsim z''' \precsim (y_j'''-4\varepsilon)_+$ for all $i,\ j$.

Since $(x_i^4-12\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$ there exist $z_{i,j}$ such that $\|z_{i,j}^*(y_j^4-4\varepsilon)_+z_{i,j} - (x_i^4-12\varepsilon)_+\| < \varepsilon$.

Since $(1-p-s)A(1-p-s)\in \mathrm{TA}\Omega$, for $H=\{x_i^4, y_j^4, z_{i,j}\}$, $1\leq i\leq 2$, $1\leq j\leq 2$, any $\varepsilon'>0$ with ε' sufficiently small, there exist a projection $t\in(1-p-s)A(1-p-s)$ and a C*-subalgebra $E\subseteq (1-p-s)A(1-p-s)$ with $E\in\Omega,\ 1_E=t$ such that

$(1)''$ $\|xt-tx\|<\varepsilon'$ for all $x\in H$,

$(2)''$ $txt\in{}'_{\varepsilon}E$ for all $x\in H$,

$(3)''$ $[1-p-s-t]\leq[(d'''-4\varepsilon)_+]$.

By $(1)''$ and $(2)''$ there exist positive elements $x_1^5,\ x_2^5,\ y_1^5,\ y_2^5\in E$ and $x_1^6,\ x_2^6,\ y_1^6,\ y_2^6\in (1-p-s-t)A(1-p-s-t)$ such that

$$\|x_1'^4 - x_1^5 - x_1^6\| < \varepsilon, \quad \|y_1^4 - y_1^5 - y_1^6\| < \varepsilon,$$
$$\|x_2'^4 - x_2^5 - x_2^6\| < \varepsilon, \quad \|y_2^4 - y_2^5 - y_2^6\| < \varepsilon.$$

Since $(x_i^4-12\varepsilon)_+ \precsim (y_j^4-4\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$, we have $(x_i^5-24\varepsilon)_+ \precsim (y_j^5-8\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$.

Since $E\in\Omega$ and E has the weak Riesz interpolation, we may assume that there exists a projection $z^4\in E_+$ such that $(x_i^5-26\varepsilon)_+ \precsim z^4 \precsim (y_j^5-8\varepsilon)_+$ for all $i,\ j$.

Therefore we have

$$\begin{aligned}(x_i-80\varepsilon)_+ &\precsim (x_i'-8\varepsilon)_+ \oplus (x_i'''-14\varepsilon)_+ \oplus (x_i^5-26\varepsilon)_+ \oplus (x_i^6-24\varepsilon)_+\\ &\precsim (x_i'-8\varepsilon)_+ \oplus (x_j'''-14\varepsilon)_+ \oplus (x_i^5-26\varepsilon)_+ \oplus (d'''-4\varepsilon)_+\\ &\precsim z'\oplus z'''\oplus z^4\\ &\precsim (y_i'-2\varepsilon)_+ \oplus (y_j'''-4\varepsilon)_+ \oplus (y_j^5-8\varepsilon)_+\\ &\precsim y_j,\end{aligned}$$

for all $1\leq i\leq 2$, $1\leq j\leq 2$.

(2.3.5.4) $(x_1''-6\varepsilon)_+$ is Cuntz equivalent to a projection, $(x_2''-6\varepsilon)_+$ is not Cuntz equivalent to a projection and $(x_1''-6\varepsilon)_+$ is equivalent to $(y_j''-2\varepsilon)_+$.

By Theorem 2.1, there is a non-zero positive element d such that $(x_2''-8\varepsilon)_+ + d \precsim (x_2''-6\varepsilon)_+$.

Since $(x_i''-6\varepsilon)_+ \precsim (y_j''-2\varepsilon)_+$ for all $1\leq i\leq 2$, $1\leq j\leq 2$ there exist $w_{i,j}$ such that $\|w_{i,j}^*(y_j''-2\varepsilon)_+w_{i,j} - (x_i''-6\varepsilon)_+\| < \varepsilon$.

Since $(1-p)A(1-p)\in\mathrm{TA}\Omega$, for $G=\{x_i'', y_j'', w_{i,j}\}$, $1\leq i\leq 2$, $1\leq j\leq 2$, any $\varepsilon'>0$ with ε' sufficiently small, there exist a projection $s\in(1-p)A(1-p)$ and a C*-subalgebra $D\subseteq(1-p)A(1-p)$ with $D\in\Omega,\ 1_D=s$ such that

$(1)'$ $\|xs - sx\| < \varepsilon'$ for all $x \in G$,

$(2)'$ $sxs \in_{\varepsilon}' D$ for all $x \in G$.

By $(1)'$ and $(2)'$ there exist positive elements $x_1''', x_2''', y_1''', y_2''', d''' \in D$ and $x_1^4, x_2^4, y_1^4, y_2^4, d^4 \in (1-p-s)A(1-p-s)$ such that

$$\|x_1'' - x_1''' - x_1^4\| < \varepsilon, \quad \|y_1'' - y_1''' - y_1^4\| < \varepsilon,$$
$$\|x_2'' - x_2''' - x_2^4\| < \varepsilon, \quad \|y_2'' - y_2''' - y_2^4\| < \varepsilon.$$

Since $(x_2''-8\varepsilon)_+ + d \lesssim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_2'''-16\varepsilon)_+ + (d'''-2\varepsilon)_+ \lesssim (y_j'''-4\varepsilon)_+$ and $(x_2^4-16\varepsilon)_+ \lesssim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $(x_1''-6\varepsilon)_+ \lesssim (y_j''-2\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_1'''-12\varepsilon)_+ \lesssim (y_j'''-4\varepsilon)_+$ and $(x_1^4-12\varepsilon)_+ \lesssim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $D \in \Omega$ and D has the weakly Riesz interpolation, we may assume that there exists a projection $z''' \in D_+$ such that $(x_2'''-18\varepsilon)_+ + (d'''-4\varepsilon)_+ \lesssim z''' \lesssim (y_j'''-4\varepsilon)_+$ for all i, j.

Since $(x_i^4-16\varepsilon)_+ \lesssim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$ there exist $z_{i,j}$ such that $\|z_{i,j}^*(y_j^4-4\varepsilon)_+ z_{i,j} - (x_i^4-16\varepsilon)_+\| < \varepsilon$.

Since $(1-p-s)A(1-p-s) \in \mathrm{TA}\Omega$, for $H = \{x_i^4, y_j^4, z_{i,j}\}$, $1 \le i \le 2$, $1 \le j \le 2$, any $\varepsilon' > 0$ with ε' sufficiently small, there exist a projection $t \in (1-p-s)A(1-p-s)$ and a C*-subalgebra $E \subseteq (1-p-s)A(1-p-s)$ with $E \in \Omega$, $1_E = t$ such that

$(1)''$ $\|xt - tx\| < \varepsilon'$ for all $x \in H$,

$(2)''$ $txt \in_{\varepsilon}' E$ for all $x \in H$,

$(3)''$ $[1-p-s-t] \le [(d'''-4\varepsilon)_+]$.

By $(1)''$ and $(2)''$ there exist positive elements $x_1^5, x_2^5, y_1^5, y_2^5 \in E$ and $x_1^6, x_2^6, y_1^6, y_2^6 \in (1-p-s-t)A(1-p-s-t)$ such that

$$\|x_1'^4 - x_1^5 - x_1^6\| < \varepsilon, \quad \|y_1^4 - y_1^5 - y_1^6\| < \varepsilon,$$
$$\|x_2'^4 - x_2^5 - x_2^6\| < \varepsilon, \quad \|y_2^4 - y_2^5 - y_2^6\| < \varepsilon.$$

Since $(x_i^4-12\varepsilon)_+ \lesssim (y_j^4-4\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$, we have $(x_i^5-24\varepsilon)_+ \lesssim (y_j^5-8\varepsilon)_+$ and $(x_i^6-24\varepsilon)_+ \lesssim (y_j^6-8\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Since $E \in \Omega$ and E has the weak Riesz interpolation, we may assume that there exists a projection $z^4 \in E_+$ such that $(x_i^5-26\varepsilon)_+ \lesssim z^4 \lesssim (y_j^5-8\varepsilon)_+$ for all $1 \le i \le 2$, $1 \le j \le 2$.

Therefore we have

$$\begin{aligned}(x_2-200\varepsilon)_+ &\lesssim (x_2'-8\varepsilon)_+ \oplus (x_2'''-18\varepsilon)_+ \oplus (x_2^5-26\varepsilon)_+ \oplus (x_2^6-24\varepsilon)_+\\ &\lesssim (x_2'-8\varepsilon)_+ \oplus (x_2'''-18\varepsilon)_+ \oplus (x_2^5-26\varepsilon)_+ \oplus (d'''-4\varepsilon)_+\\ &\lesssim z' \oplus z''' \oplus z^4 \oplus (x_1^6-24\varepsilon)_+\\ &\lesssim (y_i'-2\varepsilon)_+ \oplus (y_j'''-4\varepsilon)_+ \oplus (y_j^5-8\varepsilon)_+ \oplus (y_j^6-8\varepsilon)_+\\ &\lesssim y_j,\end{aligned}$$

for all $1 \le j \le 2$.

We also have

$$\begin{aligned}(x_1-200\varepsilon)_+ &\lesssim (x_1'-8\varepsilon)_+ \oplus (x_1'''-14\varepsilon)_+ \oplus (x_1^5-26\varepsilon)_+ \oplus (x_1^6-24\varepsilon)_+\\ &\lesssim z' \oplus z''' \oplus z^4 \oplus (x_1^6-24\varepsilon)_+\end{aligned}$$

$$\lesssim (y_i' - 2\varepsilon)_+ \oplus (y_j''' - 4\varepsilon)_+ \oplus (y_j^5 - 8\varepsilon)_+ \oplus (y_j^6 - 8\varepsilon)_+$$
$$\lesssim y_j,$$

for all $1 \le j \le 2$.

(2.3.5.5) $(x_j'' - 6\varepsilon)_+$ are Cuntz equivalent to projections and $(x_j'' - 6\varepsilon)_+$ are equivalent to $(y_j'' - 2\varepsilon)_+$.

Therefore we have

$$\begin{aligned}(x_i - 200\varepsilon)_+ &\lesssim (x_i' - 8\varepsilon)_+ \oplus (x_i'' - 6\varepsilon)_+ \\ &\lesssim z' + (x_1'' - 6\varepsilon)_+ \\ &\lesssim (y_i' - 2\varepsilon)_+ \oplus (y_j'' - 2\varepsilon)_+ \\ &\lesssim y_j,\end{aligned}$$

for all $1 \le i \le 2$, $1 \le j \le 2$. □

Corollary 3.7 *Let A be a unital simple* C*-*algebra such that A has the weak Riesz interpolation. Suppose that $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group G on A which has the tracial Rokhlin property. Then the crossed product* C*-*algebra* $\mathrm{C}^*(G, A, \alpha)$ *has the weak Riesz interpolation.*

Proof This follows from Theorems 3.2 and 3.6. □

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