Acta Mathematica Sinica, English Series
May, 2021, Vol. 37, No. 5, pp. 775–804
Published online: November 24, 2020
https://doi.org/10.1007/s10114-020-9328-y
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Hankel Operators on Bergman Spaces of Annulus Induced by Regular Weights

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Abstract This paper is devoted to studying Bergman spaces $A^p_{\omega_{1,2}}(M)\,(1<p<\infty)$ induced by regular-weight $\omega_{1,2}$ on annulus M. We characterize the function f in $L^1_{\omega_{1,2}}(M)$ for which the induced Hankel operator H_f is bounded (or compact) from $A^p_{\omega_{1,2}}(M)$ to $L^q_{\omega_{1,2}}(M)$ with $1<p,q<\infty$.

Keywords Hankel operators, regular-weight, annulus, Bergman spaces

MR(2010) Subject Classification 47B35

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $M=\{z\in\mathbb{D}: r_0<|z|<1\}$ be the annulus in $\mathbb{D}$ where $0<r_0<1$. For convenience, let $M_1=\{z\in\mathbb{D}:\frac{1+r_0}{2}<|z|<1\}$ and $M_2=\{z\in\mathbb{D}: r_0<|z|\le\frac{1+r_0}{2}\}$, then $M=M_1\cup M_2$.

Suppose $\rho(a,z)=|\frac{z-a}{1-\overline{a}z}|$ is the pseudo-hyperbolic metric on $\mathbb{D}$, then for any $a\in\mathbb{D}$ and $r\in(0,1)$ the pseudo-hyperbolic disk $\triangle(a,r)=\{z\in\mathbb{D}\,|\,\rho(a,z)<r\}$ with center a and the radius r.

Let $\{a_i\}_{i=1}^{\infty}$ be some (or any) r-lattice of M_1 under the pseudo-hyperbolic metric $\rho(z,w)$, and $\{\frac{r_0}{b_j}\}_{j=1}^{\infty}$ be some (or any) r-lattice of M_2 under the pseudo-hyperbolic metric $\rho(\frac{r_0}{z},\frac{r_0}{w})$. For any $z\in M_1$, $z\in\bigcup_{i=1}^{\infty}\triangle(a_i,r)$, and for any $z\in M_2$, $\frac{r_0}{z}\in\bigcup_{j=1}^{\infty}\triangle(\frac{r_0}{b_j},r)$.

We use the notation $A\lesssim B$ if there exists a positive constant C such that $A\le CB$ for two quantites A and B. Moreover, write $A\asymp B$ if $A\lesssim B$ and $B\lesssim A$.

Suppose $\omega\in L^1[1,0)$ is a radial weight, we will denote $\omega\in\hat{\mathcal{D}}$ for the family of radial weights such that $\hat{\omega}(z)=\int_{|z|}^1\omega(s)ds$ is doubling, i.e., there exists some constants $C=C(\omega)\ge1$ such that $\hat{\omega}(r)\le C\hat{\omega}(\frac{1+r}{2})$ for any $0\le r<1$. Futhermore, if $\omega\in\hat{\mathcal{D}}$ satisfies

$$\omega(r)\asymp\frac{\int_r^1\omega(s)ds}{1-r},\quad 0\le r<1,$$

then we call ω is regular, denoted by $\omega\in\mathcal{R}$.

If $\omega\in\mathcal{R}$, then there exists a positive constant C depending on $r\in(0,1)$, such that $C^{-1}\omega(z)<\omega(\xi)<C\omega(z)$, whenever $\xi\in\triangle(z,r)$. In other words, $\omega\in\mathcal{R}$ is equivalent to

Received August 12, 2019, accepted June 5, 2020
Supported by NNSF of China (Grant Nos. 11971125, 11471084)
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$\omega(z) \asymp \omega(\xi)$ on $\triangle(z,r)$, see [16]. From [22], if $\xi \in \triangle(z,r)$, then $1-|\xi| \asymp 1-|z|$, and $|\triangle(z,r)| \asymp (1-|z|^2)^2$. And several examples of weighs $\mathcal{R}$ are given by [13, (4.4)–(4.6)]. For the study of the regular weighted Bergman spaces, we can see [5, 13–16].

Suppose $\omega_1(z)$ and $\omega_2(z)$ are non-negative integrable functions on $\mathbb{D}$, let

$$\omega_{1,2}(z) = \omega_1(z)\chi_{M_1}(z) + \omega_2\left(\frac{r_0}{z}\right)\chi_{M_2}(z), \quad z \in \mathbb{D}.$$

For $0 < p < \infty$, define $L^p_{\omega_{1,2}}(M)$ to be the space of all Lebesgue measurable functions f satisfying the following condition

$$\|f\|_p^p = \int_M |f(z)|^p \omega_{1,2}(z) dA(z) < \infty, \tag{1.1}$$

where $dA(z) = \frac{dxdy}{\pi}$ is the normalized Lebesgue area measure.

It is easy to know that $L^p_{\omega_{1,2}}(M)$ is a Banach space when $1 \le p < \infty$. In particular, $L^2_{\omega_{1,2}}(M)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2} = \int_M f(z)\overline{g(z)}\omega_{1,2}(z) dA(z).$$

$L^p_{\omega_{1,2}}(M)$ is a Frechet space when $0 \le p < 1$. The weighted Bergman space $A^p_{\omega_{1,2}}(M)$ is defined by $A^p_{\omega_{1,2}}(M) = L^p_{\omega_{1,2}}(M) \cap H(M)$, where $H(M)$ is the set of all holomorphic functions on M.

In the weighted Bergman space $A^2_{\omega_{1,2}}(M)$, if $\omega_{1,2}$ is the radial weight, then the norm convergence implies the uniform convergence on each compact subset of M, see [14]. It follows that $A^2_{\omega_{1,2}}(M)$ is the closed subspace of $L^2_{\omega_{1,2}}(M)$. The orthogonal projection $P_{\omega_{1,2}}$ from $L^2_{\omega_{1,2}}(M)$ onto $A^2_{\omega_{1,2}}(M)$ is an integral operator given by

$$P_{\omega_{1,2}} f(z) = \int_M f(w)\overline{B_z^{\omega_{1,2}}(w)}\omega_{1,2}(w) dA(w),$$

where $B_z^{\omega_{1,2}}(w)$ is the reproducing kernel of $A^2_{\omega_{1,2}}(M)$, and $B_z^{\omega_{1,2}}(w)$ is analytic with respect to w, conjugate analytic with respect to z.

In this paper we will discuss the Hankel oprator induced by the orthogonal projection $P_{\omega_{1,2}}$. Let $f \in L^1_{\omega_{1,2}}(M)$, the Hankel operator H_f induced by f is defined as follow

$$H_f(g) = (I_d - P_{\omega_{1,2}})(fg),$$

where I_d is an identity operator.

Many researches of Hankel operators on Bergman spaces are in simply-connected domains, see [1, 5, 10, 12, 17, 22]. Arazy, Fisher and Peetre [1] discussed the properties of Hankel operator H_f on the standard weighted Bergman space, where H_f was induced by an anti-analytic function f on $\mathbb{D}$. Given $f \in L^p(\mathbb{D})$, Luecking [10] characterized the boundedness (or compactness) of the Hnakel operator $H_f : A^p(\mathbb{D}) \to L^p(\mathbb{D})\,(1 < p < \infty)$. Notice that Luecking characterized the boundedness by using the bounded distance from f to the analytic function space $A^2(\mathbb{D})$ instead of bounded distance to the constants (BMO). The distance from f to analytic function space also plays an essential role in proving our main results. For $\alpha, \beta > -1$, $1 < p \le q < \infty$ and $f \in L^q(\mathbb{B}^n, (1-|z|^2)^\beta dv(z))$, Pau, Zhao and Zhu [12] characterized the symbol f such that both the Hankel operators H_f, $H_{\bar f} : A^p(\mathbb{B}^n, (1-|z|^2)^\alpha dv(z)) \to L^q(\mathbb{B}^n, (1-|z|^2)^\beta dv(z))$ were simultaneously bounded (or compact), where $\mathbb{B}^n$ is the open unit ball in $\mathbb{C}^n$. In particular, Hu and Lu [5] studied the Bergman space induced by regular weight ω, and they considered the characterization on $f \in L^1_\omega(\mathbb{D})$ for which the Hankel operator $H_f : A^p_\omega(\mathbb{D}) \to L^q_\omega(\mathbb{D})$ is bounded (or compact) with $1 < p, q < \infty$. For more informations about researches on Hankel operators, one can refer to [4, 6, 17, 19, 20].

The researches of Bergman spaces on multi-connected domains can refer to [2, 7–9, 21]. Li [8] characterized the function $f \in L^\infty$ for which Hankel operator H_f is compact on the Bergman space A^2 acting on the multi-connected domain Ω. Furthermore, Li [9] generalized the result of [4] to the multi-connected domain Ω, and obtained the characterization on $f \in L^2(\Omega)$ such that Hankel operators H_f and $H_{\bar{f}}$ are both bounded (or compact). Recently, for given $1 < p, q < \infty$ and the positive Borel measure μ on M, He, Xia and Wang [21] discussed the boundedness (or compactness) of Toeplitz operator T_μ from the weighted Bergman space $A^p_{\omega_{1,2}}(M)$ to the weighted Bergman space $A^q_{\omega_{1,2}}(M)$.

Now, we are in the position to describe the boundedness (or compactness) of Hankel operator H_f from one regular weighted Bergman space $A^p_{\omega_{1,2}}(M)$ to another regular weighted Lebesgue space $L^q_{\omega_{1,2}}(M)$, where $f \in L^1_{\omega_{1,2}}(M)$ and $1 < p, q < \infty$. Noticing that M in here is the multi-connected domain, it is much more difficult than the simple-connected domain, because of the great difference in the topological structure between them. Many methods and techniques on the simple-connected domains didn't effect on multi-connected domains any more. In this paper, since there are two boundaries of an annulus M, we need to use two regular weights at different boundaries. So the weight $\omega_{1,2} \in \mathcal{R}$ in this paper is determined by regular weights ω_1 and ω_2.

The paper is organized as follows. In Section 2, we will give some estimates of reproducing kernels in Bergman space with regular weight, and some important lemmas. In Section 3, we will study some estimates of $\overline{\partial}$, which are used to prove the later part. Section 4 is about the proofs of main theorems.

2 Preliminaries

Let $\partial\mathbb{D}$ denote the boundary of $\mathbb{D}$. If I is an arc of $\partial\mathbb{D}$, then the Carleson square $S(I)$ is defined as

$$S(I) = \{r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D} \mid \mathrm{e}^{\mathrm{i}\theta} \in I, 1 - m(I) \le r < 1\},$$

where $m(I)$ is the normalized Lebesgue measure of I. When $w \in \mathbb{D}\backslash\{0\}$, define an arc $I_w = \{\mathrm{e}^{\mathrm{i}\theta} \in \partial\mathbb{D} \mid |\arg(w\mathrm{e}^{\mathrm{i}\theta})| \le \frac{1-|w|}{2}\}$. We will denote $S(I_w)$ as $S(w)$ for convenience.

Lemma 2.1 ([21, Theorem 3.2]) *If $\omega_{1,2} \in \mathcal{R}$, $1 < p < \infty$ and $0 < r < 1$, then*

$$\|B^{\omega_{1,2}}_w\|^p_p \asymp \frac{1}{\omega_1(S(w))^{p-1}} \asymp \frac{1}{\omega_1(\triangle(w,r))^{p-1}}, \quad |w| \to 1^-,$$

and

$$\|B^{\omega_{1,2}}_w\|^p_p \asymp \frac{1}{\omega_2(S(\frac{r_0}{w}))^{p-1}} \asymp \frac{1}{\omega_2(\triangle(\frac{r_0}{w},r))^{p-1}}, \quad |w| \to {r_0}^+.$$

Lemma 2.2 ([21, Theorem 3.3]) *Let $\omega_{1,2} \in \hat{\mathcal{D}}$. Then there exists an $r = r(\omega_1, \omega_2) \in (0,1)$ such that*

$$|B^{\omega_{1,2}}_w(z)| \asymp B^{\omega_{1,2}}_z(z), \quad z \in M_1, \quad w \in M \cap \triangle(z,r),$$

and

$$|B_w^{\omega_{1,2}}(z)| \asymp B_z^{\omega_{1,2}}(z), \quad z \in M_2, \quad \frac{r_0}{w} \in M \cap \triangle\left(\frac{r_0}{z}, r\right).$$

Lemma 2.3 ([21, Theorem 2.4]) *If $\omega_{1,2} \in \mathcal{R}$, $1 < p < \infty$, then $P_{\omega_{1,2}}$ is a bounded operator from $L^p_{\omega_{1,2}}(M)$ onto $A^p_{\omega_{1,2}}(M)$.*

The proof of [21, Theorem 2.4] implies

$$P^+_{\omega_{1,2}}(f)(z) = \int_M f(w)|B_z^{\omega_{1,2}}(w)|\omega_{1,2}(w)dA(w) < \infty.$$

If $\omega \in \mathcal{R}$, then we will write the reproducing kernel of $A^2_\omega(\mathbb{D})$ as $K_z^\omega(w)$.

Lemma 2.4 ([21, Theorem 2.3]) *Let $\omega_1, \omega_2 \in \mathcal{R}$. For $r_0 < r_2 < \frac{1+r_0}{2} < r_1 < 1$, write $U_2 = \{z \in \mathbb{D} : r_0 < |z| < r_2\}$, $U_1 = \{z \in \mathbb{D} : r_1 < |z| < 1\}$. If $(z, w) \in (M \times U_1) \cup (U_1 \times M)$, then $|B_z^{\omega_{1,2}}(w)| \lesssim |K_z^{\omega_1}(w)| + C(r_1)$; if $(z, w) \in (M \times U_2) \cup (U_2 \times M)$, then $|B_z^{\omega_{1,2}}(w)| \lesssim |K^{\omega_2}_{\frac{r_0}{z}}(\frac{r_0}{w})| + C(r_2)$.*

From Lemma 2.1, Lemma 2.2, Lemma 2.4 and [5, Lemma 2.1], we can obtain the following lemma.

Lemma 2.5 *Let $\omega_{1,2} \in \mathcal{R}$.*

(1) *For all $r > 0$, if $z \in M_1$, then*

$$\omega_{1,2}(\triangle(z, r)) = \omega_1(\triangle(z, r)) = \int_{\triangle(z,r)} \omega_1 dA \asymp (1 - |z|)^2\omega_1(z) = \breve{\omega}_1(z); \tag{2.1}$$

if $z \in M_2$, then

$$\omega_{1,2}(\triangle(z, r)) = \omega_2\left(\triangle\left(\frac{r_0}{z}, r\right)\right) = \int_{\triangle(\frac{r_0}{z},r)} \omega_2 dA \asymp \left(1 - \frac{r_0}{|z|}\right)^2 \omega_2\left(\frac{r_0}{z}\right) = \breve{\omega}_2\left(\frac{r_0}{z}\right). \tag{2.2}$$

(2) *For $1 < p < \infty$, if $\xi \in M_1$, then*

$$B_\xi^{\omega_{1,2}}(\xi) \asymp \frac{1}{(1 - |\xi|)^2\omega_1(\xi)};$$

if $\xi \in M_2$, then

$$B_\xi^{\omega_{1,2}}(\xi) \asymp \frac{1}{(1 - \frac{r_0}{|\xi|})^2\omega_2(\frac{r_0}{\xi})}.$$

By Lemma 2.2, Lemma 2.3, Lemma 2.5, and [5, Lemma 2.1], we can obtain the following useful estimates: for some positive constant α, if $\xi \in M_1, z \in \triangle(\xi, \alpha)$, then

$$|B_z^{\omega_{1,2}}(\xi)| \asymp B_\xi^{\omega_{1,2}}(\xi) \asymp \frac{1}{\breve{\omega}_1(\xi)} \asymp \frac{1}{\breve{\omega}_1(z)}; \tag{2.3}$$

if $\xi \in M_2, \frac{r_0}{z} \in \triangle(\frac{r_0}{\xi}, \alpha)$, then

$$|B_z^{\omega_{1,2}}(\xi)| \asymp B_\xi^{\omega_{1,2}}(\xi) \asymp \frac{1}{\breve{\omega}_2(\frac{r_0}{\xi})} \asymp \frac{1}{\breve{\omega}_2(\frac{r_0}{z})}. \tag{2.4}$$

This shows that there are some $\alpha > 0$ for $z \in M$, $\xi \in \triangle(z, \alpha) \cap M$ and $\frac{r_0}{\xi} \in \triangle(\frac{r_0}{z}, \alpha) \cap M$ such that $B_z^{\omega_{1,2}}(\xi) \neq 0$.

For $1 < p < \infty$, write $b_{p,z}^{\omega_{1,2}} = \frac{B_z^{\omega_{1,2}}}{\|B_z^{\omega_{1,2}}\|_p}$, then $\|b_{p,z}^{\omega_{1,2}}\|_p = 1$. By Lemma 2.5, if $z \in M_1$, then

$$\sup_{\xi\in\triangle(z,r)} |b_{p,z}^{\omega_{1,2}}(\xi)| \asymp \inf_{\xi\in\triangle(z,r)} |b_{p,z}^{\omega_{1,2}}(\xi)| \asymp \breve{\omega}_1(z)^{-\frac{1}{p}}; \tag{2.5}$$

if $z \in M_2$, then

$$\sup_{\frac{r_0}{\xi}\in\triangle(\frac{r_0}{z},r)} |b_{p,z}^{\omega_{1,2}}(\xi)| \asymp \inf_{\frac{r_0}{\xi}\in\triangle(\frac{r_0}{z},r)} |b_{p,z}^{\omega_{1,2}}(\xi)| \asymp \breve{\omega}_2\left(\frac{r_0}{z}\right)^{-\frac{1}{p}}. \tag{2.6}$$

From [21, Lemma 2.12], when $|z| \to 1^-$ or $|z| \to r_0^+$, we have $b_{p,z}^{\omega_{1,2}} \xrightarrow{w} 0$.

Lemma 2.6 ([15, Proposition 14]) *Let $1 < p < \infty, \omega \in \mathcal{R}$, and $\{z_j\}_{j=1}^{\infty} \subset \mathbb{D}\backslash\{0\}$ be a separated sequence. Then $F = \sum_{j=1}^{\infty} c_j b_{p,z_j}^{\omega} \in A_\omega^p$ with $\|F\|_{A_\omega^p} \le C\|\{c_j\}_{j=1}^{\infty}\|_{l^p}$ for all $\{c_j\}_{j=1}^{\infty} \in l^p$.*

Given $r > 0$, the local mean operator M_r on L^1_{loc} is defined as

$$M_r(f)(z) = \frac{1}{|\triangle(z,r)|}\int_{M_1} \chi_{\triangle(z,r)}(\xi)f(\xi)dA(\xi), \quad \text{where } z \in M_1;$$
$$M_r(f)(z) = \frac{1}{|\triangle(\frac{r_0}{z},r)|}\int_{M_2} \chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{\xi}\right)f(\xi)dA(\xi), \quad \text{where } z \in M_2.$$

Lemma 2.7 *Let $1 \le p \le \infty$. For any $r > 0$, M_r is a bounded linear operator on $L^p_{\omega_{1,2}}(M)$.*

Proof For $f \in L^1_{\omega_{1,2}}(M)$, if $z \in M_1$, $\xi \in \triangle(z,r)$, then $\omega_1(z) \asymp \omega_1(\xi)$ and $|\triangle(z,r)| \asymp |\triangle(\xi,r)|$. Therefore,

$$\begin{aligned}
\|M_r(f)\|_1 &= \|M_r(f)\|_{L^1_{\omega_1}(M_1)} \\
&\le \int_{M_1} \frac{\omega_1(z)}{|\triangle(z,r)|}dA(z)\int_{M_1}\chi_{\triangle(z,r)}(\xi)|f(\xi)|dA(\xi) \\
&\lesssim \int_{M_1}|f(\xi)|\omega_1(\xi)dA(\xi)\int_{M_1}\frac{\chi_{\triangle(\xi,r)}(z)}{|\triangle(z,r)|}dA(z) \\
&\asymp \|f\|_{L^1_{\omega_1}(M_1)} \\
&< \infty.
\end{aligned}$$

If $z \in M_2$, $\frac{r_0}{\xi} \in \triangle(\frac{r_0}{z},r)$, then $\omega_2(\frac{r_0}{z}) \asymp \omega_2(\frac{r_0}{\xi})$ and $|\triangle(\frac{r_0}{z},r)| \asymp (1-|\frac{r_0}{z}|^2)^2 \asymp (1-|\frac{r_0}{\xi}|^2)^2 \asymp |\triangle(\frac{r_0}{\xi},r)|$. Therefore,

$$\begin{aligned}
\|M_r(f)\|_1 &= \|M_r(f)\|_{L^1_{\omega_2}(M_2)} \\
&\le \int_{M_2}\frac{\omega_2(\frac{r_0}{z})}{|\triangle(\frac{r_0}{z},r)|}dA(z)\int_{M_2}\chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{\xi}\right)|f(\xi)|dA(\xi) \\
&\lesssim \int_{M_2}|f(\xi)|\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\int_{M_2}\frac{\chi_{\triangle(\frac{r_0}{\xi},r)}(\frac{r_0}{z})}{|\triangle(\frac{r_0}{z},r)|}dA(z) \\
&\lesssim \int_{M_2}|f(\xi)|\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi) \\
&= \|f\|_{L^1_{\omega_2}(M_2)} \\
&< \infty.
\end{aligned}$$

Obviously, if $f \in L^\infty_{\omega_{1,2}}(M)$, then $M_r(f)$ is bounded.

From the interpolation theorem, it is known that M_r is a bounded linear operator on $L^p_{\omega_{1,2}}(M)$. □

In the following sections, we use C to denote positive constants whose value may change from line to line, but do not depend on functions being considered.

3 Some $\overline{\partial}$-estimates

Given any $r \in (0, \frac{\alpha}{3}]$. Let $\{a_i\}_{i=1}^{\infty}$ be an r-lattice of M_1, $\{\frac{r_0}{b_i}\}_{i=1}^{\infty}$ be an r-lattice of M_2. Suppose $\{\psi_i\}_{i=1}^{\infty}$ and $\{\varphi_i\}_{i=1}^{\infty}$ respectively are some unity partitions of $\{\triangle(a_i, r)\}_{i=1}^{\infty}$ and $\{\triangle(\frac{r_0}{b_i}, r)\}_{i=1}^{\infty}$ with the following properties: $\psi_i \in C^{\infty}(M_1)$, $\operatorname{supp}\psi_i = \{\xi \mid \xi \in \triangle(a_i, r)\}$, $\psi_i \geq 0$, $\sum_{i=1}^{\infty}\psi_i = 1$ and $\varphi_i \in C^{\infty}(M_2)$, $\operatorname{supp}\varphi_i = \{\xi \mid \frac{r_0}{\xi} \in \triangle(\frac{r_0}{b_i}, r)\}$, $\varphi_i \geq 0$, $\sum_{i=1}^{\infty}\varphi_i = 1$.

Define the function $G_1(z, \xi)$ on $M \times M$ as

$$G_1(z,\xi) = \frac{1}{(\xi - z)(1 - |\xi|)} \sum_{i=1}^{\infty} \frac{B_{a_i}^{\omega_{1,2}}(z)\psi_i(\xi)}{B_{a_i}^{\omega_{1,2}}(\xi)}, \quad \text{where } (z,\xi) \in M_1 \times M_1,$$

$$G_1(z,\xi) = 0, \quad \text{where } (z,\xi) \notin M_1 \times M_1;$$

define the function $G_2(z, \xi)$ on $M \times M$ as

$$G_2(z,\xi) = \frac{1}{(\frac{r_0}{\xi} - \frac{r_0}{z})(1 - \frac{r_0}{|\xi|})} \sum_{i=1}^{\infty} \frac{B_{b_i}^{\omega_{1,2}}(z)\varphi_i(\xi)}{B_{b_i}^{\omega_{1,2}}(\xi)}, \quad \text{where } (z,\xi) \in M_2 \times M_2,$$

$$G_2(z,\xi) = 0, \quad \text{where } (z,\xi) \notin M_2 \times M_2.$$

The integral operators T_1 and T_2 are given by

$$T_1(f)(z) = \int_{M_1} G_1(z,\xi) f(\xi) dA(\xi) \quad \text{and} \quad T_2(f)(z) = \int_{M_2} G_2(z,\xi) f(\xi) dA(\xi).$$

Lemma 3.1 *Let $1 < p < \infty$, $\omega_{1,2} \in \mathcal{R}$. Then T_1 and T_2 are bounded linear operators on $L^p_{\omega_{1,2}}(M)$.*

Proof For $f \in L^p_{\omega_{1,2}}(M)$, we next prove that T_1 is bounded on $L^p_{\omega_{1,2}}(M)$.

When $z \in M_1$,

$$\begin{aligned} T_1(f)(z) &= \int_{M_1} G_1(z,\xi) f(\xi) dA(\xi) \\ &= \int_{M_1} \frac{1}{(\xi - z)(1 - |\xi|)} \sum_{i=1}^{\infty} \frac{B_{a_i}^{\omega_{1,2}}(z)\psi_i(\xi)}{B_{a_i}^{\omega_{1,2}}(\xi)} f(\xi) dA(\xi). \end{aligned}$$

Because $B_z^{\omega_{1,2}}(w)$ is analytic with respect to w on M_1 and conjugate analytic with respect to z on M_1, by Lemma 2.2 and the properties of $\{\psi_i\}_{i=1}^{\infty}$, we know

$$\sum_{i=1}^{\infty} \frac{|B_{a_i}^{\omega_{1,2}}(z)|\psi_i(\xi)}{|B_{a_i}^{\omega_{1,2}}(\xi)|} \asymp \frac{1}{B_{\xi}^{\omega_{1,2}}(\xi)} \sum_{i=1}^{\infty} |B_{a_i}^{\omega_{1,2}}(z)|\psi_i(\xi).$$

Since $\sum_{i=1}^{\infty} \chi_{\triangle(a_i,r)}(\xi) \leq C$, combining with the properties of the unity partition, we get

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{|B_{a_i}^{\omega_{1,2}}(z)|\psi_i(\xi)}{|B_{a_i}^{\omega_{1,2}}(\xi)|} &\leq \frac{1}{B_{\xi}^{\omega_{1,2}}(\xi)} \sum_{i \in \{k \mid \xi \in \triangle(a_k,r)\}} \frac{1}{|\triangle(a_i,r)|} \int_{\triangle(a_i,r)} |B_{\zeta}^{\omega_{1,2}}(z)| dA(\zeta) \\ &\leq C \frac{1}{B_{\xi}^{\omega_{1,2}}(\xi)} \frac{1}{|\triangle(\xi,r)|} \int_{\triangle(\xi,2r)} |B_{\zeta}^{\omega_{1,2}}(z)| dA(\zeta). \end{aligned}$$

Write

$$G_1^*(z,\xi) = \frac{1}{|\xi - z|(1 - |\xi|)^3 B_{\xi}^{\omega_{1,2}}(\xi)} \int_{\triangle(\xi,2r)} |B_{\zeta}^{\omega_{1,2}}(z)| dA(\zeta), \ (z,\xi) \in M_1 \times M_1;$$

$$G_1^*(z,\xi) = 0, \quad (z,\xi) \notin M_1 \times M_1.$$

Obviously, $|G_1(z,\xi)| \le CG_1^*(z,\xi)$.

For measureable function f and $z \in M_1$, write

$$T_1^I(f)(z) = \int_{\triangle(z,r)} G_1^*(z,\xi)f(\xi)dA(\xi)$$

and

$$T_1^{II}(f)(z) = \int_{M_1\backslash\triangle(z,r)} G_1^*(z,\xi)f(\xi)dA(\xi).$$

Thus we have only to check that T_1^I and T_1^{II} are bounded on $L^p_{\omega_{1,2}}(M)$.

For T_1^I, observe that $\sup_{(z,\zeta)\in\triangle(\xi,r)\times\triangle(\xi,2r)} |B_\zeta^{\omega_{1,2}}(z)| \le CB_\xi^{\omega_{1,2}}(\xi)$. And $1-|z| \asymp 1-|\xi|$, $|\triangle(z,r)| \asymp (1-|z|^2)^2 \asymp (1-|\xi|^2)^2 \asymp |\triangle(\xi,r)|$, when $\xi \in \triangle(z,r)$.

For $f \in L^\infty(M)$, by the definition of $G_1^*(z,\xi)$, we have $\|T_1^I(f)\|_\infty = \|T_1^I(f)\|_{L^\infty(M_1)} \le \|f\|_{L^\infty(M_1)} \cdot \sup_{z\in M_1} \int_{\triangle(z,r)} G_1^*(z,\xi)dA(\xi)$. Since

$$\begin{aligned}
&\int_{\triangle(z,r)} G_1^*(z,\xi)dA(\xi)\\
&\quad= \int_{\triangle(z,r)} \frac{1}{|\xi-z|(1-|\xi|)^3\cdot B_\xi^{\omega_{1,2}}(\xi)} \int_{\triangle(\xi,2r)} |B_\zeta^{\omega_{1,2}}(z)|dA(\zeta)dA(\xi)\\
&\quad\lesssim \int_{\triangle(z,r)} \frac{1}{|\xi-z|(1-|\xi|)^3} \int_{\triangle(\xi,2r)} dA(\zeta)dA(\xi)\\
&\quad\asymp \int_{\triangle(z,r)} \frac{1}{|\xi-z|(1-|\xi|)}dA(\xi)\\
&\quad\asymp \frac{1}{(1-|z|)} \int_{\triangle(z,r)} \frac{1}{|\xi-z|}dA(\xi)\\
&\quad\le C,
\end{aligned}$$

we get $\|T_1^I(f)\|_\infty \le C\|f\|_{L^\infty(M_1)} \lesssim \|f\|_\infty$.

For $f \in L^1_{\omega_{1,2}}(M)$, by the definition of $G_1^*(z,\xi)$, we have

$$\begin{aligned}
\|T_1^I(f)\|_1 &= \int_M |T_1^I(f)|(z)\omega_{1,2}(z)dA(z)\\
&= \int_{M_1} |T_1^I(f)|(z)\omega_1(z)dA(z)\\
&\le \int_{M_1}\int_{\triangle(z,r)} |G_1^*(z,\xi)||f(\xi)|dA(\xi)\omega_1(z)dA(z)\\
&= \int_{M_1} \omega_1(z)dA(z) \int_{\triangle(z,r)} \frac{|f(\xi)|}{|\xi-z|(1-|\xi|)^3\cdot B_\xi^{\omega_{1,2}}(\xi)}dA(\xi) \int_{\triangle(\xi,2r)} |B_\zeta^{\omega_{1,2}}(z)|dA(\zeta)\\
&\asymp \int_{M_1} \frac{|f(\xi)|\omega_1(\xi)}{(1-|\xi|)^3\cdot B_\xi^{\omega_{1,2}}(\xi)}dA(\xi) \int_{\triangle(\xi,2r)} dA(\zeta) \int_{\triangle(\xi,r)} \frac{|B_\zeta^{\omega_{1,2}}(z)|}{|\xi-z|}dA(z)\\
&\asymp \int_{M_1} \frac{|f(\xi)|\omega_1(\xi)}{(1-|\xi|)}dA(\xi) \int_{\triangle(\xi,r)} \frac{1}{|\xi-z|}dA(z)\\
&\lesssim \int_{M_1} |f(\xi)|\omega_1(\xi)dA(\xi)\\
&= \|f\|_{L^1_{\omega_1}(M_1)}\\
&\lesssim \|f\|_1.
\end{aligned}$$

The interpolation theorem shows that T_1^I is bounded on $L^p_{\omega_{1,2}}(M)$.

For T_1^{II}, observe that

$$\begin{aligned}
|T_1^{II}(f)(z)| &\le \int_{M_1\backslash\triangle(z,r)} \frac{|f(\xi)|}{|\xi-z|(1-|\xi|)^3\cdot B_\xi^{\omega_{1,2}}(\xi)}\int_{\triangle(\xi,2r)}|B_\zeta^{\omega_{1,2}}(z)|dA(\zeta)dA(\xi)\\
&= \int_{M_1\backslash\triangle(z,r)} \frac{|f(\xi)|}{|\xi-z|(1-|\xi|)^3\cdot B_\xi^{\omega_{1,2}}(\xi)}\int_{M_1}|B_\zeta^{\omega_{1,2}}(z)|\chi_{\triangle(\xi,2r)}(\zeta)dA(\zeta)dA(\xi)\\
&\asymp \int_{M_1}|B_\zeta^{\omega_{1,2}}(z)|dA(\zeta)\int_{M_1\backslash\triangle(z,r)}\frac{|f(\xi)|\chi_{\triangle(\zeta,2r)}(\xi)}{|\xi-z|(1-|\xi|)^3\cdot B_\xi^{\omega_1}(\xi)}dA(\xi)\\
&\asymp \int_{M_1}\frac{|B_\zeta^{\omega_{1,2}}(z)|}{(1-|\zeta|)^3\cdot B_\zeta^{\omega_{1,2}}(\zeta)}dA(\zeta)\int_{\{M_1\backslash\triangle(z,r)\}\cap\triangle(\zeta,2r)}\frac{|f(\xi)|}{|\xi-z|}dA(\xi)\\
&\asymp \int_{M_1}\frac{|B_\zeta^{\omega_{1,2}}(z)|}{(1-|\zeta|)^4\cdot B_\zeta^{\omega_{1,2}}(\zeta)}dA(\zeta)\int_{\{M_1\backslash\triangle(z,r)\}\cap\triangle(\zeta,2r)}\frac{|f(\xi)|(1-|\xi|)}{|\xi-z|}dA(\xi).
\end{aligned}$$

And for $\xi\in\{M_1\backslash\triangle(z,r)\}$, we have $\frac{1-|\xi|}{|1-\bar z\xi|}<1$ and $\frac{|1-\bar z\xi|}{|\xi-z|}<C$. Lemma 2.5 shows that

$$\begin{aligned}
|T_1^{II}(f)(z)| &\lesssim \int_{M_1}\frac{|B_\zeta^{\omega_{1,2}}(z)|\omega_1(\zeta)}{(1-|\zeta|)^2}dA(\zeta)\int_{\{M_1\backslash\triangle(z,r)\}\cap\triangle(\zeta,2r)}\frac{|f(\xi)|(1-|\xi|)|1-\bar z\xi|}{|\xi-z||1-\bar z\xi|}dA(\xi)\\
&< C\int_{M_1}|B_\zeta^{\omega_{1,2}}(z)|\omega_1(\zeta)dA(\zeta)\frac{1}{(1-|\zeta|)^2}\int_{\{M_1\backslash\triangle(z,r)\}\cap\triangle(\zeta,2r)}|f(\xi)|dA(\xi)\\
&\lesssim \int_{M_1}|B_\zeta^{\omega_{1,2}}(z)|\omega_1(\zeta)dA(\zeta)\frac{1}{(1-|\zeta|)^2}\int_{M_1}\chi_{\triangle(\zeta,2r)}(\xi)|f(\xi)|dA(\xi)\\
&= \int_{M_1}M_{2r}(|f|)(\zeta)|B_\zeta^{\omega_{1,2}}(z)|\omega_1(\zeta)dA(\zeta)\\
&= \int_{M}\chi_{M_1}(\zeta)M_{2r}(|f|)(\zeta)|B_\zeta^{\omega_{1,2}}(z)|\omega_{1,2}(\zeta)dA(\zeta)\\
&= P^+_{\omega_{1,2}}[\chi_{M_1}(z)M_{2r}(|f|)](z)\\
&<\infty.
\end{aligned}$$

Thus we know that T_1^{II} is bounded on $L^p_{\omega_{1,2}}(M)$. Therefore T_1 is bounded on $L^p_{\omega_{1,2}}(M)$.

For T_2, there is a similar way to T_1. Note that $G_2^*(z,\xi)$ should be written as

$$\begin{aligned}
&G_2^*(z,\xi)=\frac{1}{|\frac{r_0}{\xi}-\frac{r_0}{z}|(1-\frac{r_0}{|\xi|})^3B_\xi^{\omega_{1,2}}(\xi)}\int_{\triangle(\frac{r_0}{\xi},2r)}|B_\zeta^{\omega_{1,2}}(z)|dA(\zeta),\quad \text{where } (z,\xi)\in M_2\times M_2;\\
&G_2^*(z,\xi)=0,\quad \text{where } (z,\xi)\notin M_2\times M_2.
\end{aligned}$$

Observe that, when $(z,\xi)\in M_2\times M_2$,

$$\begin{aligned}
G_2^*(z,\xi) &= \frac{1}{|\frac{r_0}{\xi}-\frac{r_0}{z}|(1-\frac{r_0}{|\xi|})^3B_\xi^{\omega_{1,2}}(\xi)}\int_{\triangle(\frac{r_0}{\xi},2r)}|B_\zeta^{\omega_{1,2}}(z)|dA(\zeta)\\
&\asymp \frac{1}{|\frac{r_0}{\xi}-\frac{r_0}{z}|(1-\frac{r_0}{|\xi|})^3B_\xi^{\omega_{1,2}}(\xi)}\int_{M_2}\chi_{\triangle(\frac{r_0}{\xi},2r)}(\frac{r_0}{\zeta})|B_\zeta^{\omega_{1,2}}(z)|dA(\zeta). \qquad\square
\end{aligned}$$

Lemma 3.2 *Let $1<p<\infty$ and $\omega_{1,2}\in\mathcal{R}$. Suppose f satisfies $f|_{M_1}=f_1\in L^p(M_1,(1-|\cdot|)^p\omega_1(\cdot)dA(\cdot))$ and $f|_{M_2}=f_2\in L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p\omega_2(\frac{r_0}{\cdot})dA(\cdot))$. Let $u(z)=u_1(z)\chi_{M_1}(z)$*

$+ u_2(z)\chi_{M_2}(z)$, *where* $u_1(z)$ *and* $u_2(z)$ *satisfy*

$$u_1(z) = u(z)|_{M_1} = \sum_{i=1}^{\infty} B_{a_i}^{\omega_{1,2}}(z) \int_{M_1} \frac{\psi_i(\xi)}{(\xi - z)B_{a_i}^{\omega_{1,2}}(\xi)} f_1(\xi) dA(\xi), \tag{3.1}$$

$$u_2(z) = u(z)|_{M_2} = \sum_{i=1}^{\infty} B_{b_i}^{\omega_{1,2}}(z) \int_{M_2} \frac{\varphi_i(\xi)}{(\xi - z)B_{b_i}^{\omega_{1,2}}(\xi)} f_2(\xi) dA(\xi), \tag{3.2}$$

then u *is weakly solution of the equation* $\bar{\partial} u = f$ *on* M, *and there are some constants* C_1, C_2 *independent of* f *such that*

$$\|u\|_p \le C_1 \|f_1\|_{L^p(M_1,(1-|\cdot|)^p \omega_1(\,\cdot\,) dA(\,\cdot\,))} + C_2 \|f_2\|_{L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p \omega_2(\frac{r_0}{\cdot}) dA(\,\cdot\,))}.$$

Proof Since $\|u\|_p \le \|u_1\|_p + \|u_2\|_p$, by Lemma 3.1, there exist constants C_1, C_2 independent of f such that

$$\|u_1\|_p \le C_1 \|(1 - |\cdot|) f_1(\,\cdot\,)\|_{L^p_{\omega_1}(M_1)} = C_1 \|f_1\|_{L^p(M_1,(1-|\cdot|)^p \omega_1(\,\cdot\,) dA(\,\cdot\,))},$$
$$\|u_2\|_p \le C_2 \left\| \left[\left(1 - \frac{r_0}{|\cdot|}\right) \frac{r_0}{|\cdot|^2} \right] f_2(\,\cdot\,) \right\|_{L^p_{\omega_2}(M_2)} = C_2 \|f_2\|_{L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p \omega_2(\frac{r_0}{\cdot}) dA(\,\cdot\,))},$$

thus $\|u\|_p \le C_1 \|f_1\|_{L^p(M_1,(1-|\cdot|)^p \omega_1(\,\cdot\,) dA(\,\cdot\,))} + C_2 \|f_2\|_{L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p \omega_2(\frac{r_0}{\cdot}) dA(\,\cdot\,))}$.

It is a directed consequence of [3, Theorem 2.1.2] that, for $f \in C^1(\overline{M})$,

$$\frac{\partial}{\partial \bar{z}} \int_M \frac{f(\xi)}{\xi - z} dA(\xi) = f(z). \tag{3.3}$$

Then for any $\phi \in C_0^\infty(M), B_{a_i}^{\omega_{1,2}} \in H(M_1), B_{b_i}^{\omega_{1,2}} \in H(M_2)$, (3.3) shows that

$$\left\langle B_{a_i}^{\omega_{1,2}}(z) \int_{M_1} \frac{\psi_i(\xi)}{(\xi - z)B_{a_i}^{\omega_{1,2}}(\xi)} f_1(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2(M_1)} = -\langle f_1 \psi_i, \phi \rangle_{L^2(M_1)},$$
$$\left\langle B_{b_i}^{\omega_{1,2}}(z) \int_{M_2} \frac{\varphi_i(\xi)}{(\xi - z)B_{b_i}^{\omega_{1,2}}(\xi)} f_2(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2(M_2)} = -\langle f_2 \varphi_i, \phi \rangle_{L^2(M_2)}.$$

Since $\phi \in C_c^\infty(M)$, $\sup_{\xi \in M, z \in \mathrm{supp}\phi} |B_\xi^{\omega_{1,2}}(z)| < \infty$, then

$$\begin{aligned}
\int_M |u(z)| \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) &\le \int_{M_1} |u_1(z)| \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) + \int_{M_2} |u_2(z)| \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&= \int_{M_1} \left| \sum_{i=1}^{\infty} B_{a_i}^{\omega_{1,2}}(z) \int_{M_1} \frac{\psi_i(\xi)}{(\xi - z)B_{a_i}^{\omega_{1,2}}(\xi)} f_1(\xi) dA(\xi) \right| \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&\quad + \int_{M_2} \left| \sum_{i=1}^{\infty} B_{b_i}^{\omega_{1,2}}(z) \int_{M_2} \frac{\varphi_i(\xi)}{(\xi - z)B_{b_i}^{\omega_{1,2}}(\xi)} f_2(\xi) dA(\xi) \right| \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&\lesssim \int_{M_1} |B_\xi^{\omega_{1,2}}(z)| \int_{M_1} \frac{|f_1(\xi)|}{|\xi - z| B_\xi^{\omega_{1,2}}(\xi)} dA(\xi) \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&\quad + \int_{M_2} |B_\xi^{\omega_{1,2}}(z)| \int_{M_2} \frac{|f_2(\xi)| dA(\xi)}{|\xi - z| B_\xi^{\omega_{1,2}}(\xi)} \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&\lesssim \int_{M_1} \frac{|f_1(\xi)|}{B_\xi^{\omega_{1,2}}(\xi)} dA(\xi) \int_{M_1} \frac{|B_\xi^{\omega_{1,2}}(z)|}{|\xi - z|} \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z) \\
&\quad + \int_{M_2} \frac{|f_2(\xi)| dA(\xi)}{B_\xi^{\omega_{1,2}}(\xi)} dA(\xi) \int_{M_2} \frac{|B_\xi^{\omega_{1,2}}(z)|}{|\xi - z|} \left|\frac{\partial \phi}{\partial z}(z)\right| dA(z).
\end{aligned}$$

Let $c_1=\sup_{\frac{1+r_0}{2}<|z|,|\xi|<1}|B_\xi^{\omega_{1,2}}(z)||\frac{\partial\phi}{\partial z}(z)|$, $c_2=\sup_{\frac{2r_0}{1+r_0}<\frac{r_0}{|z|},\frac{r_0}{|\xi|}<1}|B_\xi^{\omega_{1,2}}(z)||\frac{\partial\phi}{\partial z}(z)|$, Lemma 2.5 shows that

$$\begin{aligned}
&\int_M |u(z)|\left|\frac{\partial\phi}{\partial z}(z)\right|dA(z)\\
&\quad\le c_1\int_{M_1}|f_1(\xi)|(1-|\xi|)^2\omega_1(\xi)dA(\xi)\int_{M_1}\frac{1}{|\xi-z|}dA(z)\\
&\qquad+c_2\int_{M_2}|f_2(\xi)|\left(1-\frac{r_0}{|\xi|}\right)^2\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\int_{M_2}\frac{1}{|\xi-z|}dA(z)\\
&\quad= c_1\int_{M_1}|f_1(\xi)|(1-|\xi|)^2\omega_1(\xi)dA(\xi)\int_{M_1}\frac{1}{|\xi-z|}dA(z)\\
&\qquad+c_2\int_{M_2}|f_2(\xi)|\left(1-\frac{r_0}{|\xi|}\right)^2\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\int_{M_2}\frac{\frac{r_0}{|z\xi|}}{|\frac{r_0}{\xi}-\frac{r_0}{z}|}dA(z)\\
&\quad\asymp \int_{M_1}|f_1(\xi)|(1-|\xi|)^2\omega_1(\xi)dA(\xi)\int_{M_1}\frac{1}{|\xi-z|}dA(z)\\
&\qquad+\int_{M_2}|f_2(\xi)|\left(1-\frac{r_0}{|\xi|}\right)^2\frac{r_0}{|\xi|^2}\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\int_{M_2}\frac{1}{|\frac{r_0}{\xi}-\frac{r_0}{z}|}dA(z)\\
&\quad\lesssim \int_{M_1}|f_1(\xi)|(1-|\xi|)\omega_1(\xi)dA(\xi)+\int_{M_2}|f_2(\xi)|\left(1-\frac{r_0}{|\xi|}\right)\frac{r_0}{|\xi|^2}\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\\
&\quad<\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
-\left\langle\frac{\partial u}{\partial\bar z},\phi\right\rangle_{L^2}&=\left\langle u,\frac{\partial\phi}{\partial z}\right\rangle_{L^2}\\
&=\sum_{i=1}^\infty\left\langle B_{a_i}^{\omega_{1,2}}(z)\int_{M_1}\frac{\psi_i(\xi)}{(\xi-z)B_{a_i}^{\omega_{1,2}}(\xi)}f_1(\xi)dA(\xi),\frac{\partial\phi}{\partial z}\right\rangle_{L^2(M_1)}\\
&\quad+\sum_{i=1}^\infty\left\langle B_{b_i}^{\omega_{1,2}}(z)\int_{M_2}\frac{\varphi_i(\xi)}{(\xi-z)B_{b_i}^{\omega_{1,2}}(\xi)}f_2(\xi)dA(\xi),\frac{\partial\phi}{\partial z}\right\rangle_{L^2(M_2)}\\
&=-\sum_{i=1}^\infty\langle f_1\psi_i,\phi\rangle_{L^2(M_1)}-\sum_{j=1}^\infty\langle f_2\varphi_i,\phi\rangle_{L^2(M_2)}\\
&=-\langle f_1+f_2,\phi\rangle_{L^2}.
\end{aligned}$$

Therefore, $\frac{\partial u}{\partial\bar z}=f_1+f_2=f|_{M_1}+f|_{M_2}=f$. □

4 $H_f: A^p_{\omega_{1,2}}(M)\mapsto L^q_{\omega_{1,2}}(M)$

Let μ be a finite positive Borel measure on M, μ is called a q-Carleson measure of $A^p_{\omega_{1,2}}(M)$ if and only if the identity operator $I_d: A^p_{\omega_{1,2}}(M)\mapsto L^q(M,d\mu)$ is bounded, and μ is called a vanishing q-Carleson measure of $A^p_{\omega_{1,2}}(M)$ if and only if the identity operator $I_d: A^p_{\omega_{1,2}}(M)\mapsto L^q(M,d\mu)$ is compact.

Lemma 4.1 *Let $\omega_{1,2}\in\mathcal{R}$, $1<p$. Suppose $f\in L^1_{\omega_{1,2}}(M)$ satisfies $\bar\partial f|_{M_1}\in L^p(M_1,(1-|\cdot|)^p\omega_1(\cdot)dA(\cdot))$ and $\bar\partial f|_{M_2}\in L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p\omega_2(\frac{r_0}{\cdot})dA(\cdot))$. For $g\in H^\infty$, let $\{a_i\}_{i=1}^\infty$ be an r-lattice of M_1, $\{\frac{r_0}{b_i}\}_{i=1}^\infty$ be an r-lattice of M_2, and $u(z)=u_1(z)\chi_{M_1}(z)+u_2(z)\chi_{M_2}(z)$,*

where $u_1(z), u_2(z)$ respectively are

$$u_1(z) = u(z)|_{M_1} = \sum_{i=1}^{\infty} B_{a_i}^{\omega_{1,2}}(z) \int_{M_1} \frac{\psi_i(\xi)}{(\xi - z)B_{a_i}^{\omega_{1,2}}(z)} g\bar{\partial} f(\xi) dA(\xi), \tag{4.1}$$

$$u_2(z) = u(z)|_{M_2} = \sum_{i=1}^{\infty} B_{b_i}^{\omega_{1,2}}(z) \int_{M_2} \frac{\varphi_i(\xi)}{(\xi - z)B_{b_i}^{\omega_{1,2}}(z)} g\bar{\partial} f(\xi) dA(\xi), \tag{4.2}$$

then $H_f(g) = u - P_{\omega_{1,2}}(u)$.

Proof For $g \in H^\infty = L^\infty(M) \cap H(M)$, then $fg \in L^1_{\omega_{1,2}}(M)$. By Lemma 3.2, we get $\bar{\partial} u = g\bar{\partial} f$ and

$$\|u\|_p \le C_1 \|(1 - |\cdot|) g(\cdot) \bar{\partial} f(\cdot)\|_{L^p_{\omega_1}(M_1)} + C_2 \left\| \left[\left(1 - \frac{r_0}{|\cdot|}\right) \frac{r_0}{|\cdot|^2} \right] g(\cdot) \bar{\partial} f(\cdot) \right\|_{L^p_{\omega_2}(M_2)} < \infty.$$

Therefore, $fg - u \in L^1_{\omega_{1,2}}(M)$ and $\bar{\partial}(fg - u) = g\bar{\partial} f - \bar{\partial} u = 0$. Hence $fg - u \in H(M)$, then we obtain $fg - u \in A^1_{\omega_{1,2}}(M)$.

And for any $f \in A^1_{\omega_{1,2}}(M)$, $P_{\omega_{1,2}} f = f$, thus $P_{\omega_{1,2}}(fg - u) = fg - u$. Furthermore, we get $H_f(g) - (u - P_{\omega_{1,2}}(u)) = (fg - u) - P_{\omega_{1,2}}(fg - u) = 0$, then $H_f(g) = u - P_{\omega_{1,2}}(u)$. □

Let $1 \le q$ and $r > 0$. For $f \in L^q_{\rm loc}$, define $G_{q,r}(f)$ as

$$G_{q,r}(f)(z) = \inf \left\{ \left(\frac{1}{|\triangle(z,r)|} \int_{\triangle(z,r)} |f - h|^q dA \right)^{\frac{1}{q}} : h \in H(\triangle(z,r)) \right\}, \quad z \in M_1;$$

$$G_{q,r}(f)(z) = \inf \left\{ \left(\frac{1}{|\triangle(\frac{r_0}{z}, r)|} \int_{\triangle(\frac{r_0}{z}, r)} |f - h|^q dA \right)^{\frac{1}{q}} : h \in H\left(\triangle\left(\frac{r_0}{z}, r \right) \right) \right\}, \quad z \in M_2.$$

Given $\omega_1, \omega_2 \in \mathcal{R}$, (2.1) and (2.2) imply:

$$G_{q,r}(f)(z) \asymp \inf \left\{ \left(\frac{1}{\omega_1(\triangle(z,r))} \int_{\triangle(z,r)} |f - h|^q \omega_1 dA \right)^{\frac{1}{q}} : h \in H(\triangle(z,r)) \right\}, \quad z \in M_1; \tag{4.3}$$

$$G_{q,r}(f)(z) \asymp \inf \left\{ \left(\frac{1}{\omega_2(\triangle(\frac{r_0}{z}, r))} \int_{\triangle(\frac{r_0}{z}, r)} |f - h|^q \omega_2 dA \right)^{\frac{1}{q}} : h \in H\left(\triangle\left(\frac{r_0}{z}, r \right) \right) \right\}, \quad z \in M_2. \tag{4.4}$$

Now we discuss the boundedness and compactness of $H_f : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ when $1 < p \le q < \infty$ or $1 < q < p < \infty$.

Theorem 4.2 *Let $\omega_{1,2} \in \mathcal{R}$ and $1 < p \le q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$ and $f \in L^1_{\omega_{1,2}}(M)$. Then the following statements are equivalent*:

(1) *$H_f : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ is bounded*;

(2) *for $\alpha \ge r > 0$, $\breve{\omega}_1^s G_{q,r}(f) \in L^\infty(M_1)$ and $\breve{\omega}_2^s G_{q,r}(f) \in L^\infty(M_2)$*;

(3) *f admits a decomposition $f = f^I + f^{II}$, where f^I satisfies $f^I|_{M_1} \in C^1(M_1)$, $f^I|_{M_2} \in C^1(M_2)$ and*

$$(1 - |\cdot|) \breve{\omega}_1^s(\cdot) |\bar{\partial} f^I(\cdot)| \in L^\infty(M_1), \tag{4.5}$$

$$\left(1 - \frac{r_0}{|\cdot|}\right) \frac{r_0}{|\cdot|^2} \breve{\omega}_2^s \left(\frac{r_0}{\cdot} \right) |\bar{\partial} f^I(\cdot)| \in L^\infty(M_2); \tag{4.6}$$

for some $r > 0$,

$$\breve{\omega}_1^s M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^\infty(M_1), \tag{4.7}$$

$$\breve{\omega}_2^s M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^\infty(M_2). \tag{4.8}$$

Moreover, for $0 < r \leq \alpha$,

$$\|H_f\|_{A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)} \asymp \|\breve{\omega}_1^s G_{q,r}(f)\|_{L^\infty(M_1)} + \|\breve{\omega}_2^s G_{q,r}(f)\|_{L^\infty(M_2)}. \tag{4.9}$$

Proof For $1 < p < \infty$, the projection $P_{\omega_{1,2}}$ is well defined on M. Therefore, $f \in L^1_{\omega_{1,2}}(M)$, H_f is well defined on $A^p_{\omega_{1,2}}(M)$.

(1)⇒(2) First we prove that $\breve{\omega}_1^s G_{q,r}(f) \in L^\infty(M_1)$ on M_1.

Fixing $r \in (0, \alpha]$, (2.5) implies that $\inf_{\xi \in \triangle(z,r)} |b^{\omega_{1,2}}_{p,z}(\xi)| \geq C|b^{\omega_{1,2}}_{p,z}(z)| \asymp \breve{\omega}_1(z)^{-\frac{1}{p}}$ for $z \in M_1$, then $\frac{1}{b^{\omega_{1,2}}_{p,z}} P_{\omega_{1,2}}(f b^{\omega_{1,2}}_{p,z}) \in H(\triangle(z,r))$. And (2.5), (4.3) tell us

$$\begin{aligned}
\|H_f(b^{\omega_{1,2}}_{p,z})\|_q^q &\geq \int_{M_1} |f(\xi) b^{\omega_{1,2}}_{p,z}(\xi) - P_{\omega_{1,2}}(f b^{\omega_{1,2}}_{p,z})(\xi)|^q \omega_1(\xi) dA(\xi) \\
&\geq \int_{\triangle(z,r)} |b^{\omega_{1,2}}_{p,z}(\xi)|^q \left| f(\xi) - \frac{1}{b^{\omega_{1,2}}_{p,z}(\xi)} P_{\omega_{1,2}}(f b^{\omega_{1,2}}_{p,z})(\xi) \right|^q \omega_1(\xi) dA(\xi) \\
&\geq C b^{\omega_{1,2}}_{p,z}(z)^q \int_{\triangle(z,r)} \left| f(\xi) - \frac{1}{b^{\omega_{1,2}}_{p,z}(\xi)} P_{\omega_{1,2}}(f b^{\omega_{1,2}}_{p,z})(\xi) \right|^q \omega_1(\xi) dA(\xi) \\
&\asymp \breve{\omega}_1^{-\frac{q}{p}}(z) \omega_1(\triangle(z,r)) G_{q,r}(f)^q(z) \\
&\asymp \breve{\omega}_1^{(sq)}(z) G_{q,r}(f)^q(z).
\end{aligned} \tag{4.10}$$

On the other hand, $\|H_f(b^{\omega_{1,2}}_{p,z})\|_q^q \leq \|H_f\|^q_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} \|b^{\omega_{1,2}}_{p,z}\|_p^q = \|H_f\|^q_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} < \infty$. Therefore, when $z \in M_1$, we get

$$\breve{\omega}_1^s G_{q,r}(f) \leq C \|H_f\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}}, \tag{4.11}$$

then $\breve{\omega}_1^s G_{q,r}(f) \in L^\infty(M_1)$.

For $z \in M_2$, we also obtain $\breve{\omega}_2^s G_{q,r}(f) \in L^\infty(M_2)$.

(2)⇒(3) Let $\{a_i\}_{i=1}^\infty$ be an $\frac{r}{2}$-lattice of M_1, $\{\psi_i\}_{i=1}^\infty$ be a partition of unity subordinate $\{\triangle(a_i, \frac{r}{2})\}_{i=1}^\infty$ with the property that $|(1-|a_i|)\bar{\partial}\psi_i| \leq C$; let $\{\frac{r_0}{b_i}\}_{i=1}^\infty$ be an $\frac{r}{2}$-lattice of M_2, $\{\varphi_i\}_{i=1}^\infty$ be a partition of unity subordinate to $\{\triangle(\frac{r_0}{b_i}, \frac{r}{2})\}_{i=1}^\infty$ with the property that $|(1 - \frac{r_0}{|b_i|})\frac{r_0}{|b_i|^2}\bar{\partial}\varphi_i| \leq C$.

If $z \in M_1$, first we have to prove that there exists some $h_i \in H(\triangle(a_i, r))$ such that

$$\frac{1}{\omega_1(\triangle(a_i, r))} \int_{\triangle(a_i,r)} |f - h_i|^q \omega_1 dA = G_{q,r}(f)^q(a_i) \tag{4.12}$$

for each i. To prove, we can choose a sequence $\{h_{i,k}\}_{k=1}^\infty \in H(\triangle(a_i, r_1))$ with the following property:

$$\frac{1}{\omega_1(\triangle(a_i, r))} \int_{\triangle(a_i,r)} |f - h_{i,k}|^q \omega_1 dA \leq G_{q,r}(f)^q(a_i) + \frac{1}{k} \quad \text{for } k = 1, 2, \ldots \tag{4.13}$$

We know $\{h_{i,k}\}_{k=1}^\infty$ is a normal family by

$$\frac{1}{\omega_1(\triangle(a_i, r))} \int_{\triangle(a_i,r)} |h_{i,k}| \omega_1 dA$$

$$
\begin{aligned}
&= \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |h_{i,k}-f+f|\omega_1 dA\\
&\le \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |h_{i,k}-f|\omega_1 dA + \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |f|\omega_1 dA\\
&\le \left\{\frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |h_{i,k}-f|^q\omega_1 dA\right\}^{\frac{1}{q}} + \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |f|\omega_1 dA\\
&\le G_{q,r}(f)^q(a_i) + 1 + \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |f|\omega_1 dA < \infty.
\end{aligned}
$$

Without loss of generality, suppose there exists a subsequence of $\{h_{i,k}\}_{k=1}^{\infty}$ that uniformly converges to $h_i \in H(\triangle(a_i,r))$ as $k\to\infty$ on any compact subset of $\triangle(a_i,r)$. Hence, the control convergence theorem and (4.3), (4.13) imply (4.12). Set $f^I(z)|_{M_1} = \sum_{i=1}^{\infty} h_i(z)\psi_i(z) \in C^1(M_1)$. For $z\in M_1$, set $I_z = \{i : z \in \triangle(a_i, \frac{r}{2})\}$. If $i \in I_z$, then $1-|z| \asymp 1-|a_i|$, and

$$
|I_z| = \sum_{i=1}^{\infty} \chi_{\triangle(a_i,\frac{r}{2})}(z) \le C.
$$

Now we are going to show

$$
|\bar\partial f^I(z)| \le \frac{C}{1-|z|}\sum_{i\in I_z} G_{q,r}(f)(a_i), \quad z\in M_1. \tag{4.14}
$$

Reference [10, pp. 254–255], set $f^I(z) = \sum_{i=1}^{\infty} h_i(z)\psi_i(z) = \sum_{i\in I_z} h_i(z)\psi_i(z)$ for $z \in M_1$. Without loss of generality, set $1 \in I_z$, then

$$
\begin{aligned}
f^I(z) &= \sum_{i\in I_z}(h_1(z)+h_i(z)-h_1(z))\psi_i(z)\\
&= h_1(z)\sum_{i\in I_z}\psi_i(z) + \sum_{i\in I_z}(h_i(z)-h_1(z))\psi_i(z)\\
&= h_1(z) + \sum_{i\in I_z}(h_i(z)-h_1(z))\psi_i(z).
\end{aligned}
$$

Therefore, $|\bar\partial f^I(z)| = |\sum_{i\in I_z}(h_i(z)-h_1(z))\bar\partial\psi_i(z)|$. We only have to check that $|h_i-h_k|$ is sup-bounded on $\triangle(a_i,r)\cap\triangle(a_k,r)$, where $i,k\in I_z$, $h_i\in H(\triangle(a_i,r)), h_k \in H(\triangle(a_k,r))$. Using the subharmonic of $|h_i-h_k|$, Hölder inequality, (2.1) and (4.12), we get

$$
\begin{aligned}
&|h_i-h_k|(z)\\
&\le \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})} |h_i-h_k|^q dA\right\}^{\frac{1}{q}}\\
&\le \left\{\frac{1}{|\triangle(a_i,\frac{r}{2})|}\int_{\triangle(a_i,r)} |h_i-f|^q dA\right\}^{\frac{1}{q}} + \left\{\frac{1}{|\triangle(a_k,\frac{r}{2})|}\int_{\triangle(a_k,r)} |h_k-f|^q dA\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)} |h_i-f|^q\omega_1 dA\right\}^{\frac{1}{q}}\left\{\frac{1}{\omega_1(\triangle(a_k,r))}\int_{\triangle(a_k,r)} |h_k-f|^q\omega_1 dA\right\}^{\frac{1}{q}}\\
&\asymp G_{q,r}(f)(a_i) + G_{q,r}(f)(a_k).
\end{aligned}
$$

Since $|(1-|a_i|)\bar\partial\psi_i| \le C$ and $1-|a_i| \asymp 1-|z|$ for $i\in I_z$, then $|\bar\partial f^I(z)| = |\sum_{i\in I_z}(h_i(z)-h_1(z))\bar\partial\psi_i(z)| \le \frac{C}{1-|z|}\sum_{i\in I_z}|h_i(z)-h_1(z)| \le \frac{C}{1-|z|}\sum_{i\in I_z} G_{q,r}(f)(a_i)$ for $z\in M_1$. Thus (4.14) holds.

Therefore, (4.14) shows that

$$(1-|z|)\breve{\omega}_1^s(z)|\bar{\partial} f^I(z)| \le C\|\breve{\omega}_1^s G_{q,r}(f)\|_{L^\infty(M_1)} \tag{4.15}$$

for $z \in M_1$.

Similarly, we get

$$\left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2}\breve{\omega}_2^s\left(\frac{r_0}{z}\right)|\bar{\partial} f^I(z)| \le C\left\|\breve{\omega}_2^s\left(\frac{r_0}{z}\right)G_{q,r}(f)(z)\right\|_{L^\infty(M_2)} \tag{4.16}$$

for $z \in M_2$. Note that when $z \in M_2$, $f^I(z) = \sum_{i=1}^\infty h_i(z)\varphi_i(z) \in C^1(M_2)$, $I_z = \{i : \frac{r_0}{z} \in \triangle(\frac{r_0}{b_i}, \frac{r}{2})\}$. (4.15) and (4.16) imply (4.5) and (4.6).

Note that (4.7) does not depend on the value of r, so we only need to prove (4.7) holds for some $r > 0$. For $f^{II} = f - f^I$, when $z \in M_1$,

$$\begin{aligned}
M_r(|f^{II}|^q)^{\frac{1}{q}}(z) &= \left\{\frac{1}{|\triangle(z,r)|}\int_{M_1}\chi_{\triangle(z,r)}|f^{II}|^q dA\right\}^{\frac{1}{q}}\\
&= \left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}\left|\sum_{i=1}^\infty (f-h_i)\psi_i\right|^q dA\right\}^{\frac{1}{q}}\\
&= \left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}\left|\sum_{i\in I_z} (f-h_i)\psi_i\right|^q dA\right\}^{\frac{1}{q}}\\
&\le \sum_{i\in I_z}\left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}|(f-h_i)\psi_i|^q dA\right\}^{\frac{1}{q}}\\
&\le \sum_{i\in I_z}\left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)\cap\triangle(a_i,\frac{r}{2})}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\lesssim \sum_{i\in I_z}\left\{\frac{1}{|\triangle(a_i,r)|}\int_{\triangle(a_i,r)}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\asymp \sum_{i\in I_z}\left\{\frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,r)}|f-h_i|^q \omega_1 dA\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4.17}$$

When $z \in M_1$, (4.12) implies

$$\breve{\omega}_1^s(z)M_r(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\|\breve{\omega}_1^s G_{q,r}(f)\|_{L^\infty(M_1)}. \tag{4.18}$$

Similarly, when $z \in M_2$, we also have

$$\begin{aligned}
M_r(|f^{II}|^q)^{\frac{1}{q}}(z) &= \left\{\frac{1}{|\triangle(\frac{r_0}{z},r)|}\int_{M_2}\chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{\xi}\right)\left|\sum_{i=1}^\infty (f(\xi)-h_i(\xi))\varphi_i(\xi)\right|^q dA(\xi)\right\}^{\frac{1}{q}}\\
&\le \sum_{i\in I_z}\left\{\frac{1}{|\triangle(\frac{r_0}{z},r)|}\int_{\triangle(\frac{r_0}{z},r)\cap\triangle(\frac{r_0}{b_i},\frac{r}{2})}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\lesssim \sum_{i\in I_z}\left\{\frac{1}{|\triangle(\frac{r_0}{b_i},r)|}\int_{\triangle(\frac{r_0}{b_i},r)}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\asymp \sum_{i\in I_z}\left\{\frac{1}{\omega_2(\triangle(\frac{r_0}{b_i},r))}\int_{\triangle(\frac{r_0}{b_i},r)}|f-h_i|^q \omega_2 dA\right\}^{\frac{1}{q}}.
\end{aligned}$$

Notice that $I_z = \{i : \frac{r_0}{z} \in \triangle(\frac{r_0}{b_i}, \frac{r}{2})\}$ for $z \in M_2$. Therefore,

$$\breve{\omega}_2^s\left(\frac{r_0}{z}\right)M_r(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\|\breve{\omega}_2^s G_{q,r}(f)\|_{L^\infty(M_2)}. \tag{4.19}$$

Therefore, (4.7) and (4.8) holds.

(3)⇒(1) Set $d\mu_1 = |f^{II}|^q\omega_1 dA$, (2.1) and the definition of f^{II} (or suppψ_i) imply

$$\begin{aligned}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}} &= \frac{\int_{\triangle(z,r)}|f^{II}|^q\omega_1 dA}{\omega_1(\triangle(z,r))^{\frac{q}{p}}} \asymp \frac{\omega_1^{(1-\frac{q}{p})}(\triangle(z,r))\int_{\triangle(z,r)}|f^{II}|^q dA}{|\triangle(z,r)|}\\ &= \frac{\omega_1^{(1-\frac{q}{p})}(\triangle(z,r))\int_{M_1}\chi_{\triangle(z,r)}|f^{II}|^q dA}{|\triangle(z,r)|}\\ &= \breve{\omega}_1^{(s\cdot q)}(z)M_r(|f^{II}|^q)(z) < \infty.\end{aligned} \tag{4.20}$$

Next we prove μ_1 is the q-Carleson measure on $A^p_{\omega_1}(M_1)$. We only need to prove that for every $g \in A^p_{\omega_{1,2}}(M)$, there exists a constant C, such that $\int_{M_1}|g(z)|^q d\mu_1(z) \le C\|g\|^q_{L^p_{\omega_1}(M_1)}$. Let $\{a_i\}_{i=1}^\infty$ be an r-lattice of M_1, then for any $z \in M_1$, we have $\sum_{i=1}^\infty \chi_{\triangle(a_i,r)}(z) = N < \infty$. Hence,

$$\int_{M_1}|g(z)|^q d\mu_1(z) \le \sum_{i=1}^\infty\int_{\triangle(a_i,r)}|g(z)|^q d\mu_1(z) \lesssim \sum_{i=1}^\infty \mu_1(\triangle(a_i,r))\sup_{z\in\triangle(a_i,r)}|g(z)|^q.$$

For $g \in A^p_{\omega_{1,2}}(M)$, there exists a constant C for all $i = 1, 2, 3, \ldots$, such that

$$\begin{aligned}\sup_{z\in\triangle(a_i,r)}|g(z)|^q &\le \frac{C}{|\triangle(a_i,r)|}\int_{\triangle(a_i,2r)}|g(z)|^q dA(z)\\ &\asymp \frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,2r)}|g(z)|^q\omega_1(z)dA(z).\end{aligned}$$

Then

$$\int_{M_1}|g(z)|^q d\mu_1(z) \le \sum_{i=1}^\infty\frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,2r)}|g(z)|^q\omega_1(z)dA(z). \tag{4.21}$$

Hölder inequality implies

$$\begin{aligned}&\frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,2r)}|g(z)|^q\omega_1(z)dA(z)\\ &\le \left\{\int_{\triangle(a_i,2r)}\left(\frac{1}{\omega_1(\triangle(a_i,r))}\right)^{\frac{p}{p-q}}\omega_1(z)dA(z)\right\}^{\frac{p-q}{p}}\left\{\int_{\triangle(a_i,2r)}|g(z)|^{q\cdot\frac{p}{q}}\omega_1(z)dA(z)\right\}^{\frac{q}{p}}\\ &\lesssim \omega_1(\triangle(a_i,r))^{-\frac{q}{p}}\left\{\int_{\triangle(a_i,2r)}|g(z)|^p\omega_1(z)dA(z)\right\}^{\frac{q}{p}}.\end{aligned}$$

Hence, with (4.21) we get

$$\begin{aligned}\int_{M_1}|g(z)|^q d\mu_1(z) &\le \sum_{i=1}^\infty\frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))^{\frac{q}{p}}}\left\{\int_{\triangle(a_i,2r)}|g(z)|^p\omega_1(z)dA(z)\right\}^{\frac{q}{p}}\\ &\lesssim \sum_{i=1}^\infty\sup_{z\in\triangle(a_i,2r)}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\left\{\int_{\triangle(a_i,2r)}|g(z)|^p\omega_1(z)dA(z)\right\}^{\frac{q}{p}}\end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\sum_{i=1}^{\infty}\left[\int_{\triangle(a_i,2r)}|g(z)|^p\omega_1(z)dA(z)\right]^{\frac{q}{p}}\\
&\le CN\sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\left\{\int_{\triangle(a_i,2r)}|g(z)|^p\omega_1(z)dA(z)\right\}^{\frac{q}{p}}\\
&\le CN\sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\left\{\int_{M_1}|g(z)|^p\omega_1(z)dA(z)\right\}^{\frac{q}{p}}\\
&\asymp \sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\|g\|^q_{L^p_{\omega_1}(M_1)}\\
&\le \sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\|g\|^q_p. \qquad (4.22)
\end{aligned}
$$

(4.20) and (4.22) imply

$$\|I_d\|^q_{A^p_{\omega_1}\mapsto L^q(d\mu_1)} \lesssim \sup_{z\in M_1}\frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}}\asymp \|\breve{\omega}_1^{s\cdot q}M_r(|f^{II}|^q)\|_{L^\infty(M_1)}<\infty.$$

Similarly, set $d\mu_2=|f^{II}|^q\omega_2 dA$, (2.2) and the definition of f^{II} (or $\operatorname{supp}\psi_i$) imply

$$
\begin{aligned}
\frac{\mu_2(\triangle(\frac{r_0}{z},r))}{\omega_1(\triangle(\frac{r_0}{z},r))^{\frac{q}{p}}}&=\frac{\int_{\triangle(\frac{r_0}{z},r)}|f^{II}|^q\omega_2 dA}{\omega_2(\triangle(\frac{r_0}{z},r))^{\frac{q}{p}}}\asymp\frac{\omega_2^{(1-\frac{q}{p})}(\triangle(\frac{r_0}{z},r))\int_{\triangle(\frac{r_0}{z},r)}|f^{II}|^q dA}{|\triangle(\frac{r_0}{z},r)|}\\
&\asymp\frac{\omega_2^{(1-\frac{q}{p})}(\triangle(\frac{r_0}{z},r))\int_{M_2}\chi_{\triangle(\frac{r_0}{z},r)}(\frac{r_0}{\xi})|f^{II}(\xi)|^q dA(\xi)}{|\triangle(\frac{r_0}{z},r)|}\\
&=\breve{\omega}_2^{(s\cdot q)}\left(\frac{r_0}{z}\right)M_r(|f^{II}|^q)(z)\\
&<\infty.
\end{aligned}
$$

We also obtain μ_2 is a q-Carleson measure on $A^p_{\omega_2}(M_2)$, and

$$\|I_d\|^q_{A^p_{\omega_2}\mapsto L^q(d\mu_2)}\lesssim\|\breve{\omega}_2^{s\cdot q}M_r(|f^{II}|^q)\|_{L^\infty(M_2)}<\infty.$$

For any $\phi\in A^p_{\omega1,2}(M)$,

$$
\begin{aligned}
\|H_{f^{II}}(\phi)\|_q&=\|(I_d-P_{\omega_{1,2}})f^{II}\phi\|_q\le C\|f^{II}\phi\|_q=C\left\{\int_M|f^{II}|^q|\phi|^q\omega_{1,2}dA\right\}^{\frac{1}{q}}\\
&\lesssim\left\{\int_{M_1}|f^{II}|^q|\phi|^q\omega_1 dA\right\}^{\frac{1}{q}}+\left\{\int_{M_2}|f^{II}|^q|\phi|^q\omega_2 dA\right\}^{\frac{1}{q}}\\
&\lesssim\|I_d\|_{A^p_{\omega_1}\mapsto L^q(d\mu_1)}\|\phi\|_{L^p_{\omega_1}(M_1)}+\|I_d\|_{A^p_{\omega_2}\mapsto L^q(d\mu_2)}\|\phi\|_{L^p_{\omega_2}(M_2)}\\
&\lesssim(\|\breve{\omega}_1^sM_r(|f^{II}|^q)^{\frac{1}{q}}\|_{L^\infty(M_1)}+\|\breve{\omega}_2^sM_r(|f^{II}|^q)^{\frac{1}{q}}\|_{L^\infty(M_2)})\|\phi\|_p. \qquad (4.23)
\end{aligned}
$$

Hence,

$$\|H_{f^{II}}\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\lesssim\|\breve{\omega}_1^sM_r(|f^{II}|^q)^{\frac{1}{q}}\|_{L^\infty(M_1)}+\|\breve{\omega}_2^sM_r(|f^{II}|^q)^{\frac{1}{q}}\|_{L^\infty(M_2)}. \qquad (4.24)$$

We know $s\le 0$ and $\breve{\omega}_1(z)\asymp\omega_1(\triangle(z,r))\le\omega_1(M_1)$, $\breve{\omega}_2(\frac{r_0}{z})\asymp\omega_2(\triangle(\frac{r_0}{z},r))\le\omega_2(M_2)$. (4.5) and (4.6) show that $(1-|z|)|\bar{\partial}f^I(z)|\in L^\infty(M_1)$ and $(1-\frac{r_0}{|z|})\frac{r_0}{|z|^2}|\bar{\partial}f^I(z)|\in L^\infty(M_2)$. Therefore, $\bar{\partial}f^I\in L^p(M_1,(1-|\cdot|)^p\omega_1(\cdot)dA(\cdot))$ and $\bar{\partial}f^I\in L^p(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^p\omega_2(\frac{r_0}{\cdot})dA(\cdot))$. By (4.7) and (4.20), we know $f^I \in L^1_{\omega_{1,2}}(M)$. For $g \in H^\infty$, define u_1 and u_2 respectively as (4.1) and (4.2):

$$u_1(z) = \sum_{i=1}^{\infty} B^{\omega_{1,2}}_{a_i}(z) \int_{M_1} \frac{\psi_i(\xi)}{(\xi - z)B^{\omega_{1,2}}_{a_i}(\xi)} g\bar{\partial} f^I(\xi) dA(\xi), \quad z \in M_1;$$
$$u_2(z) = \sum_{i=1}^{\infty} B^{\omega_{1,2}}_{b_i}(z) \int_{M_2} \frac{\varphi_i(\xi)}{(\xi - z)B^{\omega_{1,2}}_{b_i}(\xi)} g\bar{\partial} f^I(\xi) dA(\xi), \quad z \in M_2.$$

Theorem 3.2 and Theorem 4.1 show that, for M_1,

$$H_{f^I}(g) = u_1 - P_{\omega_{1,2}}(u_1) \quad \text{and} \quad \|u_1\|_{L^q_{\omega_1}(M_1)} \le C\|g(\cdot)\bar{\partial} f^I(\cdot)\|_{L^q(M_1,(1-|\cdot|)^q\omega_1(\cdot)dA(\cdot))};$$

for M_2,

$$H_{f^I}(g) = u_2 - P_{\omega_{1,2}}(u_2) \quad \text{and} \quad \|u_2\|_{L^q_{\omega_2}(M_2)} \le C\|g(\cdot)\bar{\partial} f^I(\cdot)\|_{L^q(M_2,[(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^q\omega_2(\frac{r_0}{\cdot})dA(\cdot))}.$$

By the boundedness of $P_{\omega_{1,2}}$ on $L^q_{\omega_{1,2}}(M)$, if $z \in M_1$, then

$$\|H_{f^I}(g)\|_{L^q_{\omega_1}(M_1)} \le (1 + \|P_{\omega_{1,2}}\|_{L^q_{\omega_1} \mapsto A^q_{\omega_1}})\|u_1\|_{L^q_{\omega_1}(M_1)} \le C\|(1-|z|)g(z)\bar{\partial} f^I(z)\|_{L^q_{\omega_1}(M_1)};$$

if $z \in M_2$, then

$$\|H_{f^I}(g)\|_{L^q_{\omega_2}(M_2)} \le (1 + \|P_{\omega_{1,2}}\|_{L^q_{\omega_2} \mapsto A^q_{\omega_2}})\|u_2\|_{L^q_{\omega_2}(M_2)} \le C\left\|\left(1 - \frac{r_0}{|z|}\right)\frac{r_0}{|z|^2} g(z)\bar{\partial} f^I(z)\right\|_{L^q_{\omega_2}(M_2)}.$$

Meanwhile, set $d\nu_1(\cdot) = (1-|\cdot|)^q|\bar{\partial} f^I(\cdot)|^q\omega_1(\cdot)dA(\cdot)$ and $d\nu_2(\cdot) = [(1-\frac{r_0}{|\cdot|})\frac{r_0}{|\cdot|^2}]^q|\bar{\partial} f^I(\cdot)|^q$ $\omega_2(\frac{r_0}{\cdot})dA(\cdot)$. By (2.1) and (2.2), we obtain

$$\begin{aligned}\frac{\nu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}} &= \frac{\int_{\triangle(z,r)}(1-|w|)^q|\bar{\partial} f^I(w)|^q\omega_1(w)dA(w)}{\omega_1(\triangle(z,r))^{\frac{q}{p}}} \asymp \frac{(1-|z|)^q|\bar{\partial} f^I(z)|^q(\omega_1\triangle(z,r))}{\omega_1(\triangle(z,r))^{\frac{q}{p}}} \\ &= (1-|z|)^q\breve{\omega}_1^{s\cdot q}(z)|\bar{\partial} f^I(z)|^q < \infty.\end{aligned}$$

As the proof of μ_1, we know ν_1 is a q-Carleson measure on $A^p_{\omega_1}(M_1)$, and

$$\|I_d\|_{A^p_{\omega_1} \mapsto L^q_{\nu_1}} \le C\|(1-|\cdot|)\breve{\omega}_1^s(\cdot)\bar{\partial} f^I(\cdot)\|_{L^\infty(M_1)}.$$

We can also obtain ν_2 is q-Carleson measure on $A^p_{\omega_2}(M_2)$, and

$$\|I_d\|_{A^p_{\omega_2} \mapsto L^q_{\nu_2}} \le C\left\|\left(1 - \frac{r_0}{|\cdot|}\right)\frac{r_0}{|\cdot|^2}\breve{\omega}_2^s\left(\frac{r_0}{\cdot}\right)\bar{\partial} f^I(\cdot)\right\|_{L^\infty(M_2)}.$$

Then, for any $\phi \in A^p_{\omega_{1,2}}(M)$,

$$\begin{aligned}\|H_{f^I}(\phi)\|_q &\le C\|I_d\|_{A^p_{\omega_1} \mapsto L^q(d\nu_1)}\|\phi\|_{L^p_{\omega_1}(M_1)} + C\|I_d\|_{A^p_{\omega_2} \mapsto L^q(d\nu_2)}\|\phi\|_{L^p_{\omega_2}(M_2)} \\ &\lesssim \Bigg(\|(1-|\cdot|)\breve{\omega}_1^s(\cdot)\bar{\partial} f^I(\cdot)\|_{L^\infty(M_1)} \\ &\quad + \left\|\left(1 - \frac{r_0}{|\cdot|}\right)\left(\frac{r_0}{|\cdot|^2}\right)\breve{\omega}_2^s\left(\frac{r_0}{\cdot}\right)\bar{\partial} f^I(\cdot)\right\|_{L^\infty(M_2)}\Bigg)\|\phi\|_p. \end{aligned} \tag{4.25}$$

Thus $\|H_f\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} \le \|H_{f^I}\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} + \|H_{f^{II}}\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}}$. Hence, (1) holds.

(4.9) follows from (4.11), (4.15), (4.16), (4.18), (4.19), (4.24) and (4.25). □

Theorem 4.3 *Let $\omega_{1,2} \in \mathcal{R}$ and $1 < p \le q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$ and $f \in L^1_{\omega_{1,2}}(M)$. Then the following statements are equivalent*:

(1) *$H_f : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ is compact*;

(2) *for $\alpha \ge r > 0$,*

$$\lim_{|z|\to 1^-} \breve{\omega}_1^s(z) G_{q,r}(f)(z) = 0,$$

$$\lim_{|z|\to r_0^+} \breve{\omega}_2^s\left(\frac{r_0}{z}\right) G_{q,r}(f)(z) = 0;$$

(3) *f admits a decomposition $f = f^I + f^{II}$, where f^I satisfies $f^I|_{M_1} \in C^1(M_1)$, $f^I|_{M_2} \in C^1(M_2)$, and*

$$\lim_{|z|\to 1^-} (1-|z|)\breve{\omega}_1^s(z)|\bar{\partial} f^I(z)| = 0, \tag{4.26}$$

$$\lim_{|z|\to r_0^+} \left(1 - \frac{r_0}{|z|}\right)\frac{r_0}{|z|^2}\breve{\omega}_2^s\left(\frac{r_0}{z}\right)|\bar{\partial} f^I(z)| = 0; \tag{4.27}$$

and for some $r > 0$,

$$\lim_{|z|\to 1^-} \breve{\omega}_1^s(z) M_r(|f^{II}|^q)^{\frac{1}{q}}(z) = 0, \tag{4.28}$$

$$\lim_{|z|\to r_0^+} \breve{\omega}_2^s\left(\frac{r_0}{z}\right) M_r(|f^{II}|^q)^{\frac{1}{q}}(z) = 0. \tag{4.29}$$

Proof (1)⇒(2) By [21, Lemma 2.12], we have $b^{\omega_{1,2}}_{p,z} \xrightarrow{w} 0$ as $|z| \to 1^-$ or $|z| \to r_0^+$. It is a direct consequence of (4.10) that

$$\breve{\omega}_1^s(z) G_{q,r}(f)(z) \le \|H_f(b^{\omega_{1,2}}_{p,z})\|_q \to 0, \quad |z| \to 1^-;$$

and

$$\breve{\omega}_2^s\left(\frac{r_0}{z}\right) G_{q,r}(f)(z) \le \|H_f(b^{\omega_{1,2}}_{p,z})\|_q \to 0, \quad |z| \to r_0^+.$$

(2)⇒(3) (4.15) and (4.16) imply

$$(1-|z|)\breve{\omega}_1^s(z)|\bar{\partial} f^I(z)| \le C\breve{\omega}_1^s(z) G_{q,r}(f)(z);$$

and

$$\left(1 - \frac{r_0}{|z|}\right)\frac{r_0}{|z|^2}\breve{\omega}_2^s\left(\frac{r_0}{z}\right)|\bar{\partial} f^I(z)| \le C\breve{\omega}_2^s\left(\frac{r_0}{z}\right) G_{q,r}(f)(z).$$

By (4.18) and (4.19), we get

$$\breve{\omega}_1^s(z) M_r(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\breve{\omega}_1^s(z) G_{q,r}(f)(z),$$

$$\breve{\omega}_2^s\left(\frac{r_0}{z}\right) M_r(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\breve{\omega}_2^s\left(\frac{r_0}{z}\right) G_{q,r}(f)(z).$$

(3)⇒(1) The proof is similar to that in Theorem 4.2. Firstly, we set $d\mu_1 = |f^{II}|^q \omega_1 dA$, $d\mu_2 = |f^{II}|^q \omega_2 dA$, $d\nu_1(\cdot) = (1-|\cdot|)^q |\bar{\partial} f^I(\cdot)|^q \omega_1(\cdot) dA(\cdot)$ and $d\nu_2(\cdot) = [(1-\frac{r_0}{|\cdot|})(\frac{r_0}{|\cdot|^2})]^q |\bar{\partial} f^I(\cdot)|^q$ $\omega_2(\frac{r_0}{\cdot}) dA(\cdot)$. We next prove μ_1 is a vanishing q-Carleson measure on $A^p_{\omega_1}(M_1)$, then for any $z \in M_1$, we have $\sum_{i=1}^{\infty} \chi_{\triangle(a_i,r)}(z) = N < \infty$. Moreover, $|a_i| \to 1^-$ when $i \to \infty$. Suppose

$\{g_n\}_{n=1}^{\infty}$ is a bounded sequence in $A^p_{\omega_{1,2}}(M)$ with the property that $g_n \to 0$ uniformly on any compact subset of M. (4.20) and (4.28) show that

$$\lim_{i\to\infty} \frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))^{\frac{q}{p}}} \asymp \lim_{i\to\infty} \breve{\omega}_1^{(sq)}(a_i) M_r(|f^{II}|^q)(a_i) = 0. \tag{4.30}$$

Hence, for any $\varepsilon > 0$, there exists I_0, if $i \geq I_0$, then $\frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))^{\frac{q}{p}}} < \varepsilon$. By (4.21), (4.22) and $g_n \in A^p_{\omega_{1,2}}(M)$, for any n,

$$\begin{aligned}\sum_{i=I_0}^{\infty} \int_{\triangle(a_i,r)} |g_n(z)|^q d\mu_1(z) &\leq \sum_{i=I_0}^{\infty} \frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))} \int_{\triangle(a_i,2r)} |g_n(z)|^q \omega_1(z) dA(z) \\ &\leq \sum_{i=I_0}^{\infty} \frac{\mu_1(\triangle(a_i,r))}{\omega_1(\triangle(a_i,r))^{\frac{q}{p}}} \left\{ \int_{\triangle(a_i,2r)} |g_n(z)|^p \omega_1(z) dA(z) \right\}^{\frac{q}{p}} \\ &< \varepsilon \sum_{i=I_0}^{\infty} \left\{ \int_{\triangle(a_i,2r)} |g_n(z)|^p \omega_1(z) dA(z) \right\}^{\frac{q}{p}} \\ &\leq \varepsilon N \left\{ \int_{M_1} |g_n(z)|^p \omega_1(z) dA(z) \right\}^{\frac{q}{p}} \\ &\leq \varepsilon N \|g_n\|^q_{L^p_{\omega_1}(M_1)} \\ &\leq \varepsilon N \|g_n\|^q_p. \end{aligned} \tag{4.31}$$

Since $\{g_n\}_{n=1}^{\infty}$ uniformly converges to 0 on any compact subset of M, we have

$$\lim_{n\to\infty} \sum_{i=1}^{I_0-1} \int_{\triangle(a_i,r)} |g_n(z)|^q d\mu_1(z) \to 0. \tag{4.32}$$

(4.31) and (4.32) show that

$$\begin{aligned}\lim_{n\to\infty} \int_{M_1} |g_n(z)|^q d\mu_1(z) &\leq \lim_{n\to\infty} \left(\sum_{i=I_0}^{\infty} \int_{\triangle(a_i,r)} |g_n(z)|^q d\mu_1(z) + \sum_{i=1}^{I_0-1} \int_{\triangle(a_i,r)} |g_n(z)|^q d\mu_1(z) \right) \\ &\leq \varepsilon N \|g_n\|^q_p.\end{aligned}$$

Since ε is arbitrary, we obtain

$$\lim_{n\to\infty} \|I_d(g_n)\|^q_{A^p_{\omega_1} \mapsto L^q_{\mu_1}} = \lim_{n\to\infty} \int_{M_1} |g_n(z)|^q d\mu_1(z) = 0.$$

Then $I_d : A^p_{\omega_1}(M_1) \mapsto L^q_{\mu_1}(M_1)$ is compact, i.e., μ_1 is a vanishing q-Carleson measure on $A^p_{\omega_1}(M_1)$.

Similarly, μ_2, ν_1 and ν_2 respectively are vanishing q-Carleson measures on $A^p_{\omega_2}(M_2)$, $A^p_{\omega_1}(M_1)$ and $A^p_{\omega_2}(M_2)$.

For any bounded sequence $\{\phi_m\}_{m=1}^{\infty}$ in $A^p_{\omega_{1,2}}(M)$, which uniformly converges to 0 on any compact subset of M, using (4.23), when $m \to \infty$,

$$\begin{aligned}\|H_{f^{II}}(\phi_m)\|_q &\lesssim \left\{ \int_{M_1} |f^{II}|^q |\phi_m|^q \omega_1 dA \right\}^{\frac{1}{q}} + \left\{ \int_{M_2} |f^{II}|^q |\phi_m|^q \omega_2 dA \right\}^{\frac{1}{q}} \\ &= \|I_d(\phi_m)\|_{A^p_{\omega_1} \mapsto L^q_{\mu_1}} + \|I_d(\phi_m)\|_{A^p_{\omega_2} \mapsto L^q_{\mu_2}} \to 0.\end{aligned}$$

Hence, $\lim_{m\to\infty}\|H_{f^{II}}(\phi_m)\|_q \to 0$. Using (4.25), when $m\to\infty$,

$$\begin{aligned}\|H_{f^I}(\phi_m)\|_q &\lesssim \left\{\int_{M_1}|\phi_m|^q d\nu_1(z)\right\}^{\frac{1}{q}} + \left\{\int_{M_2}|\phi_m|^q d\nu_2(z)\right\}^{\frac{1}{q}}\\ &= \|I_d(\phi_m)\|_{A^p_{\omega_1}\mapsto L^q_{\nu_1}} + \|I_d(\phi_m)\|_{A^p_{\omega_2}\mapsto L^q_{\nu_2}} \to 0.\end{aligned}$$

Hence $\lim_{m\to\infty}\|H_{f^I}(\phi_m)\|_q \to 0$. Then $\lim_{m\to\infty}\|H_f(\phi_m)\|_q \to 0$. □

Theorem 4.4 *Let $\omega_{1,2}\in\mathcal{R}$ and $1<q<p<\infty$. Set $s=\frac{1}{q}-\frac{1}{p}$ and $f\in L^1_{\omega_{1,2}}(M)$. Then the following statements are equivalent*:

(1) *$H_f : A^p_{\omega_{1,2}}(M)\mapsto L^q_{\omega_{1,2}}(M)$ is bounded*;

(2) *$H_f : A^p_{\omega_{1,2}}(M)\mapsto L^q_{\omega_{1,2}}(M)$ is compact*;

(3) *for $\frac{\alpha}{2}\geq r>0$, $G_{q,r}(f)\in L^{\frac{1}{s}}_{\omega_1}(M_1)$ and $G_{q,r}(f)\in L^{\frac{1}{s}}_{\omega_2}(M_2)$*;

(4) *f admits a decomposition $f=f^I+f^{II}$, where f^I satisfies $f^I|_{M_1}\in C^1(M_1)$, $f^I|_{M_2}\in C^1(M_2)$ and*

$$(1-|\cdot|)|\bar{\partial}f^I(\cdot)|\in L^{\frac{1}{s}}_{\omega_1}(M_1), \tag{4.33}$$

$$\left(1-\frac{r_0}{|\cdot|}\right)\frac{r_0}{|\cdot|^2}|\bar{\partial}f^I(\cdot)|\in L^{\frac{1}{s}}_{\omega_2}(M_2); \tag{4.34}$$

for some $r>0$,

$$M_r(|f^{II}|^q)^{\frac{1}{q}}\in L^{\frac{1}{s}}_{\omega_1}(M_1) \tag{4.35}$$

and

$$M_r(|f^{II}|^q)^{\frac{1}{q}}\in L^{\frac{1}{s}}_{\omega_2}(M_2). \tag{4.36}$$

Moreover, for $0<r\leq\frac{\alpha}{2}$,

$$\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}} \asymp \|G_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_1}(M_1)} + \|G_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_2}(M_2)}. \tag{4.37}$$

Proof It is clear that (2)⇒(1) holds, we only have to prove (1)⇒(3), (3)⇒(4) and (4)⇒(2).

(1)⇒(3) For $r\in(0,\alpha]$, let $\{a_i\}_{i=1}^\infty$ be an $\frac{r}{4}$-lattice of M_1, $\{\frac{r_0}{b_i}\}_{i=1}^\infty$ be an $\frac{r}{4}$-lattice of M_2. By Lemma 2.6, there exists $\{\alpha_i\}_{i=1}^\infty$, $\{\beta_i\}_{i=1}^\infty\in l^p$, such that $F_1(z)=\sum_{i=1}^\infty\alpha_i b^{\omega_{1,2}}_{p,a_i}(z)$, $F_2(z)=\sum_{i=1}^\infty\beta_i b^{\omega_{1,2}}_{p,b_i}(z)\in A^p_{\omega_{1,2}}(M)$, and

$$\|F_1\|_p=\left\|\sum_{i=1}^\infty\alpha_i b^{\omega_{1,2}}_{p,a_i}\right\|_p\leq C\|\{\alpha_i\}_{i=1}^\infty\|_{l^p},\quad \|F_2\|_p=\left\|\sum_{i=1}^\infty\beta_i b^{\omega_{1,2}}_{p,b_i}\right\|_p\leq C\|\{\beta_i\}_{i=1}^\infty\|_{l^p}.$$

As in [11], let the sequence $\{\phi_i\}_{i=1}^\infty$ be the *Rademacher* functions on $[0,1]$. By *Khintchine's* inequality, we have

$$\int_0^1\left|\sum_{i=1}^\infty\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}(z)\right|^q dt\asymp\left(\sum_{i=1}^\infty|\alpha_i|^2|b^{\omega_{1,2}}_{p,a_i}(z)|^2\right)^{\frac{q}{2}},$$

and

$$\begin{aligned}\left\|H_f\left(\sum_{i=1}^\infty\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}(z)\right)\right\|_q &\leq\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\left\|\sum_{i=1}^\infty\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}(z)\right\|_p\\ &\leq C\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\|\{\alpha_i\}_{i=1}^\infty\|_{l^p}\end{aligned}$$

$$= C\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\|\{|\alpha_i|^q\}_{i=1}^{\infty}\|^{\frac{1}{q}}_{l^{\frac{p}{q}}}.$$

Moreover, we have

$$\begin{aligned}\int_0^1\Big\|H_f\Big(\sum_{i=1}^{\infty}\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}(z)\Big)\Big\|_q^q dt &= \int_0^1\int_M\Big|H_f\Big(\sum_{i=1}^{\infty}\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}\Big)(z)\Big|^q\omega_{1,2}(z)dA(z)dt\\ &= \int_M\omega_{1,2}(z)dA(z)\int_0^1\Big|H_f\Big(\sum_{i=1}^{\infty}\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}\Big)(z)\Big|^q dt\\ &= \int_M\omega_{1,2}(z)dA(z)\int_0^1\Big|\sum_{i=1}^{\infty}\alpha_i\phi_i(t)H_f(b^{\omega_{1,2}}_{p,a_i})(z)\Big|^q dt\\ &\asymp \int_M\Big(\sum_{i=1}^{\infty}|\alpha_i|^2|H_f(b^{\omega_{1,2}}_{p,a_i})(z)|^2\Big)^{\frac{q}{2}}\omega_{1,2}(z)dA(z).\end{aligned}$$

Then

$$\begin{aligned}\|H_f\|^q_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\|\{|\alpha_i|^q\}_{i=1}^{\infty}\|_{l^{\frac{p}{q}}} &\geq \int_0^1\Big\|H_f\Big(\sum_{i=1}^{\infty}\alpha_i\phi_i(t)b^{\omega_{1,2}}_{p,a_i}(z)\Big)\Big\|_q^q dt\\ &\geq \int_{M_1}\Big(\sum_{i=1}^{\infty}|\alpha_i|^2|H_f(b^{\omega_{1,2}}_{p,a_i})(z)|^2\Big)^{\frac{q}{2}}\omega_1(z)dA(z)\\ &\asymp \sum_{j=1}^{\infty}\int_{\triangle(a_j,r)}\Big(\sum_{i=1}^{\infty}|\alpha_i|^2|H_f(b^{\omega_{1,2}}_{p,a_i})(z)|^2\Big)^{\frac{q}{2}}\omega_1(z)dA(z)\\ &\geq C\sum_{j=1}^{\infty}\int_{\triangle(a_j,r)}(|\alpha_j||H_f(b^{\omega_{1,2}}_{p,a_j})(z)|)^q\omega_1(z)dA(z)\\ &\geq C\sum_{j=1}^{\infty}|\alpha_j|^q\int_{\triangle(a_j,r)}|H_f(b^{\omega_{1,2}}_{p,a_j})(z)|^q\omega_1(z)dA(z).\end{aligned}$$

For $z\in M_1$, (2.1), (2.5) and (4.3) imply that

$$\begin{aligned}\int_{\triangle(a_j,r)}|H_f(b^{\omega_{1,2}}_{p,a_j})(z)|^q\omega_1(z)dA(z) &= \int_{\triangle(a_j,r)}|f(z)b^{\omega_{1,2}}_{p,a_j}(z)-P(fb^{\omega_{1,2}}_{p,a_j})(z)|^q\omega_1(z)dA(z)\\ &\geq Cb^{\omega_{1,2}}_{p,a_j}(a_j)^q\int_{\triangle(a_j,r)}\left|f(z)-\frac{P(fb^{\omega_{1,2}}_{p,a_j})(z)}{b^{\omega_{1,2}}_{p,a_j}(z)}\right|^q\omega_1(z)dA(z)\\ &\geq Cb^{\omega_{1,2}}_{p,a_j}(a_j)^q\omega_1(\triangle(a_j,r))G_{q,r}(f)^q(a_j)\\ &\geq C\omega_1(\triangle(a_j,r))^{1-\frac{q}{p}}G_{q,r}(f)^q(a_j).\end{aligned}$$

To sum up, we have

$$\sum_{j=1}^{\infty}|\alpha_j|^q\omega_1(\triangle(a_j,r))^{1-\frac{q}{p}}G_{q,r}(f)^q(a_j)\leq C\|H_f\|^q_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}\|\{|\alpha_i|^q\}_{i=1}^{\infty}\|_{l^{\frac{p}{q}}}.$$

From the duality theory, we get

$$\sum_{j=1}^{\infty}\omega_1(\triangle(a_j,r))G_{q,r}(f)^{\frac{pq}{p-q}}(a_j)\leq C\|H_f\|_{A^p_{\omega_1}\mapsto L^q_{\omega_1}}.$$

Then

$$\begin{aligned}\int_{M_1} G_{q,r}(f)^{\frac{pq}{p-q}}(w)\omega_1(w)dA(w) &\leq \sum_{j=1}^{\infty}\int_{\triangle(a_j,r)} G_{q,r}(f)^{\frac{pq}{p-q}}(w)\omega_1(w)dA(w)\\ &\leq \sum_{j=1}^{\infty}\omega_1(\triangle(a_j,r))G_{q,r}(f)^{\frac{pq}{p-q}}(a_j)\\ &\leq C\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}}. \end{aligned}\tag{4.38}$$

Hence, $G_{q,r}(f)\in L^{\frac{1}{s}}_{\omega_1}(M_1)$.

Similarly, when $z\in M_2$, we have $G_{q,r}(f)\in L^{\frac{1}{s}}_{\omega_2}(M_2)$.

(3)⇒(4) Similar to the proof of (2)⇒(3) in Theorem 4.2. Let $\{\frac{r_0}{b_i}\}_{i=1}^{\infty}$ be an $\frac{r}{2}$-lattice of M_2, $\{\varphi_i\}_{i=1}^{\infty}$ to be the unity partitions of $\{\triangle(\frac{r_0}{b_i},\frac{r}{2})\}_{i=1}^{\infty}$ with the propety $|(1-\frac{r_0}{|b_i|})\frac{r_0}{|b_i|^2}\bar\partial\varphi_i|\leq C$. Set $f^I(z)=\sum_{i=1}^{\infty}h_i(z)\varphi_i(z)\in C^{\infty}(M_2), z\in M_2$. Since $\rho(\frac{r_0}{b_i},\frac{r_0}{u})<\frac{r}{2}$, then $G_{q,\frac{r}{2}}(f)(b_i)\leq CG_{q,r}(f)(u)$. Furthermore,

$$\begin{aligned}G_{q,\frac{r}{2}}(f)(b_i) &= \frac{1}{|\triangle(\frac{r_0}{b_i},\frac{r}{2})|}\int_{\triangle(\frac{r_0}{b_i},\frac{r}{2})} G_{q,\frac{r}{2}}(f)(b_i)dA(u)\\ &\leq C\frac{1}{|\triangle(\frac{r_0}{b_i},\frac{r}{2})|}\int_{\triangle(\frac{r_0}{b_i},\frac{r}{2})} G_{q,r}(f)(u)dA(u).\end{aligned}$$

By (4.14), when $z\in M_2$, we know $|\bar\partial f^I(z)|\leq\frac{C}{1-\frac{r_0}{|z|}}\cdot\frac{|z|^2}{r_0}\sum_{i\in I_z}G_{q,\frac{r}{2}}(f)(b_i)$, where $I_z=\{i:\frac{r_0}{z}\in\triangle(\frac{r_0}{b_i},\frac{r}{2})\}$. Then

$$\begin{aligned}\left[\left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2}|\bar\partial f^I(z)|\right]^{\frac{pq}{p-q}} &\leq C\sum_{i\in I_z}G_{q,\frac{r}{2}}(f)^{\frac{pq}{p-q}}(b_i)\\ &\leq C\sum_{i\in I_z}\frac{1}{|\triangle(\frac{r_0}{b_i},\frac{r}{2})|}\int_{\triangle(\frac{r_0}{b_i},\frac{r}{2})}G_{q,r}(f)^{\frac{pq}{p-q}}(u)dA(u)\\ &\lesssim\frac{1}{\omega_2(\triangle(\frac{r_0}{z},\frac{r}{2}))}\int_{\triangle(\frac{r_0}{b_i},r)}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\omega_2\left(\frac{r_0}{u}\right)dA(u).\end{aligned}$$

Integrate both sides on M_2 against the measure $\omega_2 dA$, and apply Fubini theorem to get

$$\begin{aligned}&\int_{M_2}\left[\left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2}|\bar\partial f^I(z)|\right]^{\frac{pq}{p-q}}\omega_2\left(\frac{r_0}{z}\right)dA(z)\\ &\lesssim\int_{M_2}\frac{1}{\omega_2(\triangle(\frac{r_0}{z},\frac{r}{2}))}\int_{\triangle(\frac{r_0}{b_i},r)}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\omega_2\left(\frac{r_0}{u}\right)dA(u)\omega_2\left(\frac{r_0}{z}\right)dA(z)\\ &\lesssim\int_{M_2}\frac{\omega_2(\frac{r_0}{z})}{\omega_2(\triangle(\frac{r_0}{z},\frac{r}{2}))}dA(z)\int_{M_2}\chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{u}\right)G_{q,r}(f)^{\frac{pq}{p-q}}(u)\omega_2\left(\frac{r_0}{u}\right)dA(u)\\ &\asymp\int_{M_2}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\omega_2\left(\frac{r_0}{u}\right)dA(u)\int_{M_2}\frac{\chi_{\triangle(\frac{r_0}{u},r)}(\frac{r_0}{z})\omega_2(\frac{r_0}{z})}{\omega_2(\triangle(\frac{r_0}{z},\frac{r}{2}))}dA(z)\\ &\lesssim\int_{M_2}G_{q,r}(f)^{\frac{pq}{p-q}}(u)\omega_2\left(\frac{r_0}{u}\right)dA(u)\\ &=\|G_{q,r}(f)\|^{\frac{1}{s}}_{L^{\frac{1}{s}}_{\omega_2}(M_2)}\\ &<\infty.\end{aligned}\tag{4.39}$$

Thus (4.34) holds. We can also obtain (4.33) in the same way. For $z \in M_1$, (4.17) and Hölder inequality show that

$$\begin{aligned}
M_r(|f^{II}|^q)^{\frac{1}{q}}(z) &\le \sum_{i\in I_z}\left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)\cap\triangle(a_i,\frac{r}{2})}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\lesssim \sum_{i\in I_z}\left\{\frac{1}{\omega_1(\triangle(a_i,r))}\int_{\triangle(a_i,\frac{r}{2})}|f-h_i|^q\omega_1 dA\right\}^{\frac{1}{q}}\\
&\le CG_{q,\frac{r}{2}}(f)(z)\\
&= CG_{q,\frac{r}{2}}(f)(z)\cdot\frac{1}{\omega_1(\triangle(z,r))}\int_{\triangle(z,r)}\omega_1(\xi)dA(\xi)\\
&\le C\frac{1}{\omega_1(\triangle(z,r))}\int_{\triangle(z,r)}G_{q,r}(f)(\xi)\omega_1(\xi)dA(\xi)\\
&\le C\left\{\frac{1}{\omega_1(\triangle(z,r))}\int_{\triangle(z,r)}G_{q,r}(f)^{\frac{1}{s}}(\xi)\omega_1(\xi)dA(\xi)\right\}^{s}.
\end{aligned}$$

Integrating both sides on M_1 against the measure $\omega_1 dA$, we have $I_z = \{i : \frac{r_0}{z} \in \triangle(\frac{r_0}{b_i}, \frac{r}{2})\}$ and

$$\begin{aligned}
&\int_{M_1}|M_r(|f^{II}|^q)^{\frac{1}{q}}(z)|^{\frac{1}{s}}\omega_1(z)dA(z)\\
&\le C\int_{M_1}\frac{1}{\omega_1(\triangle(z,r))}\int_{\triangle(z,r)}G_{q,r}(f)^{\frac{1}{s}}(\xi)\omega_1(\xi)dA(\xi)\omega_1(z)dA(z)\\
&\le C\int_{M_1}\frac{1}{\omega_1(\triangle(z,r))}\int_{M_1}\chi_{\triangle(z,r)}(\xi)G_{q,r}(f)^{\frac{1}{s}}(\xi)\omega_1(\xi)dA(\xi)\omega_1(z)dA(z)\\
&\le C\|G_{q,r}(f)\|^{\frac{1}{s}}_{L^{\frac{1}{s}}_{\omega_1}(M_1)}\\
&< \infty. \qquad (4.40)
\end{aligned}$$

Similarly, when $z \in M_2$, we have

$$\begin{aligned}
M_r(|f_2^{II}|^q)^{\frac{1}{q}}(z) &\le \sum_{i\in I_z}\left\{\frac{1}{|\triangle(\frac{r_0}{z},r)|}\int_{\triangle(\frac{r_0}{z},r)\cap\triangle(\frac{r_0}{b_i},\frac{r}{2})}|f-h_i|^q dA\right\}^{\frac{1}{q}}\\
&\le C\left\{\frac{1}{\omega_2(\triangle(\frac{r_0}{z},r))}\int_{\triangle(\frac{r_0}{z},r)}G_{q,r}(f)^{\frac{1}{s}}(\xi)\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\right\}^{s}.
\end{aligned}$$

Integrating both sides on M_2 against the measure $\omega_2 dA$, we have

$$\int_{M_2}|M_r(|f^{II}|^q)^{\frac{1}{q}}(z)|^{\frac{1}{s}}\omega_2(\frac{r_0}{z})dA(z) \le C\|G_{q,r}(f)\|^{\frac{1}{s}}_{L^{\frac{1}{s}}_{\omega_2}(M_2)} < \infty \qquad (4.41)$$

(4.40) and (4.41) imply (4.35) and (4.36).

(4)⇒(2) First we prove H_{f^I} is compact. For $z \in M_2$, set $d\nu_2(z) = [(1-\frac{r_0}{|z|})(\frac{r_0}{|z|^2})]^q|\bar\partial f^I(z)|^q \cdot \omega_2(\frac{r_0}{z})dA(z)$. Now we prove ν_2 is a q-Carleson measure on $A^p_{\omega_2}(M_2)$. For ν_2, we have

$$\begin{aligned}
&\left\|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\right\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}_{\omega_2}(M_2)}\\
&= \left\{\int_{M_2}\left|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\right|^{\frac{p}{p-q}}\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{s}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \int_{M_2} \left| \frac{[(1-\frac{r_0}{|z|})\frac{r_0}{|z|^2}]^q |\bar\partial f^I(z)|^q \omega_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))} \right|^{\frac{p}{p-q}} \omega_2\left(\frac{r_0}{z}\right) dA(z) \right\}^s \\
&= \left\{ \int_{M_2} \left| \left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2} \bar\partial f^I(z) \right|^{\frac{pq}{p-q}} \omega_2\left(\frac{r_0}{z}\right) dA(z) \right\}^s \\
&= \left\{ \int_{M_2} \left| \left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2} \bar\partial f^I(z) \right|^{\frac{1}{s}} \omega_2\left(\frac{r_0}{z}\right) dA(z) \right\}^s \\
&= \left\| \left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2} |\bar\partial f^I(z)| \right\|_{L^{\frac{1}{s}}_{\omega_2}(M_2)} \\
&< \infty.
\end{aligned}
$$

If $g \in A^p_{\omega_{1,2}}(M)$, when $z \in M_2$, we get

$$
\begin{aligned}
\|I_d(g)\|^q_{A^p_{\omega_2} \mapsto L^q_{\nu_2}} &= \int_{M_2} |g(z)|^q d\nu_2(z) \\
&\le \int_{M_2} d\nu_2(z) \frac{1}{\omega_2(\triangle(\frac{r_0}{z},r))} \int_{M_2} |g(w)|^q \chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{w}\right) \omega_2\left(\frac{r_0}{w}\right) dA(w) \\
&\lesssim \int_{M_2} \frac{\nu_2(\triangle(\frac{r_0}{w},r))}{\omega_2(\triangle(\frac{r_0}{w},r))} |g(w)|^q \omega_2\left(\frac{r_0}{w}\right) dA(w) \\
&\le \left\{ \int_{M_2} |g(w)|^p \omega_2\left(\frac{r_0}{w}\right) dA(w) \right\}^{\frac{q}{p}} \left\{ \int_{M_2} \left| \frac{\nu_2(\triangle(\frac{r_0}{w},r))}{\omega_2(\triangle(\frac{r_0}{w},r))} \right|^{\frac{p}{p-q}} \omega_2\left(\frac{r_0}{w}\right) dA(w) \right\}^{\frac{p-q}{p}} \\
&= \|g\|^q_{L^p_{\omega_2}(M_2)} \left\| \frac{\nu_2(\triangle(\frac{r_0}{w},r))}{\omega_2(\triangle(\frac{r_0}{w},r))} \right\|_{L^{\frac{p}{p-q}}_{\omega_2}(M_2)}.
\end{aligned}
$$

Then $I_d : A^p_{\omega_2}(M_2) \mapsto L^q_{\nu_2}(M_2)$ is bounded and $\|I_d\|_{A^p_{\omega_2} \mapsto L^q_{\nu_2}} \le \|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}_{\omega_2}(M_2)} < \infty$, thus ν_2 is a q-Carleson measure for $A^p_{\omega_2}(M_2)$.

Next we are going to prove ν_2 is a vanishing q-Carleson measure on $A^p_{\omega_2}(M_2)$, we only need to check that $I_d : A^p_{\omega_2}(M_2) \mapsto L^q_{\nu_2}(M_2)$ is compact. For any bounded sequence $\{g_n\}_{n=1}^\infty$ in $A^p_{\omega_{1,2}}(M)$ with the property that uniformly converges to 0 on any compact subset of M, when $z \in M_2$, we know $(1-\frac{r_0}{|z|})\frac{r_0}{|z|^2}|\bar\partial f^I(z)| \in L^{\frac{1}{s}}_{\omega_1}(M_2)$, for any $\varepsilon > 0$, there exists $r_0 < r_2 < \frac{1+r_0}{2}$, such that

$$
\begin{aligned}
&\int_{r_0<|z|<r_2} \left| \frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))} \right|^{\frac{p}{p-q}} \omega_2\left(\frac{r_0}{z}\right) dA(z) \\
&\quad = \int_{r_0<|z|<r_2} \left| \frac{[(1-\frac{r_0}{|z|})\frac{r_0}{|z|^2}]^q |\bar\partial f^I(z)|^q \omega_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))} \right|^{\frac{p}{p-q}} \omega_2\left(\frac{r_0}{z}\right) dA(z) \\
&\quad = \int_{r_0<|z|<r_2} \left| \left(1-\frac{r_0}{|z|}\right)\frac{r_0}{|z|^2} \bar\partial f^I(z) \right|^{\frac{pq}{p-q}} \omega_2\left(\frac{r_0}{z}\right) dA(z) < \varepsilon.
\end{aligned}
$$

When $z \in M_2$, we have

$$
\begin{aligned}
\|I_d(g_n)\|^q_{L^q_{\nu_2}(M_2)} &= \int_{M_2} |g_n(z)|^q d\nu_2(z) \\
&\le \int_{r_0<|z|<r_2} |g_n(z)|^q d\nu_2(z) + \int_{r_2 \le |z| \le \frac{1+r_0}{2}} |g_n(z)|^q d\nu_2(z)
\end{aligned}
$$

$$\leq \|g_n\|^q_{L^p_{\omega_2}(M_2)}\left\{\int_{r_0<|z|<r_2}\left|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\right|^{\frac{p}{p-q}}\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{\frac{p-q}{p}}$$
$$+\left\|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\right\|_{L^{\frac{p}{p-q}}_{\omega_2}(M_2)}\left\{\int_{r_2\leq|z|\leq\frac{1+r_0}{2}}|g_n(z)|^{\frac{1}{p}}\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{\frac{p}{q}}.$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty}\|I_d(g_n)\|^q_{L^q_{\nu_2}(M_2)}\\&\quad\leq\varepsilon\|g_n\|^q_{L^p_{\omega_2}(M_2)}\\&\qquad+\left\|\frac{\nu_2(\triangle(\frac{r_0}{z},r))}{\omega_2(\triangle(\frac{r_0}{z},r))}\right\|_{L^{\frac{p}{p-q}}_{\omega_2}(M_2)}\lim_{n\to\infty}\left\{\int_{r_2\leq|z|\leq\frac{1+r_0}{2}}|g_n(z)|^{\frac{1}{p}}\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{\frac{p}{q}}\to 0\end{aligned}$$

It is clear that, $I_d : A^p_{\omega_2}(M_2)\mapsto L^q_{\nu_2}(M_2)$ is compact, when $z\in M_2$. We also get I_d : $A^p_{\omega_1}(M_1)\mapsto L^q_{\nu_1}(M_1)$ is compact, when $z\in M_1$, $d\nu_1(z)=(1-|z|)^q|\bar\partial f^I(z)|^q\omega_1(z)dA(z)$.

Finally, we are going to prove H_{f^I} is compact on M_2. For any bounded sequence $\{\phi_m\}_{m=1}^\infty$ in $A^p_{\omega_{1,2}}(M)$ with the property that uniformly converges to 0 on any compact subset of M, for each m take some $t_m\in(1,1+\frac{1}{m})$, such that $\|\phi_m(z)-\phi_m(t_mz)\|_{L^p_{\omega_2}(M_2)}<\frac{1}{m}$. Define $h_m(z)=\phi_m(t_mz)\in H^\infty$. Since $\omega_2\in\mathcal{R}$, using variable replacement, we get

$$\|h_m\|^p_{L^p_{\omega_2}(M_2)}=\int_{M_2}|h_m(z)|^p\omega_2\left(\frac{r_0}{z}\right)dA(z)=\int_{M_2}|\phi_m(t_mz)|^p\omega_2\left(\frac{r_0}{z}\right)dA(z)\leq C\|\phi_m\|^p_{L^p_{\omega_2}(M_2)}.$$

Thus $\|h_m\|_{L^p_{\omega_2}(M_2)}\leq C\|\phi_m\|_{L^p_{\omega_2}(M_2)}$. If $z\in M_2$, letting $\{\frac{r_0}{b_i}\}_{i=1}^\infty$ be the lattice of M_2, from (4.2), we set

$$u_m(z)=\sum_{i=1}^\infty B^{\omega_{1,2}}_{b_i}(z)\int_{M_2}\frac{\varphi_i(\xi)}{(\xi-z)B^{\omega_{1,2}}_{b_i}(\xi)}h_m(\xi)\bar\partial f^I(\xi)dA(\xi).$$

Lemma 3.2 implies that $\bar\partial u_m=h_m\bar\partial f^I$ and

$$\|u_m\|_{L^q_{\omega_2}(M_2)}\leq C\|h_m\|_{L^q_{\nu_2}(M_2)}.$$

Since $I_d : A^p_{\omega_2}(M_2)\mapsto L^q_{\nu_2}(M_2)$ is compact, then

$$\lim_{m\to\infty}\|u_m\|_{A^q_{\omega_2}(M_2)}\leq C\lim_{m\to\infty}\|h_m\|_{L^q_{\nu_2}(M_2)}=0.$$

On the other hand, by $M_r(|f^{II}|^q)^{\frac{1}{q}}\in L^{\frac{1}{s}}_{\omega_2}(M_2)$, the proof of Lemma 2.7, Hölder inequality, the definition of f^{II} (or supp ψ_i), with dual number $\frac{p}{p-q}$ and $\frac{p}{q}$, for M_2, we have

$$\begin{aligned}&\int_{M_2}|f^{II}(\xi)|^q\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\\&\quad\asymp\int_{M_2}|f^{II}(\xi)|^q\omega_2\left(\frac{r_0}{\xi}\right)dA(\xi)\frac{1}{|\triangle(\frac{r_0}{z},r)|}\int_{M_2}\chi_{\triangle(\frac{r_0}{\xi},r)}\left(\frac{r_0}{z}\right)dA(z)\\&\quad\asymp\int_{M_2}\frac{\omega_2(\frac{r_0}{z})}{|\triangle(\frac{r_0}{z},r)|}\int_{M_2}\chi_{\triangle(\frac{r_0}{z},r)}\left(\frac{r_0}{\xi}\right)|f^{II}(\xi)|^qdA(\xi)dA(z)\\&\quad=\int_{M_2}M_r(|f^{II}|^q)(z)\omega_2\left(\frac{r_0}{z}\right)dA(z)\\&\quad\leq\left\{\int_{M_2}M_r(|f^{II}|^q)^{\frac{p}{p-q}}(z)\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{\frac{p-q}{p}}\left\{\int_{M_2}\omega_2\left(\frac{r_0}{z}\right)dA(z)\right\}^{\frac{q}{p}}\end{aligned}$$

$$
\begin{aligned}
&\leq \left\{ \int_{M_2} M_r(|f^{II}|^q)^{\frac{1}{q}\frac{1}{s}}(z)\omega_2\left(\frac{r_0}{z}\right) dA(z) \right\}^{qs} \\
&= \|M_r(|f^{II}|^q)^{\frac{1}{q}}\|^q_{L^{\frac{1}{s}}_{\omega_2}(M_2)} \\
&< \infty.
\end{aligned}
$$

If $z \in M_2$, then $f^I(z) = f(z) - f^{II}(z) \in L^1_{\omega_2}(M_2)$. Hence, $h_m(z)f^I(z) \in L^1_{\omega_2}(M_2)$. It follows from Lemma 4.1 that $H_{f^I}(h_m) = u_m - P_{\omega_{1,2}}(u_m)$. For M_2, we have

$$
\lim_{m\to\infty} \|H_{f^I}(h_m)\|_{L^q_{\omega_2}(M_2)} \leq (1 + \|P_{\omega_{1,2}}\|_{L^p_{\omega_2}\mapsto L^q_{\omega_2}}) \lim_{m\to\infty} \|u_m\|_{L^q_{\omega_2}(M_2)} = 0. \tag{4.42}
$$

Since

$$
\lim_{m\to\infty} \|H_{f^I}(\phi_m - h_m)\|_{L^q_{\omega_2}(M_2)} \leq \|H_{f^I}\|_{A^p_{\omega_2}\mapsto L^q_{\omega_2}} \cdot \lim_{m\to\infty} \|\phi_m - h_m\|_{L^q_{\omega_2}(M_2)} = 0,
$$

then

$$
\lim_{m\to\infty} \|H_{f^I}(\phi_m)\|_{L^q_{\omega_2}(M_2)} \leq \lim_{m\to\infty} \|H_{f^I}(\phi_m - h_m)\|_{L^q_{\omega_2}(M_2)} + \lim_{m\to\infty} \|H_{f^I}(h_m)\|_{L^q_{\omega_2}(M_2)} = 0.
$$

Therefore, H_{f^I} is compact on M_2. Similarly, H_{f^I} is compact on M_1.

Now we are going to prove $H_{f^{II}}$ is compact. For M_1, we set $d\mu_1 = |f^{II}|^q\omega_1 dA$, similar to the proof of ν_2, we also get μ_1 is q-Carleson measure on $A^p_{\omega_1}(M_1)$, $I_d : A^p_{\omega_1}(M_1) \mapsto L^q_{\mu_1}(M_1)$ is compact, and

$$
\begin{aligned}
\|H_{f^{II}}(g)\|_{L^q_{\omega_1}(M_1)} &\leq C\|f^{II}g\|_{L^q_{\omega_1}(M_1)} \\
&= \|I_d(g)\|_{L^q_{\mu_1}(M_1)} \\
&\leq \left\| \frac{\mu_1(\triangle(z,r))}{\omega_1(\triangle(z,r))} \right\|^{\frac{1}{q}}_{L^{\frac{p}{p-q}}_{\omega_1}(M_1)} \|g\|_{L^p_{\omega_1}(M_1)} \\
&\asymp \|M_r(|f^{II}|^q)^{\frac{1}{q}}\|_{L^{\frac{1}{s}}_{\omega_1}(M_1)} \|g\|_{L^p_{\omega_1}(M_1)}.
\end{aligned} \tag{4.43}
$$

Then $H_{f^{II}}$ is compact on M_1. Similarly, we get $H_{f^{II}}$ is compact on M_2, where $d\mu_2 = |f^{II}|^q\omega_2 dA$. Hence, $H_{f^{II}}$ is compact.

To sum up, we know H_{f^I} and $H_{f^{II}}$ are compact. Since $H_f = H_{f^I} + H_{f^{II}}$, then H_f is compact.

(4.39), (4.40), (4.41), (4.42) and (4.43) imply

$$
\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}} \lesssim \|G_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_1}(M_1)} + \|G_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_2}(M_2)}.
$$

Combining with (4.38), we get (4.37). □

The following theorems are applications of Theorems 4.2–4.4. First we introduce a new natation.

For $f \in L^p_{\text{loc}}$ and $r > 0$, set

$$
MO_{p,r}(f)(z) = \left\{ \frac{1}{|\triangle(z,r)|} \int_{\triangle(z,r)} |f - M_r(f)(z)|^p dA \right\}^{\frac{1}{p}}, \quad \text{where } z \in M_1,
$$

$$
MO_{p,r}(f)(z) = \left\{ \frac{1}{|\triangle(\frac{r_0}{z},r)|} \int_{\triangle(\frac{r_0}{z},r)} |f - M_r(f)(z)|^p dA \right\}^{\frac{1}{p}}, \quad \text{where } z \in M_2;
$$

and

$$Osc_r(f)(z) = \sup_{\xi\in\triangle(z,r)} |f(\xi) - f(z)|, \quad \text{where } z \in M_1,$$
$$Osc_r(f)(z) = \sup_{\frac{r_0}{\xi}\in\triangle(\frac{r_0}{z},r)} |f(\xi) - f(z)|, \quad \text{where } z \in M_2.$$

Theorem 4.5 *Let $\omega_{1,2} \in \mathcal{R}$ and $1 < p \le q < \infty$. Set $f \in L^1_{\omega_{1,2}}(M)$. Then the following statements are equivalent*:

(1) $H_f, H_{\bar f} : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ *are simultaneously bounded*;

(2) *for some $r > 0$, $\breve\omega_1^s MO_{q,r}(f) \in L^\infty(M_1)$ and $\breve\omega_2^s MO_{q,r}(f) \in L^\infty(M_2)$*;

(3) *f admits a decomposition $f = f^I + f^{II}$ where f^I satisfies $f^I \in C^1(M)$, and for some $r > 0$,*

$$\breve\omega_1^s Osc_r(f^I) \in L^\infty(M_1), \tag{4.44}$$
$$\breve\omega_2^s Osc_r(f^I) \in L^\infty(M_2), \tag{4.45}$$
$$\breve\omega_1^s M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^\infty(M_1), \tag{4.46}$$
$$\breve\omega_2^s M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^\infty(M_2). \tag{4.47}$$

Morover, for some $r > 0$,

$$\|H_f\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}} + \|H_{\bar f}\|_{A^p_{\omega_{1,2}}\mapsto L^q_{\omega_{1,2}}} \asymp \|\breve\omega_1^s MO_{q,r}(f)\|_{L^\infty(M_1)} + \|\breve\omega_2^s MO_{q,r}(f)\|_{L^\infty(M_2} \tag{4.48}$$

Proof (2)⇒(3) If $z \in M_1$, set $f^I = M_{\frac{r}{2}}(f)$ and $f^{II} = f - f^I$. If $\rho(z,\xi) < \frac{r}{2}$, then $\triangle(\xi, \frac{r}{2}) \subset \triangle(z,r)$. By Hölder inequality,

$$\begin{aligned}
&|f^I(z) - f^I(\xi)| \\
&\le |f^I(z) - M_r(f)(z)| + |M_r(f)(z) - f^I(\xi)| \\
&\le \frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})} |f(w) - M_r(f)(z)|dA(w) + \frac{1}{|\triangle(\xi,\frac{r}{2})|}\int_{\triangle(\xi,\frac{r}{2})} |f(w) - M_r(f)(z)|dA(w) \\
&\le \frac{C}{|\triangle(z,r)|}\int_{\triangle(z,r)} |f(w) - M_r(f)(z)|dA(w) \\
&\le C\Big\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)} |f(w) - M_r(f)(z)|^q dA(w)\Big\}^{\frac{1}{q}} \\
&\asymp MO_{q,r}(f)(z) \\
&< \infty.
\end{aligned}$$

Therefore,

$$\breve\omega_1^s(z) Osc_r(f^I)(z) \le C\breve\omega_1^s(z) MO_{q,r}(f)(z) < \infty. \tag{4.49}$$

When $z \in M_1$, observe that

$$\begin{aligned}
|M_{\frac{r}{2}}(f)(z) - M_r(f)(z)| &\le \frac{1}{|\triangle(z,\frac{r}{2})|}\int_{M_1} \chi_{\triangle(z,\frac{r}{2})}(\xi)|f - M_r(f)(z)|dA \\
&\le C\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)} |f - M_r(f)(z)|dA \\
&= C\cdot MO_{q,r}(f)(z),
\end{aligned}$$

then

$$MO_{q,\frac{r}{2}}(f)(z) \le \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})}|f-M_r(f)(z)|^q dA(\xi)\right\}^{\frac{1}{q}} + \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})}|M_r(f)(z)-M_{\frac{r}{2}}(f)(z)|^q dA(\xi)\right\}^{\frac{1}{q}} \asymp MO_{q,r}(f)(z).$$

If $z \in M_1$,

$$\begin{aligned} M_{\frac{r}{2}}(|f^{II}|^q)^{\frac{1}{q}}(z) &= M_{\frac{r}{2}}(|f-f^I|^q)^{\frac{1}{q}}(z) \\ &= \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{M_1}\chi_{\triangle(z,\frac{r}{2})}(\xi)|f-f^I|^q(\xi)dA(\xi)\right\}^{\frac{1}{q}} \\ &\le \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})}|f(\xi)-f^I(z)|^q dA(\xi)\right\}^{\frac{1}{q}} \\ &\quad + \left\{\frac{1}{|\triangle(z,\frac{r}{2})|}\int_{\triangle(z,\frac{r}{2})}|f^I(z)-f^I(\xi)|^q dA(\xi)\right\}^{\frac{1}{q}} \\ &\le MO_{q,\frac{r}{2}}(f)(z) + Osc_{\frac{r}{2}}(f^I)(z) \\ &\le C\cdot MO_{q,\frac{r}{2}}(f)(z) \\ &\le C\cdot MO_{q,r}(f)(z). \end{aligned}$$

Then

$$\breve{\omega}_1^s(z)M_{\frac{r}{2}}(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\breve{\omega}_1^s(z)MO_{q,r}(f)(z) < \infty. \tag{4.50}$$

When $z \in M_2$ (it should be discussed at $\triangle(\frac{r_0}{z}, r)$), we can also get

$$\begin{aligned} &\breve{\omega}_2^s(z)Osc_r(f^I)(z) \le C\breve{\omega}_2^s(z)MO_{q,r}(f)(z) < \infty, \\ &\breve{\omega}_2^s(z)M_{\frac{r}{2}}(|f^{II}|^q)^{\frac{1}{q}}(z) \le C\breve{\omega}_2^s(z)MO_{q,r}(f)(z) < \infty. \end{aligned} \tag{4.51}$$

(4.49), (4.50) and (4.51) show that (4.44)–(4.47) hold.

(3)⇒(2) For f, set $f = f^I + f^{II}$, we have

$$\begin{aligned} MO_{q,r}(f^I)(z) &= \left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}|f^I(w)-M_r(f)(z)|^q dA(w)\right\}^{\frac{1}{q}} \\ &\le \left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}\left|f^I(w)-\frac{\int_{\triangle(z,r)}|f^1(\xi)|dA(\xi)}{|\triangle(z,r)|}\right|^q dA(w)\right\}^{\frac{1}{q}} \\ &\le C\left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}\left|\frac{\int_{\triangle(z,r)}|f^I(w)-f^1(\xi)|dA(\xi)}{|\triangle(z,r)|}\right|^q dA(w)\right\}^{\frac{1}{q}} \\ &= C\cdot Osc_r(f^I)(z), \end{aligned}$$

for $z \in M_1$. Then $\breve{\omega}_1^s(z)MO_{q,r}(f^I)(z) \le \breve{\omega}_1^s(z)Osc_r(f^I)(z)$. Since

$$MO_{q,r}(f^{II})(z) = \left\{\frac{1}{|\triangle(z,r)|}\int_{\triangle(z,r)}|f^{II}-M_r(f^{II})(z)|^q dA\right\}^{\frac{1}{q}}$$

$$\leq \left\{ \frac{1}{|\triangle(z,r)|} \int_{M_1} \chi_{\triangle(z,r)} |f^{II}|^q dA \right\}^{\frac{1}{q}} + M_r(f^{II})(z)$$
$$\leq 2M_r(|f^{II}|^q)^{\frac{1}{q}}(z),$$

then $\breve{\omega}_1^s(z)MO_{q,r}(f^{II})(z) \leq C\breve{\omega}_1^s(z)M_r(|f^{II}|^q)^{\frac{1}{q}}(z)$. Therefore, for $z \in M_1$, $\breve{\omega}_1^s MO_{q,r}(f) \in L^\infty(M_1)$. For $z \in M_2$, we also get $\breve{\omega}_2^s MO_{q,r}(f) \in L^\infty(M_2)$.

(1)$\Leftrightarrow$(2) Let $0 < r < \alpha$, H_f and $H_{\bar{f}}$ are bounded. By (4.9), we have

$$\breve{\omega}_1^s G_{q,r}(f) + \breve{\omega}_2^s G_{q,r}(f) \asymp \|H_f\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}},$$
$$\breve{\omega}_1^s G_{q,r}(\bar{f}) + \breve{\omega}_2^s G_{q,r}(\bar{f}) \asymp \|H_{\bar{f}}\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}}.$$

Similar to [6, Proposition 2.4 and Proposition 2.5], we have

$$\breve{\omega}_1^s MO_{q,r}(f) + \breve{\omega}_2^s MO_{q,r}(f) \asymp \breve{\omega}_1^s G_{q,r}(f) + \breve{\omega}_1^s G_{q,r}(\bar{f} + \breve{\omega}_2^s G_{q,r}(f)) + \breve{\omega}_2^s G_{q,r}(\bar{f})$$
$$\asymp \|H_f\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} + \|H_{\bar{f}}\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}}. \qquad \square$$

Theorem 4.6 *Let $\omega_{1,2} \in \mathcal{R}$ and $1 < p \leq q < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$, $f \in L^1_{\omega_{1,2}}(M)$. Then the following statements are equivalent*:

(1) *$H_f, H_{\bar{f}} : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ are simultaneously compact*;

(2) *for some $r > 0$, $\lim_{|z| \to 1^-} \breve{\omega}_1^s MO_{q,r}(f) = 0$ and $\lim_{|z| \to r_0^+} \breve{\omega}_2^s MO_{q,r}(f) = 0$*;

(3) *f admits a decomposition $f = f^I + f^{II}$, where f^I satisfies $f^I|_{M_1} \in C^1(M_1)$, $f^I|_{M_2} \in C^1(M_2)$, and for some $r > 0$,*

$$\lim_{|z| \to 1^-} \breve{\omega}_1^s Osc_r(f^I) = 0, \quad \lim_{|z| \to r_0^+} \breve{\omega}_2^s Osc_r(f^I) = 0,$$
$$\lim_{|z| \to 1^-} \breve{\omega}_1^s M_r(|f^{II}|^q)^{\frac{1}{q}} = 0, \quad \lim_{|z| \to r_0^+} \breve{\omega}_2^s M_r(|f^{II}|^q)^{\frac{1}{q}} = 0.$$

Theorem 4.7 *Let $\omega_{1,2} \in \mathcal{R}$ and $1 < q < p < \infty$. Set $s = \frac{1}{q} - \frac{1}{p}$, then for $f \in L^1_{\omega_{1,2}}(M)$, the following statements are equivalent*:

(1) *$H_f, H_{\bar{f}} : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ are bounded*;

(2) *$H_f, H_{\bar{f}} : A^p_{\omega_{1,2}}(M) \mapsto L^q_{\omega_{1,2}}(M)$ are compact*;

(3) *for some $r > 0$, $MO_{q,r}(f) \in L^{\frac{1}{s}}_{\omega_1}(M_1)$ and $MO_{q,r}(f) \in L^{\frac{1}{s}}_{\omega_2}(M_2)$*;

(4) *f admits a decomposition $f = f^I + f^{II}$, where f^I satisfies $f^I \in C^1(M)$, and for some $r > 0$,*

$$Osc_r(f^I) \in L^{\frac{1}{s}}_{\omega_1}(M_1), \quad Osc_r(f^I) \in L^{\frac{1}{s}}_{\omega_2}(M_2),$$
$$M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^{\frac{1}{s}}_{\omega_1}(M_1), \quad M_r(|f^{II}|^q)^{\frac{1}{q}} \in L^{\frac{1}{s}}_{\omega_2}(M_2).$$

Moreover, for $0 < r \leq \frac{\alpha}{2}$,

$$\|H_f\|_{A^p_{\omega_{1,2}} \mapsto L^q_{\omega_{1,2}}} \asymp \|MO_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_1}(M_1)} + \|MO_{q,r}(f)\|_{L^{\frac{1}{s}}_{\omega_2}(M_2)}.$$

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