Acta Mathematica Sinica, English Series
Jan., 2021, Vol. 37, No. 1, pp. 59–72
Published online: March 24, 2020
https://doi.org/10.1007/s10114-020-8213-z
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A Note on Integral Structures in Some Locally Algebraic Representations of $\mathbf{GL_2}$

Yong Quan HU
Morningside Center of Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100190, *P. R. China*
and
Chinese Academy of Sciences, University of Chinese Academy of Sciences,
Beijing 100049, *P. R. China*
E-mail: *yhu@amss.ac.cn*

Abstract In the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, the following theorem of Berger and Breuil has played an important role: the locally algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to crystabelline Galois representations admit a unique unitary completion. In this note, we give a new proof of the weaker statement that the locally algebraic representations admit *at most one* unitary completion and such a completion is automatically admissible. Our proof is purely representation theoretic, involving neither (φ, Γ)-module techniques nor global methods. When F is a finite extension of $\mathbb{Q}_p$, we also get a simpler proof of a theorem of Vignéras for the existence of integral structures for (locally algebraic) special series and for (smooth) tamely ramified principal series.

Keywords Invariant norm, diagram

MR(2010) Subject Classification 22E50

1 Introduction

Let p be a prime number and F be a finite extension of $\mathbb{Q}_p$ with $\mathcal{O}_F$ the ring of integers. We also fix a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$, which will serve as the coefficient field and be sufficiently large (in particular L contains the Galois closure of F).

Let Π be a locally algebraic representation of $\mathrm{GL}_n(F)$ defined over L. It is a central and difficult question whether there exist integral structures in Π. Here, by an *integral structure* we mean an $\mathcal{O}_L$-submodule $\mathcal{L}$ of Π which is stable under $\mathrm{GL}_n(F)$, spans Π over L and contains no L-line (see for example [16, Def. 1.1]); this is equivalent to asking whether Π admits non-zero p-adic unitary completions. The first non-trivial examples were found by Breuil [5] in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. One obvious necessary condition for the existence of integral structures is that the central character of Π is unitary. In fact, Emerton's theory of Jacquet functor on locally analytic representations (in particular applicable to locally algebraic representations) provides other necessary conditions and, conjecturally, these conditions together with the unitarity of the

Received May 17, 2018, accepted October 18, 2019
Supported by National Natural Science Foundation of China (Grant No. 11688101), China's Recruitement Program of Global Experts, National Center for Mathematics and Interdisciplinary Sciences and Hua Loo-Keng Center for Mathematical Sciences of Chinese Academy of Sciences

central character are also *sufficient*. This is related to the so-called *Breuil–Schneider* conjecture, see [13], which turns out to be very difficult to prove in general. Here is a list of works surrounding this problem:1)

(1) $G = \mathrm{GL}_2(\mathbb{Q}_p)$, see the work of Colmez [10], Berger and Breuil [3] (both of the proofs use Fontaine's theory of (φ, Γ)-modules).

(2) $G = \mathrm{GL}_2(F)$, see the work of de Ieso [11], Vignéras [24], Kazhdan and de Shalit [16], and Assaf, Kazhdan and de Shalit [1]; the proofs are local and representation theoretic.

(3) $G = \mathrm{GL}_n(F)$, see the work of Sorensen [22] and Caraiani et al. [8] (both of the proofs use global methods). Note that, when $F \neq \mathbb{Q}_p$, the integral structures constructed in (2) do not give *admissible* unitary completions (see [21] for the notion of admissibility).

In this note, we (re)prove the following results (see below for the notation), firstly proved by Vignéras for (i) and (ii), and by Berger and Breuil for (iii).

Theorem 1.1 (Theorems 2.8, 2.10, 3.9) *Let* $G = \mathrm{GL}_2(F)$ *and* $\Pi = \Pi_{\mathrm{sm}} \otimes \Pi_{\mathrm{alg}}$ *be an irreducible locally algebraic* L*-representation of* G*. Assume that the central character of* Π *is unitary.*

(i) *Assume* Π_{sm} *is a special series representation. Then* Π *admits an integral structure.*

(ii) *Assume* $\Pi = \Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ *is an irreducible principal series with* χ_1, χ_2 *being tamely ramified characters such that* $\chi_1|_{\mathcal{O}_F^\times} \neq \chi_2|_{\mathcal{O}_F^\times}$*. Then* Π *admits an integral structure if and only if* $1 \leq |\chi_1(\varpi_F)| \leq |q^{-1}|$*.*

(iii) *Assume* $F = \mathbb{Q}_p$ *and* $\Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ (*irreducible*)*. If* Π *admits an integral structure, say* $\mathcal{L}$*, then* $\mathcal{L}$ *is necessarily finitely generated as an* $\mathcal{O}_L[G]$*-module and is residually of finite length. Moreover, the universal unitary completion of* Π *is irreducible.*

Remark 1.2 Note that in [9, §5], another proof of (iii) is given, but under mild restrictions. The proof, although local, involves certain projective envelopes of $\mathrm{GL}_2(\mathbb{Z}_p)$-representations. Our proof is elementary and simpler, using only basic results on diagrams and a key observation found in [15, Prop. 4.1]. More interestingly, our proof provides an interpretation of the p-adic Hodge coincidence that there exists only one weakly admissible filtration with given jumps on the underlying Weil–Deligne representation attached to a two-dimensional crystabelline Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; see §3.

Notation We fix a uniformizer ϖ_F of $\mathcal{O}_F$ and let val_F be the p-adic valuation on F normalized as $\mathrm{val}_F(\varpi_F) := 1$; set $q := |\mathcal{O}_F/\varpi_F|$. Similarly, we fix a uniformizer $\varpi = \varpi_L$ of $\mathcal{O}$ and let val_L be the normalized p-adic valuation on L. Write $\mathcal{O} = \mathcal{O}_L$ and let $k = k_L$ be the residue field of $\mathcal{O}$.

Let $G := \mathrm{GL}_2(F)$ and $K := \mathrm{GL}_2(\mathcal{O}_F)$ and Z be the center of G. Define the following subgroups of G (where $m \geq 1$):

$$I := \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \varpi_F \mathcal{O}_F & \mathcal{O}_F^\times \end{pmatrix}, \quad I_m := \begin{pmatrix} 1 + \varpi_F^m \mathcal{O}_F & \varpi_F^{m-1} \mathcal{O}_F \\ \varpi_F^m \mathcal{O}_F & 1 + \varpi_F^m \mathcal{O}_F \end{pmatrix},$$
$$K_m := \begin{pmatrix} 1 + \varpi_F^m \mathcal{O}_F & \varpi_F^m \mathcal{O}_F \\ \varpi_F^m \mathcal{O}_F & 1 + \varpi_F^m \mathcal{O}_F \end{pmatrix}.$$

1) The list may not be complete and we refer to the cited papers for the precise conditions imposed. See also [23] for a nice exposition about this problem.

Let $\mathfrak{N}_0$ be the G-normalizer of K so that $\mathfrak{N}_0 = KZ$, and $\mathfrak{N}_1$ be the G-normalizer of I so that $\mathfrak{N}_1$ is generated by IZ and $t := \left(\begin{smallmatrix} 0 & 1 \\ \varpi_F & 0 \end{smallmatrix}\right)$ as a group. One checks that $\mathfrak{N}_0 \cap \mathfrak{N}_1 = IZ$ with

$$[\mathfrak{N}_0 : IZ] = q + 1, \quad [\mathfrak{N}_1 : IZ] = 2. \tag{1.1}$$

Let B be the upper Borel subgroup of G. Given two characters $\chi_1, \chi_2 : F^\times \to L^\times$, we consider $\chi_1 \otimes \chi_2$ as a character of B sending $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\chi_1(a)\chi_2(d)$ and let $\mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ denote the principal series representation of G.

Finally, if H is a group, A is a commutative ring, W is an $A[H]$-module and $W_1 \subset W$ is any subset, we let $\langle H.W_1 \rangle$ denote the sub-$A[H]$-module of W generated by W_1.

2 Diagrams

Let A be a topological commutative ring, typically $A = L, \mathcal{O}, k$. By a diagram D (for GL_2) of continuous A-modules, we mean the data (D_0, D_1, r), where D_0 (resp. D_1) is a topological A-module with a continuous[2)] linear action of $\mathfrak{N}_0$ (resp. $\mathfrak{N}_1$), and $r : D_1 \to D_0$ is an IZ-equivariant continuous morphism of A-modules. Diagrams of continuous A-modules with obvious morphisms form an abelian category. Attached to a diagram, we can define a G-equivariant morphism $\partial : \text{c-Ind}_{\mathfrak{N}_1}^G D_1 \otimes \delta_{-1} \to \text{c-Ind}_{\mathfrak{N}_0}^G D_0$ (see [7, §9] or [18, §3]), where δ_{-1} denotes the (continuous) character of $\mathfrak{N}_1$ (to A) of order 2 sending g to $(-1)^{\mathrm{val}_F(\det g)}$, and ∂ is the G-equivariant morphism determined by

$$\partial([\mathrm{Id}, x]) = [\mathrm{Id}, r(x)] - [t, r(t^{-1} \cdot x)] \in \text{c-Ind}_{\mathfrak{N}_0}^G D_0, \quad \forall x \in D_1 \otimes \delta_{-1}, \tag{2.1}$$

where $[g, x]$ denotes the unique function supported on $\mathfrak{N}_i g^{-1}$ and taking value x on g^{-1}. The kernel and cokernel of ∂ are denoted by $H_1(D)$ and $H_0(D)$ respectively, so that we have an exact sequence

$$0 \to H_1(D) \to \text{c-Ind}_{\mathfrak{N}_1}^G D_1 \otimes \delta_{-1} \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{N}_0}^G D_0 \to H_0(D) \to 0.$$

By definition, a short exact sequence of diagrams of A-modules $0 \to D' \to D \to D'' \to 0$ gives a long exact sequence

$$0 \to H_1(D') \to H_1(D) \to H_1(D'') \to H_0(D') \to H_0(D) \to H_0(D'') \to 0. \tag{2.2}$$

Note that, if π is a continuous A-representation of G, we get trivially a diagram $\mathcal{K}(\pi) := (\pi|_{\mathfrak{N}_0}, \pi|_{\mathfrak{N}_1}, \mathrm{Id})$. One has that $H_0(\mathcal{K}(\pi)) \cong \pi$ by [17, Lem. 5.4.2] and $H_1(\mathcal{K}(\pi)) = 0$ by Lemma 2.1 below.

Convention In the rest, we only consider diagrams D admitting a central character, namely Z acts on D_i via some character $Z \to A^\times$.

2.1 Mod p Diagrams with Trivial H_0

In this subsection, we only consider diagrams of k-modules. Since k is equipped with the discrete topology, the action of $\mathfrak{N}_i$ on D_i is smooth. We first recall the following result.

Lemma 2.1 *Let D be a diagram of k-modules such that D_0 is an admissible $\mathfrak{N}_0$-representation and r is injective. Then $H_0(D) \neq 0$ and $H_1(D) = 0$.*

Proof The first assertion is [17, Lem. 5.3.2] and the second is [24, Lem. 1.3]. □

2) It means that $\mathfrak{N}_i \times D_i \to D_i$ is continuous.

Proposition 2.2 *Let $D = (D_0, D_1, r)$ be a diagram of k-modules, not necessarily finite dimensional, such that $H_0(D) = 0$. Then D has a filtration by sub-diagrams such that each graded piece has one of the following three forms (Q_0, Q_1, r) :*

(I) $Q_1 = k \cdot v$, $Q_0 = 0$, $r = 0$;

(II) $Q_1 = k \cdot v \oplus k \cdot t(v)$ *where IZ acts on v via some character ψ,* $Q_0 \cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot t(v))$, $r|_{k \cdot v} = 0$ *and* $r|_{k \cdot t(v)}$ *is the natural map sending* $t(v)$ *to* $[\mathrm{Id}, t(v)]$;

(III) $Q_1 = k \cdot v \oplus k \cdot t(v)$ *where IZ acts on v via some character ψ, Q_0 is a quotient of* $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot t(v))$ *such that* $\dim_k Q_0 \le q$ *(possibly* 0*),* $r|_{k \cdot v} = 0$ *and* $r|_{k \cdot t(v)}$ *is the natural map.*

In particular, if D_1 is of finite dimension, then $\dim_k D_0 \le \dim_k D_1 \cdot \frac{q+1}{2}$ *and the equality holds if and only if only diagrams of type* (II) *appear as graded pieces of the filtration, see* (1.1).

Remark 2.3 Consider a diagram Q of type (III). Since $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot t(v))$ has dimension $q+1$, the condition $\dim_k Q_0 \le q$ is equivalent to demanding that Q_0 is a proper quotient of $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot t(v))$. When $F = \mathbb{Q}_p$, this is again equivalent to demanding that Q_0 is irreducible or zero, since $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}\psi$ has length 2 for any smooth character $\psi : IZ \to k^\times$; see [7, §2].

Proof Since $H_0(D) = 0$, Lemma 2.1 implies that r is not injective; since I_1 is a pro-p group, we deduce that $(\ker(r))^{I_1}$ is non-zero by [2, Lem. 3]. Choose a non-zero vector $v \in (\ker(r))^{I_1}$ and write $M = k \cdot v$. Since the order of I/I_1 is prime to p, we may choose v to be an eigenvector for IZ, i.e. M is stable under IZ. Consider the sub-diagram $Q := (Q_0, Q_1, r_Q)$ of D defined by

$$Q_1 = M + t(M), \quad Q_0 = \langle \mathfrak{R}_0.r(t(M)) \rangle, \quad r_Q = r|_{Q_1}.$$

In particular, $r_Q = 0$ on M. Remark that we *do not* guarantee that v and $t(v)$ are linearly independent over k; indeed, Q_0 could be zero and r_Q be identically zero, in which case Q is of type (I) in the statement. If v and $t(v)$ are linearly independent, then $\dim_k Q_1 = 2$. By Frobenius reciprocity, Q_0 is a quotient of $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0} t(M)$ and Q is of type (II) if $Q_0 \cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0} t(M)$, or equivalently $\dim_k Q_0 = q+1$, and of type (III) otherwise. Note that, in case of type (III), it can happen that r is identically zero, i.e. $Q_0 = 0$.

Since $H_0(D/Q) = 0$ by (2.2), we can continue the above construction for D/Q and in this way get a filtration of D by sub-diagrams whose graded pieces are one of the three types (I)–(III). If D_1 is of finite dimension, the filtration is also finite. The last assertion follows from the corresponding dimension inequality for the graded pieces Q. □

2.2 Naive Diagrams

In this subsection, we classify diagrams of k-modules with trivial H_0 and H_1.

Definition 2.4 *Let $D = (D_0, D_1, r)$ be a diagram of k-modules such that D_0 and D_1 are both finite dimensional. We say that D satisfies the dimension relation if there exists $d \in \mathbb{Z}_{\ge 0}$ such that*

$$\dim_k D_1 = 2d, \quad \dim_k D_0 = d(q+1).$$

We give some examples of diagrams which satisfy the dimension relation. For an absolutely irreducible k-representation σ of $\mathfrak{R}_0$, $\lambda \in k$ and $\chi : F^\times \to k^\times$ a smooth character, we recall the usual notation [4]:

$$\pi(\sigma, \lambda, \chi) := (\text{c-Ind}_{\mathfrak{R}_0}^G \sigma/(T - \lambda)) \otimes \chi \circ \det,$$

where $T \in \mathrm{End}_G(\text{c-Ind}_{\mathfrak{R}_0}^G \sigma)$ is the Hecke operator defined in [2].

Example 2.5 Let $\pi = \pi(\sigma, \lambda, \chi)$ for some σ, λ, χ as above and assume $F = \mathbb{Q}_p$ if $\lambda = 0$. Then the canonical diagram (see [14]) $D(\pi) := (D_0(\pi), D_1(\pi), \mathrm{can})$ defined by

$$D_1(\pi) := \pi^{I_1}, \quad D_0(\pi) := \langle \mathfrak{R}_0.D_1(\pi)\rangle \subset \pi, \quad \mathrm{can} : D_1(\pi) \hookrightarrow D_0(\pi)$$

satisfies the dimension relation. In fact, using results of [2] and [4] (when $\lambda = 0$ and $F = \mathbb{Q}_p$), one checks easily that $\dim_k D_1(\pi) = 2$ and $\dim_k D_0(\pi) = q + 1$ (resp. $p + 1$ when $\lambda = 0$ in which case $F = \mathbb{Q}_p$).

Note that the canonical diagram $D(\mathrm{Sp})$ (resp. $D(\mathbf{1})$) of the Steinberg representation Sp (resp. the trivial representation $\mathbf{1}$) does not satisfy the dimension relation (but $D(\mathrm{Sp}) \oplus D(\mathbf{1})$ does). Another example of diagrams satisfying the dimension relation is a diagram of type (II) in Proposition 2.2. We give it a name for convenience.

Definition 2.6 *A diagram* $D = (D_0, D_1, r)$ *of* k*-modules is said to be naive if it is of type* (II) *as in Proposition* 2.2.

By definition, if $D = (D_0, D_1, r)$ is a naive diagram, then $\dim_k D_1 = 2$ and $\dim_k D_0 = q+1$, hence D satisfies the dimension relation with $d = 1$ in Definition 2.4.

Lemma 2.7 (i) *If* D *is a naive diagram, then* $H_0(D) = H_1(D) = 0$.

(ii) *Conversely, if* $D = (D_0, D_1, r)$ *is a diagram of* k*-modules such that* $H_0(D) = H_1(D) = 0$, *then* D *can be written as a successive extension of naive diagrams. In particular, if* D_0 *and* D_1 *are finite dimensional, then* D *satisfies the dimension relation.*

Proof (i) By definition of D, there exists some $D_1^+ \subset D_1$, a sub-IZ-representation, such that

$$\text{c-Ind}_{\mathfrak{R}_1}^G D_1 \otimes \delta_{-1} \cong \text{c-Ind}_{IZ}^G t(D_1^+) \cong \text{c-Ind}_{\mathfrak{R}_0}^G D_0.$$

Moreover, one checks that if we identify both the source and the target with $\text{c-Ind}_{IZ}^G t(D_1^+)$, then ∂ is exactly the identity morphism. The result follows.

(ii) We may assume D is non-zero. First, by Proposition 2.2, D admits a sub-diagram Q which is one of the three types (I)–(III). It suffices to show that Q is naive. Since $H_1(Q) \hookrightarrow H_1(D)$ and $H_1(D) = 0$ by assumption, we have $H_1(Q) = 0$. Therefore, it suffices to show that diagrams of type (I) or (III) always have non-zero H_1. This is an easy exercise. □

2.3 Diagrams in Characteristic 0

Let Π_{sm} be a finite length smooth L-representation of G admitting a central character. Let $c \geq 1$ be an integer such that Π_{sm} is generated by its K_c-invariants. To Π_{sm} one may associate a diagram $\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}$. As a special case of a theorem of Schneider and Stuhler [20, Thm. V.1], we know that

$$H_0(\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}) \cong \Pi_{\mathrm{sm}}.$$

If moreover, Π_{alg} is an irreducible algebraic L-representation of G, we set $\Pi = \Pi_{\mathrm{sm}} \otimes_L \Pi_{\mathrm{alg}}$ and

$$X = (X_1 \xrightarrow{r} X_0) := (\Pi_{\mathrm{sm}}^{I_c} \hookrightarrow \Pi_{\mathrm{sm}}^{K_c}) \otimes \Pi_{\mathrm{alg}}.$$

Then we have (see [24, Prop. 0.4])

$$H_0(X) \cong \Pi. \tag{2.3}$$

By a diagram of sub-$\mathcal{O}$-lattices $\mathcal{X}$ in X, we mean that $\mathcal{X}_0$ (resp. $\mathcal{X}_1$) is an $\mathfrak{R}_0$-invariant (resp. $\mathfrak{R}_1$-invariant) $\mathcal{O}$-lattice inside X_0 (resp. X_1) and the morphism $\mathcal{X}_1 \to \mathcal{X}_0$ is the restriction of

$r : X_1 \to X_0$. We have a natural morphism $H_0(\mathcal{X}) \to H_0(X) \cong \Pi$ which, however, need not be injective.

Starting from a diagram of $\mathcal{O}$-lattices $\mathcal{X}$ in X, Vignéras constructed in [24] a sequence of diagrams of $\mathcal{O}$-lattices $(\mathcal{X}^{(n)})_{n\geq 0}$ with $\mathcal{X}^{(0)} = \mathcal{X}$ (denoted by $(z^n(\mathcal{X}))_{n\geq 1}$ in *loc. cit.*). The construction is as follows: knowing $\mathcal{X}^{(n)}$, we let inductively

$$\mathcal{X}_1^{(n+1)} = X_1 \cap \mathcal{X}_0^{(n)} + t(X_1 \cap \mathcal{X}_0^{(n)});$$
$$\mathcal{X}_0^{(n+1)} = \langle \mathfrak{R}_0.\mathcal{X}_1^{(n+1)} \rangle.$$

By construction the natural map $H_0(\mathcal{X}^{(n)}) \to H_0(\mathcal{X}^{(n+1)})$ is surjective for any n. Moreover, by [24, Cor. 0.3], Π admits an integral structure if and only if the sequence $(\mathcal{X}^{(n)})_{n\geq 0}$ stabilizes.

2.4 Application I

Our first application of the techniques developed above is a simple proof of the following result of Vignéras [24, Prop. 0.9]. Let St denote the smooth Steinberg L-representation of G.

Theorem 2.8 *Let* $\Pi = \mathrm{St} \otimes \mathrm{Sym}^m L^2 \otimes |\det|^{m/2}$ *for some integer* $m \geq 0$*. Then* Π *admits an integral structure.*

Proof In the notation of §2.3, we may take $c = 1$ so that

$$X_0 = \mathrm{St}^{K_1} \otimes \mathrm{Sym}^m L^2 \otimes |\det|^{m/2}, \quad X_1 = \mathrm{St}^{I_1} \otimes \mathrm{Sym}^m L^2 \otimes |\det|^{m/2}. \tag{2.4}$$

It is clear that the central character of Π is unitary. Since $\mathfrak{R}_1/Z$ is compact, there exist open bounded $\mathfrak{R}_1$-stable $\mathcal{O}$-lattices in X_1. We fix such a lattice $\mathcal{X}_1$ and let $\mathcal{X}_0 := \langle \mathfrak{R}_0.\mathcal{X}_1 \rangle$, which is an open bounded $\mathcal{O}$-lattice in X_0. Let $\mathcal{X}^{(0)} := \mathcal{X}$ and $(\mathcal{X}^{(n)})_{n\geq 0}$ be the sequence of diagrams of $\mathcal{O}$-modules obtained by applying Vignéras' algorithm. If the sequence is finite, we are done; so we assume it is infinite in the rest of the proof. Since X_1 is irreducible as an $\mathfrak{R}_1$-representation[3] and since the coefficient field L is discretely valued, there are only finitely many homothety classes of $\mathfrak{R}_1$-invariant $\mathcal{O}$-lattices in X_1. Therefore there exist integers $n < n'$ such that $\mathcal{X}_1^{(n)}$ and $\mathcal{X}_1^{(n')}$ lie in the same homothety class, that is, there exists $\lambda \in L^\times$ such that

$$\mathcal{X}_1^{(n')} = \lambda \mathcal{X}_1^{(n)}.$$

Since $\mathcal{X}_0^{(n)}$ (resp. $\mathcal{X}_0^{(n')}$) is generated by $\mathcal{X}_1^{(n)}$ (resp. $\mathcal{X}_1^{(n')}$), we get

$$\mathcal{X}^{(n')} = \lambda \mathcal{X}^{(n)}.$$

Moreover, since $\mathcal{X}^{(n)} \subsetneq \mathcal{X}^{(n')}$, we have $\mathrm{val}_L(\lambda) < 0$.

To simplify the notation we assume $\mathcal{X} = \mathcal{X}^{(n)}$, i.e. $n = 0$. Since the natural morphism $H_0(\mathcal{X}) \to H_0(\lambda\mathcal{X})$ is surjective, we have $H_0(\lambda\mathcal{X}/\mathcal{X}) = 0$. Noting that $\lambda\mathcal{X} \cong \varpi^{\mathrm{val}_L(\lambda)}\mathcal{X}$, we deduce by *dévissage* that $H_0(\varpi^{-1}\mathcal{X}/\mathcal{X}) = 0$, equivalently $H_0(\mathcal{X} \otimes_{\mathcal{O}} k) = 0$. By Proposition 2.2, this implies that

$$\dim_k(\mathcal{X}_0 \otimes_{\mathcal{O}} k) \leq \dim_k(\mathcal{X}_1 \otimes_{\mathcal{O}} k) \cdot \frac{q+1}{2},$$

but this is not the case by (2.4) because $\dim_k(\mathcal{X}_i \otimes_{\mathcal{O}} k) = \mathrm{rank}_{\mathcal{O}}\mathcal{X}_i = \dim_L X_i$ for $i \in \{0, 1\}$. □

Remark 2.9 It is known, at least in the case $F = \mathbb{Q}_p$ and $m \geq 1$, that the universal unitary completion of $\mathrm{St} \otimes \mathrm{Sym}^m L^2 \otimes |\det|^{m/2}$ is not admissible (in the sense of Schneider and Teitelbaum [21]).

3) Indeed, St^{I_1} is 1-dimensional and $\mathrm{Sym}^k L^2$ is irreducible as an $\mathfrak{R}_1$-representation.

2.5 Application II

In this subsection, we reprove (under a mild extra condition) a result of Vignéras [24, Thm. 0.10] about the existence of integral structures in (smooth) tamely ramified principal series. Kazhdan and de Shalit have given another proof using Kirillov models, see [16, Thm. 1.2]. Our proof is motivated by Vignéras', but has the advantage that the computation needed is very small.

Theorem 2.10 *Let* $\Pi = \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ *be a smooth principal series* L*-representation* (*i.e. the algebraic part is trivial*). *Assume that* $\chi_1, \chi_2 : F \to L^\times$ *are tamely ramified and* $\chi_1|_{\mathcal{O}_F^\times} \neq \chi_2|_{\mathcal{O}_F^\times}$. *Then* Π *admits an integral structure if and only if* $\chi_1\chi_2$ *is unitary and* $1 \leq |\chi_1(\varpi_F)| \leq |q^{-1}|$.

Proof The necessity is well-known, see [24] or [16, §3.1].

For the sufficiency, note that we may take

$$X_0 = \Pi^{K_1}, \quad X_1 = \Pi^{I_1}$$

in the notation of §2.3. In particular, we have $\dim_L X_1 = 2$ and $\dim_L X_0 = q+1$. Assume that Π does not admit an integral structure. Since X_1 is irreducible as an $\mathfrak{R}_1$-representation by the assumption $\chi_1|_{\mathcal{O}_F^\times} \neq \chi_2|_{\mathcal{O}_F^\times}$, the proof of Theorem 2.8 produces a diagram of $\mathcal{O}$-modules $\mathcal{X} \subset X$ such that $H_0(\mathcal{X} \otimes_{\mathcal{O}} k) = 0$. Write $D = \mathcal{X} \otimes_{\mathcal{O}} k$. The assumption on χ_1, χ_2 also implies that D_1 is irreducible as an $\mathfrak{R}_1$-representation. Since the dimension relation holds for D, Proposition 2.2 implies that D is a naive diagram.

Again using the assumption $\chi_1 \neq \chi_2$ on $\mathcal{O}_F^\times$, we have a decomposition $\mathcal{X}_1 = \mathcal{O} \cdot v \oplus \mathcal{O} \cdot t(v)$, where v is a non-zero vector on which I acts via the character given by $\left(\begin{smallmatrix} a & b \\ \varpi_F c & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$. As a consequence, $D_1 = k \cdot v \oplus k \cdot \overline{t(v)}$. Since D is naive, *exactly* one of $\overline{v}$ and $\overline{t(v)}$ is sent to zero via the natural morphism $\overline{r} : D_1 \to D_0$. Without loss of generality, we assume that $\overline{r}(\overline{t(v)}) = 0$, i.e. $t(v) \in \varpi \mathcal{X}_0$. Then we obtain

$$D_0 = \langle \mathfrak{R}_0.\overline{v} \rangle \cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(k \cdot \overline{v}).$$

Using Nakayama's lemma, this implies that $\mathcal{X}_0$ is generated by v as an $\mathcal{O}[\mathfrak{R}_0]$-module, hence an isomorphism $\mathcal{X}_0 \cong \mathrm{Ind}_{IZ}^{\mathfrak{R}_0}(\mathcal{O} \cdot v)$ by Frobenius reciprocity and by comparing their $\mathcal{O}$-ranks. In particular, by lifting a k-basis of D_0 (see [7, Lem. 2.7]), $\mathcal{X}_0$ has an $\mathcal{O}$-basis given by

$$v, \quad \sum_{\lambda \in \mathbb{F}_q} [\lambda]^i \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} v, \quad 0 \leq i \leq q-1.$$

However, an easy computation shows that

$$t(v) = \chi_1(\varpi_F) \sum_{\lambda \in \mathbb{F}_q} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} v,$$

so $t(v) \in \varpi \mathcal{X}_0$ if and only if $|\chi_1(\varpi_F)| < 1$. This contradicts the assumption on $\chi_1(\varpi_F)$, hence finishes the proof. □

3 The Case of $\mathbf{GL_2}(\mathbb{Q}_p)$

In this section, we assume $F = \mathbb{Q}_p$ so that $G = \mathrm{GL}_2(\mathbb{Q}_p)$. In [3], Berger and Breuil proved that the locally algebraic representations associated to crystabelline Galois representations admit a unique non-zero unitary completion. This fact, very important in the p-adic local Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$, corresponds to the p-adic Hodge theoretic coincidence that there exists only one weakly admissible filtration on the underlying Weil–Deligne representation with given jumps (determined by the Hodge–Tate weights of the Galois representation). This phenomenon only happens for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ and crystabelline representations. In [9, §5], the uniqueness part is reproved under mild conditions. We give a new proof here, based on the techniques developed in the last section. We will see that the coincidence can be interpreted as a certain dimension relation.

3.1 Standard Diagrams

Let π be a smooth k-representation of G *of finite length* and D be a sub-diagram of $\mathcal{K}(\pi)$. Note that $r : D_1 \to D_0$ is injective and $H_1(D) = 0$.

Definition 3.1 *We say that D is a standard diagram*[4)] *of π if D_0 is finite dimensional and the natural morphism $H_0(D) \to \pi$ is an isomorphism.*

Lemma 3.2 *Let π and D be as above. If D is a standard diagram of π, then $D_0 = \langle \mathfrak{R}_0.D_1\rangle$ and $D_1 = D_0 \cap t(D_0)$.*

Proof Letting $D_0' := \langle \mathfrak{R}_0.D_1\rangle$, we obtain a short exact sequence of diagrams

$$0 \to (D_0', D_1, r) \to D \to (D_0/D_0', 0, 0) \to 0,$$

which induces a surjection $H_0(D) \twoheadrightarrow \text{c-Ind}_{\mathfrak{R}_0}^G(D_0/D_0')$. Since $H_0(D)$ is of finite length by assumption, so is $\text{c-Ind}_{\mathfrak{R}_0}^G(D_0/D_0')$, but this forces $D_0/D_0' = 0$ by [2, Prop. 18] (the proof of [15, Lem. 4.2]).

Letting $D_1' := D_0 \cap t(D_0)$, then $D_1 \subseteq D_1'$ because D_1 is stable under t. As above, we have a short exact sequence of diagrams

$$0 \to D \to (D_0, D_1', r) \to (0, D_1'/D_1, 0) \to 0,$$

which induces an injection $\text{c-Ind}_{\mathfrak{R}_1}^G(D_1'/D_1) \hookrightarrow H_0(D)$ by Lemma 2.1. Again, since $H_0(D)$ is of finite length, so is $\text{c-Ind}_{\mathfrak{R}_1}^G(D_1'/D_1)$, which forces $D_1'/D_1 = 0$ by [15, Lem. 4.2]. □

The assumption $G = \mathrm{GL}_2(\mathbb{Q}_p)$ guarantees the existence of standard diagrams D of π; see [10, Chap. III]. When π is irreducible, we know that

$$D(\pi) = (D_0(\pi), D_1(\pi), \mathrm{can}) := (\langle \mathfrak{R}_0.\pi^{I_1}\rangle, \pi^{I_1}, \mathrm{can})$$

is a standard diagram of π ([7, §10] or [10, Chap. III]). It is also called the canonical diagram associated to π in [14], in the sense that $D(\pi)$ is the *smallest* standard diagram of π. We give a proof of this fact for completeness.

Lemma 3.3 *Let π be an absolutely irreducible smooth k-representation of G with a central character and D be a standard diagram of π. Then D contains the diagram $D(\pi)$.*

Proof We need to show the inclusions (i) $D_1(\pi) \subseteq D_1$ and (ii) $D_0(\pi) \subseteq D_0$. First remark that it suffices to check either of them. In fact, since $D_0(\pi) = \langle \mathfrak{R}_0.D_1(\pi)\rangle$ and $D_0 = \langle \mathfrak{R}_0.D_1\rangle$ by Lemma 3.2, (ii) follows from (i); on the other hand, we have $D_1(\pi) = D_0(\pi) \cap t(D_0(\pi))$ and $D_1 = D_0 \cap t(D_0)$, so (i) follows from (ii). Note that, we always have $D_1 \cap D_1(\pi) \neq 0$ (as

4) The notion comes from Colmez's *"présentation standard"*, see [10].

$D_1^{I_1} \neq 0$) and $D_0 \cap D_0(\pi) \neq 0$ as we can check that $D_0(\pi) \supseteq \mathrm{soc}_K \pi$, where $\mathrm{soc}_K \pi$ denotes the K-socle of π.

We refer to [10, §III.3] or [7, §10] for the explicit structure of $D(\pi)$. If π is a special series representation or a character, then $D_1(\pi) = \pi^{I_1}$ is 1-dimensional, hence is contained in D_1 (since $D_1^{I_1} \neq 0$) and the proof is finished. If π is a ramified principal series representation (see [7, §10 (iv)]), then $D_1(\pi) = \pi^{I_1}$ is of the form $\psi \oplus \psi^s$ with $\psi \neq \psi^s$. Since $D_1^{I_1}$ is non-zero and stable under t, we must have $D_1(\pi) \subseteq D_1$ and the result follows again. Finally, in all the other cases, namely π is either supersingular or an unramified principal series representation, we have $D_0(\pi)|_K = \mathrm{soc}_K \pi \cong \sigma_1 \oplus \sigma_2$ is the direct sum of two *non-isomorphic* irreducible K-representations and $D_1(\pi) = \sigma_1^{I_1} \oplus \sigma_2^{I_1}$ is two-dimensional (however, the eigen-characters of I acting on $\sigma_i^{I_1}$ are possibly equal). Since $D_1 \cap \pi^{I_1} \neq 0$, D_1 contains a non-zero vector $v \in \pi^{I_1}$. If $v \in \sigma_1$ (resp. $v \in \sigma_2$), then by the explicit description of $D(\pi)$ (see [7, §10 (iii), (iv)]), we have $t(v) \in \sigma_2$ (resp. $t(v) \in \sigma_1$) so that the inclusion (i) holds. If $v \notin \sigma_1$ and $v \notin \sigma_2$, then $\langle \mathfrak{R}_0.v \rangle$ is equal to $\sigma_1 \oplus \sigma_2$ because σ_1 and σ_2 are non-isomorphic, which implies the inclusion (ii). This finishes the proof. □

Lemma 3.4 *Let π be a smooth k-representation of G of finite length and D be a standard diagram of π. Let $\pi' \subset \pi$ be a sub-G-representation and π'' be the corresponding quotient. Then $D \cap \mathcal{K}(\pi')$ is a standard diagram of π' and $D/(D \cap \mathcal{K}(\pi'))$ is a standard diagram of π''.*

Proof Write $D' := D \cap \mathcal{K}(\pi')$ and $D'' := D/D'$. By definition, we know that D' (resp. D'') is a sub-diagram of $\mathcal{K}(\pi')$ (resp. $\mathcal{K}(\pi'')$). In particular, we have $H_1(D') = H_1(D'') = 0$. The exact sequence $0 \to D' \to D \to D'' \to 0$ then gives a short exact sequence

$$0 \to H_0(D') \to H_0(D) \to H_0(D'') \to 0.$$

In particular, $H_0(D')$ and $H_0(D)$ are both of finite length since $H_0(D)$ is. It then follows from [15, Prop. 4.1] that the natural morphisms $H_0(D') \to \pi'$ and $H_0(D'') \to \pi''$ are both injective, hence are also surjective for the reason of lengths. □

We introduce one more notion. If π is a k-representation of G of finite length and if τ is an irreducible k-representation of G, we set

$$[\pi : \tau] := \dim_k \mathrm{Hom}_G(\tau, \pi^{\mathrm{ss}}),$$

i.e., the multiplicity with which τ appears in the semisimplification π^{ss}. Similarly we have the notion for $\mathfrak{R}_0$-representations.

Recall that Sp denotes the Steinberg k-representation of G; let st denote the Steinberg k-representation of $\mathrm{GL}_2(\mathbb{F}_p)$.

Proposition 3.5 *Let π be a smooth k-representation of G of finite length and with central character χ. Let $D \hookrightarrow \mathcal{K}(\pi)$ be a standard diagram of π. The following statements hold.*

(i) *There exists $r \in \mathbb{N}$ such that*

$$\begin{cases} \dim_k D_1 = 2r + ([\pi : \chi \circ \det] - [\pi : \mathrm{Sp} \otimes \chi \circ \det]), \\ \dim_k D_0 = (p+1)r + ([\pi : \chi \circ \det] - [\pi : \mathrm{Sp} \otimes \chi \circ \det]). \end{cases} \tag{3.1}$$

(ii) *Let σ be an absolutely irreducible smooth k-representation of $\mathfrak{R}_0$. If $\sigma \notin \{\chi \circ \det, \mathrm{st} \otimes$*

$\chi \circ \det\}$*, then* $[D_0 : \sigma] = [D_0 : \sigma^{[s]}]$*; otherwise we have*

$$[D_0 : \chi \circ \det] - [D_0 : \mathrm{st} \otimes \chi \circ \det] = [\pi : \chi \circ \det] - [\pi : \mathrm{Sp} \otimes \chi \circ \det].$$

Here, we denote by $\sigma^{[s]}$ *the unique irreducible* k*-representation of* $\mathfrak{R}_0$ *such that* $\sigma \oplus \sigma^{[s]}$ *is isomorphic to* $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0} \psi$ *for some character* ψ*.*[5)]

Proof Using Lemma 3.4, we may assume that π is semi-simple, say $\pi \cong \bigoplus_{i=1}^s \pi_i$. Moreover, by twisting we may assume the central character χ is trivial.

(i) For each π_i, let $D(\pi_i)$ be the associated canonical diagram. Then $D(\pi) := \bigoplus_{i=0}^s D(\pi_i)$ is a standard diagram of π. We claim that the equalities (3.1) hold for $D(\pi)$. In fact, an induction shows that we may assume π is irreducible, in which case the assertion is obvious by Example 2.5 and the explicit description of $D(\mathbf{1})$ and $D(\mathrm{Sp})$ (see [7, §10]).

Now, by Lemmas 3.3 and 3.4, D contains $D(\pi)$ as a sub-diagram. If we denote by Q the quotient $D/D(\pi)$, then the long exact sequence (2.2) associated to $0 \to D(\pi) \to D \to Q \to 0$ shows that $H_1(Q) = H_0(Q) = 0$, hence Q satisfies the dimension relation by Lemma 2.7 (ii). This implies the equalities (3.1) for D.

(ii) The proof is similar as in (i) using the following two facts: (a) the statement holds for $D(\pi)$; (b) for a naive diagram Q, one has $[Q_0 : \mathbf{1}] = [Q_0 : \mathrm{st}]$. □

We record an obvious corollary of Proposition 3.5.

Corollary 3.6 *With notation in Proposition* 3.5*, we have*

$$\dim_k D_0 \leq \dim_k D_1 \cdot (p+1)/2, \quad (\text{resp.} \dim_k D_0 \geq \dim_k D_1 \cdot (p+1)/2)$$

if and only if

$$[\pi : \chi \circ \det] \geq [\pi : \mathrm{Sp} \otimes \chi \circ \det], \quad (\text{resp.} [\pi : \chi \circ \det] \leq [\pi : \mathrm{Sp} \otimes \chi \circ \det]).$$

3.2 Criteria

In this subsection we give two criteria for a diagram to be standard.

Theorem 3.7 *Let* π *be a smooth* k*-representation of* G *of finite length. Let* $W = (W_0, W_1, r)$ *be a sub-diagram of* $\mathcal{K}(\pi)$ *such that* W_0 *is of finite dimension and the natural morphism* $\theta : H_0(W) \to \pi$ *is surjective. Assume that*

(i)

$$\dim_k W_0 \leq \dim_k W_1 \cdot (p+1)/2; \tag{$\star$}$$

(ii) *there exists some* (*hence any*) *standard diagram* D *of* π *such that* $\dim_k D_0 \geq \dim_k D_1 \cdot (p+1)/2$.

Then W *is a standard diagram of* π*. In particular,* $H_0(W)$ *is of finite length and the inequalities in* (i) *and* (ii) *are both equalities.*

Proof By [10, Cor. III.1.15], we can choose a standard diagram D of π containing W. Let Q be the quotient D/W. Then we have an exact sequence

$$0 \to H_1(Q) \to H_0(W) \xrightarrow{\theta} \pi \to H_0(Q) \to 0.$$

5) This coincides with the definition in [7, p.9] under our assumption $F = \mathbb{Q}_p$.

Since θ is assumed to be surjective, we get $H_0(Q)=0$. Write $Q=(Q_0,Q_1,r_Q)$, then $\dim_k Q_0 \le \dim_k Q_1\cdot\frac{p+1}{2}$ by Proposition 2.2. By $(\star)$, we deduce the same inequality for $\dim_k D_i$. Hence, by (ii) we have

$$\dim_k D_0=\dim_k D_1\cdot\frac{p+1}{2}$$

and that $(\star)$ is in fact an equality. Moreover, Q also satisfies the dimension relation, hence $H_0(Q)=H_1(Q)=0$ by Proposition 2.2, and θ is an isomorphism. □

For application later, we need a variant of Theorem 3.7 as follows. The advantage is that we do not need to fix a (finite length) representation π of G.

Theorem 3.8 *Let $W=(W_0,W_1,r)$ be a diagram of k-modules with central character such that r is an injection and that W_0 is of finite dimension. Assume the following conditions*:

(a) $(\star)$ *holds*;

(b) *up to semi-simplification, W_0 is isomorphic to a direct sum of* $\mathrm{Ind}_{IZ}^{\mathfrak{R}_0}\psi_i$, *for a finite set of smooth characters* $\psi_i: I\to k^\times$.

Then $H_0(W)$ is of finite length and $(\star)$ is an equality.

Proof The idea of the proof is as follows: starting with a finite dimensional diagram W, we produce π via the construction of Breuil and Paškūnas [7], then verify the condition (ii) of Theorem 3.7 under the assumption (b) on W_0.

Up to twist we assume that the central character of W is trivial. By [7, §9], we can embed W into $\mathcal{K}(\Omega)$, where Ω is a smooth G-representation such that $\Omega|_K$ is isomorphic to an injective envelope of $\mathrm{soc}_K W_0$ in the category of smooth k-representations of K with central character. Let $\pi\subset\Omega$ be the sub-G-representation generated by W_0; equivalently, π is the image of the natural morphism $H_0(W)\to H_0(\mathcal{K}(\Omega))\cong\Omega$. Since Ω is admissible and W_0 is of finite dimension, π is of finite length;[6] see for example [12, Cor. 4.9]. Write $m_{\mathbf{1}}=[\pi:\mathbf{1}]$, the multiplicity of $\mathbf{1}$ in π^{ss}, and $m_{\mathrm{Sp}}=[\pi:\mathrm{Sp}]$. We will show that $m_{\mathbf{1}}=m_{\mathrm{Sp}}$, hence the result follows from Theorem 3.7 using Proposition 3.5.

Let D be a standard diagram of π containing W. We claim that $m_{\mathbf{1}}\ge m_{\mathrm{Sp}}$. Indeed, if $m_{\mathbf{1}}<m_{\mathrm{Sp}}$, then we would have $\dim_k D_0>\dim_k D_1\cdot(p+1)/2$ by Corollary 3.6, which would contradict Theorem 3.7. So we assume $m_{\mathbf{1}}\ge m_{\mathrm{Sp}}$ in the rest of the proof. Note that Proposition 3.5 (i) implies the following equality

$$\dim_k D_1=\frac{2}{p-1}(\dim_k D_0-\dim_k D_1)+(m_{\mathbf{1}}-m_{\mathrm{Sp}}). \tag{3.2}$$

Since $H_0(D/W)=0$, we can find a finite filtration of D/W whose graded pieces Q has one of the shapes (I)–(III) in Proposition 2.2. In all cases, the (analogue) condition $(\star)$ holds for Q. We let s_1 (resp. s_2) be the number of Q of type (I) (resp. type (II)), and $s_{3,\sigma}$ (resp. $s_{3,0}$) be the number of Q of type (III) with $Q_0\cong\sigma$ for each irreducible σ (resp. $Q_0=0$); note that Q_0 is either irreducible or zero in case of type (III), see Remark 2.3. The assumption (b) implies that $[W_0:\sigma]=[W_0:\sigma^{[s]}]$ for any irreducible σ. By definition, we also have $[Q_0:\sigma]=[Q_0:\sigma^{[s]}]$ if Q is of type (I) or (II). Thus, we deduce from Proposition 3.5 (ii) that

$$s_{3,\sigma}=s_{3,\sigma^{[s]}},\qquad \text{if } \sigma\notin\{\mathbf{1},\mathrm{st}\} \tag{3.3}$$

6) This is a special property for smooth k-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$; it is unknown whether it remains true if $F\ne\mathbb{Q}_p$.

$$m_{\mathbf{1}} - m_{\mathrm{Sp}} = s_{3,\mathbf{1}} - s_{3,\mathrm{st}}. \tag{3.4}$$

On the other hand, if we let $s = \frac{1}{2}\dim W_1 + s_2 \in \mathbb{Q}_{\geq 0}$ and $s' = s_1 + 2s_{3,0}$, then

$$\begin{aligned}\dim_k D_1 &= \dim_k W_1 + s_1 + 2s_2 + \sum_{\sigma} 2s_{3,\sigma} + 2s_{3,0} \\ &= 2s + s' + 2\sum_{\sigma} s_{3,\sigma},\end{aligned}$$

and using (3.3) and the fact $\dim_k\sigma + \dim_k\sigma^{[s]} = p+1$,

$$\dim_k D_0 = \dim_k W_0 + (p+1)s_2 + \frac{(p+1)}{2}\sum_{\sigma\notin\{\mathbf{1},\mathrm{st}\}} s_{3,\sigma} + s_{3,\mathbf{1}} + p\cdot s_{3,\mathrm{st}}.$$

Using $(\star)$, we deduce

$$\begin{aligned}\dim_k D_0 - \dim_k D_1 &\leq (p-1)s - s' + \frac{p-3}{2}\cdot\Bigg(\sum_{\sigma\notin\{\mathbf{1},\mathrm{st}\}} s_{3,\sigma}\Bigg) - s_{3,\mathbf{1}} + (p-2)s_{3,\mathrm{st}} \\ &= (p-1)\Bigg(s + \frac{s'}{2} + \sum_{\sigma} s_{3,\sigma}\Bigg) - \frac{p+1}{2}\Bigg(s' + \sum_{\sigma\notin\{\mathbf{1},\mathrm{st}\}} s_{3,\sigma}\Bigg) \\ &\quad - ps_{3,\mathbf{1}} - s_{3,\mathrm{st}}.\end{aligned}$$

Using the relations (3.2), (3.4), we get

$$0 \leq -\frac{p+1}{p-1}\Bigg(s' + \sum_{\sigma\notin\{\mathbf{1},\mathrm{st}\}} s_{3,\sigma}\Bigg) - \frac{p+1}{p-1}s_{3,\mathbf{1}} - \frac{p+1}{p-1}s_{3,\mathrm{st}}.$$

This implies $s' = 0$ and $s_{3,\sigma} = 0$ for all σ, that is, only Type (II) diagrams appear in the filtration of D/W. Hence $H_0(D/W) = H_1(D/W) = 0$ and W is a standard diagram. □

3.3 Application III

Assume $\Pi_{\mathrm{sm}} = \mathrm{Ind}_B^G\chi_1\otimes\chi_2$ is an irreducible principal series and $\Pi = \Pi_{\mathrm{sm}}\otimes\Pi_{\mathrm{alg}}$ is an irreducible locally algebraic representation of G. The following theorem is a part of a result of Berger and Breuil [3]. It is reproved under mild conditions in [9, Thm. 5.1].

Theorem 3.9 *Let $c \geq 1$ be such that $\Pi_{\mathrm{sm}}^{K_c} \neq 0$ and define X as in* §2.3.

(i) *If $\mathcal{L}$ is an integral structure in Π and $\mathcal{X} \subset X$ is the induced diagram of $\mathcal{O}$-modules, then $H_0(\mathcal{X}) \cong \mathcal{L}$ and $\mathcal{L}$ is residually of finite length.*

(ii) *The universal unitary completion of* Π, *if non-zero, is automatically admissible* (*in the sense of* [21]).

(iii) *The universal unitary completion of* Π, *if non-zero, is topologically irreducible.*

Proof (i) We first show that $H_0(\mathcal{X}\otimes k)$ is of finite length as a G-representation. It suffices to check that the conditions in Theorem 3.8 hold for $\mathcal{X}\otimes k$: indeed the inequality $(\star)$ follows from the fact that

$$\Pi_{\mathrm{sm}}^{K_c} \cong \mathrm{Ind}_{J_c}^{K}\theta, \quad \Pi_{\mathrm{sm}}^{I_c} \cong \mathrm{Ind}_{J_c}^{I}\theta,$$

where $\theta := \chi_1\otimes\chi_2$ and $J_c := (K\cap B)K_c = (I\cap B)K_c$; the condition (b) is verified[7] using the isomorphism

$$X_0 = \Pi_{\mathrm{sm}}^{K_c}\otimes\Pi_{\mathrm{alg}} \cong \mathrm{Ind}_{J_c}^{K}(\theta\otimes\Pi_{\mathrm{alg}}) \cong \mathrm{Ind}_I^K\mathrm{Ind}_{J_c}^{I}(\theta\otimes\Pi_{\mathrm{alg}})$$

7) Alternatively, we may apply [6, Prop. 4.2].

so that, up to semisimplification, $\mathcal{X}_0 \otimes k$ is isomorphic to Ind_I^K of the mod ϖ reduction of $\mathrm{Ind}_{J_c}^I(\theta \otimes \Pi_{\mathrm{alg}})$, and using the fact that irreducible k-representations of I are characters.

By construction the morphisms $\mathcal{X}_1 \otimes k \hookrightarrow \mathcal{X}_0 \otimes k \hookrightarrow \mathcal{L} \otimes k$ are all injective. So [15, Prop. 4.1] is applicable and implies that the morphism $H_0(\mathcal{X} \otimes k) \to \mathcal{L} \otimes k$ is injective. Hence, we get $H_0(\mathcal{X}) = \mathcal{L}$ by [15, Lem. 4.5].

(ii) By (i), any G-invariant open bounded $\mathcal{O}$-lattice (if exists) in Π is finitely generated as an $\mathcal{O}[G]$-module. Therefore, any two such lattices are commensurable and the universal unitary completion $\hat{\Pi}$ of Π is exactly the completion of Π with respect to any such lattice, say $\mathcal{L}$. Moreover, the admissibility of $\hat{\Pi}$ is equivalent to that $\mathcal{L} \otimes k$ is (smooth) admissible in usual sense. Thus, the result follows from (i).

(iii) If $\hat{\Pi}$ is not topologically irreducible, it admits a non-trivial quotient, say $\hat{\Pi}'$. Since Π is itself absolutely irreducible, the composition $\Pi \to \hat{\Pi}'$ is still injective[8]. Let $\mathcal{L}' := \Pi \cap \hat{\Pi}'^0$ be the induced lattice of Π, where $\hat{\Pi}'^0$ denotes the unit ball of $\hat{\Pi}'$, and let $\mathcal{X}' := (X_0 \cap \mathcal{L}', X_1 \cap \mathcal{L}', \mathrm{can})$. Then $\mathcal{L}' \cong H_0(\mathcal{X}')$ by (i). It is clear that $\mathcal{L}'/\varpi\mathcal{L}' \cong \hat{\Pi}'^0/\varpi\hat{\Pi}'^0$. Since $\mathcal{L}$ and $\mathcal{L}'$ are commensurable, $\mathcal{L}/\varpi\mathcal{L}$ and $\mathcal{L}'/\varpi\mathcal{L}'$ have the same length as G-representations. But this would contradict the assumption that $\hat{\Pi}'$ is a non-trivial quotient of $\hat{\Pi}$ ([19, Lem. 5.5]). □

Remark 3.10 (1) Keep the notation in Theorem 3.9. In [15, Thm. 4.6], the authors proved the isomorphism $H_0(\mathcal{X}) \cong \mathcal{L}$ by *assuming* $\mathcal{L}$ is residually of finite length (i.e., theorem of Berger and Breuil). The proof also used crucially [15, Prop. 4.1]. The main innovation of Theorem 3.9 (i) is to prove $H_0(\mathcal{X}) \cong \mathcal{L}$ without this assumption: in fact we deduce it as a byproduct.

(2) Note that our proof of topological irreducibility of $\hat{\Pi}$ is different of the original proof of Berger and Breuil which uses (φ, Γ)-modules ([3, Cor. 5.3.2]).

Acknowledgements We would like to thank V. Paškūnas for his comments, especially for his encouragement to publish this note. We also thank the anonymous referee for the careful reading of the manuscript and pertinent comments.

References

[1] Assaf, E., Kazhdan, D., de Shalit, E.: Kirillov models and the Breuil–Schneider conjecture for $\mathrm{GL}_2(F)$, preprint 2013, arXiv:1302.3060v1

[2] Barthel, L., Livné, R.: Irreducible modular representations of GL_2 of a local field. *Duke Math. J.*, **75**, 261–292 (1994)

[3] Berger, L., Breuil, C.: Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$. *Astérisque*, **339**, 157–180 (2011)

[4] Breuil, C.: Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$ I. *Compositio Math.* **138**, 165–188 (2003)

[5] Breuil, C.: Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$ II. *J. Inst. Math. Jussieu*, **2**, 1–36 (2003)

[6] Breuil, C., Diamond, F.: Formes modulaires de Hilbert modulo p et valeurs d'extensions entre caractères galoisiens. *Ann. Scient. de l'E.N.S.*, **47**, 905–974 (2014)

[7] Breuil, C., Paškūnas, V.: Towards a Modulo p Langlands Correspondence for GL_2, Memoirs of Amer. Math. Soc., 216, 2012

8) Otherwise, Π would be contained in the kernel, say $\hat{\Pi}''$, which is a Banach sub-representation of $\hat{\Pi}$. The universal property of $\hat{\Pi}$ gives a morphism $\hat{\Pi} \to \hat{\Pi}''$ such that the composition $\hat{\Pi} \to \hat{\Pi}'' \hookrightarrow \hat{\Pi}$ is identity, hence the equality $\hat{\Pi}'' = \hat{\Pi}$, a contradiction.

[8] Caraiani, A., Emerton, M., Geraghty, D., et al.: Patching and the p-adic local Langlands correspondence. *Cambridge Journal of Mathematics*, **4**(2), 197–287 (2016)

[9] Caraiani, A., Emerton, M., Geraghty, D., et al.: Patching and the p-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$. *Compositio Math.*, **154**, 503–548 (2018)

[10] Colmez, P.: Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. *Astérisque*, **330**, 281–509 (2010)

[11] De Ieso, M.: Analyse p-adique et complétés unitaires universels pour $\mathrm{GL}_2(F)$, Ph. D. Thesis, Orsay, 2013

[12] Emerton, M.: On a class of coherent rings, with applications to the smooth representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ in characteristic p, preprint, 2008

[13] Hu, Y.: Normes invariantes et existence de filtrations admissibles. *J. Reine Angew. Math.*, **634**, 107–141 (2009)

[14] Hu, Y.: Diagrammes canoniques et représentations modulo p de $\mathrm{GL}_2(F)$. *J. Inst. Math. Jussieu*, **11**, 67–118 (2012)

[15] Hu, Y., Paškūnas, V.: On Crystabelline Deformation Rings of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (with an appendix by Jack Shotton), Math. Ann., 373, 2019, 421–487

[16] Kazhdan, D., de Shalit, E.: Kirillov models and integral structures in p-adic smooth representations of $GL_2(F)$. *J. Algebra*, **353**, 212–223 (2012)

[17] Paškūnas, V.: Coefficient systems and supersingular representations of $\mathrm{GL}_2(F)$. *Mémoirs de la SMF*, **99**, (2004)

[18] Paškūnas, V.: On some crystalline representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. *Algebra and Number Theory*, **3**(4), 411–421 (2009)

[19] Paškūnas, V.: Admissible unitary completions of locally $\mathbb{Q}_p$-rational representations of $\mathrm{GL}_2(F)$. *Representation Theory*, **14**, 324–354 (2010)

[20] Schneider, P., Stuhler, U.: Representation theory and sheaves on the Bruhat–Tits building. *Publ. Math. Inst. Hautes Études Sci.*, **85**, 97–191 (1997)

[21] Schneider, P., Teitelbaum, J.: Banach space representations and Iwasawa theory. *Israel J. Math.*, **127**, 359–380 (2002)

[22] Sorensen, C.: A proof of the Breuil–Schneider conjecture in the indecomposable case. *Ann. of Math.*, **177**, 1–16 (2013)

[23] Sorensen, C.: The Breuil–Schneider conjecture: a survey. *Advances in the Theory of Numbers*, **77**, 219–235 (2015)

[24] Vignéras, M. F.: A criterion for integral structures and coefficient systems on the tree of $\mathrm{PGL}(2, F)$. *Pure Appl. Math. Q.*, **4**(4), 1291–1316 (2008)