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Transportation Cost Inequalities for Stochastic Reaction-Diffusion Equations with Lévy Noises and Non-Lipschitz Reaction Terms

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Abstract For stochastic reaction-diffusion equations with Lévy noises and non-Lipschitz reaction terms, we prove that W_1H transportation cost inequalities hold for their invariant probability measures and for their process-level laws on the path space with respect to the L^1-metric. The proofs are based on the Galerkin approximations.

Keywords Stochastic reaction-diffusion equation, poisson random measure, transportation cost inequality

MR(2010) Subject Classification 28A35, 60E15

1 Introduction

Let (E,d) be a metric space equipped with σ-field $\mathcal{B}$ such that $d(\cdot,\cdot)$ is $\mathcal{B}\times\mathcal{B}$-measurable. Given $p\geq 1$ and two probability measures μ and ν on E, define the L^p-Wasserstein distance between μ and ν:

$$W_{p,d}(\mu,\nu)=\inf\left(\int\int_{E\times E}d(x,y)^p\pi(dx,dy)\right)^{1/p},$$

where the infimum is taken over all probability measures π on the product space $E\times E$ with marginal distribution μ and ν. The relative entropy of ν with respect to μ is defined by

$$\mathbf{H}(\nu|\mu)=\begin{cases}\displaystyle\int_E \log\frac{d\nu}{d\mu}d\nu, & \text{if } \nu\ll\mu;\\ +\infty, & \text{otherwise.}\end{cases}\tag{1.1}$$

We say that the probability μ satisfies a W_pH transportation cost-information inequality on (E,d) if there exists a constant $C>0$ such that for any probability measure $\nu\in\mathcal{M}_1(E)$

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(the space of all probability measures on E),

$$W_{p,d}(\mu,\nu) \le \sqrt{2C\mathbf{H}(\nu|\mu)}. \tag{1.2}$$

Let $\alpha : [0,\infty) \to [0,\infty]$ be a non-decreasing left-continuous convex function, with $\alpha(0) = 0$. μ is said to satisfy the $\alpha - W_pH$ if for all probability measure ν on E,

$$\alpha(W_{p,d}(\mu,\nu)) \le \mathbf{H}(\nu|\mu). \tag{1.3}$$

The inequality (1.2) is a particular case of (1.3) with $\alpha(t) = t^{2/p}/(2C)$ for any $t \ge 0$.

The properties $W_pH, p = 1, 2$ are of particular interest. They have been brought into relation with the phenomenon of measure concentration, functional inequalities, Hamilton–Jacobi's equation, optimal transport problem, large deviations, (e.g., [2, 3, 5, 10, 11, 14–16, 28, 30, 31]) and references therein. For example, we give Gozlan–Léonard's characterization for $\alpha - W_1H$ transportation cost inequality.

Theorem 1.1 (Gozlan–Léonard [15]) *Let $\alpha : [0,\infty) \to [0,\infty]$ be a non-decreasing left continuous convex function with $\alpha(0) = 0$. The following properties are equivalent*:

(i) *The $\alpha - W_1H$ inequality below holds*

$$\alpha(W_{1,d}(\nu,\mu)) \le H(\nu|\mu), \quad \forall \nu \in \mathcal{M}_1(E);$$

(ii) *For every $f : (E, d) \to \mathbb{R}$ bounded and Lipschitzian with $\|f\|_{\mathrm{Lip}} \le 1$,*

$$\int_E \mathrm{e}^{\lambda(f-\mu(f))} d\mu \le \mathrm{e}^{\alpha^*(\lambda)}, \quad \lambda > 0, \tag{1.4}$$

where $\alpha^*(\lambda) := \sup_{r\ge 0}(r\lambda - \alpha(r))$ is the semi-Legendre transformation;

(iii) *Let $(\xi_k)_{k\ge 1}$ be a sequence of independent and identically distributed random variables taking values in E of common law μ. For every $f : E \to \mathbb{R}$ with $\|f\|_{\mathrm{Lip}} \le 1$,*

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n f(\xi_k) - \mu(f) > r\right) \le \mathrm{e}^{-n\alpha(r)}, \quad r > 0,\ n \ge 1.$$

The equivalence of (i) and (ii) is a generalization of Bobkov-Götze's criterion [5] for quadratic α, and (iii) gives a probability meaning to the $\alpha - W_1H$ inequality.

The W_2H inequalities on the path spaces of stochastic (partial) differential equations driven by Gaussian noises have been investigated by many authors, for example, [4, 10, 12, 23] for stochastic differential equations (SDEs) and [6, 26, 33] for stochastic partial differential equations (SPDEs).

The $\alpha - W_1H$ inequalities on the path spaces of SDEs with jumps have also been investigated, see [32] for SDEs driven by pure jump processes, [19] for SDEs driven by both Gaussian and jump noises, and [27] for regime-switching diffusion processes.

The transportation inequalities for non-globally dissipative SDEs with jumps were studied in Majka [21], by using the mirror coupling for the jump part and the reflection coupling for the Brownian part, for bounding Malliavin derivatives of solutions of SDEs with both jump and Gaussian noise. We would also like to mention the works of [18] and [20] for the exponential convergence with respect to the L^1-Wasserstein distance when the drift is dissipative outside a compact set.

The aim of this paper is to prove that the $\alpha - W_1H$ transportation cost inequalities hold for stochastic reaction-diffusion equations driven by both Gaussian and Lévy noises under the L^1 distance in the path space. The reaction term can be chosen to be Lipschitz continuous, or to be a polynomial, for example $f(x) = -x^3 + C_1 x$ for some $C_1 \in \mathbb{R}$. The main ingredient in our study is the finite dimensional approximations, which is more or less standard in the literature for the Lipschitz case, but it is difficult in the non-Lipschitz case.

This paper is organized as follows. In Section 2, we present the framework for the stochastic reaction-diffusion equations with jumps and with Lipschitz reaction terms, and then prove the transportation cost inequalities. In Section 3, we first establish some tightness results for approximating processes of the system with non-Lipschitz reaction terms, and then prove the transportation cost inequalities.

2 Transportation Cost Inequalities for SPDE with Lévy Noise and Lipschitz Reaction Term

2.1 SPDE with Lévy Noise and Lipschitz Reaction Term

Let $\mathbb{H} := L^2(0,1)$ be the space of square integrable real-valued functions on $[0,1]$. The norm and the inner product on $\mathbb{H}$ are denoted by $\|\cdot\|_{\mathbb{H}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, respectively. Let $\mathbb{H}^k(0,1)$ be the Sobolev space of all functions in $\mathbb{H}$ whose derivatives up to order k also belong to $\mathbb{H}$. $\mathbb{H}_0^1(0,1)$ is the subspace of $\mathbb{H}^1(0,1)$ of all functions whose values at 0 and 1 vanish. Let Δ be the Laplace operator on $\mathbb{H}$:

$$\Delta x := \frac{\partial^2}{\partial \xi^2} x(\xi), \quad x \in \mathbb{H}^2(0,1) \cap \mathbb{H}_0^1(0,1).$$

It is well known that Δ is the infinitesimal generator of a strongly continuous semigroup $S(t) := \mathrm{e}^{t\Delta}, t \geq 0$. $\{e_k(\xi) := \sqrt{2}\sin(k\pi\xi)\}_{k\geq 1}$ is an orthonormal basis of $\mathbb{H}$ consisting of the eigenvectors of Δ, i.e.,

$$\Delta e_k = -\lambda_k e_k \quad \text{with } \lambda_k = k^2\pi^2.$$

For any $\theta \in \mathbb{R}$, let

$$\mathbb{H}_\theta := \Big\{ x = \sum_{k\geq 1} x_k e_k : (x_k)_{k\geq 1} \in \mathbb{R}, \sum_{k\geq 1} \lambda_k^\theta |x_k|^2 < \infty \Big\},$$

endowed with norm

$$\|x\|_{\mathbb{H}_\theta} := \Big(\sum_{k\geq 1} \lambda_k^\theta |x_k|^2 \Big)^{1/2}.$$

Then, for any $\theta > 0$, $\mathbb{H}_\theta$ is densely and compactly embedded in $\mathbb{H}$. Particularly, denote $\mathbb{V} := \mathbb{H}_1 = \mathbb{H}_0^1(0,1)$, whose dual space is $\mathbb{V}^* = \mathbb{H}_{-1}$. The norm and the inner product on $\mathbb{V}$ are denoted by $\|\cdot\|_{\mathbb{V}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{V}}$, respectively. If ${}_{\mathbb{V}^*}\langle\cdot,\cdot\rangle_{\mathbb{V}}$ denotes the duality between $\mathbb{V}$ and its dual space $\mathbb{V}^*$, we have

$${}_{\mathbb{V}^*}\langle u, v\rangle_{\mathbb{V}} = \langle u, v\rangle_{\mathbb{H}}, \quad \text{for any } u \in \mathbb{H}, v \in \mathbb{V}.$$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ be a measurable space, and ϑ a σ-finite measure on it. Let $N(dt, du)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{X}$ with intensity measure $dt\vartheta(du)$, $\widetilde{N}(dt, du) = N(dt, du) - dt\vartheta(du)$ the compensated Poisson random measure, and $(\beta^k)_{k\geq 1}$ a sequence of independent and identically distributed one dimensional standard

Brownian motions on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then $\beta_t := \sum_{k\geq 1} \beta_t^k e_k$ is an $\mathbb{H}$-cylindrical Brownian motion.

Consider the following SPDE on the Hilbert space $\mathbb{H}$:

$$\begin{cases} dX_t = \Delta X_t dt + f(X_t)dt + \sigma(X_t)d\beta_t + \displaystyle\int_{\mathbb{X}} G(X_{t-}, v)\widetilde{N}(dt, dv); \\ X_0 = x \in \mathbb{H}, \end{cases} \tag{2.1}$$

where x is $\mathcal{F}_0$-measurable. The coefficients $f : \mathbb{H} \to \mathbb{H}$, $\sigma : \mathbb{H} \to \mathcal{L}_2(\mathbb{H}; \mathbb{H})$ (the space of all Hilbert–Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$), $G : \mathbb{H}\times\mathbb{X} \to \mathbb{H}$ are Fréchet continuously differentiable, and they satisfy the following conditions:

(H1) The reaction term f is Lipschitz continuous, i.e., there exists a positive constant $C_f > 0$ such that

$$\|f(x) - f(y)\|_{\mathbb{H}} \leq C_f \|x - y\|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H}.$$

(H2) σ is Lipschitz continuous, i.e., there exists a positive constant $C_\sigma > 0$ such that

$$\|\sigma(x) - \sigma(y)\|_{\mathrm{HS}} \leq C_\sigma \|x - y\|_{\mathbb{H}}, \quad \forall x, y \in \mathbb{H}.$$

(H3) G satisfies the following conditions:

$$\int_{\mathbb{X}} \|G(x, v) - G(y, v)\|_{\mathbb{H}}^2 \vartheta(dv) \leq C_G \|x - y\|_{\mathbb{H}}^2; \tag{2.2}$$

$$\int_{\mathbb{X}} \|G(x, v)\|_{\mathbb{H}}^2 \vartheta(dv) \leq C_G(1 + \|x\|_{\mathbb{H}}^2). \tag{2.3}$$

Let $\mathbb{D}([0, T]; \mathbb{H})$ be the space of all right continuous with left limits $\mathbb{H}$-valued functions on $[0, T]$, endowed with the Skorokhod topology. Recall the following results about equation (2.1) from [25, Theorem 3.3] and [35, Lemma 3.13].

Theorem 2.1 ([25, 35]) *Under Conditions* (H1)–(H3)*, for any* $x \in L^2(\Omega; \mathbb{H})$*, there exists a unique* $\mathbb{H}$*-valued progressively measurable process* $\{X_t\}_{t\in[0,T]} \in \mathbb{D}([0, T]; \mathbb{H}) \cap L^2((0, T]; \mathbb{V})$ *for any* $T > 0$*, and for any* $\phi \in \mathbb{V}$*, it holds that a.s.*

$$\begin{aligned} \langle X_t, \phi\rangle_{\mathbb{H}} = \langle x, \phi\rangle_{\mathbb{H}} &+ \int_0^t {}_{\mathbb{V}^*}\langle \Delta X_s, \phi\rangle_{\mathbb{V}} ds + \left\langle \int_0^t \sigma(X_s)d\beta_s, \phi \right\rangle_{\mathbb{H}} \\ &+ \int_0^t \int_{\mathbb{X}} \langle G(X_{s-}, v), \phi\rangle_{\mathbb{H}} \widetilde{N}(ds, dv). \end{aligned} \tag{2.4}$$

Furthermore, we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|X_t\|_{\mathbb{H}}^2 + \int_0^T \|X_t\|_{\mathbb{V}}^2 dt\right] < \infty. \tag{2.5}$$

Remark 2.2 Recall that λ_1 is the first eigenvalue of $-\Delta$. Let

$$K := 2\lambda_1 - (2C_f + C_\sigma^2 + C_G). \tag{2.6}$$

By (H1)–(H3), we know that for all $x_1, x_2 \in \mathbb{H}$,

$$\begin{aligned} &2\langle x_1 - x_2, \Delta(x_1 - x_2)\rangle_{\mathbb{H}} + 2\langle x_1 - x_2, f(x_1) - f(x_2)\rangle_{\mathbb{H}} + \|\sigma(x_1) - \sigma(x_2)\|_{\mathrm{HS}}^2 \\ &\quad + \int_{\mathbb{X}} \|G(x_1, v) - G(x_2, v)\|_{\mathbb{H}}^2 \vartheta(dv) \end{aligned}$$

$$\leq -K\|x_1 - x_2\|_{\mathbb{H}}^2. \tag{2.7}$$

When $K > 0$, (2.7) is the globally dissipative condition, which is used to guarantee the existence of invariant measure and further to obtain the transportation cost inequality for this invariant measure. For example, under the globally dissipative condition, the following results hold: the Markov process $\{X_t\}_{t\geq 0}$ admits an invariant measure (e.g. [24, Chapter 16]); for the SPDE driven by additive Gaussian noise (i.e., $G = 0$ and σ is a constant matrix), Da Prato *et al.* [9] obtained the log-Sobolev inequality for the invariant measure of $\{X_t\}_{t\geq 0}$, and Wu and Zhang [33] obtained the log-Sobolev inequalities for the process-level law on the continuous path space with respect to the L^2-metric, which are stronger than the transportation cost inequality W_2H by [22]; for finite dimensional stochastic differential equations with jumps, the transportation cost inequalities W_1H were obtained for their invariant probability measure as well as for their process-level law on the right continuous paths space with respect to the L^1-metric, see [32] and [19].

2.2 Transportation Cost Inequalities for SPDE with Lévy Noise and Lipschitz Reaction Term

Recall the following result, which tells that the W_pH-inequality is stable under the weak convergence.

Lemma 2.3 ([10, Lemma 2.2]) *Let (E, d) be a metric, separable and complete space and $(\mu_n, \mu)_{n\in\mathbb{N}}$ a family of probability measures on E. Assume that $\mu_n \in W_pH(C)$ for all $n \in \mathbb{N}$ and $\mu_n \to \mu$ weakly. Then $\mu \in W_pH(C)$.*

The first named author [19] proved the transportation cost inequalities for SDE with Lévy noises under the globally dissipative condition. Now, we use the finite dimensional approximation's technique and Lemma 2.3 to prove the transportation cost inequalities for SPDE (2.1).

Theorem 2.4 *Assume Conditions* (H1)–(H3) *hold, $K > 0$ in* (2.6), *$\|\sigma(x)\|_{\mathrm{HS}} \leq \bar{\sigma}$ for any $x \in \mathbb{H}$ and there is some Borel-measurable function $\bar{G}(u)$ on $\mathbb{X}$ such that $|G(x, v)| \leq \bar{G}(v)$ for all $x \in \mathbb{H}, v \in \mathbb{X}$ and*

$$\exists\lambda > 0 : \Lambda(\lambda) := \int_{\mathbb{X}} (\mathrm{e}^{\lambda\bar{G}(v)} - \lambda\bar{G}(v) - 1)\vartheta(dv) < \infty. \tag{2.8}$$

The following properties hold:

(1) *$\{X_t\}_{t\geq 0}$ admits a unique invariant probability measure μ, and for any $p \in [1, 2]$, $t > 0$, $\nu \in \mathcal{M}_1(\mathbb{H})$,*

$$W_{p,d}(\nu P_t, \mu) \leq \mathrm{e}^{-Kt} W_{p,d}(\nu, \mu), \tag{2.9}$$

where $d(x, y) = \|x - y\|_{\mathbb{H}}$.

(2) *For each $T > 0, x \in \mathbb{H}$, the Markov transition probability $P_T(x, dy)$ satisfies the following $\alpha - W_1H$ transportation inequality*:

$$\alpha_T(W_{1,d}(\nu, P_T(x, dy))) \leq \mathbf{H}(\nu|P_T(x, dy)), \quad \forall\nu \in \mathcal{M}_1(\mathbb{H}), \tag{2.10}$$

where

$$\begin{aligned}\alpha_T(r) &:= \sup_{\lambda>0}\left\{r\lambda - \int_0^T \Lambda(\mathrm{e}^{-Kt}\lambda)dt - \frac{\bar{\sigma}^2\lambda^2}{4K}(1 - \mathrm{e}^{-2KT})\right\} \\ &\geq \frac{1}{K}\gamma_{1/2}^*(Kr),\end{aligned}$$

with $\gamma_a(\lambda) := \Lambda(\lambda) + a\bar{\sigma}^2\lambda^2/2$ *and* $\gamma_a^*(r) := \sup_{\lambda\geq 0}(r\lambda - \gamma_a(\lambda))$, $r \geq 0$. *In particular, for the invariant probability measure* μ,

$$\frac{1}{K}\gamma_{1/2}^*(KW_{1,d}(\nu,\mu)) \leq \alpha_\infty(W_{1,d}(\nu,\mu)) \leq \mathbf{H}(\nu|\mu), \tag{2.11}$$

for all $\nu \in \mathcal{M}_1(\mathbb{H})$.

(3) *For each* $T > 0$, *the law* $\mathbb{P}_x$ *of* $X_{[0,T]}$, *the solution of* (2.1) *with* $X_0 = x$ *being a fixed point in* $\mathbb{H}$, *satisfies the* W_1H *on the space* $\mathbb{D}([0,T];\mathbb{H})$,

$$\alpha_T^P(W_{1,d_{L^1}}(\mathbb{Q},\mathbb{P}_x)) \leq \mathbf{H}(\mathbb{Q}|\mathbb{P}_x), \quad \forall\ \mathbb{Q} \in \mathcal{M}_1(\mathbb{D}([0,T];\mathbb{H})) \tag{2.12}$$

and

$$\begin{aligned}\alpha_T^P(r) &:= \sup_{\lambda>0}\left(\lambda r - \int_0^T \Lambda(\eta(t)\lambda)dt - \frac{\bar{\sigma}^2\lambda^2}{2}\int_0^T \eta^2(t)dt\right)\\ &\geq T\gamma_1^*(rK/T),\end{aligned} \tag{2.13}$$

where $\eta(t) := (1-\mathrm{e}^{-Kt})/K$ *and* $d_{L^1}(\gamma_1,\gamma_2) = \int_0^T \|\gamma_1(t)-\gamma_2(t)\|_{\mathbb{H}}dt$ *for any* $\gamma_1,\gamma_2 \in \mathbb{D}([0,T];\mathbb{H})$.

According to the proof of [32, Corollary 2.7], we can apply part (3) of Theorem 2.4 to obtain the following result.

Corollary 2.5 *In the framework of Theorem* 2.4, *let* $\mathcal{A}$ *be a family of real Lipschitzian functions* f *on* $\mathbb{H}$ *with* $\|f\|_{\mathrm{Lip}} := \sup_{x\neq y\in\mathbb{H}}|f(x)-f(y)|/\|x-y\|_{\mathbb{H}} \leq 1$, *and*

$$Z_T := \sup_{f\in\mathcal{A}}\left(\frac{1}{T}\int_0^T f(X_s)ds - \mu(f)\right).$$

We have for all $r, T > 0$,

$$\log\mathbb{P}(Z_T > \mathbb{E}[Z_T] + r) \leq -\alpha_T^P(Tr) \leq -T\gamma_1^*(Kr).$$

The same inequality holds for $Z_T = W_{1,d}(L_T,\mu)$, *where* $L_T := \frac{1}{T}\int_0^T \delta_{X_s}ds$ *is the empirical measure.*

Proof of Theorem 2.4 Recall that $\{e_1, e_2, \ldots\}$ is an orthonormal basis of $\mathbb{H}$. Let $P_n : \mathbb{V}^* \to \mathbb{H}_n$ be defined by

$$P_n y := \sum_{i=1} {}_{\mathbb{V}^*}\langle y, e_i\rangle_{\mathbb{V}} e_i, \quad y \in \mathbb{V}^*. \tag{2.14}$$

Then $P_n|_{\mathbb{H}}$ is also the orthogonal projection onto $\mathbb{H}_n$ in $\mathbb{H}$ and we have

$${}_{\mathbb{V}^*}\langle P_n\Delta u, v\rangle_{\mathbb{V}} = \langle P_n\Delta u, v\rangle_{\mathbb{H}} = {}_{\mathbb{V}^*}\langle \Delta u, v\rangle_{\mathbb{V}}, \quad \text{for all } u \in \mathbb{V}, v \in \mathbb{H}_n,$$

and $\|v\|_{\mathbb{H}_n} = \|v\|_{\mathbb{H}}$ for all $v \in \mathbb{H}_n$.

Let $\beta_t^{(n)} = \sum_{i=1}^n \beta_i e_i$. Then for any $x \in \mathbb{H}$, we have

$$P_n\sigma(x)d\beta_t = P_n\sigma(x)d\beta_t^{(n)}.$$

Consider the following Galerkin approximations: $X^{(n)} \in \mathbb{H}_n$ denotes the solution of the following stochastic differential equation:

$$\begin{aligned}dX_t^{(n)} &= P_n\Delta X_t^{(n)}dt + P_n f(X_t^{(n)})dt + P_n\sigma(X_t^{(n)})d\beta_t^{(n)}\\ &\quad + \int_{\mathbb{X}} P_n G(X_{t-}^{(n)}, v)\widetilde{N}(dt,dv),\end{aligned} \tag{2.15}$$

with initial condition $X_0^{(n)} = P_n X_0 = P_n x \in \mathbb{H}_n$. By the Lipschitz continuity of f, σ and G, we know that Equation (2.15) admits a unique strong solution $X_n \in \mathbb{D}([0,T];\mathbb{H}_n) \cap L^2([0,T];\mathbb{V}_n)$. Furthermore, we have

$$\sup_{n\geq 1} \mathbb{E}\bigg[\sup_{0\leq t\leq T} \|X_t^{(n)}\|_{\mathbb{H}_n}^2 + \int_0^T \|X_t^{(n)}\|_{\mathbb{V}_n}^2 dt\bigg] < \infty. \tag{2.16}$$

Since λ_1 is the first eigenvalue of $-\Delta$, by (H1)–(H3), we have that for any $x_1, x_2 \in \mathbb{H}_n$,

$$\begin{aligned}
&2\langle x_1 - x_2, P_n\Delta(x_1 - x_2)\rangle + 2\langle x_1 - x_2, P_n f(x_1) - P_n f(x_2)\rangle \\
&\quad + \|P_n(\sigma(x_1) - \sigma(x_2))\|_{\mathrm{HS}}^2 + \int_{\mathbb{X}} \|P_n G(x_1, v) - P_n G(x_2, v)\|_{\mathbb{H}_n}^2 \vartheta(dv) \\
&\leq -K\|x_1 - x_2\|_{\mathbb{H}_n}^2.
\end{aligned} \tag{2.17}$$

If $K > 0$, by [19, Theorem 2.2], we know that all the results in Theorem 2.4 replacing X by $X^{(n)}$ hold. Hence, Proposition 2.6 below together with Lemma 2.3, implies that Theorem 2.4 holds. The proof is complete. □

Proposition 2.6 *Under Conditions* (H1)–(H3)*, for any* $t \geq 0$*, we have*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\sup_{0\leq s\leq t} \|X_s - X_s^{(n)}\|_{\mathbb{H}}^2 + \int_0^t \|X_s - X_s^{(n)}\|_{\mathbb{V}}^2 ds\bigg] = 0. \tag{2.18}$$

Proof Applying Itô's formula to $\|X_t - X_t^{(n)}\|_{\mathbb{H}}^2$, we obtain that

$$\begin{aligned}
&\|X_t - X_t^{(n)}\|_{\mathbb{H}}^2 + 2\int_0^t \|X_s - X_s^{(n)}\|_{\mathbb{V}}^2 ds \\
&= \|(I - P_n)x\|_{\mathbb{H}}^2 + 2\int_0^t \langle X_s - X_s^{(n)}, f(X_s) - P_n f(X_s^{(n)})\rangle_{\mathbb{H}} ds \\
&\quad + 2\int_0^t \langle X_s - X_s^{(n)}, [\sigma(X_s) - P_n\sigma(X_s^{(n)})] d\beta_s\rangle_{\mathbb{H}} \\
&\quad + \int_0^t \|\sigma(X_s) - P_n\sigma(X_s^{(n)})\|_{\mathrm{HS}}^2 ds \\
&\quad + 2\int_0^t \int_{\mathbb{X}} \langle X_{s-} - X_{s-}^{(n)}, G(X_{s-}, v) - P_n G(X_{s-}^{(n)}, v)\rangle_{\mathbb{H}} \widetilde{N}(ds, dv) \\
&\quad + \int_0^t \int_{\mathbb{X}} \|G(X_{s-}, v) - P_n G(X_{s-}^{(n)}, v)\|_{\mathbb{H}}^2 N(ds, dv).
\end{aligned} \tag{2.19}$$

Taking the supremum up to t in (2.19), and then taking the expectation, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq s\leq t} \|X_s - X_s^{(n)}\|_{\mathbb{H}}^2\Big] + 2\mathbb{E}\int_0^t \|X_s - X_s^{(n)}\|_{\mathbb{V}}^2 ds \\
&\leq \mathbb{E}[\|(I - P_n)x\|_{\mathbb{H}}^2] + 2\mathbb{E}\int_0^t |\langle X_s - X_s^{(n)}, f(X_s) - P_n f(X_s^{(n)})\rangle_{\mathbb{H}}| ds \\
&\quad + 2\mathbb{E}\bigg[\sup_{0\leq s\leq t} \bigg|\int_0^s \langle X_r - X_r^{(n)}, [\sigma(X_r) - P_n\sigma(X_r^{(n)})] d\beta_r\rangle_{\mathbb{H}}\bigg|\bigg] \\
&\quad + \mathbb{E}\int_0^t \|\sigma(X_s) - P_n\sigma(X_s^{(n)})\|_{\mathrm{HS}}^2 ds \\
&\quad + 2\mathbb{E}\bigg[\sup_{0\leq s\leq t} \bigg|\int_0^s \int_{\mathbb{X}} \langle X_{r-} - X_{r-}^{(n)}, G(X_{r-}, v) - P_n G(X_{r-}^{(n)}, v)\rangle_{\mathbb{H}} \widetilde{N}(dr, dv)\bigg|\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\Big[\sup_{0\le s\le t}\int_0^s\int_{\mathbb{X}}\|G(X_{r-},v)-P_nG(X_{r-}^{(n)},v)\|_{\mathbb{H}}^2N(dr,dv)\Big]\\
=:{}&\mathbb{E}[\|(I-P_n)x\|_{\mathbb{H}}^2]+I_1^{(n)}(t)+I_2^{(n)}(t)+\cdots+I_5^{(n)}(t).
\end{aligned}\tag{2.20}
$$

For $I_1^{(n)}$, by the Lipschitz continuity of f and the elementary inequality $2ab\le a^2+b^2$ for all $a,b>0$, we have

$$
\begin{aligned}
I_1^{(n)}(t)&\le 2\mathbb{E}\int_0^t|\langle X_s-X_s^{(n)},f(X_s)-P_nf(X_s)\rangle_{\mathbb{H}}|ds\\
&\quad+2\mathbb{E}\int_0^t|\langle X_s-X_s^{(n)},P_n(f(X_s)-f(X_s^{(n)}))\rangle_{\mathbb{H}}|ds\\
&\le(1+2C_f)\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds\\
&\quad+\mathbb{E}\int_0^t\|(I-P_n)f(X_s)\|_{\mathbb{H}}^2ds.
\end{aligned}\tag{2.21}
$$

By Burkholder–Davis–Gundy's inequality and the Lipschitz continuity of σ, we have

$$
\begin{aligned}
I_2^{(n)}(t)&\le 2\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s\langle X_r-X_r^{(n)},[\sigma(X_r)-P_n\sigma(X_r)]d\beta_r\rangle_{\mathbb{H}}\Big|\Big]\\
&\quad+2\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s\langle X_r-X_r^{(n)},P_n[\sigma(X_r)-\sigma\ (X_r^{(n)})]d\beta_r\rangle_{\mathbb{H}}\Big|\Big]\\
&\le 4\mathbb{E}\Big[\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\cdot\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds\Big]^{\frac12}\\
&\quad+4C_\sigma\mathbb{E}\Big[\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^4ds\Big]^{\frac12}\\
&\le 2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds+2\mathbb{E}\int_0^t\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds\\
&\quad+4C_\sigma\mathbb{E}\Big[\sup_{0\le s\le t}\|X_s-X_s^{(n)}\|_{\mathbb{H}}\cdot\Big(\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds\Big)^{\frac12}\Big]\\
&\le 2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds+2\mathbb{E}\int_0^t\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds\\
&\quad+\frac14\mathbb{E}\Big[\sup_{0\le s\le t}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\Big]\\
&\quad+16C_\sigma^2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds.
\end{aligned}\tag{2.22}
$$

For the third term $I_3^{(n)}$, by the Lipschitz continuity of σ, we have

$$
\begin{aligned}
I_3^{(n)}(t)&\le\mathbb{E}\int_0^t\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds+\mathbb{E}\int_0^t\|P_n\sigma(X_s)-P_n\sigma(X_s^{(n)})\|_{\mathrm{HS}}^2ds\\
&\le\mathbb{E}\int_0^t\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds+C_\sigma^2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds.
\end{aligned}\tag{2.23}
$$

By Burkholder–Davis–Gundy's inequality and the Lipschitz continuity of G, we have

$$
I_4^{(n)}(t)\le 2\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s\int_{\mathbb{X}}\langle X_{r-}-X_{r-}^{(n)},(I-P_n)G(X_{r-},v)\rangle_{\mathbb{H}}\widetilde{N}(dr,dv)\Big|\Big]
$$

$$
\begin{aligned}
&+ 2\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s\int_{\mathbb{X}}\langle X_{r-}-X_{r-}^{(n)}, P_nG(X_{r-},v)-P_nG(X_{r-}^{(n)},v)\rangle_{\mathbb{H}}\widetilde{N}(dr,dv)\Big|\Big]\\
&\le 4\mathbb{E}\Big[\int_0^t\int_{\mathbb{X}}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\cdot\|(I-P_n)G(X_s,v)\|_{\mathbb{H}}^2\vartheta(dv)ds\Big]^{\frac{1}{2}}\\
&\quad + 4\mathbb{E}\Big[\int_0^t\int_{\mathbb{X}}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\cdot\|P_nG(X_s,v)-P_nG(X_s^{(n)},v)\|_{\mathbb{H}}^2\vartheta(dv)ds\Big]^{\frac{1}{2}}\\
&\le 4\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds + 4\mathbb{E}\int_0^t\int_{\mathbb{X}}\|(I-P_n)G(X_s,v)\|_{\mathbb{H}}^2\vartheta(dv)ds\\
&\quad + 4C_G\mathbb{E}\Big[\int_0^t\sup_{0\le r\le s}\|X_r-X_r^{(n)}\|_{\mathbb{H}}^2\cdot\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds\Big]^{\frac{1}{2}}\\
&\le 4\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds\\
&\quad + 4\mathbb{E}\int_0^t\int_{\mathbb{X}}\|(I-P_n)G(X_s,v)\|_{\mathbb{H}}^2\vartheta(dv)ds\\
&\quad + \frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\Big]\\
&\quad + 16C_G^2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds. \qquad (2.24)
\end{aligned}
$$

For the last term, we have

$$
\begin{aligned}
I_5^{(n)}(t) &= \mathbb{E}\int_0^t\int_{\mathbb{X}}\|G(X_{s-},v)-P_nG(X_{s-}^{(n)},v)\|_{\mathbb{H}}^2N(ds,dv)\\
&= \mathbb{E}\int_0^t\int_{\mathbb{X}}\|G(X_s,v)-P_nG(X_s^{(n)},v)\|_{\mathbb{H}}^2\vartheta(dv)ds\\
&\le \mathbb{E}\int_0^t\int_{\mathbb{X}}\|(I-P_n)G(X_s,v)\|_{\mathbb{H}}^2\vartheta(dv)ds\\
&\quad + C_G\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds. \qquad (2.25)
\end{aligned}
$$

Putting the above inequalities together, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2\Big]\\
&\quad + 2\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{V}}^2ds\\
&\le \|(I-P_n)x\|_{\mathbb{H}}^2 + 2(7+2C_f+17C_\sigma^2+17C_G)\mathbb{E}\int_0^t\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2ds\\
&\quad + 2\mathbb{E}\int_0^t\|(I-P_n)f(X_s)\|_{\mathbb{H}}^2ds\\
&\quad + 6\mathbb{E}\int_0^t\|(I-P_n)\sigma(X_s)\|_{\mathrm{HS}}^2ds\\
&\quad + 10\mathbb{E}\int_0^t\int_{\mathbb{X}}\|(I-P_n)G(X_s,v)\|_{\mathbb{H}}^2\vartheta(dv)ds. \qquad (2.26)
\end{aligned}
$$

By (2.5) and (2.16), we know that $\mathbb{E}[\sup_{0\le s\le t}\|X_s-X_s^{(n)}\|_{\mathbb{H}}^2] < \infty$. Hence, by Gronwall's inequality, Fatou's lemma and (2.26), we obtain the desired result (2.18). The proof is complete. □

3 Transportation Cost Inequalities for SPDE with Lévy Noise and Non-Lipschitz Reaction Term

3.1 SPDE with Lévy Noise and Non-Lipschitz Reaction Term

Let $\mathbb{H}, \mathbb{V}, \beta, \sigma, \widetilde{N}$ be the same as those in precedent section. In this section, we extend the reaction term f from the Lipschitz case to the non-Lipschitz case, for example, one can take $f(x) = -x^3 + C_1 x$ for some $C_1 \in \mathbb{R}$.

Consider the following SPDE on the Hilbert space $\mathbb{H}$:

$$\begin{cases} dX_t = \Delta X_t dt + f(X_t)dt + \sigma(X_t)d\beta_t + \displaystyle\int_{\mathbb{X}} G(X_{t-}, v)\widetilde{N}(dt, dv); \\ X_0 = x \in \mathbb{H}. \end{cases} \tag{3.1}$$

Suppose that

(H4) the reaction term f is a third degree polynomial with the negative leading coefficient,

$$f(x) = -x^3 + C_1 x, \quad \forall x \in \mathbb{H}, \tag{3.2}$$

where $C_1 \in \mathbb{R}$.

(H5) G satisfies the following condition:

$$\int_{\mathbb{X}} \|G(x,v)\|_{\mathbb{H}}^6 \vartheta(dv) \leq C'_G(1 + \|x\|_{\mathbb{H}}^6). \tag{3.3}$$

Definition 3.1 *An $\mathbb{H}$-valued right continuous with left limits $(\mathcal{F}_t)$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of* (3.1)*, if for its $dt \times \mathbb{P}$-equivalent class $\hat{X}$, we have $\hat{X} \in \mathbb{D}([0,T]; \mathbb{H}) \cap L^2((0,T]; \mathbb{V}), \mathbb{P}$-a.s. and the following equality holds $\mathbb{P}$-a.s.:*

$$\begin{aligned} X_t = x + \int_0^t \Delta \bar{X}_s ds + \int_0^t f(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)d\beta_s \\ + \int_0^t \int_{\mathbb{X}} G(\bar{X}_{s-}, v)\widetilde{N}(ds, dv), \quad t \in [0,T], \end{aligned}$$

where $\bar{X}$ is any $\mathbb{V}$-valued progressively measurable $dt \times \mathbb{P}$ version of $\hat{X}$.

Brzeźniak et al. [8] proved the following result for the solution of Eq. (3.1).

Theorem 3.2 ([8, Theorem 1.2 and Example 2.2]) *Under* (H2)–(H5)*, for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H})$,* (3.1) *admits a unique solution $\{X_t\}_{t\in[0,T]}$, and there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|X_t\|_{\mathbb{H}}^6\Big) + \mathbb{E}\int_0^T \|X_t\|_{\mathbb{H}}^4 \cdot \|X_t\|_{\mathbb{V}}^2 dt \leq C(1 + \mathbb{E}\|x\|_{\mathbb{H}}^6). \tag{3.4}$$

Theorem 3.3 *Assume* (H2)–(H5) *hold with $2C_1 + C_\sigma^2 + C_G < 2\lambda_1$, $\|\sigma(x)\|_{\mathrm{HS}} \leq \bar{\sigma}$ for any $x \in \mathbb{H}$ and there is some Borel-measurable function $\bar{G}(u)$ on $\mathbb{X}$ such that $|G(x,u)| \leq \bar{G}(u)$ for all $x \in \mathbb{H}, u \in \mathbb{X}$ and* (2.8) *holds. Then the statements in Theorem* 2.4 *hold with $K = 2\lambda_1 - 2C_1 - C_\sigma^2 - C_G$.*

Remark 3.4 The condition $2C_1 + C_\sigma^2 + C_G < 2\lambda_1$ guarantees the global dissipation for the system (3.1), and we could apply finite dimensional SDE's results in [19]. When C_1 is large, the system (3.1) is dissipative outside a bounded set. By Majka's work [21], one expects that the transportation cost inequalities should hold. However, in [21], the non-degenerated conditions of the noises are assumed to make the mirror coupling successful. Thus, to remove the restriction of C_1 in Theorem 3.3, we need some extra non-degenerated conditions of the noises. This is not the task of this paper, and we hope to study it in future.

Proof of Theorem 3.3 Recall that P_n is the projection mapping from $\mathbb{V}^*$ into $\mathbb{H}_n$ defined by (2.14). For any $n \ge 1$, consider the following stochastic differential equation on $\mathbb{H}_n$:

$$dX_t^{(n)} = P_n\Delta X_t^{(n)}dt + P_n f(X_t^{(n)})dt + P_n\sigma(X_t^{(n)})d\beta_t^{(n)} + \int_{\mathbb{X}} P_n G(X_{t-}^{(n)}, v)\widetilde{N}(dt, dv), \tag{3.5}$$

with initial condition $X_0^{(n)} = P_n x$. According to [1, Theorem 3.1], (3.5) admits a unique strong solution $X^{(n)}$ satisfying that

$$\begin{aligned} X_t^{(n)} = P_n x &+ \int_0^t P_n\Delta X_s^{(n)}ds + \int_0^t P_n f(X_s^{(n)})ds + \int_0^t P_n\sigma(X_s^{(n)})d\beta_s^{(n)} \\ &+ \int_0^t\int_{\mathbb{X}} P_n G(X_{s-}^{(n)}, v)\widetilde{N}(ds, dv), \quad t\in[0,T]. \end{aligned} \tag{3.6}$$

Furthermore, using the same method in the proof of (3.4), we have

$$\sup_{n\ge1}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|X_t^{(n)}\|_{\mathbb{H}_n}^6 + \int_0^T \|X_t^{(n)}\|_{\mathbb{H}_n}^4\cdot\|X_t^{(n)}\|_{\mathbb{V}_n}^2dt\bigg) \le C(1+\mathbb{E}\|x\|_{\mathbb{H}}^6). \tag{3.7}$$

According to [19, Theorem 2.2] and Lemma 2.3, Theorem 2.4 is established once the following statements are proved:

(C1) $\{X^{(n)}\}_{n\ge1}$ converges in distribution to X in $L^2([0,T];\mathbb{H})$ as $n\to\infty$;

(C2) $\{X_T^{(n)}\}_{n\ge1}$ converges in distribution to $\{X_T\}$ in $\mathbb{H}$ as $n\to\infty$.

In the sequel, we will prove Conditions (C1) and (C2). The proof is complete. □

Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be a separable metric space. Given $p>1$, $\alpha\in(0,1)$, let $W^{\alpha,p}([0,T];\mathbb{U})$ be the Sobolev space of all $u\in L^p([0,T];\mathbb{U})$ such that

$$\int_0^T\int_0^T \frac{\|u(t)-u(s)\|_{\mathbb{U}}^p}{|t-s|^{1+\alpha p}}dtds < \infty,$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}([0,T];\mathbb{U})}^p = \int_0^T \|u(t)\|_{\mathbb{U}}^p dt + \int_0^T\int_0^T \frac{\|u(t)-u(s)\|_{\mathbb{U}}^p}{|t-s|^{1+\alpha p}}dtds.$$

Lemma 3.5 ([17, Sect. 5, Ch. I], [29, Sect. 13.3]) *Let $\mathbb{U}\subset\mathbb{Y}\subset\mathbb{U}^*$ be Banach spaces, $\mathbb{U}$ and $\mathbb{U}^*$ reflexive, with compact embedding of $\mathbb{U}$ in $\mathbb{Y}$. For any $p\in(1,\infty)$ and $\alpha\in(0,1)$, let $\Gamma = L^p([0,T];\mathbb{U})\cap W^{\alpha,p}([0,T];\mathbb{U}^*)$ endowed with the natural norm. Then the embedding of Γ in $L^p([0,T];\mathbb{Y})$ is compact.*

We first give a priori estimates for $X^{(n)}$.

Lemma 3.6 *Under* (H2)–(H5)*, we have*

$$\sup_{n\ge1}\mathbb{E}\bigg[\sup_{0\le t\le T}\|X_t^{(n)}\|_{\mathbb{H}}^2 + \int_0^T \|X_t^{(n)}\|_{\mathbb{V}}^2dt\bigg] < \infty, \tag{3.8}$$

and for any $\alpha \in (0, 1/2)$,

$$\sup_{n\geq 1} \mathbb{E}[\|X^{(n)}\|_{W^{\alpha,2}([0,T],\mathbb{V}^*)}] < \infty. \tag{3.9}$$

Proof Applying Itô's formula with $p = 2$ (instead of taking $p = \beta + 2$) in the proof of [8, Lemma 4.2]), one can obtain the estimate (3.8). The details are omitted here. Next, we prove (3.9). Notice that

$$\begin{aligned} X_t^{(n)} &= P_n x + \int_0^t P_n \Delta X_s^{(n)} ds + \int_0^t P_n f(X_s^{(n)}) ds \\ &\quad + \int_0^t P_n \sigma(X_s^{(n)}) d\beta_s^{(n)} + \int_0^t \int_{\mathbb{U}} P_n G(X_{s-}^{(n)}, v) \widetilde{N}(ds, dv) \\ &=: J_1^{(n)} + J_2^{(n)}(t) + J_3^{(n)}(t) + J_4^{(n)}(t) + J_5^{(n)}(t). \end{aligned} \tag{3.10}$$

By the same arguments as in the proof of [13, Theorem 3.1], we have

$$\sup_{n\geq 1} \mathbb{E}\|J_1^{(n)}\|_{\mathbb{H}}^2 < \infty, \quad \sup_{n\geq 1} \mathbb{E}\|J_2^{(n)}\|_{W^{1,2}([0,T];\mathbb{V}^*)}^2 < \infty. \tag{3.11}$$

Since for $t > s$,

$$\begin{aligned} \mathbb{E}\|J_3^{(n)}(t) - J_3^{(n)}(s)\|_{\mathbb{H}}^2 &= \mathbb{E}\bigg\| \int_s^t P_n f(X_r^{(n)}) dr \bigg\|_{\mathbb{H}}^2 \\ &\leq C\mathbb{E}\bigg(\int_s^t \sqrt{1 + \|X_r^{(n)}\|_{\mathbb{H}}^6} dr \bigg)^2 \\ &\leq C\mathbb{E}\Big(1 + \sup_{r\in[0,T]} \|X_r^{(n)}\|_{\mathbb{H}}^6\Big)(t-s), \end{aligned}$$

we have

$$\mathbb{E}\int_0^T \|J_3^{(n)}(t)\|_{\mathbb{H}}^2 dt \leq C\mathbb{E}\Big(1 + \sup_{r\in[0,T]} \|X_r^{(n)}\|_{\mathbb{H}}^6\Big) T^2, \tag{3.12}$$

and

$$\mathbb{E}\int_0^T \int_0^T \frac{\|J_3^{(n)}(t) - J_3^{(n)}(s)\|_{\mathbb{H}}^2}{|t-s|^{1+2\alpha}} dt ds \leq C(\alpha, T)\mathbb{E}\Big(1 + \sup_{r\in[0,T]} \|X_r^{(n)}\|_{\mathbb{H}}^6\Big). \tag{3.13}$$

By (3.7), (3.12) and (3.13), we obtain

$$\sup_{n\geq 1} \mathbb{E}\|J_3^{(n)}\|_{W^{\alpha,2}([0,T];\mathbb{V}^*)}^2 < \infty. \tag{3.14}$$

Now for $J_4^{(n)}$, since for $t > s$,

$$\begin{aligned} \mathbb{E}\|J_4^{(n)}(t) - J_4^{(n)}(s)\|_{\mathbb{H}}^2 &= \mathbb{E}\bigg\| \int_s^t P_n \sigma(X_r^{(n)}) d\beta_r^{(n)} \bigg\|_{\mathbb{H}}^2 \\ &\leq C\mathbb{E}\bigg(\int_s^t \|\sigma(X_r^{(n)})\|_{\mathrm{HS}}^2 dr \bigg) \\ &\leq CC_\sigma^2 \mathbb{E}\bigg(\int_s^t (1 + \|X_r^{(n)}\|_{\mathbb{H}}^2) dr \bigg) \\ &\leq CC_\sigma^2 \mathbb{E}\Big(1 + \sup_{r\in[0,T]} \|X_r^{(n)}\|_{\mathbb{H}}^2\Big)(t-s), \end{aligned}$$

similarly to (3.14), we have

$$\sup_{n\geq 1}\mathbb{E}\|J_4^{(n)}\|^2_{W^{\alpha,2}([0,T];\mathbb{V}^*)}<\infty. \tag{3.15}$$

For $J_5^{(n)}$, we also have

$$\begin{aligned}\mathbb{E}\|J_5^{(n)}(t)-J_5^{(n)}(s)\|^2_{\mathbb{H}}&=\mathbb{E}\bigg\|\int_s^t\int_{\mathbb{X}}G(X_{r-}^{(n)},v)\widetilde{N}(dr,dv)\bigg\|^2_{\mathbb{H}}\\&\leq C\mathbb{E}\int_s^t\int_{\mathbb{X}}\|G(X_r^{(n)},v)\|^2_{\mathbb{H}}\vartheta(dv)dr\\&\leq CC_G\mathbb{E}\Big(1+\sup_{r\in[0,T]}\|X_r^{(n)}\|^2_{\mathbb{H}}\Big)(t-s).\end{aligned}$$

Similarly to (3.14), we have

$$\sup_{n\geq 1}\mathbb{E}\|J_5^{(n)}\|^2_{W^{\alpha,2}([0,T];\mathbb{V}^*)}<\infty. \tag{3.16}$$

Putting above inequalities together, we get (3.9). The proof is complete. □

Proposition 3.7 *Under* (H2)–(H5)*, for any* $T>0$*,*

(a) $\{X^{(n)}\}_{n\geq 1}$ *converges in distribution to* X *in* $L^2([0,T];\mathbb{H})$ *as* $n\to\infty$*;*

(b) $\{X_T^{(n)}\}_{n\geq 1}$ *converges in distribution to* $\{X_T\}$ *in* $\mathbb{H}$ *as* $n\to\infty$*.*

Proof (a) For any subsequence $\{X^{(n_k)}\}_{k\geq 1}\subset\{X^{(n)}\}_{n\geq 1}$, by Lemmas 3.5 and 3.6, we know that $\{X^{(n_k)}\}_{k\geq 1}$ is tight in the space $L^2([0,T];\mathbb{H})$. Hence, there exists a subsequence $\{X^{(n_k')}\}_{k\geq 1}\subset\{X^{(n_k)}\}_{k\geq 1}$, which converges in distribution as random variables in the space $L^2([0,T];\mathbb{H})$. By the uniqueness of the limit (see the proof of [8, Theorem 4.1]) and the arbitrariness of the subsequence $\{X^{(n_k)}\}_{k\geq 1}$, we know that $\{X^{(n)}\}_{n\geq 1}$ converges in distribution to X in $L^2([0,T];\mathbb{H})$ as $n\to\infty$.

(b) Recall that $\{S(t)\}_{t\geq 0}$ is the analytic semigroup associated with Δ. Let $S^{(n)}(t)=P_nS(t)$. According to [24, Chapter 9.3], the solution $\{X_t\}_{t\geq 0}$ to (3.1) is equivalent to the following form:

$$\begin{aligned}X_t=S(t)x&+\int_0^t S(t-s)f(X_s)ds+\int_0^t S(t-s)\sigma(X_s)d\beta_s\\&+\int_0^t\int_{\mathbb{X}}S(t-s)G(X_{s-},v)\widetilde{N}(ds,dv),\end{aligned} \tag{3.17}$$

and the solution $\{X_t^{(n)}\}_{t\geq 0}$ to (3.5) is equivalent to following form:

$$\begin{aligned}X_t^{(n)}=S^{(n)}(t)P_nx&+\int_0^t S^{(n)}(t-s)f(X_s^{(n)})ds+\int_0^t S^{(n)}(t-s)\sigma(X_s^{(n)})P_nd\beta_s\\&+\int_0^t\int_{\mathbb{X}}S^{(n)}(t-s)P_nG(X_{s-}^{(n)},v)\widetilde{N}(ds,dv).\end{aligned} \tag{3.18}$$

Applying a generalized version of the Skorokhod representation theorem (e.g., [7, Theorem C.1]), there exist a stochastic basis $(\bar{\Omega},\bar{\mathcal{F}},(\bar{\mathcal{F}}_t)_{t\geq 0},\bar{\mathbb{P}})$ and the random variables

$$\{(\bar{x}^{(n)},\bar{X}^{(n)},\bar{x},\bar{X},\bar{\beta},\bar{N})\}_{n\geq 1}$$

on this basis satisfying that $(\bar{x}^{(n)},\bar{X}^{(n)},\bar{x},\bar{X},\bar{\beta},\bar{N})$ has the same law as $(x^{(n)},X^{(n)},x,X,\beta,N)$ for any $n\geq 1$, $\bar{x}^{(n)}\to\bar{x}$ in $\mathbb{H}$, $\bar{\mathbb{P}}$-a.s., and $\bar{X}^{(n)}\to\bar{X}$ in $L^2([0,T];\mathbb{H})$, $\bar{\mathbb{P}}$-a.s.. Next, we prove that $\bar{X}_T^{(n)}$ converges to $\bar{X}_T$ in probability under $\bar{\mathbb{P}}$.

For any $n \geq 1, M > 0$, let

$$\bar{\Omega}_{n,M} = \Big\{\bar{\omega};\ \sup_{t\in[0,T]} \|\bar{X}_t^{(n)}(\bar{\omega})\|_{\mathbb{H}} \vee \|\bar{X}_t(\bar{\omega})\|_{\mathbb{H}} \leq M\Big\}. \tag{3.19}$$

Then by (3.4), (3.8) and Fatou's lemma, we know that

$$\lim_{M\to\infty} \sup_{n\geq 1} \bar{\mathbb{P}}(\bar{\Omega}_{n,M}^c) = 0, \tag{3.20}$$

and for any $M > 0$, by the dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathbb{E}^{\bar{\mathbb{P}}}\bigg(\int_0^T \|\bar{X}_t - \bar{X}_t^{(n)}\|_{\mathbb{H}}^2 dt \cdot 1_{\bar{\Omega}_{n,M}}\bigg) = 0. \tag{3.21}$$

Next, we will prove that

$$\lim_{n\to\infty} \mathbb{E}^{\bar{\mathbb{P}}}(\|\bar{X}_T - \bar{X}_T^{(n)}\|_{\mathbb{H}}^2 \cdot 1_{\bar{\Omega}_{n,M}}) = 0. \tag{3.22}$$

This, together with (3.20), implies (b).

By (3.17) and (3.18), we have for any $t \in [0, T]$,

$$\begin{aligned}
\|\bar{X}_t - \bar{X}_t^{(n)}\|_{\mathbb{H}} \leq{}& \|S(t)\bar{x} - S^{(n)}(t)P_n\bar{x}\|_{\mathbb{H}} \\
&+ \bigg\|\int_0^t [S(t-s)f(\bar{X}_s) - S^{(n)}(t-s)f(\bar{X}_s^{(n)})]ds\bigg\|_{\mathbb{H}} \\
&+ \bigg\|\int_0^t [S(t-s)\sigma(\bar{X}_s) - S^{(n)}(t-s)\sigma(\bar{X}_s^{(n)})]d\beta_s\bigg\|_{\mathbb{H}} \\
&+ \bigg\|\int_0^t \int_{\mathbb{X}} [S(t-s)G(\bar{X}_{s-}, v) - S^{(n)}(t-s)P_nG(\bar{X}_{s-}^{(n)}, v)]\tilde{\bar{N}}(ds, dv)\bigg\|_{\mathbb{H}} \\
=:{}& J_{1,n}(t) + J_{2,n}(t) + J_{3,n}(t) + J_{4,n}(t).
\end{aligned} \tag{3.23}$$

By the dominated convergence theorem, we can prove that for $k = 1, \ldots, 4, t \in [0, T]$,

$$\lim_{n\to\infty} \mathbb{E}^{\bar{\mathbb{P}}}[J_{k,n}(t) \cdot 1_{\bar{\Omega}_{n,M}}] = 0. \tag{3.24}$$

Here, we will only prove (3.24) for $k = 2$ and the other term can be proved similarly but more easily. Notice that

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{P}}}[J_{2,n}(t) \cdot 1_{\bar{\Omega}_{n,M}}] \leq{}& \mathbb{E}^{\bar{\mathbb{P}}}\bigg\|\int_0^t (I - P_n)S(t-s)f(\bar{X}_s)ds \cdot 1_{\bar{\Omega}_{n,M}}\bigg\|_{\mathbb{H}} \\
&+ \mathbb{E}^{\bar{\mathbb{P}}}\bigg\|\int_0^t S^{(n)}(t-s)[f(\bar{X}_s) - f(\bar{X}_s^{(n)})]ds \cdot 1_{\bar{\Omega}_{n,M}}\bigg\|_{\mathbb{H}}.
\end{aligned} \tag{3.25}$$

By (2.10) in [34] and the Sobolev embedding theorem, we have

$$\|f(x)\|_{\mathbb{H}} \leq C(1 + \|x\|_{\mathbb{H}_{1/6}}^3) \leq C(1 + \|x\|_{\mathbb{H}}^2 \cdot \|x\|_{\mathbb{V}}). \tag{3.26}$$

Since $P_n \to I$ as $n \to \infty$, by (3.4) and the dominated convergence theorem, we have

$$\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}} \int_0^t \|(I - P_n)S(t-s)f(\bar{X}_s)\|_{\mathbb{H}} ds \\
&\leq C\mathbb{E}^{\bar{\mathbb{P}}} \int_0^t \|(I - P_n)\| \cdot (1 + \|\bar{X}_s\|_{\mathbb{H}}^2 \cdot \|\bar{X}_s\|_{\mathbb{V}})ds \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.27}$$

By (2.8) in [34] and the Sobolev embedding theorem, we have

$$\|f(x) - f(y)\|_{\mathbb{H}} \leq C(1 + \|x\|_{\mathbb{H}_{1/4}}^2 + \|y\|_{\mathbb{H}_{1/4}}^2)\|x - y\|_{\mathbb{H}}$$

$$\leq C(1 + \|x\|_{\mathbb{H}} \cdot \|x\|_{\mathbb{V}} + \|y\|_{\mathbb{H}} \cdot \|x\|_{\mathbb{V}})\|x - y\|_{\mathbb{H}}. \tag{3.28}$$

Then, by (3.4), (3.7) and (3.21), we have

$$\begin{aligned}
&\mathbb{E}\bigg\| \int_0^t S^{(n)}(t-s)[f(\bar{X}_s) - f(\bar{X}_s^{(n)})]ds \cdot 1_{\bar{\Omega}_{n,M}} \bigg\|_{\mathbb{H}} \\
&\quad \leq C\mathbb{E}\bigg(\int_0^t (1 + \|\bar{X}_s\|_{\mathbb{H}} \cdot \|\bar{X}_s\|_{\mathbb{V}} + \|\bar{X}_s^{(n)}\|_{\mathbb{H}} \cdot \|\bar{X}_s^{(n)}\|_{\mathbb{V}})\|\bar{X}_s - \bar{X}_s^{(n)}\|_{\mathbb{H}} ds \cdot 1_{\bar{\Omega}_{n,M}} \bigg) \\
&\quad \leq C\bigg[\mathbb{E}\int_0^t (1 + \|\bar{X}_s\|_{\mathbb{H}} \cdot \|\bar{X}_s\|_{\mathbb{V}} + \|\bar{X}_s^{(n)}\|_{\mathbb{H}} \cdot \|\bar{X}_s^{(n)}\|_{\mathbb{V}})^2 \cdot 1_{\bar{\Omega}_{n,M}} ds\bigg]^{\frac{1}{2}} \\
&\qquad \cdot \bigg[\mathbb{E}\int_0^t \|\bar{X}_s - \bar{X}_s^{(n)}\|_{\mathbb{H}}^2 ds \cdot 1_{\bar{\Omega}_{n,M}}\bigg]^{\frac{1}{2}} \\
&\quad \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.29}$$

The proof is complete. □

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