Acta Mathematica Sinica, English Series
Jun., 2020, Vol. 36, No. 6, pp. 691–710
Published online: June 15, 2020
https://doi.org/10.1007/s10114-020-9282-8
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Construction of Generalized Diffusion Processes: the Resolvent Approach

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Abstract In this paper, we define the generalized diffusion operator $L = \frac{d}{dM}\frac{d}{dS}$ for two suitable measures on the line, which includes the generators of the birth-death processes, the one-dimensional diffusion and the gap diffusion among others. Via the standard resolvent approach, the associated generalized diffusion processes are constructed.

Keywords Generalized diffusion operator, birth-death processes, diffusion, gap diffusion, resolvent, generalized diffusion processes

MR(2010) Subject Classification 26A24, 47A10, 60G22

1 Introduction

Diffusion processes and birth-death processes are two important stochastic processes. Interestingly, these two processes share similar analytical and probability properties [3]. Let us start with the infinitesimal generator.

For birth-death processes with birth rates $(b_i)_{i\in\mathbb{N}}$ and death rates $(a_i)_{i\in\mathbb{N}}$, Feller [7] proves that the birth-death Q-matrix has the representation of $D_\mu f^+$ with $\mu_0 = 1$, $\mu_n = \frac{b_0 b_1\cdots b_{n-1}}{a_1 a_2\cdots a_n}$ ($n \geq 1$) and for $x_n = \sum_{k=1}^n \frac{1}{\mu_k a_k} + \frac{1}{a_0} I_{\{a_0\neq 0\}}$,

$$f^+(x_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}, \quad D_\mu g(x_n) = \frac{g(x_n) - g(x_{n-1})}{\mu_n}.$$

For one dimensional diffusion operator $L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ with derivative of a, b continuous and $a > 0$, Feller [6] establishes that $L = D_m D_s$ with

$$s(x) = \int_c^x \exp\left[-\int_c^y \frac{b(z)}{a(z)}dz\right]dy, \quad D_s f(x) = \frac{f'(x)}{s'(x)},$$

Received July 7, 2019, revised October 21, 2019, accepted December 5, 2019
Supported in part by NSFC (Grant No. 11771047) and Hu Xiang Gao Ceng Ci Ren Cai Ju Jiao Gong Cheng-Chuang Xin Ren Cai (Grant No. 2019RS1057)
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$$m(x)=\int_c^x \exp\left[\int_c^y \frac{b(z)}{a(z)}dz\right]\frac{dy}{a(y)},\quad D_m f(x)=\frac{f'(x)}{m'(x)}.$$

It is obvious that both m and s are strictly increasing and continuous with respect to x.

There are many works in generalizing the diffusion operator D_mD_s. Feller [6] extends it to $D_m^+D_s$, with m strictly increasing and right-continuous and D_m^+ in the sense of Radon–Nikodym. Itô and Mckean [9] consider m being strictly increasing, right-continuous or left-continuous. Kotani and Watanabe [11] generalizes m to be increasing and right-continuous by means of Krein's correspondence.

In the present paper, we proceed to define an operator so that it is not limited to the above cases (also include Brownian motion on fractals, see Example 5.4 in Section 5). Furthermore, we give the construction of the associated processes with the resolvent approach dating from [5, 7, 8].

Now, we define the generalized diffusion operators.

Let M and S be increasing functions from $\mathbb{R}$ to $\overline{\mathbb{R}}$. Assume M is left-continuous and S is right-continuous. We shall identify the functions M and S with the Borel measures M and S as $M((-\infty,x)):=M(x)$ and $S((-\infty,x]):=S(x)$ for any $x\in\mathbb{R}$. The support of measure μ is given by

$$E_\mu=\{x:\forall r>0, 0<\mu((x-r,x+r))<+\infty\}.$$

Throughout this paper, we use the following assumptions.

Assumption (1) $M\ll S$, so that $E_M\subset E_S=:E$.

(2) $-\infty<\inf E=:l_1\in E$, $+\infty\geq\sup E=:l_2\notin E$.

Remark 1.1 Assumption (2) is not determined by the definition of the generalized diffusion operators. However, when establishing the construction of the generalized diffusion processes, we only consider the case of $l_1\in E$ and $l_2\notin E$ in this paper.

Suppose the Borel (signed) measure $\hat{\mu}$ satisfies $\hat{\mu}\ll M$. Define

$$\hat{f}(x)=\hat{\mu}([l_1,x)),\quad f(x)=\int_{(l_1,x]}\hat{f}(y)S(dy),\quad x\in\mathbb{R}.$$

Then $\hat{f}$ is left-continuous and f is right-continuous, and both with bounded variation. For any $x,y\in\mathbb{R}$, let $\mu_f((x,y]):=f(y)-f(x)$ be the Borel (signed) measure induced by f. It is clear that μ_f is absolutely continuous with respect to S. Define

$$D_Sf(x):=\frac{d\mu_f}{dS}=\hat{f}(x),\quad S\text{-a.e.}.$$

Since $\hat{\mu}\ll M$, define

$$D_MD_Sf:=D_M\hat{f}:=\frac{d\hat{\mu}}{dM},\quad M\text{-a.e.}.$$

Some examples of the generalized diffusion operators are given. Please see Section 5 for details.

In this paper, we use the resolvent approach in [5, 7, 8] to construct the generalized diffusion processes corresponding to $L=D_MD_S$, which guarantees the Markov transition semi-group by Hille–Yosida theorem. That is, our task is to solve the inhomogeneous equation $(\alpha-D_MD_S)g=f$ for any (or all) $\alpha>0$. For this, we do some preparations.

To obtain the monotone solutions of the homogeneous equation, we need Feller's boundary classification. For any $x \in \mathbb{R}$, define

$$\Sigma(x) = \int_{(0,x]} S(dy) \int_{[0,y)} M(dz),$$
$$N(x) = \int_{[0,x)} M(dy) \int_{(0,y]} S(dz).$$

Analogy to [4], we call the boundary l_i $(i = 1, 2)$ to be

$$\begin{array}{lll} \text{regular}, & \text{if } \Sigma(l_i) < \infty, & N(l_i) < \infty; \\ \text{exit}, & \text{if } \Sigma(l_i) < \infty, & N(l_i) = \infty; \\ \text{entrance}, & \text{if } \Sigma(l_i) = \infty, & N(l_i) < \infty; \\ \text{natural}, & \text{if } \Sigma(l_i) = \infty, & N(l_i) = \infty, \end{array}$$

where $\Sigma(l_2)$ and $N(l_2)$ are the left limits of Σ and N at the point of l_2 respectively. Please refer to Proposition 3.3 for the detail boundary behaviours of M and S determined by boundary classification.

Let (v_1, v_2) solve the following equations:

$$\begin{cases} D_M D_S v_1 = \alpha v_1, & v_1(0) = 1, \quad D_S v_1(0) = 0, \\ D_M D_S v_2 = \alpha v_2, & v_2(0) = 0, \quad D_S v_2(0) = 1. \end{cases} \tag{1.1}$$

Now, we state the main theorems concerning the construction of generalized diffusion processes. First, we present the monotone and positive solutions of the homogeneous equation.

Theorem 1.2 *For each $\alpha > 0$, the homogeneous equation*

$$(\alpha - D_M D_S)u = 0 \tag{1.2}$$

has monotone and positive solutions which have the form of

$$u(x) = v_1(x) - \gamma v_2(x), \quad x \in E$$

up to a multiplicative positive constant, where γ is a constant to be determined. Moreover, the above monotone and positive solutions u are increasing when

$$\frac{v_1}{v_2}(l_1) \le \gamma \le \frac{\alpha \int_{[l_1,0)} v_1(y)M(dy)}{1 + \alpha \int_{[l_1,0)} v_2(y)M(dy)},$$

and are decreasing when

$$\lim_{x \uparrow l_2} \frac{\alpha \int_{[0,x)} v_1(y)M(dy)}{1 + \alpha \int_{[0,x)} v_2(y)M(dy)} \le \gamma \le \lim_{x \uparrow l_2} \frac{v_1}{v_2}(x).$$

Denote by u_1 and u_2 the increasing and decreasing positive solutions respectively. If l_i $(i = 1, 2)$ is regular, then there are infinite many u_i; Otherwise, u_i is unique.

Now for the general measures M and S, proving Theorem 1.2 depends heavily on the measure theory, see Section 2 for instance.

To obtain the solutions of the inhomogeneous equation, we introduce the Wronskian of two functions:

$$W(u, v)(x) = v(x) \cdot \left(D_S u(0) + \alpha \int_{[0,x)} u(t)M(dt) \right)$$

$$- u(x) \cdot \left(D_S v(0) + \alpha \int_{[0,x)} v(t) M(dt) \right), \quad \forall x \in E. \tag{1.3}$$

Lemma 3.1 indicates that the Wronskian of solutions of homogeneous equation (1.2) is identically equal to a constant.

The following theorem establishes the solutions of the inhomogeneous equation.

Theorem 1.3 *Suppose that u_1 and u_2 are given in Theorem 1.2. If the Wronskian of u_1 and u_2 satisfies $W(u_1, u_2) \equiv 1$, then the solutions of the inhomogeneous equation*

$$(\alpha - D_M D_S) g = f \quad (\alpha > 0) \tag{1.4}$$

are given by

$$\begin{aligned} g(x) = c_1 u_1(x) + c_2 u_2(x) + u_1(x) \int_{[x,l_2)} u_2(y) f(y) M(dy) \\ + u_2(x) \int_{[l_1,x)} u_1(y) f(y) M(dy), \quad x \in E, \end{aligned} \tag{1.5}$$

where c_1 and c_2 are arbitrary constants.

Denote by $\mathscr{A}$ the space of bounded right-continuous functions on E with the maximal module $\|f\| := \sup_{x \in E} |f(x)|$, and $\mathscr{A}^+$ the non-negative functions in $\mathscr{A}$.

The resolvent operators, which are determined by the minimal non-negative solution of the inhomogeneous equation $(\alpha - D_M D_S) g = f$ $(\alpha > 0)$, give the construction of generalized diffusion processes by Hille–Yosida theorem.

Theorem 1.4 *Suppose that u_1 and u_2 are given in Theorem 1.2. Define*

$$K(x, y) = \begin{cases} u_1(x) u_2(y), & x \le y \quad \text{and} \quad x, y \in E, \\ u_2(x) u_1(y), & y \le x \quad \text{and} \quad x, y \in E. \end{cases}$$

If $f \in \mathscr{A}^+$, then the minimal non-negative solution of the inhomogeneous equation $(\alpha - D_M D_S) g = f$ $(\alpha > 0)$ is given by

$$R_\alpha f(x) = \int_{[l_1,l_2)} K(x, y) f(y) M(dy), \quad x \in E. \tag{1.6}$$

Furthermore, R_α is a bounded linear operator on $\mathscr{A}$ which has the following properties:

(1) *Norm condition.* $\|\alpha R_\alpha\| \le 1$.

(2) *Resolvent equation.* $R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0$, $\alpha, \beta > 0$.

(3) *$\mathscr{D} = R_\alpha \mathscr{A} (\subset \mathscr{A})$ is independent of α. Moreover, $R_\alpha \mathscr{D}$ is dense in $\mathscr{D}$.*

The rest of the paper is organized as follows. In Section 2, we present some formulas for D_M and D_S, and in Sections 3–4, we prove the main results, Theorems 1.2–1.4. In Section 5, some examples are considered, including the Brownian motion on Cantor set.

To conclude this section, we make the following conventions.

(1) If f is right-continuous (or left-continuous) with bounded variation, we regard the Borel (signed) measure induced by f as $\mu_f((a, b]) := f(b) - f(a)$ (or $\mu_f([a, b)) := f(b) - f(a)$) for any $a, b \in \mathbb{R}$.

(2) When taking $(x, y]$ and $[x, y)$ as the integration interval, we read them as $(x, y] \cap E$ and $[x, y) \cap E$ respectively.

(3) If $x > y$, we make a convention that the integrations $\int_{(x,y]}$ and $\int_{[x,y)}$ represent $-\int_{(y,x]}$ and $-\int_{[y,x)}$ respectively.

2 Preliminaries

To facilitate the computation behind, we need some formulas of D_M and D_S.

Proposition 2.1 *For any $x, y \in E$ and $c \in \mathbb{R}$,*

(1) *if F is right-continuous, then $\int_{(x,y]} D_S F(z) S(dz) = F(y) - F(x)$;*
(2) *if G is left-continuous, then $\int_{[x,y)} D_M G(z) M(dz) = G(y) - G(x)$;*
(3) *suppose that $F(x) = \int_{(c,x]} f(y)S(dy)$, then we have $D_S F(x) = f(x)$;*
(4) *suppose that $G(x) = \int_{[c,x)} g(y)M(dy)$, then we have $D_M G(x) = g(x)$;*
(5) $D_S F(x) = D_S(F(\cdot-))(x)$, $D_M G(x) = D_M(G(\cdot+))(x)$;
(6) *if $M \ll S$, then $D_S F(x) = D_S M(x) \cdot D_M F(x)$.*

Proof (1)–(4) follows from Radon–Nikodym theorem, and (5) follows from Lemma 2.2 below. (6) is clear since it is known that for any σ-finite signed measure ν and σ-finite measures μ and ρ, if $\mu \ll \rho$ and $\nu \ll \mu$ then

$$\frac{d\nu}{d\rho} = \frac{d\nu}{d\mu}\frac{d\mu}{d\rho}, \quad \mu\text{-a.e.}$$ □

To obtain Proposition 2.1 (5), we prove the invariance of the measure by changing the continuity of the corresponding cumulative distribution functions.

Lemma 2.2 *Suppose h is right-continuous (or left-continuous). For any $x \in \mathbb{R}$, let $\widetilde{h}(x) = h(x-)$ (or $\widetilde{h}(x) = h(x+)$). Then h and $\widetilde{h}$ induce the same Borel (signed) measure.*

Proof Assume h is right-continuous. For any $a,\ b \in \mathbb{R}$, let μ_h and $\mu_{\widetilde{h}}$ be the Borel (signed) measures induced by h and $\widetilde{h}$ respectively. Then, for any $[a,b) \subset \mathcal{B}(\mathbb{R})$,

$$\mu_{\widetilde{h}}([a,b)) = \widetilde{h}(b) - \widetilde{h}(a) = h(b-) - h(a-) = \mu_h([a,b)).$$

Hence $\mu_h = \mu_{\widetilde{h}}$, by the measure extension theorem. □

The formulas of "D" acting on product and quotient are important, see the proof of Lemmas 3.1–3.2 for instance.

Proposition 2.3 *Suppose that μ is a Borel measure, both f and g are right-continuous (or left-continuous) with bounded variation. It holds μ-a.e. that*

(1) $D_\mu(fg)(x) = f(x+)D_\mu g(x) + g(x-)D_\mu f(x)$.
(2) *If $1/g$ is also of bounded variation, then*

$$D_\mu\left(\frac{f}{g}\right)(x) = \frac{g(x+)D_\mu f(x) - f(x+)D_\mu g(x)}{g(x+)g(x-)}.$$

To show Proposition 2.3, we need a representation of "D", which follows immediately from [1, Theorem 5.8.8].

Proposition 2.4 *Let f be right-continuous (or left-continuous) with bounded variation, μ be a Borel measure on $\mathbb{R}$. If the Borel (signed) measure μ_f induced by f is absolutely continuous with respect to μ, then*

$$D_\mu f(x) = \lim_{r\downarrow 0} \frac{\mu_f([x-r, x+r])}{\mu([x-r, x+r])} < \infty, \quad \mu\text{-a.e.}$$

Now, we can give a proof of Proposition 2.3 as follows.

Proof of Proposition 2.3 (1) Assume both f and g are right-continuous with bounded variation, then so is $f \cdot g$. It follows from Proposition 2.4 that

$$\begin{aligned}
D_\mu(f\cdot g)(x) &= \lim_{r\downarrow 0} \frac{\mu_{f\cdot g}([x-r,x+r])}{\mu([x-r,x+r])} \\
&= \lim_{r\downarrow 0} \frac{\lim_{h\downarrow 0}\mu_{f\cdot g}((x-r-h,x+r])}{\mu([x-r,x+r])} \\
&= \lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{f(x+r)g(x+r)-f(x-r-h)g(x-r-h)}{\mu([x-r,x+r])} \\
&= \lim_{r\downarrow 0}\lim_{h\downarrow 0} \\
&= f(x+)\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{g(x+r)-g(x-r-h)}{\mu([x-r,x+r])} + g(x-)\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{f(x+r)-f(x-r-h)}{\mu([x-r,x+r])} \\
&= f(x+)\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{\mu_g((x-r-h,x+r])}{\mu([x-r,x+r])} + g(x-)\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{\mu_f((x-r-h,x+r])}{\mu([x-r,x+r])} \\
&= f(x+)\lim_{r\downarrow 0} \frac{\mu_g([x-r,x+r])}{\mu([x-r,x+r])} + g(x-)\lim_{r\downarrow 0} \frac{\mu_f([x-r,x+r])}{\mu([x-r,x+r])} \\
&= f(x+)D_\mu g(x) + g(x-)D_\mu f(x).
\end{aligned}$$

(2) Similarly, we have that

$$\begin{aligned}
D_\mu(1/g)(x) &= \lim_{r\downarrow 0} \frac{\mu_{1/g}([x-r,x+r])}{\mu([x-r,x+r])} \\
&= \lim_{r\downarrow 0} \frac{\lim_{h\downarrow 0}\mu_{1/g}((x-r-h,x+r])}{\mu([x-r,x+r])} \\
&= \lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{1/g(x+r)-1/g(x-r-h)}{\mu([x-r,x+r])} \\
&= -\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{1}{g(x+r)g(x-r-h)} \cdot \frac{g(x+r)-g(x-r-h)}{\mu([x-r,x+r])} \\
&= -\frac{1}{g(x+)g(x-)}\lim_{r\downarrow 0}\lim_{h\downarrow 0} \frac{\mu_g((x-r-h,x+r])}{\mu([x-r,x+r])} \\
&= -\frac{D_\mu g(x)}{g(x+)g(x-)}.
\end{aligned}$$

Together with (1), we have thus proved (2). □

3 Proof of Theorem 1.2

Without loss of generality, we assume that $0 \in E_M = E_S = E$ and $M(0) = S(0) = 0$ in this section. To prove Theorem 1.2, we first construct a system of the homogeneous equation, and then obtain the monotone solutions by some lemmas.

Fix $\alpha > 0$, integrating the homogeneous equation $(\alpha - D_M D_S)v = 0$ with respect to the measure M and from Proposition 2.1 (2), we derive that

$$D_S v(y) = D_S v(0) + \alpha \int_{[0,y)} v(z)M(dz), \quad y \in E. \tag{3.1}$$

Integrating it with respect to the measure S and from Proposition 2.1 (1), we have

$$v(x) = v(0) + D_S v(0) \cdot S(x) + \alpha \int_{(0,x]} S(dy) \int_{[0,y)} v(z) M(dz), \quad x \in E. \tag{3.2}$$

Define

$$\begin{cases} v_1(x) = 1 + \alpha \int_{(0,x]} S(dy) \int_{[0,y)} v_1(z) M(dz), \\ v_2(x) = S(x) + \alpha \int_{(0,x]} S(dy) \int_{[0,y)} v_2(z) M(dz). \end{cases} \tag{3.3}$$

For any $x \in E$, take

$$\begin{cases} v_1^{(0)}(x) = 1, \\ v_1^{(n+1)}(x) = \alpha \int_{(0,x]} S(dy) \int_{[0,y)} v_1^{(n)}(z) M(dz), \quad n \geq 0. \end{cases}$$

and

$$\begin{cases} v_2^{(0)}(x) = S(x), \\ v_2^{(n+1)}(x) = \alpha \int_{(0,x]} S(dy) \int_{[0,y)} v_2^{(n)}(z) M(dz), \quad n \geq 0. \end{cases}$$

By the second iteration method (see [2, Theorem 2.9]), we have that

$$v_1(x) = \sum_{n=0}^{\infty} v_1^{(n)}(x), \quad v_2(x) = \sum_{n=0}^{\infty} v_2^{(n)}(x),$$

solving (1.1). More specifically, $\{v_1, v_2\}$ consists of a system of fundamental solutions of the homogeneous equation $(\alpha - D_M D_S)v = 0$ $(\alpha > 0)$.

Now, let us calculate the Wronskian of the solutions of the homogeneous equation.

Lemma 3.1 *If u and v are two solutions of the homogeneous equation $(\alpha - D_M D_S)v = 0$ $(\alpha > 0)$, then the Wronskian $W(u,v)(x)$ is independent of x. In particular, $W(v_2, v_1)|_E(x) = v_1(x) D_S v_2(x) - v_2(x) D_S v_1(x) \equiv 1$ with v_1 and v_2 given by* (3.3).

Proof We will verify that $W(u,v)(x) = W(u,v)(y)$ for any $x, y \in E$. First, we do the following transformation so that the sign "D" can work.

$$\begin{aligned} W(u,v)(z) = {} & v(z-) \cdot \Big(D_S u(0) + \alpha \int_{[0,z)} u(t) M(dt) \Big) \\ & - u(z-) \cdot \Big(D_S v(0) + \alpha \int_{[0,z)} v(t) M(dt) \Big) \\ & + (v(z) - v(z-)) \cdot \Big(D_S u(0) + \alpha \int_{[0,z)} u(t) M(dt) \Big) \\ & - (u(z) - u(z-)) \cdot \Big(D_S v(0) + \alpha \int_{[0,z)} v(t) M(dt) \Big). \end{aligned}$$

From (3.2), we have

$$v(z) - v(z-) = S(\{z\}) \Big(D_S v(0) + \alpha \int_{[0,z)} v(t) M(dt) \Big),$$

$$u(z)-u(z-)=S(\{z\})\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big),$$

which yields to

$$\begin{aligned}&(v(z)-v(z-))\cdot\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big)\\&\quad=S(\{z\})\Big(D_S v(0)+\alpha\int_{[0,z)}v(t)M(dt)\Big)\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big)\\&\quad=(u(z)-u(z-))D_S v(z).\end{aligned}$$

Consequently,

$$\begin{aligned}W(u,v)(z)=v(z-)\cdot\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big)\\-u(z-)\cdot\Big(D_S v(0)+\alpha\int_{[0,z)}v(t)M(dt)\Big).\end{aligned}$$

From Proposition 2.3 (1) and Proposition 2.1 (5)–(6), it follows that, M-a.e.

$$\begin{aligned}&D_M\Big(v(\cdot-)\cdot\Big(D_S u(0)+\alpha\int_{[0,\cdot)}u(t)M(dt)\Big)\Big)(z)\\&\quad=D_S u(0)D_M(v(\cdot-))(z)+v(z)\cdot\alpha u(z)+\alpha\int_{[0,z)}u(t)M(dt)\cdot D_M(v(\cdot-))(z)\\&\quad=\alpha u(z)v(z)+\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big)D_M v(z)\\&\quad=\alpha u(z)v(z)+\Big(D_S u(0)+\alpha\int_{[0,z)}u(t)M(dt)\Big)D_M S(z)D_S v(z)\\&\quad=\alpha u(z)v(z)+D_S u(z)D_S v(z)D_M S(z)\\&\quad=D_M\Big(u(\cdot-)\cdot\Big(D_S v(0)+\alpha\int_{[0,\cdot)}v(t)M(dt)\Big)\Big)(z),\end{aligned}$$

which yields that $(D_M W)(z)=0$ for any $z\in E$. Hence, for any $x,y\in E$,

$$0=\int_{[x,y)}D_M W(z)M(dz)=W(y)-W(x).$$

In particular,

$$W(v_2,v_1)(x)=W(v_2,v_1)(0)=1. \qquad \square$$

The next lemma gives the monotonicity of v_2/v_1 and "$D_S v_1/D_S v_2$".

Lemma 3.2 *Let v_1 and v_2 be given by* (3.3). *Then both v_2/v_1 and*

$$\frac{\alpha\int_{[0,x)}v_1(t)M(dt)}{1+\alpha\int_{[0,x)}v_2(t)M(dt)}$$

are increasing on E.

Proof (i) Clearly, both v_2 and $1/v_1$ are right-continuous with bounded variation. Then it follows from Proposition 2.3 (2) and Lemma 3.1 that, for any $z\in E$,

$$D_S\Big(\frac{v_2}{v_1}\Big)(z)=\frac{v_1(z+)D_S v_2(z)-v_2(z+)D_S v_1(z)}{v_1(z+)v_1(z-)}$$

$$
\begin{aligned}
&= \frac{v_1(z)D_S v_2(z) - v_2(z)D_S v_1(z)}{v_1(z)v_1(z-)}\\
&= \frac{1}{v_1(z)v_1(z-)}\\
&> 0.
\end{aligned}
$$

Thus, for any $x,\ y \in E$ with $x < y$, Proposition 2.1 (1) implies that

$$0 \le \int_{(x,y]} D_S\left(\frac{v_2}{v_1}\right)(z)S(dz) = \frac{v_2(y)}{v_1(y)} - \frac{v_2(x)}{v_1(x)}.$$

Hence, v_2/v_1 is increasing on E.

(ii) According to Proposition 2.3 (2), Proposition 2.1 (4) and Lemma 3.1, for any $z \in E$, we have

$$
\begin{aligned}
&D_M\left(\frac{\alpha\int_{[0,\cdot)} v_1(t)M(dt)}{1+\alpha\int_{[0,\cdot)} v_2(t)M(dt)}\right)(z)\\
&= \frac{(1+\alpha\int_{[0,z]} v_2(t)M(dt))\cdot \alpha v_1(z) - \alpha\int_{[0,z]} v_1(t)M(dt)\cdot \alpha v_2(z)}{(1+\alpha\int_{[0,z]} v_2(t)M(dt))(1+\alpha\int_{[0,z)} v_2(t)M(dt))}\\
&= \frac{\alpha W(v_2,v_1)(z) + \alpha v_2(z)M(\{z\})\cdot \alpha v_1(z) - \alpha v_1(z)M(\{z\})\cdot \alpha v_2(z)}{(1+\alpha\int_{[0,z]} v_2(t)M(dt))(1+\alpha\int_{[0,z)} v_2(t)M(dt))}\\
&= \frac{\alpha W(v_2,v_1)(z)}{(1+\alpha\int_{[0,z]} v_2(t)M(dt))(1+\alpha\int_{[0,z)} v_2(t)M(dt))}\\
&= \frac{\alpha}{(1+\alpha\int_{[0,z]} v_2(t)M(dt))(1+\alpha\int_{[0,z)} v_2(t)M(dt))}.
\end{aligned}
$$

The denominator of the last formula is positive because of the second iteration of v_2. Thus, by Proposition 2.1 (2), we have that for any $x, y \in E$ with $x < y$,

$$
\begin{aligned}
0 &\le \int_{[x,y)} D_M\left(\frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)}\right)(z)M(dz)\\
&= \frac{\alpha\int_{[0,y)} v_1(t)M(dt)}{1+\alpha\int_{[0,y)} v_2(t)M(dt)} - \frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)}.
\end{aligned}
$$

Hence, $\frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)}$ is also increasing on E. □

Next, we present the relationships of boundary behaviours between M, S and Σ, N.

Proposition 3.3 *For $i = 1, 2$,*

(1) *if l_i is regular, then $|S(l_i)| < +\infty$ and $|M(l_i)| < +\infty$;*

(2) *if l_i is exit, then $|S(l_i)| < +\infty$ and $|M(l_i)| = +\infty$;*

(3) *if l_i is entrance, then $|S(l_i)| = +\infty$ and $|M(l_i)| < +\infty$;*

(4) *if l_i is nature, then $|S(l_i)| = +\infty$ and $|M(l_i)| = +\infty$.*

Proof First, we show that $\Sigma(l_i) < +\infty$ implies $|S(l_i)| < +\infty$. Take $l_1 < x_0 < 0$ such that $M(x_0) < 0$ and $S(x_0) > -\infty$. Since $M(0) = 0$, we have

$$\Sigma(l_1) = \int_{(l_1,0]} S(dy)\int_{[y,0)} M(dz)$$

$$= -\int_{(l_1,0]} M(y)S(dy)$$
$$\geq -\int_{(l_1,x_0]} M(y)S(dy)$$
$$\geq -M(x_0)(S(x_0) - S(l_1)).$$

Thus $\Sigma(l_1) < +\infty$ implies $S(l_1) > -\infty$. Similarly, $\Sigma(l_2) < +\infty$ implies $S(l_2) < +\infty$.

Second, we show that $N(l_i) < +\infty$ implies $|M(l_i)| < +\infty$. Take $0 < x_0 < l_2$ such that $S(x_0) > 0$ and $M(x_0) < +\infty$. Since $S(0) = 0$, we have

$$\begin{aligned} N(l_2) &= \int_{[0,l_2)} M(dy) \int_{(0,y]} S(dz) \\ &= \int_{[0,l_2)} S(y)M(dy) \\ &\geq \int_{[x_0,l_2)} S(y)M(dy) \\ &\geq S(x_0)(M(l_2) - M(x_0)). \end{aligned}$$

Thus, $N(l_2) < +\infty$ implies $M(l_2) < +\infty$. In the same way, $N(l_1) < +\infty$ implies $M(l_1) > -\infty$.

Third, we show that if $N(l_i) = +\infty$ and $|S(l_i)| < +\infty$, then $|M(l_i)| = +\infty$. This is obtained from

$$\begin{aligned} N(l_2) &= \int_{[0,l_2)} M(dy) \int_{(0,y]} S(dz) \\ &= \int_{[0,l_2)} S(y)M(dy) \\ &\leq \int_{[0,l_2)} S(l_2)M(dy) \\ &= S(l_2)M(l_2), \end{aligned}$$

and similarly, $N(l_1) \leq S(l_1)M(l_1)$.

Finally, if $\Sigma(l_i) = +\infty$ and $|M(l_i)| < +\infty$, then we have that $|S(l_i)| = +\infty$ in the same way. $\square$

The following lemma is due to [12], which is an application of the integral transform theorem.

Lemma 3.4 *Suppose that μ is a Borel measure. Define its "cumulative distribution function" and "inverse cumulative distribution function" as*:

$$\varphi(x) := \mu((-\infty, x]), \quad \varphi^{-1}(y) := \inf\{x : \varphi(x) \geq y\},$$

respectively. Then for any Borel set Γ and measurable function f,

$$\int_\Gamma f d\mu = \int_{\{y:\ \varphi^{-1}(y) \in \Gamma\}} f \circ \varphi^{-1}(y)dy.$$

The following lemma gives the estimates of v_1 and v_2.

Lemma 3.5 *Let v_1 and v_2 be given by* (3.3). *Then for any $x \in E$,*

(1) $\alpha\Sigma(x) \leq v_1(x) \leq e^{\alpha\Sigma(x)}$;

(2) $\alpha \int_{(0,x]} S(dy) \int_{[0,y)} |S(z)|M(dz) \leq |v_2(x)| \leq |S(x)|e^{\alpha\Sigma(x)}$.

Proof (1) The first inequality is direct since

$$v_1(x) \geq v_1^{(1)}(x) = \alpha\Sigma(x), \quad \forall x \in E.$$

For the second inequality, it suffices to show that

$$v_1^{(n)}(x) \leq \frac{(\alpha\Sigma(x))^n}{n!}, \quad \forall n \geq 1. \tag{3.4}$$

For this, we use the mathematical induction. The case of $n = 1$ is clear. Suppose (3.4) holds for $n = k$. For $n = k+1$, since $\Sigma(z) \leq \Sigma(y-)$ for $z < y$ and $D_S\Sigma(y) = \int_{[0,y)} M(dz)$ for $y \in E$, we have that

$$\begin{aligned} v_1^{(k+1)}(x) &\leq \alpha \int_{(0,x]} S(dy) \int_{[0,y)} \frac{(\alpha\Sigma(z))^k}{k!} M(dz) \\ &= \frac{\alpha^{k+1}}{k!} \int_{(0,x]} S(dy) \int_{[0,y)\cap E} \Sigma(z)^k M(dz) \\ &\leq \frac{\alpha^{k+1}}{k!} \int_{(0,x]} \Sigma(y-)^k D_S\Sigma(y) S(dy) \\ &= \frac{\alpha^{k+1}}{k!} \int_{(0,x]} \Sigma(y-)^k \Sigma(dy). \end{aligned}$$

Define $\Sigma^{-1}(y) = \inf\{x : \Sigma(x) \geq y\}$. Take $\Gamma = (0,x]$ and $f(x) = \Sigma(x-)^k$ ($x \in E$) in Lemma 3.4 to derive that

$$\begin{aligned} v_1^{(k+1)}(x) &\leq \frac{\alpha^{k+1}}{k!} \int_{(0,x]} \Sigma(y-)^k \Sigma(dy) \\ &= \frac{\alpha^{k+1}}{k!} \int_{(\Sigma(0),\Sigma(x)]} \Sigma(\Sigma^{-1}(y)-)^k dy \\ &\leq \frac{\alpha^{k+1}}{k!} \int_{(0,\Sigma(x)]} y^k dy \\ &= \frac{(\alpha\Sigma(x))^{k+1}}{(k+1)!}, \end{aligned}$$

where the last inequality follows from $\Sigma((\Sigma^{-1}(y))-) \leq y$. Thus, (3.4) holds.

(2) The first inequality is clear since $|v_2(x)| \geq |v_2^{(1)}(x)|$. For the second inequality, it suffices to show inductively that

$$v_2^{(n)}(x) \leq |S(x)| \frac{(\alpha\Sigma(x))^n}{n!}, \quad \forall n \geq 1. \tag{3.5}$$

When $n = 1$, we have

$$\begin{aligned} v_2^{(1)}(x) &= \alpha \int_{(0,x]} S(dy) \int_{[0,y)} S(z) M(dz) \\ &\leq \alpha |S(x)| \int_{(0,x]} S(dy) \int_{[0,y)} M(dz) \\ &= \alpha |S(x)| \Sigma(x). \end{aligned}$$

Suppose (3.5) holds for $n = k$. According to (1), we have

$$|v_2^{(k+1)}(x)| \leq \alpha \int_{(0,x]} S(dy) \int_{[0,y)} |S(z)| \cdot \frac{(\alpha\Sigma(z))^k}{k!} M(dz)$$

$$
\begin{aligned}
&\le |S(x)|\cdot \frac{\alpha^{k+1}}{k!}\int_{(0,x]} S(dy)\int_{[0,y)} \Sigma(z)^k M(dz)\\
&\le |S(x)|\frac{(\alpha\Sigma(x))^{k+1}}{(k+1)!},
\end{aligned}
$$

which gives (3.5). □

The limits of $\frac{v_1}{v_2} - \frac{D_S v_1}{D_S v_2}$ at the end points depend on the boundary classification.

Lemma 3.6 *Let v_1 and v_2 given by* (3.3). *Then*

$$
\frac{v_1}{v_2}(l_1) - \frac{\alpha\int_{[l_1,0)} v_1(y)M(dy)}{1+\alpha\int_{[l_1,0)} v_2(y)M(dy)} = \begin{cases} c_1 < 0, & l_1 \text{ is regular},\\ 0, & \text{otherwise};\end{cases}
$$

and

$$
\lim_{x\uparrow l_2}\left(\frac{v_1}{v_2}(x) - \frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)}\right) = \begin{cases} c_2 > 0, & l_2 \text{ is regular},\\ 0, & \text{otherwise}.\end{cases}
$$

Proof For $x \in E$ big or small enough (so that $v_2(x)$ is not zero), since $W(v_2, v_1)|_E \equiv 1$, we have

$$
\frac{v_1}{v_2}(x) - \frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)} = \frac{1}{v_2(x)\cdot(1+\alpha\int_{[0,x)} v_2(t)M(dt))}.
$$

When l_i is regular, from Proposition 3.3 we have $|S(l_i)| < \infty$, $|M(l_i)| < \infty$ and $\Sigma(l_i) < \infty$ for $i = 1, 2$. This, together with Lemma 3.5 and $M(0) = 0$, derives that

$$
\begin{aligned}
&|v_2(l_i)| \le |S(l_i)|\mathrm{e}^{\alpha\Sigma(l_i)} < \infty,\\
&1+\alpha\int_{[0,l_i)} v_2(t)M(dt) \le 1+\alpha\int_{[0,l_i)} v_2(l_i)M(dt) = 1+|v_2(l_i)M(l_i)| < +\infty.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{v_1}{v_2}(l_1) - \frac{\alpha\int_{[l_1,0)} -v_1(t)M(dt)}{1+\alpha\int_{[l_1,0)} -v_2(t)M(dt)} = c_1 < 0,\\
&\lim_{x\uparrow l_2}\left(\frac{v_1}{v_2}(x) - \frac{\alpha\int_{[0,x)} v_1(t)M(dt)}{1+\alpha\int_{[0,x)} v_2(t)M(dt)}\right) = c_2 > 0.
\end{aligned}
$$

When l_i is not regular, either $|S(l_i)|$ or $|M(l_i)|$ is infinity. If $S(l_1) = -\infty$, take $l_1 < x_0 < 0$ such that $\int_{[x_0,0)} |S(z)|M(dz) > 0$ and $|S(x_0)| < +\infty$. Then we have from Lemma 3.5 that

$$
\begin{aligned}
|v_2(l_1)| &\ge \alpha\int_{(l_1,0]} S(dy)\int_{[y,0)} |S(z)|M(dz)\\
&\ge \alpha\int_{(l_1,x_0]} S(dy)\int_{[x_0,0)} |S(z)|M(dz)\\
&= \alpha(S(x_0)-S(l_1))\int_{[x_0,0)} |S(z)|M(dz)\\
&= +\infty.
\end{aligned}
$$

Similarly, if $S(l_2) = +\infty$, then $v_2(l_2) = +\infty$.

When $|M(l_i)| = \infty$, take x_0 such that $|v_2(x_0)| > 0$. We have

$$\begin{aligned}1 + \alpha \int_{[0,l_i)} v_2(t)M(dt) &\geq 1 + \alpha \int_{[x_0,l_i)} v_2(t)M(dt) \\ &\geq 1 + \alpha v_2(x_0)(M(l_i) - M(x_0)) \\ &= +\infty.\end{aligned}$$

This completes the proof. □

Now, it is ready to prove Theorem 1.2.

Proof of Theorem 1.2 Suppose that the positive solution u has the form of

$$u = v_1 - \gamma v_2.$$

According to Lemma 3.2, we have

$$\frac{v_1(l_1)}{v_2(l_1)} < \gamma \leq \lim_{x \uparrow l_2} \frac{v_1}{v_2}(x). \tag{3.6}$$

Since u_1 is increasing, it holds that S-a.e.

$$D_S u_1 = D_S v_1 - \gamma D_S v_2 \geq 0,$$

which yields that $\gamma \leq \frac{\alpha \int_{[0,x)} v_1(t)M(dt)}{1+\alpha \int_{[0,x)} v_2(t)M(dt)}$ for any $x \in \mathbb{R}$. By Lemma 3.2 again,

$$\gamma \leq \frac{\alpha \int_{[l_1,0)} v_1(t)M(dt)}{1 + \alpha \int_{[l_1,0)} v_2(t)M(dt)} \leq 0.$$

Combining with (3.6) and $\lim\limits_{x \uparrow l_2} \frac{v_1}{v_2}(x) \geq 0$, we obtain that

$$\frac{v_1(l_1)}{v_2(l_1)} \leq \gamma \leq \frac{D_S v_1(l_1)}{D_S v_2(l_1)}.$$

Since u_2 is decreasing, it follows that S-a.e.

$$D_S u_2 = D_S v_1 - \gamma D_S v_2 \leq 0,$$

which yields that $\gamma \geq \frac{\alpha \int_{[0,x)} v_1(t)M(dt)}{1+\alpha \int_{[0,x)} v_2(t)M(dt)}$ for any $x \in \mathbb{R}$. Using Lemma 3.2 again, we have

$$\gamma \geq \lim_{x \uparrow l_2} \frac{\alpha \int_{[0,x)} v_1(t)M(dt)}{1 + \alpha \int_{[0,x)} v_2(t)M(dt)} \geq 0.$$

Combining with (3.6) and $\frac{v_1}{v_2}(l_1) \leq 0$, we obtain that

$$\lim_{x \uparrow l_2} \frac{\alpha \int_{[0,x)} v_1(t)M(dt)}{1 + \alpha \int_{[0,x)} v_2(t)M(dt)} \leq \gamma \leq \lim_{x \uparrow l_2} \frac{v_1}{v_2}(x).$$ □

4 Proof of Theorems 1.3–1.4

First, we present the solutions of the inhomogeneous equation.

Proof of Theorem 1.3 Theorem 1.2 implies that u_1 and u_2 are linearly independent. Thus, $\{u_1, u_2\}$ consists of a system of fundamental solutions of the homogeneous equation (1.2). Suppose that the inhomogeneous equation (1.4) has solutions with the form

$$g(x) = c_1(x)u_1(x) + c_2(x)u_2(x),$$

where $c_1(x)$ and $c_2(x)$ are right-continuous functions to be determined later.

Proposition 2.3 (1) implies that for any $x \in E$,

$$\begin{aligned} D_S g(x) &= c_1(x-)D_S u_1(x) + u_1(x+)D_S c_1(x) + c_2(x-)D_S u_2(x) + u_2(x+)D_S c_2(x) \\ &= c_1(x-)D_S u_1(x) + c_2(x-)D_S u_2(x) + u_1(x)D_S c_1(x) + u_2(x)D_S c_2(x). \end{aligned}$$

By setting

$$u_1(x)D_S c_1(x) + u_2(x)D_S c_2(x) = 0, \tag{4.1}$$

we have that

$$D_S g(x) = c_1(x-)D_S u_1(x) + c_2(x-)D_S u_2(x).$$

Moreover, $\forall\, x \in E$,

$$\begin{aligned} D_M D_S g(x) &= c_1(x)D_M D_S u_1(x) + D_S u_1(x-)D_M(c_1(\cdot-))(x) \\ &\quad + c_2(x)D_M D_S u_2(x) + D_S u_2(x-)D_M(c_2(\cdot-))(x) \\ &= \alpha(c_1(x)u_1(x) + c_2(x)u_2(x)) + D_S u_1(x)D_M c_1(x) + D_S u_2(x)D_M c_2(x) \\ &= \alpha g(x) + D_S u_1(x)D_M c_1(x) + D_S u_2(x)D_M c_2(x). \end{aligned}$$

By setting

$$D_S u_1(x)D_M c_1(x) + D_S u_2(x)D_M c_2(x) = -f(x)$$

and multiplying both sides by $D_S M(x)$, we have that

$$D_S u_1(x)D_S c_1(x) + D_S u_2(x)D_S c_2(x) = -f(x)D_S M(x). \tag{4.2}$$

First, multiply (4.1) by $D_S u_1(x)$, and multiply (4.2) by $u_1(x)$, then make difference to derive that

$$D_S c_2(x) = W(u_1, u_2)D_S c_2(x) = u_1(x)f(x)D_S M(x),$$

where the last equality is due to $W(u_1, u_2)|_E = 1$.

Integrating both sides with respect to S, since c_2 is right-continuous, we have

$$c_2(x) = c_2(l_1-) + \int_{[l_1,x]} u_1(y)f(y)M(dy).$$

Similarly,

$$c_1(x) = c_1(l_2-) + \int_{(x,l_2)} u_2(y)f(y)M(dy).$$

By taking $c_1 = c_1(l_2-),\ c_2 = c_2(l_1-)$,

$$u_2 \int_{\{x\}} u_1(y)f(y)M(dy) = u_1 \int_{\{x\}} u_2(y)f(y)M(dy)$$

implies (1.5). □

The proof of Theorem 1.4 (1) is due to the monotonicity of u_1 and u_2, the rest results of Theorem 1.4 essentially come from [13, Lemmas 1.4–1.5], and we include here for completeness.

Proof of Theorem 1.4 (1) By the monotonicity of u_1 and u_2, we have

$$D_S u_1(0) + \alpha \int_{[0,l_1)} u_1(y)M(dy) \geq 0,$$

$$D_S u_2(0) + \alpha \int_{[0,l_2)} u_2(y)M(dy) \leq 0,$$

which yields that

$$\begin{aligned} 0 &\leq u_2(x)\Big(D_S u_1(0) + \alpha \int_{[0,l_1)} u_1(y)M(dy)\Big) - u_1(x)\Big(D_S u_2(0) + \alpha \int_{[0,l_2)} u_2(y)M(dy)\Big) \\ &= W(u_1,u_2)(x) + \alpha u_2(x) \int_{[x,l_1)} u_1(y)M(dy) - \alpha u_1(x) \int_{[x,l_2)} u_2(y)M(dy) \\ &= 1 - \alpha u_2(x) \int_{[l_1,x)} u_1(y)M(dy) - \alpha u_1(x) \int_{[x,l_2)} u_2(y)M(dy) \\ &= 1 - \alpha R_\alpha 1. \end{aligned}$$

That is, $\alpha R_\alpha 1 \leq 1$. Hence,

$$\|\alpha R_\alpha\| = \sup_{\|f\|\leq 1} \|\alpha R_\alpha f\| \leq \sup_{\|f\|\leq 1} \|f\| \cdot \|\alpha R_\alpha 1\| \leq 1.$$

(2) First, we show that R_α satisfies the resolvent equation on $\mathscr{A}^+$.

Since on $\mathscr{A}^+$

$$(\alpha - D_M D_S)R_\alpha = \mathrm{I},$$

for any $\alpha,\ \beta > 0$, we have

$$(\beta - D_M D_S)R_\alpha = (\alpha - D_M D_S)R_\alpha + (\beta-\alpha)R_\alpha = \mathrm{I} + (\beta-\alpha)R_\alpha. \tag{4.3}$$

Now that $R_\beta f$ is the minimal non-negative solution of $(\beta - D_M D_S)g = f$ $(\beta > 0)$, we have

$$R_\beta(\beta - D_M D_S) \leq \mathrm{I}. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$R_\alpha \geq R_\beta(\beta - D_M D_S)R_\alpha = R_\beta + (\beta-\alpha)R_\beta R_\alpha,$$

which implies that

$$R_\alpha - R_\beta + (\alpha-\beta)R_\beta R_\alpha \geq 0.$$

Similarly, we have

$$R_\beta - R_\alpha + (\beta-\alpha)R_\alpha R_\beta \geq 0.$$

Thus, it suffices to show $R_\beta R_\alpha = R_\alpha R_\beta$. Indeed, we have from (4.3) that

$$\begin{aligned} (\alpha - D_M D_S)(\beta - D_M D_S)R_\alpha &= (\alpha - D_M D_S)\mathrm{I} + (\beta-\alpha)(\alpha - D_M D_S)R_\alpha \\ &= \alpha\mathrm{I} - D_M D_S + (\beta-\alpha)\mathrm{I} \\ &= \beta - D_M D_S. \end{aligned}$$

Furthermore,

$$(\alpha - D_M D_S)(\beta - D_M D_S)R_\alpha R_\beta = (\beta - D_M D_S)R_\beta = \mathrm{I}.$$

Together with (4.4), we have that

$$R_\beta R_\alpha = R_\beta R_\alpha(\alpha - D_M D_S)(\beta - D_M D_S)R_\alpha R_\beta$$

$$
\begin{aligned}
&= R_\beta(R_\alpha(\alpha - D_M D_S))(\beta - D_M D_S) R_\alpha R_\beta \\
&\le (R_\beta(\beta - D_M D_S)) R_\alpha R_\beta \\
&\le R_\alpha R_\beta.
\end{aligned}
$$

In the same way, we have $R_\alpha R_\beta \le R_\beta R_\alpha$. Hence $R_\beta R_\alpha = R_\alpha R_\beta$.

Next, we generalize $R.$ to $\mathscr{A}$. For any $f \in \mathscr{A}$, we have $f = f^+ - f^-$ with $f^+, f^- \in \mathscr{A}^+$. So,

$$
\begin{aligned}
(R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta) f &= (R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta)(f^+ - f^-) \\
&= (R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta) f^+ - (R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta) f^- \\
&= 0.
\end{aligned}
$$

(3) According to the resolvent condition and $R_\beta R_\alpha = R_\alpha R_\beta$,

$$
\begin{aligned}
R_\alpha &= R_\beta[\mathrm{I} + (\beta - \alpha) R_\alpha], \\
R_\beta &= R_\alpha[\mathrm{I} + (\alpha - \beta) R_\beta],
\end{aligned}
$$

which implies that $R_\alpha \mathscr{A}$ is independent of α.

To prove $R_\alpha \mathscr{D}$ is dense in $\mathscr{D}$, for any $h \in \mathscr{A}$,

$$
\begin{aligned}
D_M D_S R_\alpha (R_\alpha h) &= \alpha R_\alpha R_\alpha h - R_\alpha h, \\
R_\alpha D_M D_S (R_\alpha h) &= R_\alpha(\alpha R_\alpha h - h) = \alpha R_\alpha R_\alpha h - R_\alpha h,
\end{aligned}
$$

which yields to $R_\alpha D_M D_S = D_M D_S R_\alpha$ on $\mathscr{D}$. Then for any $f \in \mathscr{D}$,

$$
f = (\alpha - D_M D_S) R_\alpha f = \alpha R_\alpha f - R_\alpha D_M D_S f.
$$

Consequently,

$$
\|f - \alpha R_\alpha f\| = \|R_\alpha D_M D_S f\| \le \frac{1}{\alpha} \|D_M D_S f\| \to 0 \quad (\alpha \to +\infty). \qquad \square
$$

5 Examples

We give several examples of generalized diffusion operators and establish the resolvents of Brownian motion on fractals.

The first example is diffusion operator.

Example 5.1 (Diffusion operator) Let $-\infty < l_1 < l_2 \le +\infty$. For any $x \in [l_1, l_2)$, suppose $a(x) > 0$ and the derivatives of $a(x)$ and $b(x)$ are continuous. Let (see for example [8])

$$
M(x) = \int_c^x \frac{1}{a(y)} \exp\bigg\{ \int_c^y \frac{b(z)}{a(z)} dz \bigg\} dy, \quad S(x) = \int_c^x \exp\bigg\{ - \int_c^y \frac{b(z)}{a(z)} dz \bigg\} dy.
$$

Since both M and S are absolutely continuous with respect to the Lebesgue measure, it follows that

$$
\begin{aligned}
D_M &= \frac{1}{M'(x)} \cdot \frac{d}{dx}, \quad D_S = \frac{1}{S'(x)} \cdot \frac{d}{dx}, \\
D_M D_S &= a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}.
\end{aligned}
$$

The second example is Birth-death Q-matrix.

Example 5.2 (Birth-death Q-matrix) Consider a birth-death Q-matrix with birth rates $(b_i)_{i\in\mathbb{N}}$ and death rates $(a_i)_{i\in\mathbb{N}}$. Take (please refer to [7])

$$M(\{0\})=1,\quad M(\{i\})=\frac{b_0b_1\cdots b_{i-1}}{a_1a_2\cdots a_i},\quad i\ge 1.$$
$$S(\{0\})=1/a_0,\quad S(\{i\})=\frac{1}{M(\{i\})a_i},\quad i\ge 1.$$

Then for any $i\in\mathbb{N}$,

$$\begin{aligned}
D_MD_Sf(i)&=\lim_{r\downarrow 0}\frac{\mu_{D_Sf}([i-r,i+r])}{M([i-r,i+r])}\\
&=\frac{\mu_{D_Sf}(\{i\})}{M(\{i\})}\\
&=\frac{\mu_{D_Sf}([i,i+1))}{M(\{i\})}\\
&=\frac{D_Sf(i+1)-D_Sf(i)}{M(\{i\})}\\
&=(M(\{i\}))^{-1}\lim_{r\downarrow 0}\left[\frac{\mu_f([i+1-r,i+1+r])}{S([i+1-r,i+1+r])}-\frac{\mu_f([i-r,i+r])}{S([i-r,i+r])}\right]\\
&=(M(\{i\}))^{-1}\left[\frac{\mu_f(\{i+1\})}{S(\{i+1\})}-\frac{\mu_f(\{i\})}{S(\{i\})}\right]\\
&=(M(\{i\}))^{-1}\left[\frac{\mu_f((i,i+1])}{S(\{i+1\})}-\frac{\mu_f((i-1,i])}{S(\{i\})}\right]\\
&=(M(\{i\}))^{-1}\left[\frac{f(i+1)-f(i)}{S(\{i+1\})}-\frac{f(i)-f(i-1)}{S(\{i\})}\right]\\
&=b_i(f(i+1)-f(i))-a_i(f(i)-f(i-1)).
\end{aligned}$$

The third example is the so-called gap diffusion.

Example 5.3 (Gap diffusion [11]) Suppose that $M:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ is increasing, right-continuous and $M(\infty)=\infty$. Take $l_1=\inf\{x: M(x)>-\infty\}$ and $l_2=\sup\{x: M(x)<+\infty\}$. Let $(B_t)_{t\ge 0}$ be a one-dimensional standard Brownian motion and $l(t,x),\ t\ge 0, x\in\mathbb{R}$, be its local time. Set $\phi(t)=\int l(t,x)M(dx)$, ϕ^{-1} be the inverse of ϕ. Then $\{B(\phi^{-1}(t)):\ t\ge 0\}$ is a time homogeneous strong Markov process corresponding to the operator D_MD_x.

The last example is the Brownian motion on Cantor set. Since the "cumulative distribution function" determined by Cantor measure is increasing and singularly continuous, the corresponding operator $D_\lambda D_\lambda$ is different from those in [6, 7, 9, 11].

Example 5.4 (Brownian motion on Cantor set [10]) First, we recall the standard Bernoulli measure on Cantor set in $\mathbb{R}$. Let $\Omega_i=\{0,1\}$, $i=0,1,\ldots$, and ρ_m be the uniform probability measure on $\Omega^m:=\prod_{i=0}^m\Omega_i$, that is, $\rho_m(\{x\})=2^{-(m+1)}$ for any $(x_0,x_1,\ldots,x_m)\in\Omega^m$. Consider the map $J:\Omega^m\to[0,1]$,

$$J(x):=a_0^mx_0+a_1^mx_1+\cdots+a_m^mx_m,\quad \forall x=(x_0,x_1,\ldots,x_m)\in\Omega^m,$$

where $a_k^m=3^{-m}b_k, b_0=1, b_k=2\cdot 3^{k-1}$. Let $K_m=J(\Omega^m)$. Then the closure of $\bigcup_{m=0}^{+\infty}K_m$ is Cantor set in $[0,1]$ and is denoted by $\mathbb{K}$. Let $\lambda_m=\rho_m\circ J^{-1}$. Then $\lambda_m(\{p\})=2^{-(m+1)}$, $\forall p\in K_m$.

Following [10], we know that there exists a unique probability measure λ on $\mathbb{K}$ such that $\lambda_m \Rightarrow \lambda$, that is, $\forall f \in C(\mathbb{K})$, $\lim_{m\to+\infty}\int_{K_m} f d\lambda_m = \int_{\mathbb{K}} f d\lambda$. λ is called the standard Bernoulli probability measure on $\mathbb{K}$. Let $\widetilde{\mathbb{K}} = \bigcup_{n=0}^{+\infty}(n+\mathbb{K})$ be Cantor set on $[0,+\infty)$ and denote again by λ the extended Bernoulli measure on $\widetilde{\mathbb{K}}$.

We will give a system of fundamental solutions of the homogeneous equation

$$(\alpha - D_\lambda D_\lambda)v = 0 \quad (\alpha > 0) \tag{5.1}$$

by Picard iteration.

Fix $c \in \widetilde{\mathbb{K}}$ and let $\Lambda(x) = \lambda([0,x]) - \lambda([0,c])$, $x \in [0,+\infty)$. Then Λ is an increasing continuous function and $\Lambda(c) = 0$. Define $\Lambda^{-1}(y) = \inf\{x : \Lambda(x) \geq y\}$, it is clear that $\Lambda(\Lambda^{-1}(y)) = y$. Lemma 3.4 implies that for any Borel measurable function g and Borel set Γ,

$$\int_\Gamma g(\Lambda(t))\lambda(dt) = \int_{\{t:\Lambda^{-1}(t)\in\Gamma\}} g(\Lambda(\Lambda^{-1}(t)))dt = \int_{\{t:\Lambda^{-1}(t)\in\Gamma\}} g(t)dt. \tag{5.2}$$

Define

$$\begin{cases} v_1^{(0)}(x) = 1, \\ v_1^{(n+1)}(x) = \alpha \int_{(c,x]} \lambda(dy) \int_{[c,y)} v_1^{(n)}(z)\lambda(dz), \quad n \geq 0. \end{cases}$$

Suppose $v_1^{(n)}(x) = \alpha^n \frac{\Lambda(x)^{2n}}{(2n)!}$, it follows by (5.2) that for any $n \geq 0$,

$$\begin{aligned} v_1^{(n+1)}(x) &= \alpha \int_{(c,x]} \lambda(dy) \int_{[c,y)} \alpha^n \frac{\Lambda(z)^{2n}}{(2n)!}\lambda(dz) \\ &= \frac{\alpha^{n+1}}{(2n)!} \int_{(c,x]} \lambda(dy) \int_0^{\Lambda(y)} z^{2n}\lambda(dz) \\ &= \frac{\alpha^{n+1}}{(2n+1)!} \int_{(c,x]} \Lambda(y)^{2n+1}\lambda(dy) \\ &= \frac{\alpha^{n+1}}{(2n+1)!} \int_0^{\Lambda(x)} y^{2n+1}dy \\ &= \alpha^{n+1}\frac{\Lambda(x)^{2n+2}}{(2n+2)!}. \end{aligned}$$

By induction, we have

$$v_1(x) = \sum_{n=0}^{+\infty} \frac{[\sqrt{\alpha}\Lambda(x)]^{2n}}{(2n)!} = \cosh(\sqrt{\alpha}\Lambda(x)).$$

Define

$$\begin{cases} v_2^{(0)}(x) = \Lambda(x), \\ v_2^{(n+1)}(x) = \alpha \int_{(c,x]} \lambda(dy) \int_{[c,y)} v_2^{(n)}(z)\lambda(dz), \quad n \geq 0, \end{cases}$$

In the same way, we obtain

$$v_2(x) = \alpha^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{[\sqrt{\alpha}\Lambda(x)]^{2n+1}}{(2n+1)!} = \alpha^{-\frac{1}{2}}\sinh(\sqrt{\alpha}\Lambda(x)).$$

Hence, $\{\cosh(\sqrt{\alpha}\Lambda(x)), \alpha^{-1/2}\sinh(\sqrt{\alpha}\Lambda(x))\}$ consists of a system of fundamental solutions of the homogeneous equation (5.1). Moreover, we have

$$\frac{v_1}{v_2}(0) = \alpha^{\frac{1}{2}}\frac{\cosh(\sqrt{\alpha}\Lambda(0))}{\sinh(\sqrt{\alpha}\Lambda(0))} = \alpha^{\frac{1}{2}}\coth(\sqrt{\alpha}\Lambda(0)) =: \gamma_1,$$
$$\frac{\alpha\int_{[0,c)} v_1(t)\lambda(dt)}{1+\alpha\int_{[0,c)} v_2(t)\lambda(dt)} = \alpha\cdot\frac{v_2}{v_1}(0) = \alpha^{\frac{1}{2}}\tanh(\sqrt{\alpha}\Lambda(0)) =: \gamma_2$$

with $\Lambda(0) = -\lambda([0,x]) < 0$. In addition,

$$\lim_{x\uparrow+\infty}\frac{v_1}{v_2}(x) = \alpha^{\frac{1}{2}}\lim_{x\uparrow+\infty}\frac{e^{2\sqrt{\alpha}\Lambda(x)}+1}{e^{2\sqrt{\alpha}\Lambda(x)}-1} = \alpha^{\frac{1}{2}},$$
$$\lim_{x\uparrow+\infty}\frac{\alpha\int_{[c,x)} v_1(t)\lambda(dt)}{1+\alpha\int_{[c,x)} v_2(t)\lambda(dt)} = \alpha\cdot\lim_{x\uparrow+\infty}\frac{v_2}{v_1}(x) = \alpha^{\frac{1}{2}}.$$

Then $W(v_1-\gamma v_2, v_1-\alpha^{1/2}v_2) = \alpha^{1/2}-\gamma$ holds with $\gamma_1\le\gamma\le\gamma_2<0$. By taking $u_1 = v_1-\gamma v_2$ and $u_2 = (v_1-\alpha^{1/2}v_2)/(\alpha^{1/2}-\gamma)$, we have $W(u_1,u_2)|_{\widetilde{\mathbb{K}}}\equiv 1$. According to Theorem 1.2, there are infinitely many u_1 and there is a unique u_2 up to a multiplicative positive constant, which indicates that 0 is regular while $+\infty$ is not regular. In fact, from (5.2), we have

$$\Sigma(x) = N(x) = \frac{\Lambda(x)^2}{2}.$$

Consequently,

$$\Sigma(0) = N(0) = \lambda([0,c])^2 < +\infty \quad\text{and}\quad \Sigma(+\infty) = N(+\infty) = +\infty.$$

Hence 0 is regular and $+\infty$ is nature.

The Green function determined by u_1 and u_2 is

$$K(x,y) = \begin{cases} \dfrac{e^{\sqrt{\alpha}\Lambda(y)}[(1-\gamma\alpha^{-1/2})e^{\sqrt{\alpha}\Lambda(x)}+(1+\gamma\alpha^{-1/2})e^{-\sqrt{\alpha}\Lambda(x)}]}{2(\alpha^{1/2}-\gamma)}, & x\le y \quad\text{and}\quad x,y\in\widetilde{\mathbb{K}}, \\ \dfrac{e^{\sqrt{\alpha}\Lambda(x)}[(1-\gamma\alpha^{-1/2})e^{\sqrt{\alpha}\Lambda(y)}+(1+\gamma\alpha^{-1/2})e^{-\sqrt{\alpha}\Lambda(y)}]}{2(\alpha^{1/2}-\gamma)}, & y\le x \quad\text{and}\quad x,y\in\widetilde{\mathbb{K}}, \end{cases}$$

where $\gamma_1\le\gamma\le\gamma_2<0$.

For the inhomogeneous equation

$$(\alpha - D_\lambda D_\lambda)g = f \quad (\alpha>0), \tag{5.3}$$

the minimal non-negative solution is

$$R_\alpha f(x) := \int_0^{+\infty} K(x,y)f(y)\lambda(dy), \quad x\in\widetilde{\mathbb{K}},$$

which indicates that there are infinitely many resolvents.

Remark 5.5 When both M and S are the Lebesgue measure, the associated process is Brownian motion and the solutions of the corresponding homogeneous equation and inhomogeneous equation are well known (see for example [8]). When both M and S are the Cantor measure, we use integral transform theorem (or Lemma 3.4) to deal with the integration and Picard iteration to solve the corresponding homogeneous equation. Both the system of fundamental solutions of the homogeneous equation (5.1) and the minimal non-negative solution of the inhomogeneous equation (5.3) are novel as far as we know. When taking one measure as the Lebesgue measure and another as the Cantor measure, we have no idea to solve the corresponding homogeneous equation.

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