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On the Character Sum of Polynomials and the Two-term Exponential Sums

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Abstract The main purpose of this paper is using the analytic method and the properties of the character sums to study the computational problem of one kind hybrid power mean involving the character sums of polynomials and the two-term exponential sums, and give several interesting identities and asymptotic formulae for them.

Keywords Character sums of polynomials, two-term exponential sums, hybrid power mean, analytic method, asymptotic formula

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1 Introduction

As usual, let q be an integer with $q \geq 2$. For any integers m and n, the two-term exponential sums $K(m,n,k,h;q)$ is defined as follows:

$$K(m,n,k,h;q) = \sum_{a=1}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),$$

where $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$, k and h denote the integers with $k > h$.

If $q = p$ is a prime, $k = 1$ and $h = -1$, then $K(m,n,1,-1;p) = S(m,n;p)$ becomes the famous Kloostermann sum. That is,

$$S(m,n;q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} e\left(\frac{ma + n\overline{a}}{q}\right),$$

where the summation over all $1 \leq a \leq q$ such that $(a,q) = 1$, $\overline{a}$ denotes the solution of the congruence equation $xa \equiv 1 \bmod q$.

It is clear that these sums occupy a very important position in the research of analytic number theory, and many classical problems in analytic number theory are closely related to them. Therefore, any substantial progress in these fields will certainly promote the development of analytic number theory. For these reasons, many scholars have studied the properties of $K(m,n,k,h;q)$ and $S(m,n;q)$, and obtained a series of important results. For example, Zhang

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and Zhang [21] proved a precise formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-p^2 & \text{if } 3\nmid p-1, \\ 2p^3-7p^2 & \text{if } 3\mid p-1, \end{cases}$$

where p indicate an odd prime, n is any integer with $(n,p)=1$.

Zhang [22] used the analytic methods to study the properties of the fourth power mean of $S(m,n;q)$, and proved the identity

$$\sum_{m=1}^{q}|S(m,n;q)|^4 = 3^{\omega(q)}\cdot q^2\cdot\phi(q)\cdot\prod_{p\|q}\left(\frac{2}{3}-\frac{1}{3p}-\frac{4}{3p(p-1)}\right),$$

where $\phi(q)$ is Euler function, $\omega(q)$ denotes the number of all different prime divisors of q, $\prod_{p\|q}$ denotes the product over all prime divisors of q with $p\mid q$ and $p^2\nmid q$, n is any integer with $(n,q)=1$.

Lv and Zhang [12] first introduced a sum analogous to Kloosterman sum as follows:

$$K(m,n,r,\chi;q)=\sum_{a=1}^{q}\chi(ma+n\overline{a})e\left(\frac{ra}{q}\right),$$

where m, n and r are integers, χ denotes any Dirichlet character mod q.

Then, for any odd prime p with $p\equiv 3 \bmod 4$ and integers m and n with $(mn,p)=1$, Lv and Zhang [12] proved the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(b+\overline{b})\right|^2=(p-1)(3p^2-6p-1)$$

and

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(b+\overline{b})e\left(\frac{nb}{p}\right)\right|^2$$
$$=(p-1)(p^2-2p-1)+p(p-1)\left(\sum_{b=2}^{p-2}e\left(\frac{n(b+\overline{b})}{p}\right)+\sum_{b=2}^{p-2}e\left(\frac{n(b-\overline{b})}{p}\right)\right).$$

From the second formula and the estimate for Kloosterman sum, Lv and Zhang [12] deduced the asymptotic formula

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\cdot\left|\sum_{b=1}^{p-1}\chi(b+\overline{b})e\left(\frac{nb}{p}\right)\right|^2=p^3+O(p^{\frac{5}{2}}).$$

Shane [5] also studied the properties of $K(m,n,r,\chi;q)$, and obtained the following identity:

$$\sum_{\chi \bmod p}\left|\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\chi(ma+n\overline{a})e\left(\frac{ka}{p}\right)\right|^2\right|^2=(p-1)(p^4-7p^3+17p^2-5p-25),$$

where p is an odd prime, n and k are integers with $(nk,p)=1$.

On the other hand, Zhang and Han [25] studied the fourth power mean of the 2-dimensional Kloostermann sums, and proved the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}e\left(\frac{ma+b+\overline{ab}}{p}\right)\right|^4=\begin{cases}7p^5-18p^4-(b_p+6)p^3-6p^2-3p & \text{if } p\equiv 1 \bmod 6;\\ 7p^5-22p^4-(b_p-14)p^3-6p^2-3p & \text{if } p\equiv 5 \bmod 6,\end{cases}$$

where b_p is an integer satisfying $|b_p| < 2p^{\frac{3}{2}}$.

Zhang and Li [26] studied the fourth power mean of the general 2-dimensional Kloostermann sums, and obtained the identity

$$\sum_{\chi \bmod p} \sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(ab) e\left(\frac{a+b+m\overline{ab}}{p} \right) \right|^4 = (p-1)(2p^5 - 7p^4 + 2p^3 + 8p^2 + 4p + 1).$$

Many papers related to character sums of polynomials and two-term exponential sums can also be found in [6–11, 13–24], here we will not list them one by one.

In this paper, we are interested in the following two hybrid power means

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a+b+m\overline{a}\overline{b}) \right|^2 \cdot \left| \sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p} \right) \right|^2 \tag{1.1}$$

and

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a+b+m\overline{a}\overline{b}) \right|^2 \cdot \left| \sum_{b=1}^{p-1} e\left(\frac{mb^3+b}{p} \right) \right|^2 . \tag{1.2}$$

About these kind hybrid power means, Han [7] also studied some related contents, and proved the following interesting conclusion:

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^k+na}{p} \right) \right|^2 \cdot \left| \sum_{a=1}^{p-1} \chi(ma+\overline{a}) \right|^2 = \begin{cases} 2p^3 + O(|k|p^2), & \text{if } 2 \mid k; \\ 2p^3 + O(|k|p^{\frac{5}{2}}), & \text{if } 2 \nmid k, \end{cases}$$

where p is an odd prime, χ denotes any non-principal even Dirichlet character mod p, and $\overline{a}$ denotes the multiplicative inverse of a mod p.

Taking $k = -1$ in this theorem, one can deduce the asymptotic formula

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p} \right) \right|^2 \cdot \left| \sum_{b=1}^{p-1} \chi(mb+\overline{b}) \right|^2 = 2p^3 + O(p^2).$$

But for the hybrid power means (1.1) and (1.2), so far no one seems to study them, at least we have not seen any related paper before. These problems are meaningful, at least they can reflection the properties of the character sums of polynomials.

The main purpose of this paper is using the analytic methods, the properties of the classical Gauss sum and character sums to study the computational problems of (1.1) and (1.2), and give two identities and two sharp asymptotic formulae for them. That is, we will prove the following two theorems:

Theorem 1.1 *Let p be an odd prime with $3 \nmid (p-1)$. Then for any non-principal character χ mod p, we have the identities*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a+b+m\overline{a}\overline{b}) \right|^2 \cdot \left| \sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p} \right) \right|^2 = p^2 \cdot (p^2 - p - 1)$$

and

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a+b+m\overline{a}\overline{b}) \right|^2 \cdot \left| \sum_{b=1}^{p-1} e\left(\frac{mb^3+b}{p} \right) \right|^2 = p^2 \cdot (p^2 - p - 1).$$

Theorem 1.2 *Let p be an odd prime with $3 \mid (p-1)$. Then for any three-th character χ mod p (i.e., there exists a character χ_1 mod p such that $\chi = \chi_1^3$), then we have the asymptotic formulae*

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2 \cdot \left|\sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p}\right)\right|^2 = 3p^4 + E(p)$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2 \cdot \left|\sum_{b=1}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 = 3p^4 + E_1(p),$$

where $E(p)$ and $E_1(p)$ satisfy $|E(p)| \leq 9 \cdot p^{\frac{7}{2}}$ and $|E_1(p)| \leq 15 \cdot p^3$ respectively.

Some Notes We only discussed the special case that χ is a three-th character mod p in Theorem 1.2. If χ is not a three-th character mod p, then for any integer m with $(m,p)=1$, we have the identity

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab}) = 0.$$

So in this case, we have the identites

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2 \cdot \left|\sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p}\right)\right|^2 = 0$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2 \cdot \left|\sum_{b=1}^{p-1} e\left(\frac{mb^3+b}{p}\right)\right|^2 = 0.$$

It seems that using our methods we can not obtain an asymptotic formula for the fourth power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^4 \quad \text{and} \quad \sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^4.$$

Whether there exits a sharp asymptotic formula for these fourth power means are two open problems. Interested readers are suggested to study with us.

2 Some Lemmas

In many places of this paper, we need to use the definition and properties of the Gauss sums $\tau(\chi)$ and character sums, these contents can be found in some analytic number theory books, such as [10, 14, 16], here we will not repeat the related contents. First we have the following:

Lemma 2.1 *Let p be an odd prime and $3 \mid (p-1)$, χ is any non-principal character* mod p, *and m is an integer with $(m,p)=1$. If χ is not a three-th character* mod p, *then we have the identity*

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2 = 0;$$

If χ is a three-th character mod p, *then we have the identity*

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab})\right|^2$$

$$= 3p^2 + \lambda(m)\frac{\tau^3(\overline{\lambda})}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\lambda(c-1)\lambda(a^2bc-1)\lambda(ab^2c-1)$$
$$+ \overline{\lambda}(m)\frac{\tau^3(\lambda)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1),$$

where λ denotes the third-order character mod p (*i.e.*, $\chi \neq \chi_0$ *and* $\chi^3 = \chi_0$, χ_0 *is the principal character*).

Proof If $3 \mid (p-1)$, and χ is not a three-th character mod p, then there exists an integer $1 < r < p-1$ such that $r^3 \equiv \overline{r}^3 \equiv 1 \bmod p$ and $\chi(r) \neq 1$. So from the properties of the reduced residue system mod p we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab}) &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(ra+rb+m\overline{r}^2\overline{ab}) \\ &= \chi(r)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{r}^3\overline{ab}) \\ &= \chi(r)\cdot\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab}). \end{aligned} \tag{2.1}$$

Since $\chi(r) \neq 1$, so from (2.1) we have the identity

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{ab}) = 0. \tag{2.2}$$

If $3 \mid (p-1)$, and χ is a three-th character mod p, then for any integer m with $(m,p)=1$, note that the identity

$$\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right) = 1 + \sum_{a=1}^{p-1}(1+\lambda(a)+\overline{\lambda}(a))e\left(\frac{ma}{p}\right) = \lambda(m)\tau(\overline{\lambda}) + \overline{\lambda}(m)\tau(\lambda),$$

from the properties of the reduced residue system mod p and the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n, \end{cases} \tag{2.3}$$

we have

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2 \\ &= \left|\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(c)e\left(\frac{c(a+b+m\overline{ab})}{p}\right)\right|^2 \\ &= \left|\frac{1}{\tau(\overline{\chi})}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)e\left(\frac{c(a^2b+b^2a+m)}{p}\right)\right|^2 \\ &= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=1}^{p-1}e\left(\frac{ge(a^2bc-1)+gf(ab^2c-1)+g\overline{ef}m(c-1)}{p}\right)\end{aligned}$$

$$= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)+g^3\overline{ef}m(c-1)}{p}\right)$$
$$= \frac{p-1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e(a^2b-1)+f(ab^2-1)}{p}\right)$$
$$+ \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=2}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)+g^3\overline{ef}m(c-1)}{p}\right)$$
$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e(a^2b-1)+f(ab^2-1)}{p}\right)$$
$$- \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)}{p}\right)$$
$$+ \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=2}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=0}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)+g^3\overline{ef}m(c-1)}{p}\right)$$
$$= p^2 \sum_{\substack{a=1\\ a^2b\equiv b^2a\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab) - p\sum_{\substack{a=1\\ a^2bc\equiv b^2ac\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc) + 2\sum_{\substack{a=1\\ a^2bc\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)$$
$$+ \frac{\tau(\overline{\lambda})}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\lambda(\overline{ef}m(c-1))e\left(\frac{e(a^2bc-1)+f(ab^2c-1)}{p}\right)$$
$$+ \frac{\tau(\lambda)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\overline{\lambda}(\overline{ef}m(c-1))e\left(\frac{e(a^2bc-1)+f(ab^2c-1)}{p}\right)$$
$$= 3p^2 + \lambda(m)\frac{\tau^3(\overline{\lambda})}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\lambda(c-1)\lambda(a^2bc-1)\lambda(ab^2c-1)$$
$$+ \overline{\lambda}(m)\frac{\tau^3(\lambda)}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1). \tag{2.4}$$

Now Lemma 2.1 follows from (2.2) and (2.4). □

Lemma 2.2 *Let p be an odd prime with $3 \nmid (p-1)$. Then for any non-principal even character χ mod p and integer m with $(m,p)=1$, we have the identity*

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2 = p^2.$$

Proof Note that $3 \nmid (p-1)$, so if a pass through a reduced residue system mod p, then a^3 also pass through a reduced residue system mod p. Then from the method of proving Lemma 1.1 and the trigonometric identity (2.3) we have

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2$$
$$= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)+g^3\overline{ef}m(c-1)}{p}\right)$$

$$
\begin{aligned}
&= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{g=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)+g(c-1)}{p}\right)\\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e(a^2b-1)+f(ab^2-1)}{p}\right)\\
&\quad - \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e(a^2bc-1)+f(ab^2c-1)}{p}\right)\\
&= p^2 \sum_{\substack{a=1\\ a^2b\equiv b^2a\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab) - 2\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\chi}(ab)\sum_{e=1}^{p-1} e\left(\frac{e(a^2b-1)}{p}\right)\\
&\quad - p \sum_{\substack{a=1\\ a^2bc\equiv ab^2c\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc) + \frac{2}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\sum_{e=1}^{p-1} e\left(\frac{e(a^2bc-1)}{p}\right)\\
&= p^2 - 3p\sum_{a=1}^{p-1}\chi(a) + 2\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\\
&= p^2.
\end{aligned}
$$

This proves Lemma 2.2. □

Lemma 2.3 *Let p be an odd prime with $3 \mid (p-1)$. Then for any third-order character $\lambda \bmod p$, we have the estimate*

$$
\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right|^2 \le p^{\frac{3}{2}}
$$

and

$$
\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 \le p+2\sqrt{p}.
$$

Proof Note that $\lambda(-1)=1$, $\lambda^2=\overline{\lambda}$, from the definition and properties of the classical Gauss sums we have

$$
\begin{aligned}
\sum_{m=1}^{p-1}\lambda(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma+\overline{a}}{p}\right)\right|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\lambda(m)e\left(\frac{m(a-b)+\overline{a}-\overline{b}}{p}\right)\\
&= \tau(\lambda)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\overline{\lambda}(a-b)e\left(\frac{\overline{a}-\overline{b}}{p}\right)\\
&= \tau^2(\lambda)\sum_{a=1}^{p-1}\overline{\lambda}(a-1)\overline{\lambda}(\overline{a}-1)\\
&= \tau^2(\lambda)\sum_{a=1}^{p-1}\lambda(a)\overline{\lambda}^2(a-1)\\
&= \tau^2(\lambda)\sum_{a=1}^{p-1}\lambda(a)\lambda(a-1). \qquad (2.5)
\end{aligned}
$$

On the other hand, we also have the identity

$$\sum_{a=1}^{p-1} \lambda(a)\lambda(a-1) = \frac{1}{\tau(\overline{\lambda})} \sum_{a=1}^{p-1} \lambda(a) \sum_{b=1}^{p-1} \overline{\lambda}(b) e\left(\frac{b(a-1)}{p}\right) = \frac{\tau^2(\lambda)}{\tau\left(\overline{\lambda}\right)}. \tag{2.6}$$

Note that $|\tau(\lambda)| = \sqrt{p}$, from (2.5) and (2.6) we have the estimate

$$\sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma + \overline{a}}{p}\right) \right|^2 = \frac{\tau^4(\lambda)}{\tau(\overline{\lambda})} \leq p^{\frac{3}{2}}. \tag{2.7}$$

From the method of proving (2.7) we also have

$$\begin{aligned} \sum_{m=1}^{p-1} \lambda(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \lambda(m) e\left(\frac{m(a^3 - b^3) + a - b}{p}\right) \\ &= \tau(\lambda) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \overline{\lambda}(a^3 - 1) e\left(\frac{b(a-1)}{p}\right) \\ &= -\tau(\lambda) \sum_{a=1}^{p-1} \overline{\lambda}(a^3 - 1) \\ &= -\tau(\lambda) \sum_{a=1}^{p-1} (1 + \lambda(a) + \overline{\lambda}(a)) \overline{\lambda}(a - 1) \\ &= \tau^2(\lambda) - 2\tau(\lambda) \\ &\leq p + 2\sqrt{p}. \end{aligned} \tag{2.8}$$

Now Lemma 2.3 follows from the estimates (2.7) and (2.8). □

Lemma 2.4 *Let p be an odd prime. Then we have the identities*

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 = \begin{cases} p^2 - 3p - 1 & \text{if } 3 \mid (p-1), \\ p^2 - p - 1 & \text{if } 3 \nmid (p-1), \end{cases}$$

and

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} e\left(\frac{ma + \overline{a}}{p}\right) \right|^2 = p^2 - p - 1.$$

Proof These two identities can easily be obtained by using the trigonometric identity (2.3), so the details of the proof are omitted. □

Lemma 2.5 *Let p be an odd prime with $3 \mid (p-1)$. Then for any non-principal character $\chi \bmod p$, we have the estimate*

$$\left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc) \overline{\lambda}(c-1) \overline{\lambda}(a^2bc - 1) \overline{\lambda}(ab^2c - 1) \right| \leq 3 \cdot p^{\frac{3}{2}}.$$

Proof If χ is not a three-th character mod p, then from the method of proving (2.1) we have the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc) \overline{\lambda}(c-1) \overline{\lambda}(a^2bc - 1) \overline{\lambda}(ab^2c - 1) = 0. \tag{2.9}$$

If χ is a three-th character mod p, then for any non-three-th residue $h \bmod p$, note that $1 + \lambda(h) + \overline{\lambda}(h) = 0$, from Lemma 2.1 we have

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh\overline{a}\overline{b})\right|^2 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh^2\overline{a}\overline{b})\right|^2 = 9p^2. \tag{2.10}$$

So for any integer m with $(m,p)=1$, from (2.10) we have the estimate

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^4 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh\overline{a}\overline{b})\right|^4 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh^2\overline{a}\overline{b})\right|^4 \le 81\cdot p^4. \tag{2.11}$$

On the other hand, from Lemma 2.1 we also have

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^4 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh\overline{a}\overline{b})\right|^4 + \left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+mh^2\overline{a}\overline{b})\right|^4$$
$$= 27p^4 + 6p\cdot\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1)\right|^2. \tag{2.12}$$

Combining (2.11) and (2.12) we have the estimate

$$\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1)\right| \le 3\cdot p^{\frac{3}{2}}. \tag{2.13}$$

Now Lemma 2.5 follows from (2.9) and (2.13). □

3 Proofs of the Theorems

By using the basic lemmas of the previous section, we can easily complete the proofs of our theorems. First we prove Theorem 1.1.

Proof If $3 \nmid (p-1)$, then for any non-principal $\chi \bmod p$, from Lemmas 2.2 and 2.4 we have the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\overline{b}}{p}\right)\right|^2 = p^2\cdot(p^2-p-1)$$

and

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb^3+b}{p}\right)\right|^2 = p^2\cdot(p^2-p-1).$$

This proves Theorem 1.1. □

Proofs of Theorem 1.2.

Proof If $3 \mid (p-1)$, then for any three-th non-principal character $\chi \bmod p$, from Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5 we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a+b+m\overline{a}\overline{b})\right|^2\cdot\left|\sum_{b=1}^{p-1}e\left(\frac{mb+\overline{b}}{p}\right)\right|^2$$

$$
\begin{aligned}
&= 3p^2 \sum_{m=1}^{p-1} \left| \sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p} \right) \right|^2 + \frac{\tau^3(\lambda)}{p} \sum_{m=1}^{p-1} \overline{\lambda}(m) \left| \sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p} \right) \right|^2 \\
&\quad \times \left(\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1) \right) \\
&\quad + \frac{\tau^3(\overline{\lambda})}{p} \sum_{m=1}^{p-1} \lambda(m) \left| \sum_{b=1}^{p-1} e\left(\frac{mb+\overline{b}}{p} \right) \right|^2 \\
&\quad \times \left(\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc)\lambda(c-1)\lambda(a^2bc-1)\lambda(ab^2c-1) \right) \\
&= p^2 \frac{\tau(\overline{\lambda})}{\tau(\lambda)} \left(\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc)\lambda(c-1)\lambda(a^2bc-1)\lambda(ab^2c-1) \right) \\
&\quad + p^2 \frac{\tau(\lambda)}{\tau(\overline{\lambda})} \left(\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \overline{\chi}(abc)\overline{\lambda}(c-1)\overline{\lambda}(a^2bc-1)\overline{\lambda}(ab^2c-1) \right) \\
&\quad + 3p^2(p^2-p-1) \\
&= 3p^4 + E(p), \qquad (3.1)
\end{aligned}
$$

where $E(p)$ satisfy the estimate $|E(p)| \leq 9 \cdot p^{\frac{7}{2}}$.

Similarly, from Lemma 2.1, Lemma 2.3, Lemma 2.4 and Lemma 2.5 we also have

$$
\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a+b+m\overline{a}\overline{b}) \right|^2 \cdot \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p} \right) \right|^2 = 3p^4 + E_1(p), \qquad (3.2)
$$

where $E_1(p)$ satisfy the estimate $|E_1(p)| \leq 15 \cdot p^3$.

It is clear that Theorem 1.2 follows from the asymptotic formulae (3.1) and (3.2). This completes the proof of our all results. □

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