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Some Existence Theorems on Path Factors with Given Properties in Graphs

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Abstract A path factor of G is a spanning subgraph of G such that its each component is a path. A path factor is called a $P_{\geq n}$-factor if its each component admits at least n vertices. A graph G is called $P_{\geq n}$-factor covered if G admits a $P_{\geq n}$-factor containing e for any $e \in E(G)$, which is defined by [*Discrete Mathematics*, **309**, 2067–2076 (2009)]. We first define the concept of a $(P_{\geq n}, k)$-factor-critical covered graph, namely, a graph G is called $(P_{\geq n}, k)$-factor-critical covered if G-D is $P_{\geq n}$-factor covered for any $D \subseteq V(G)$ with $|D| = k$. In this paper, we verify that (i) a graph G with $\kappa(G) \geq k+1$ is $(P_{\geq 2}, k)$-factor-critical covered if $\text{bind}(G) > \frac{2+k}{3}$; (ii) a graph G with $|V(G)| \geq k+3$ and $\kappa(G) \geq k+1$ is $(P_{\geq 3}, k)$-factor-critical covered if $\text{bind}(G) \geq \frac{4+k}{3}$.

Keywords Graph, binding number, $P_{\geq 2}$-factor, $P_{\geq 3}$-factor, $(P_{\geq 2}, k)$-factor-critical covered graph, $(P_{\geq 3}, k)$-factor-critical covered graph

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1 Introduction

All graphs discussed here are finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $x \in V(G)$, we use $d_G(x)$ to denote the degree of x in G. For a vertex subset X of G, we denote by $G[X]$ the subgraph of G induced by X, and write $G - X$ for $G[V(G) \setminus X]$. For any $E' \subseteq E(G)$, we use $G - E'$ to denote the graph which is obtained from G by deleting edges of E'. A vertex subset X of G is called independent if $G[X]$ has no edges. Let $i(G)$ and $\omega(G)$ denote the number of isolated vertices and connected components in G, respectively. We write $\kappa(G)$ for the vertex connectivity of G.

The path with n vertices is denoted by P_n, where $n \geq 2$ is an integer. A path factor of G is a spanning subgraph of G such that its each component is a path. A path factor is called a $P_{\geq n}$-factor if its each component admits at least n vertices.

The path factors of graphs were studied by Kawarabayashi et al. [11], Asratian and Casselgren [2], Kano et al. [9], Johnson et al. [6], Zhou [21], Zhou et al. [27], Kano et al. [10], Matsubara et al. [14]. For some other results on graph factors, see [3–5, 20, 22, 24, 26].

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Next, we list some of the known results concerning a $P_{\geq n}$-factor. Akiyama et al. [1] demonstrated the following classical result.

Theorem 1.1 ([1]) *A graph G possesses a $P_{\geq 2}$-factor if and only if*

$$i(G-X)\leq 2|X|$$

holds for any vertex subset X of G.

In order to characterize a graph possessing a $P_{\geq 3}$-factor, Kaneko [7] put forward the concept of a sun. A graph R is factor-critical if $R-\{x\}$ possesses a perfect matching for any $x\in V(R)$. Let R be a factor-critical graph with vertex set $V(R)=\{x_1,x_2,\dots,x_n\}$. By adding new vertices $y_1,y_2,\dots,y_n$ together with new edges $x_1y_1,x_2y_2,\dots,x_ny_n$ to R, a new graph is derived. Then the resulting graph is defined as a sun. In terms of Kaneko, K_1 and K_2 are also suns. A sun with at least six vertices is said to be a big sun. A component of a graph G is called a sun component if it is isomorphic to a sun. We write $\mathrm{sun}(G)$ for the number of sun components of G.

Kaneko [7] showed a characterization for a graph possessing a $P_{\geq 3}$-factor. Kano et al. [8] posed a shorter proof.

Theorem 1.2 ([7, 8]) *A graph G contains a $P_{\geq 3}$-factor if and only if*

$$\mathrm{sun}(G-X)\leq 2|X|$$

holds for any vertex subset X of G.

A graph G is called $P_{\geq n}$-factor covered if G admits a $P_{\geq n}$-factor containing e for any $e\in E(G)$, which is first defined by Zhang and Zhou [17]. Furthermore, they acquired two necessary and sufficient conditions for the existence of a $P_{\geq 2}$-factor covered graph and a $P_{\geq 3}$-factor covered graph.

Theorem 1.3 ([17]) *A connected graph G is $P_{\geq 2}$-factor covered if and only if*

$$i(G-X)\leq 2|X|-\varepsilon_1(X)$$

for any vertex subset X of G, where $\varepsilon_1(X)$ is defined by

$$\varepsilon_1(X)=\begin{cases}2, & \text{if } X \text{ is not an independent set;}\\ 1, & \text{if } X \text{ is a nonempty independent set and}\\ & G-X \text{ possesses a nontrivial component;}\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 1.4 ([17]) *A connected graph G is $P_{\geq 3}$-factor covered if and only if*

$$\mathrm{sun}(G-X)\leq 2|X|-\varepsilon_2(X)$$

for any vertex subset X of G, where $\varepsilon_2(X)$ is defined by

$$\varepsilon_2(X)=\begin{cases}2, & \text{if } X \text{ is not an independent set;}\\ 1, & \text{if } X \text{ is a nonempty independent set and}\\ & G-X \text{ possesses a non-sun component;}\\ 0, & \text{otherwise.}\end{cases}$$

The binding number was first introduced by Woodall [16]. We denote by $N_G(x)$ the set of vertices adjacent to a vertex x in G, and write $N_G(X)$ for $\bigcup_{x\in X} N_G(x)$. The binding number of G is the minimum value of $\frac{|N_G(X)|}{|X|}$ taken over all nonempty subsets X of $V(G)$ with $N_G(X) \neq V(G)$, and is denoted by $\mathrm{bind}(G)$, that is,

$$\mathrm{bind}(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

The relationships between binding numbers and graph factors were studied by Katerinis and Woodall [12], Zhou [18, 19], Plummer and Saito [15]. Zhou et al. [25] posed three sufficient conditions for graphs to be $P_{\geq 3}$-factor covered. Zhou [23] acquired two binding number conditions for a graph to be $P_{\geq 2}$-factor covered and $P_{\geq 3}$-factor covered, which are shown in the following.

Theorem 1.5 ([23]) *Let G be a connected graph. Then G is $P_{\geq 2}$-factor covered if*

$$\mathrm{bind}(G) > \frac{2}{3}.$$

Theorem 1.6 ([23]) *Let G be a connected graph. Then G is $P_{\geq 3}$-factor covered if*

$$\mathrm{bind}(G) \geq \frac{3}{2}.$$

We generalize the concept of a $P_{\geq n}$-factor covered graph, and define first the concept of a $(P_{\geq n}, k)$-factor-critical covered graph. A graph G is called $(P_{\geq n}, k)$-factor-critical covered if $G - D$ is $P_{\geq n}$-factor covered for any $D \subseteq V(G)$ with $|D| = k$. In this paper, we show two sufficient conditions for a graph to be $(P_{\geq 2}, k)$-factor-critical covered and $(P_{\geq 3}, k)$-factor-critical covered, which are given in Sections 2 and 3.

2 Binding Number and $(P_{\geq 2}, k)$-factor-critical Covered Graphs

Next, we give a binding number condition for a graph being $(P_{\geq 2}, k)$-factor-critical covered, which is a generalization of Theorem 1.5.

Theorem 2.1 *Let k be an integer with $k \geq 0$, and let G be a graph with $\kappa(G) \geq k+1$. Then G is $(P_{\geq 2}, k)$-factor-critical covered if* $\mathrm{bind}(G) > \frac{2+k}{3}$.

Proof Theorem 2.1 holds for $k = 0$ by Theorem 1.5. Next, we consider $k \geq 1$. Set $H = G - D$ for any $D \subseteq V(G)$ with $|D| = k$. It is obvious that H is connected. In order to demonstrate Theorem 2.1, it suffices to show that H is $P_{\geq 2}$-factor covered. On the contrary, we assume that H is not $P_{\geq 2}$-factor covered. Then it follows from Theorem 1.3 that

$$i(H - X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{2.1}$$

for some vertex subset X of H.

We shall consider three cases by the value of $|X|$.

Case 1 $|X| = 0$.

Obviously, $\varepsilon_1(X) = 0$. In terms of (2.1), we get

$$i(H) \geq 1. \tag{2.2}$$

Note that H is connected, which implies $i(H) = 0$, contradicting (2.2).

Case 2 $|X| = 1$.

We write $Y=\{x:d_{H-X}(x)=0,x\in V(H)\setminus X\}$.

Subcase 2.1 $H-X$ does not possess a nontrivial component.

Clearly, $\varepsilon_1(X)=0$. According to (2.1), we derive

$$i(H-X)\geq 2|X|-\varepsilon_1(X)+1=2|X|+1=3. \tag{2.3}$$

Obviously, $|Y|=i(H-X)\geq 2|X|+1=3$ by (2.3). We easily see that $Y\neq\emptyset$ and $|N_G(Y)|\leq|D\cup X|=|D|+|X|=k+1$. Combining these with the definition of bind(G), we have

$$\text{bind}(G)\leq\frac{|N_G(Y)|}{|Y|}\leq\frac{1+k}{3},$$

which conflicts that bind$(G)>\frac{2+k}{3}$.

Subcase 2.2 $H-X$ possesses a nontrivial component Q.

In this case, $\varepsilon_1(X)=1$. From (2.1), we acquire

$$i(H-X)\geq 2|X|-\varepsilon_1(X)+1=2|X|=2.$$

Note that $|Y|=i(H-X)\geq 2$ and $|V(Q)|\geq 2$. Then we easily see that $|N_G(Y\cup V(Q))|\leq|D|+|X|+|V(Q)|=k+1+|V(Q)|$. By the definition of bind(G), we get

$$\begin{aligned}\text{bind}(G)&\leq\frac{|N_G(Y\cup V(Q))|}{|Y\cup V(Q)|}\\&\leq\frac{k+1+|V(Q)|}{|Y|+|V(Q)|}\\&\leq\frac{k+1+|V(Q)|}{2+|V(Q)|}\\&=1+\frac{k-1}{2+|V(Q)|}\\&\leq 1+\frac{k-1}{4}\\&=\frac{3+k}{4},\end{aligned}$$

which contradicts that bind$(G)>\frac{2+k}{3}$ by $k\geq 1$.

Case 3 $|X|\geq 2$.

Note that $\varepsilon_1(X)\leq 2$. From (2.1), we get

$$i(H-X)\geq 2|X|-\varepsilon_1(X)+1=2|X|-1=3. \tag{2.4}$$

Let $W=\{x:d_{H-X}(x)=0,x\in V(H)\setminus X\}$. It follows from (2.4) that $W\neq\emptyset$ and $|N_G(W)|\leq|D\cup X|=|D|+|X|=k+|X|$. In light of (2.4) and the definition of bind(G), we derive

$$\begin{aligned}\text{bind}(G)&\leq\frac{|N_G(W)|}{|W|}\\&=\frac{|N_G(W)|}{i(H-X)}\\&\leq\frac{k+|X|}{2|X|-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(1+\frac{2k+1}{2|X|-1}\right)\\
&\leq \frac{1}{2}\left(1+\frac{2k+1}{4-1}\right)\\
&= \frac{2+k}{3},
\end{aligned}
$$

which contradicts that $\text{bind}(G) > \frac{2+k}{3}$. Theorem 2.1 is verified. □

Remark 2.2 Now, we claim that the condition $\text{bind}(G) > \frac{2+k}{3}$ in Theorem 2.1 is sharp.

Set $G = K_{k+2} \vee (3K_1)$, where k is a nonnegative integer, and $\vee$ denotes "join". It is easily seen that $\text{bind}(G) = \frac{2+k}{3}$ and $\kappa(G) = k+2 > k+1$. For any $D \subseteq V(K_{k+2})$ with $|D| = k$, let $H = G - D$. We select $X = V(K_{k+2}) \setminus D \subseteq V(H)$, and so $|X| = 2$. Note that X is not an independent set. Then we admit $\varepsilon_1(X) = 2$. Thus, we acquire

$$i(H - X) = 3 > 2 = 2|X| - \varepsilon_1(X).$$

Using Theorem 1.3, H is not $P_{\geq 2}$-factor covered, that is, G is not $(P_{\geq 2}, k)$-factor-critical covered.

Remark 2.3 Now, we claim that the condition $\kappa(G) \geq k+1$ in Theorem 2.1 cannot be replaced by $\kappa(G) \geq k$.

Let $G = K_k \vee (2K_1)$, where k is an integer with $k \geq 5$. Obviously, $\kappa(G) = k$ and $\text{bind}(G) = \frac{k}{2} = \frac{3k}{6} \geq \frac{2k+5}{6} > \frac{2k+4}{6} = \frac{2+k}{3}$. For $D = V(K_k)$, let $H = G - D$. Obviously, H is not $P_{\geq 2}$-factor covered, and so G is not $(P_{\geq 2}, k)$-factor-critical covered.

3 Binding Number and $(P_{\geq 3}, k)$-factor-critical Covered Graphs

In this section, we pose a binding number condition for a graph to be $(P_{\geq 3}, k)$-factor-critical covered, which is an extension of Theorem 1.6.

Theorem 3.1 *Let k be an integer with $k \geq 1$, and let G be a graph with $\kappa(G) \geq k+1$ and $|V(G)| \geq k+3$. If $\text{bind}(G) \geq \frac{4+k}{3}$, then G is $(P_{\geq 3}, k)$-factor-critical covered.*

Proof For any $D \subseteq V(G)$ with $|D| = k$, we write $H = G - D$. Clearly, H is connected. To justify Theorem 3.1, it suffices to verify that H is $(P_{\geq 3}, k)$-factor covered. Next, we assume that H is not $(P_{\geq 3}, k)$-factor covered. Then by Theorem 1.4, we acquire

$$\text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 \tag{3.1}$$

for some vertex subset X of H.

Claim 1 $X \neq \emptyset$.

Proof Let $X = \emptyset$. Then $\varepsilon_2(X) = 0$. Using (3.1), we admit

$$\text{sun}(H) = \text{sun}(H - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq 1. \tag{3.2}$$

Note that H is connected. Combining this with (3.2),

$$1 \leq \text{sun}(H) \leq \omega(H) = 1,$$

that is,

$$\text{sun}(H) = \omega(H) = 1. \tag{3.3}$$

It follows from (3.3), $H = G - D$ and $|V(G)| \geq k+3$ that H is a big sun (otherwise, $H = K_1$ or K_2. Then $|V(H)| \leq 2$. Thus, we possess that $|V(G)| = |V(H)| + |D| \leq 2 + k$, which contradicts that $|V(G)| \geq k+3$). We write R for the factor-critical subgraph of H, and so $|V(H) \setminus V(R)| = |V(R)| \geq 3$. Thus, we acquire

$$\begin{aligned}\text{bind}(G) &\leq \frac{|N_G(V(H) \setminus V(R))|}{|V(H) \setminus V(R)|} \\ &= \frac{|N_G(V(G) \setminus (D \cup V(R)))|}{|V(R)|} \\ &\leq \frac{|D \cup V(R)|}{|V(R)|} \\ &= \frac{|D| + |V(R)|}{|V(R)|} \\ &= 1 + \frac{k}{|V(R)|} \\ &\leq 1 + \frac{k}{3},\end{aligned}$$

which contradicts that $\text{bind}(G) \geq \frac{4+k}{3}$. Claim 1 is verified. □

Assume that there exist a isolated vertices, b K_2's and c big sun components $Q_1, Q_2, \dots, Q_c$, where $|V(Q_i)| \geq 6$ for $1 \leq i \leq c$, in $H - X$. By (3.1), we get

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1. \tag{3.4}$$

Case 1 $|X| = 1$.

Subcase 1.1 $H - X$ admits a non-sun component Y.

We easily see that $\varepsilon_2(X) = 1$ and $|V(Y)| \geq 3$ (otherwise, $Y = K_1$ or K_2, which is a sun component, a contradiction). Then from (3.4), we obtain

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 = 2|X| = 2. \tag{3.5}$$

Subcase 1.1.1 $a \geq 1$.

Let $W = V(Y) \cup V(aK_1) \cup V(bK_2) \cup V(Q_1) \cup \cdots \cup V(Q_c)$. Then

$$\begin{aligned}|N_G(W)| &\leq |D| + |X| + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| \\ &= k + 1 + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)|\end{aligned}$$

and

$$|W| = |V(Y)| + a + 2b + \sum_{i=1}^{c} |V(Q_i)| \geq 3 + a + 2b + 6c.$$

Combining these with (3.5) and the definition of $\text{bind}(G)$,

$$\begin{aligned}\text{bind}(G) &\leq \frac{|N_G(W)|}{|W|} \\ &\leq \frac{k + 1 + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)|}{|V(Y)| + a + 2b + \sum_{i=1}^{c} |V(Q_i)|}\end{aligned}$$

$$\begin{aligned}&= 1 + \frac{k+1-a}{|V(Y)| + a + 2b + \sum_{i=1}^{c} |V(Q_i)|}\\&\le 1 + \frac{k}{3+a+2b+6c}\\&\le 1 + \frac{k}{3+a+b+c}\\&\le 1 + \frac{k}{5},\end{aligned}$$

which contradicts that $\mathrm{bind}(G) \ge \frac{4+k}{3}$ since $k \ge 1$.

Subcase 1.1.2 $a = 0$.

Clearly, $b + c \ge 2$ by (3.5). Setting $M = Y \cup (bK_2) \cup Q_1 \cup \cdots \cup Q_c$. Then there exist $u, v \in V(M)$ with $d_M(u) = 1$ and $uv \in E(M)$. Thus, we derive

$$\begin{aligned}|N_G(V(M) \setminus \{v\})| &\le |D| + |X| + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1\\&= k + 1 + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1\\&= k + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)|\end{aligned}$$

and

$$|V(M) \setminus \{v\}| = |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1 \ge 3 + 2b + 6c - 1 = 2 + 2b + 6c.$$

Combining these with $b + c \ge 2$, $\mathrm{bind}(G) \ge \frac{4+k}{3}$ and the definition of $\mathrm{bind}(G)$,

$$\begin{aligned}\frac{4+k}{3} &\le \mathrm{bind}(G)\\&\le \frac{|N_G(V(M) \setminus \{v\})|}{|V(M) \setminus \{v\}|}\\&\le \frac{k + |V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)|}{|V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1}\\&= 1 + \frac{k+1}{|V(Y)| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1}\\&\le 1 + \frac{k+1}{2+2b+6c}\\&\le 1 + \frac{k+1}{2+2b+2c}\\&\le 1 + \frac{k+1}{6},\end{aligned}$$

which implies

$$\frac{1}{3} \le \frac{1}{6},$$

it is a contradiction.

Subcase 1.2 $H - X$ does not admit a non-sun component.

Clearly, $\varepsilon_2(X) = 0$ by the definition of $\varepsilon_2(X)$. It follows from (3.4) that

$$\mathrm{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 = 2|X| + 1 = 3. \tag{3.6}$$

Subcase 1.2.1 $a \geq 1$.

We write $Z = (aK_1) \cup (bK_2) \cup Q_1 \cup \cdots \cup Q_c$. In light of $\mathrm{bind}(G) \geq \frac{4+k}{3}$ and the definition of $\mathrm{bind}(G)$, we have

$$\begin{aligned}\frac{4+k}{3} &\leq \mathrm{bind}(G)\\ &\leq \frac{|N_G(V(Z))|}{|V(Z)|}\\ &\leq \frac{|D| + |X| + 2b + \sum_{i=1}^{c} |V(Q_i)|}{a + 2b + \sum_{i=1}^{c} |V(Q_i)|}\\ &= \frac{k + 1 + 2b + \sum_{i=1}^{c} |V(Q_i)|}{a + 2b + \sum_{i=1}^{c} |V(Q_i)|}\\ &\leq \frac{k + a + 2b + \sum_{i=1}^{c} |V(Q_i)|}{a + 2b + \sum_{i=1}^{c} |V(Q_i)|}\\ &= 1 + \frac{k}{a + 2b + \sum_{i=1}^{c} |V(Q_i)|},\end{aligned}$$

that is,

$$0 \geq (k+1)\left(a + 2b + \sum_{i=1}^{c} |V(Q_i)|\right) - 3k. \tag{3.7}$$

Using (3.6), (3.7) and $|V(Q_i)| \geq 6$, we get

$$\begin{aligned}0 &\geq (k+1)\left(a + 2b + \sum_{i=1}^{c} |V(Q_i)|\right) - 3k\\ &\geq (k+1)(a + 2b + 6c) - 3k\\ &\geq (k+1)(a + b + c) - 3k\\ &\geq 3(k+1) - 3k\\ &= 3,\end{aligned}$$

which is a confliction.

Subcase 1.2.2 $a = 0$.

Let $T = (bK_2) \cup Q_1 \cup \cdots \cup Q_c$. Then there exist $x, y \in V(T)$ with $d_T(x) = 1$ and $xy \in E(T)$. By $\mathrm{bind}(G) \geq \frac{4+k}{3}$ and the definition of $\mathrm{bind}(G)$,

$$\begin{aligned}\frac{4+k}{3} &\leq \mathrm{bind}(G)\\ &\leq \frac{|N_G(V(T) \setminus \{y\})|}{|V(T) \setminus \{y\}|}\\ &\leq \frac{|D| + |X| + 2b + \sum_{i=1}^{c} |V(Q_i)| - 1}{2b + \sum_{i=1}^{c} |V(Q_i)| - 1}\\ &= \frac{k + 2b + \sum_{i=1}^{c} |V(Q_i)|}{2b + \sum_{i=1}^{c} |V(Q_i)| - 1}\end{aligned}$$

$$= 1 + \frac{k+1}{2b + \sum_{i=1}^{c} |V(Q_i)| - 1},$$

which implies

$$0 \geq 2b + \sum_{i=1}^{c} |V(Q_i)| - 4. \tag{3.8}$$

According to (3.6), (3.8), $a = 0$ and $|V(Q_i)| \geq 6$, we obtain

$$\begin{aligned} 0 &\geq 2b + \sum_{i=1}^{c} |V(Q_i)| - 4 \\ &\geq 2b + 6c - 4 \\ &\geq 2b + 2c - 4 \\ &= 2(a + b + c) - 4 \\ &\geq 6 - 4 \\ &= 2, \end{aligned}$$

a contradiction.

Case 2 $|X| \geq 2$.

It is obvious that $\varepsilon_2(X) \leq 2$. It follows from (3.4) that

$$\text{sun}(H - X) = a + b + c \geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \geq 3. \tag{3.9}$$

Subcase 2.1 $b + c = 0$.

In this case, we have $a \geq 3$ by (3.9). Thus, we derive by (3.9) and the definition of $\text{bind}(G)$

$$\begin{aligned} \text{bind}(G) &\leq \frac{|N_G(V(aK_1))|}{|V(aK_1)|} \\ &\leq \frac{|D| + |X|}{a} \\ &= \frac{2k + 2|X|}{2a} \\ &\leq \frac{2k + a + b + c + 1}{2a} \\ &= \frac{2k + a + 1}{2a} \\ &< \frac{2k + 2a}{2a} \\ &= 1 + \frac{k}{a} \\ &\leq 1 + \frac{k}{3}, \end{aligned}$$

which conflicts that $\text{bind}(G) \geq \frac{4+k}{3}$.

Subcase 2.2 $b + c \geq 1$.

Let $N = (bK_2) \cup Q_1 \cup \cdots \cup Q_c$. It is easily seen that there exist $u, v \in V(N)$ with $d_N(u) = 1$ and $uv \in E(N)$. In terms of $\text{bind}(G) \geq \frac{4+k}{3}$ and the definition of $\text{bind}(G)$, we acquire

$$\frac{4+k}{3} \leq \text{bind}(G)$$

$$\begin{aligned}
&\le \frac{|N_G(V(aK_1)\cup(V(N)\setminus\{v\}))|}{|V(aK_1)\cup(V(N)\setminus\{v\})|}\\
&\le \frac{|D|+|X|+2b+\sum_{i=1}^{c}|V(Q_i)|-1}{a+2b+\sum_{i=1}^{c}|V(Q_i)|-1}\\
&= \frac{k+|X|+2b+\sum_{i=1}^{c}|V(Q_i)|-1}{a+2b+\sum_{i=1}^{c}|V(Q_i)|-1}\\
&= 1+\frac{k+|X|-a}{a+2b+\sum_{i=1}^{c}|V(Q_i)|-1}\\
&\le 1+\frac{k+|X|-a}{a+2b+6c-1},
\end{aligned}$$

which implies

$$3|X| \ge 4a+2b+6c-1+k(a+2b+6c-4).$$

Combining this with (3.9), $b+c\ge 1$ and $k\ge 1$, we admit

$$\begin{aligned}
3|X| &\ge 4a+2b+6c-1+k(a+2b+6c-4)\\
&\ge 4a+2b+6c-1+(a+2b+6c-4)\\
&= 5a+4b+12c-5\\
&\ge 4(a+b+c)-5\\
&= 4\mathrm{sun}(H-X)-5\\
&\ge 4(2|X|-1)-5\\
&= 8|X|-9,
\end{aligned}$$

that is,

$$|X| \le \frac{9}{5} < 2,$$

which contradicts that $|X|\ge 2$. Theorem 3.1 is demonstrated. □

We immediately derive the following result when setting $k=1$ in Theorem 3.1.

Corollary 3.2 *Let G be a graph with $\kappa(G)\ge 2$ and $|V(G)|\ge 4$. If $\mathrm{bind}(G)\ge \frac{5}{3}$, then $G-\{x\}$ is $P_{\ge 3}$-factor covered for any $x\in V(G)$.*

A claw is a graph isomorphic to $K_{1,3}$. A graph is said to be claw-free if it does not include induced claw. The following result on the existence of $\{P_3\}$-factors in vertex deleted graphs is known, which is similar to Corollary 3.2.

Theorem 3.3 ([13]) *Let G be a 2-connected claw-free graph of order n. If $n\equiv 1 \pmod 3$, then $G-\{x\}$ has a $\{P_3\}$-factor for any $x\in V(G)$.*

Remark 3.4 Next, we claim that the assumption on binding number in Theorem 3.1 is best possible, that is, the condition $\mathrm{bind}(G)\ge\frac{4+k}{3}$ in Theorem 3.1 cannot be replaced by $\mathrm{bind}(G)\ge\frac{4+k}{4}$.

Let $k\ge 1$ be an integer. We construct a graph $G=K_{k+2}\vee(2K_1\cup K_2)$, where $\vee$ denotes "join". Obviously, $\mathrm{bind}(G)=\frac{4+k}{4}$ and $\kappa(G)=k+2>k+1$. Set $H=G-D$, where $D\subseteq V(K_{k+2})$ with $|D|=k$. Select $X=V(K_{k+2})\setminus D\subseteq V(H)$, and so $|X|=2$. Note that X is not an independent set, and so $\varepsilon_2(X)=2$. Therefore, we have

$$\mathrm{sun}(H-X)=3>2=2|X|-\varepsilon_2(X).$$

In light of Theorem 1.4, H is not $P_{\geq 3}$-factor covered, and so G is not $(P_{\geq 3}, k)$-factor-critical covered.

Remark 3.5 Next, we claim that the condition $\kappa(G) \geq k+1$ in Theorem 3.1 cannot be replaced by $\kappa(G) \geq k$.

We construct a graph $G = K_k \vee (2K_1)$, where k is an integer with $k \geq 8$. Apparently, $\kappa(G) = k$ and $\text{bind}(G) = \frac{k}{2} = \frac{3k}{6} \geq \frac{8+2k}{6} = \frac{4+k}{3}$. For $D = V(K_k)$, set $H = G - D$. It is obvious that H is not $P_{\geq 3}$-factor covered, and so G is not $(P_{\geq 3}, k)$-factor-critical covered.

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