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Diffeomorphisms with the $\mathscr{M}_0$-shadowing Property

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Abstract This paper studies the $\mathscr{M}_0$-shadowing property for the dynamics of diffeomorphisms defined on closed manifolds. The C^1 interior of the set of all two dimensional diffeomorphisms with the $\mathscr{M}_0$-shadowing property is described by the set of all Anosov diffeomorphisms. The C^1-stably $\mathscr{M}_0$-shadowing property on a non-trivial transitive set implies the diffeomorphism has a dominated splitting.

Keywords Anosov diffeomorphism, average shadowing property, $\mathscr{M}_\alpha$-shadowing

MR(2010) Subject Classification 37C50, 37D20, 37H99

1 Introduction

In the classical theory of hyperbolic systems, the pseudo-orbits and the shadowing orbits play very important roles. Inspired by the classical work of Anosov [1] and Sinai [22], Bowen [4, 5] applied the shadowing properties to study the hyperbolic invariant sets. The shadowing properties are used in the proof of the Anosov closing lemma, spectral decomposition theorem, Markov partitions for a hyperbolic invariant set and so on. Further, the existence of shadowing orbits for pseudo-orbits for systems with hyperbolic invariant sets have different proofs (see [6, 9, 15, 16]).

On the other hand, there exist various concepts of pseudo-orbits and the corresponding shadowing properties as the development of the theories of topological dynamics [25]. For the definitions of several pseudo-orbits and their corresponding shadowing properties, see Definitions 2.3, 2.4, and 2.5 in the second section. Recently, Wu et al. [25] proved that for a dynamical

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system:

$$\text{AASP} \Longrightarrow \text{WAASP} \Longleftrightarrow \text{ASP} \Longleftrightarrow \mathscr{M}_\alpha\text{-shadowing } (\forall \alpha \in [0,1)) \Longrightarrow \overline{d} + \underline{d}\text{-shadowing}.$$

The shadowing properties are closely related to the dynamics of the systems. Honary and Bahabadi [11] proved that if a diffeomorphism on a two dimensional manifold M belongs to the C^1 interior of the set of all diffeomorphisms having the asymptotic average shadowing property, then it is Anosov. Sakai [21] showed that the case of the average shadowing property as the result of [11]. Then, Lee [12] verified that if a diffeomorphism has the C^1 stable asymptotic average shadowing property, then it admits a dominated splitting. For more results on shadowing properties, one is referred to [17, 18, 23, 24, 27] and references therein.

The motivation for this paper is that whether it is possible that the weaker shadowing properties can bring new dynamics or new understanding of the hyperbolic systems, the non-uniformly hyperbolic systems, or partially hyperbolic systems and others. In this paper, we investigate this problem for two kinds of diffeomorphisms with hyperbolic dynamics by using $\mathscr{M}_0$-shadowing, since this is the weakest shadowing property as far as we know. First, we show that the C^1 interior of the set of all diffeomorphisms with $\mathscr{M}_0$-shadowing can be described by the set of all Anosov diffeomorphisms, generalizing the main results in [21]. Second, we verify that the $\mathscr{M}_0$-stably shadowing property on a non-trivial transitive set can ensure that the diffeomorphism has a dominated splitting, improving the main results in [13]. For a linear transformation $\mathscr{A} = Ax$ defined on $\mathbb{C}^n$, we proved that $\mathscr{A}$ has the (asymptotic) average shadowing property is equivalent to that A is a non-singular hyperbolic matrix, however this linear hyperbolic system does not have the d-shadowing property or the ergodic shadowing property [26]. This, together with the results in our current work, implies that the linear and nonlinear hyperbolic systems are different since they have different shadowing properties.

The rest of this work is organized as follows. In Section 2, several basic definitions are introduced. In Section 3, the diffeomorphisms defined on two dimensional manifolds are considered. In Section 4, the diffeomorphisms with non-trivial transitive sets are studied.

2 Preliminary

In this section, some basic concepts concerned with the dynamics and the shadowing properties are introduced. Throughout this paper, denote by $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. A *dynamical system* is a pair (X, f), where X is a compact metric space with a metric d and $f : X \to X$ is a continuous map.

A (*Furstenberg*) *family* $\mathscr{F}$ is a collection of subsets of $\mathbb{Z}^+$ which is *upwards hereditary*, that is, $F_1 \in \mathscr{F}$ and $F_1 \subset F_2$ imply $F_2 \in \mathscr{F}$.

For any $A \subset \mathbb{Z}^+$, the *upper density* of A is defined by

$$\overline{\text{dens}}(A) := \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}|. \tag{2.1}$$

Replacing lim sup with lim inf in (2.1) gives the definition of $\underline{d}(A)$, the *lower density* of A. If there exists a number $\text{dens}(A)$ such that $\overline{\text{dens}}(A) = \underline{\text{dens}}(A) = \text{dens}(A)$, then we say that the set A *has density* $\text{dens}(A)$. Fix any $\alpha \in [0, 1)$ and denote by $\mathscr{M}_\alpha$ (resp. $\mathscr{M}^\alpha$) the family consisting of sets $A \subset \mathbb{Z}^+$ with $\underline{\text{dens}}(A) > \alpha$ (resp. $\overline{\text{dens}}(A) > \alpha$). Denote by $\hat{\mathscr{M}}_\alpha$ the family of sets with $\underline{\text{dens}}(A) \geq \alpha$. Clearly, $\hat{\mathscr{M}}_1$ consists of sets A with $\text{dens}(A) = 1$.

Definition 2.1 ([20]) *Assume that f is a C^1 diffeomorphism defined on a compact manifold M. A point $x \in M$ is called a periodic point if $f^n(x) = x$ for some $n \in \mathbb{N}$, the period of x, denoted by $\pi(x)$, is the minimal positive integer n with $f^n(x) = x$, the set of periodic points is denoted by $Per(f)$. A point x is called ω-limit (α-limit point) if there exists y such that x is an accumulation point of the forward (backwards) orbit of y. Denote the set of ω-limit points (α-limit points) of x by $\omega(x)$ ($\alpha(x)$), that is, the collection of all the accumulation points of the forward (backwards) orbit of x. A point x is said to be non-wandering if for every open neighborhood U of x, there exists some $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points is denoted by $\Omega(f)$. A point x is called a chain recurrent point if for any $\varepsilon > 0$, there exists a sequence of points $x_0 = x, x_1, \dots, x_{n-1}, x_n = x$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, \dots, n-1$. Let $\mathcal{R}(f)$ be the set of chain recurrent points. The limit set is*

$$L(f) = \mathrm{cl}\bigg(\bigcup_{p\in M} \omega(p) \cup \alpha(p)\bigg),$$

where $\mathrm{cl}(U)$ *is the closure of the set U. So,* $\mathrm{Per}(f) \subset L(f) \subset \Omega(f) \subset \mathcal{R}(f)$. *If $\Omega(f)$ has a hyperbolic structure and* $\mathrm{cl}(\mathrm{Per}(f)) = \Omega(f)$, *then we call f satisfies Axiom A.*

Lemma 2.2 ([20, Anosov closing lemma]) *Assume f is a C^1 diffeomorphism defined on a compact manifold. If f has hyperbolic structure on the chain recurrent set, then the periodic orbits are dense in the chain recurrent set, and* $\mathrm{cl}(\mathrm{Per}(f)) = \mathcal{R}(f) = L(f) = \Omega(f)$.

Definition 2.3 ([25]) *For a dynamical system (X, f), let $\delta > 0$ and $\xi = \{x_i\}_{i=0}^{\infty} \subset X$. We say that ξ is*

(1) *a* δ-ergodic pseudo-orbit (*of f*) *if*

$$\mathrm{dens}(\{n \in \mathbb{Z}^+ : d(f(x_n), x_{n+1}) \geq \delta\}) = 0;$$

(2) *a* δ-average-pseudo-orbit (*of f*) *if there exists $N > 0$ such that for all $n \geq N$ and any $k \in \mathbb{Z}^+$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta;$$

(3) *a* δ-asymptotic-average-pseudo-orbit (*of f*) *if*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \delta;$$

(4) *an* asymptotic average pseudo-orbit (*of f*) *if*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0.$$

Definition 2.4 ([25]) *A dynamical system (X, f) has* (ergodic) $\mathscr{F}$-shadowing property *if for any $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-ergodic pseudo-orbit ξ is $\mathscr{F}$-ε-shadowed by some point $z \in X$, i.e. $\{n \in \mathbb{Z}^+ : d(f^n(z), x_n) < \varepsilon\} \in \mathscr{F}$.*

In the special case of $\mathscr{F} = \hat{\mathscr{M}}_1$ (resp., $\mathscr{F} = \mathscr{M}_0$ and $\mathscr{M}^{1/2}$), we say that (X, f) has the ergodic shadowing property (abbrev. ESP) (resp., $\underline{d}$-shadowing property and $\overline{d}$-shadowing property) (*see* [7]).

Definition 2.5 ([25]) *A dynamical system (X, f) has*

(1) *the average shadowing property (abbrev. ASP) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed on average by a point $z \in X$, i.e.*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon;$$

(2) *the asymptotic average shadowing property (abbrev. AASP) if every asymptotic average pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is asymptotically shadowed on average by a point $z \in X$, i.e.*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) = 0;$$

(3) *the weak asymptotic average shadowing property (abbrev. WAASP) if for any $\varepsilon > 0$ and any asymptotic average pseudo-orbit $\{x_i\}_{i=0}^{\infty}$, there exists $z \in X$ such that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \varepsilon.$$

Remark 2.6 For a smooth manifold M, the metric is induced by the Riemannian metric. For the shadowing properties, the pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ and its corresponding shadowing property can be studied similarly. For convenience, we only consider the pseudo-orbit $\{x_i\}_{i=0}^{\infty}$.

A subset M is a *closed manifold* if it is C^∞ compact connected and the boundary is empty. Throughout the following discussions, assume that M is a closed manifold. Let $\mathrm{Diff}^1(M)$ be the set of all C^1 diffeomorphisms defined on M.

3 $\mathscr{M}_0$-shadowing on Two Dimensional Closed Manifolds

In this section, the diffeomorphisms defined on two dimensional manifolds are considered.

Set

$$\mathscr{F}^1(M) = \{f \in \mathrm{Diff}^1(M) : \text{there is a } C^1 \text{ neighborhood } \mathcal{U}(f) \subset \mathrm{Diff}^1(M) \\ \text{such that for any } g \in \mathcal{U}(f),\ \text{every } p \in \mathrm{Per}(g) \text{ is hyperbolic}\big\}.$$

Remark 3.1 It is proved that $f \in \mathscr{F}^1(M)$ if and only if f is an Axiom A system with no cycles [8, 10].

Lemma 3.2 ([20]) *Assume $f : M \to M$ is a C^1 diffeomorphism for which the chain component $\mathcal{R}(f)$ has a hyperbolic structure. By the spectral decomposition theorem $\mathcal{R}(f) = \Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_l$, there exists a filtration $M = M_l \supset M_{l-1} \supset \cdots \supset M_1 \supset M_0 = \emptyset$ such that for any $1 \le j \le l$,*

(1) *$f(M_j) \subset \mathrm{int}(M_j)$, so each M_j is a trapping region,*

(2) *$\Lambda_j \subset \mathrm{int}(M_j \setminus M_{j-1})$,*

(3) *$\Lambda_j = \bigcap_{k=-\infty}^{\infty} f^k(M_j \setminus M_{j-1})$, and*

(4) *$\bigcap_{k=0}^{\infty} f^k(M_j) = \bigcup_{i\le j} W^u(\Lambda_i) = \bigcup_{i\le j} \mathrm{cl}(W^u(\Lambda_i))$.*

Lemma 3.3 ([8, Lemma 1.1], [21, Lemma 2]) *Let $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$ be a neighborhood of f and $f^n(p) = p$ for some $n > 0$. Then, there exist two positive constants ε_0 and δ_0 such that if $\mathcal{O}_p : T_pM \to T_pM$ is a linear isomorphism with $\|\mathcal{O}_p - I\| < \delta_0$, then there exists a $g \in \mathcal{U}(f)$ satisfying*

(1) $B_{4\varepsilon_0}(f^i(p)) \cap B_{4\varepsilon_0}(f^j(p)) = \emptyset$ *for* $0 \le i \ne j \le n-1$;

(2) $g(x) = f(x)$ *for* $x \in \{p, f(p), \ldots, f^{n-1}(p)\} \cup \{M \setminus \bigcup_{i=0}^{n-1} B_{4\varepsilon_0}(f^i(p))\}$;

(3) $g(x) = \exp_{f^{i+1}(p)} \circ D_{f^i(p)} f \circ \exp^{-1}_{f^i(p)}(x)$ *for* $x \in B_{\varepsilon_0}(f^i(p))$, $0 \le i \le n-2$;

(4) $g(x) = \exp_p \circ \mathcal{O}_p \circ D_{f^{n-1}(p)} f \circ \exp^{-1}_{f^{n-1}(p)}(x)$ *for* $x \in B_{\varepsilon_0}(f^{n-1}(p))$,

where $I : T_pM \to T_pM$ *is the identity map and* $B_\varepsilon(x) = \{y \in M : \ d(x,y) \le \varepsilon\}$ *for* $\varepsilon > 0$.

Lemma 3.4 ([21, Lemma 2]) *For any neighborhood* $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$ *of* f *and* $f^n(p) = p$ *for some* $n > 0$, *let* ε_0 *and* δ_0 *be as in Lemma* 3.3. *If* p *is not hyperbolic, then there exists a linear isomorphism* $\mathcal{O}_p : T_pM \to T_pM$ *with* $\|\mathcal{O}_p - I\| < \delta_0$ *such that for the diffeomorphism* $g \in \mathcal{U}(f)$, $g^n(p) = p$ *given by Lemma* 3.3 *for this* $\mathcal{O}_p$, *there exists a* $D_p g^{nL}$*-invariant splitting* $T_pM = E \oplus F$ *with* $\dim E = \dim F = 1$, *satisfying* $D_p g^{nL}(v) = v$ *for all* $v \in E$ *and some* $L > 0$.

Let $\mathscr{M}_0\mathcal{S}(M) = \{f \in \mathrm{Diff}^1(M) : f$ has the $\mathscr{M}_0$-shadowing property$\}$.

Lemma 3.5 *Let* M *be a two dimensional closed manifold. Then,* $\mathscr{M}_0\mathcal{S}(M) \subset \mathscr{F}^1(M)$.

Proof We will show the statement by contradiction and apply Lemma 3.4. The notations of Lemma 3.4 will be used.

Suppose p is a non-hyperbolic periodic point for f. Recall that the exponential map is defined on a small neighborhood of the origin of the tangent bundle $E \oplus F$ and $\exp_p(0) = p$, where 0 is the origin. Then, choose $0 \ne \zeta \in E$ properly (note that ζ should be taken from the domain of the exponential map), and denote by $\zeta_0 = \exp_p(-\zeta)$ and $\zeta_1 = \exp_p(\zeta)$. Take $\varepsilon_1 := d(\zeta_0, \zeta_1)$. It is evident that $\varepsilon_1 > 0$.

Let $\mathscr{L}_0 = 0$, $L_1 = \mathscr{L}_1 = 2$ and define a sequence of positive integers inductively:

$$L_n = 2^{\mathscr{L}_{n-1}} \text{ and } \mathscr{L}_n = \mathscr{L}_{n-1} + L_n, \ n \ge 2.$$

Define a sequence $\xi := \{\xi_n\}_{n=0}^{+\infty}$ as following:

$$\xi_n = \begin{cases} \zeta_0, & n \in \bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1}), \\ \zeta_1, & \text{otherwise}. \end{cases}$$

It can be verified that ξ is a δ-ergodic pseudo-orbit for any $\delta > 0$ and that

$$\overline{\mathrm{dens}}\Big(\bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1})\Big) = \overline{\mathrm{dens}}\Big(\mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1})\Big) = 1.$$

The $\mathscr{M}_0$-shadowing property implies that there exists $z \in M$ such that $\underline{\mathrm{dens}}(\{n \in \mathbb{Z} : d(g^n(z), \xi_n) < \frac{\varepsilon_1}{10}\}) > 0$. We claim that $z \notin \mathcal{I} := \{x \in B_{\varepsilon_1}(p) : g(x) = x\}$. In fact, suppose on contrary that $z \in \mathcal{I}$, then $d(g^n(z), \zeta_0) \ge \frac{\varepsilon_1}{2}$ or $d(g^n(z), \zeta_1) \ge \frac{\varepsilon_1}{2}$ for all $n \in \mathbb{Z}$. This implies that

$$\bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1}) \subset \{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) \ge \varepsilon_1/10\},$$

or

$$\mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1}) \subset \{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) \ge \varepsilon_1/10\}.$$

Therefore,

$$\begin{aligned} &\underline{\mathrm{dens}}(\{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) < \varepsilon_1/10\}) \\ &\quad = 1 - \overline{\mathrm{dens}}(\{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) \ge \varepsilon_1/10\}) \end{aligned}$$

$$= 0,$$

which is a contradiction and thus $z \notin \mathcal{I}$.

Consider the following three cases:

(1) if $\mu = 1$, then $\mathcal{I} = B_{\varepsilon_1}(p)$, which is a contradiction;

(2) if $\mu > 1$, it follows from $\dim M = 2$ and $z \notin \mathcal{I}$ that there exists some $m \in \mathbb{N}$ such that $g^n(z) \notin B_{\varepsilon_1}(p)$ for all $n \geq m$, implying that $d(g^n(z), \xi_n) \geq \frac{\varepsilon_1}{5}$, which is a contradiction;

(3) if $\mu < 1$, let us choose $m' = \min\{n \in \mathbb{Z}^+ : g^n(z) \in B_{\varepsilon_1}(p)\}$, i.e., $m' \in \mathbb{Z}^+$ such that $g^n(z) \notin B_{\varepsilon_1}(p)$ for $0 \leq n \leq m'$ and $g^{m'+1}(z) \in B_{\varepsilon_1}(p)$. Then, there exist $N \in \mathbb{N}$ and $j \in \{0, 1\}$ such that $n \geq N$ implies that $d(g^{m'+n}(z), \zeta_j) \geq \frac{2\varepsilon_1}{5}$. This implies that

$$\begin{aligned}&\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) < \varepsilon_1/10\})\\&= 1 - \overline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(g^n(z), \xi_n) \geq \varepsilon_1/10\})\\&= 0,\end{aligned}$$

which is a contradiction. □

Theorem 3.6 *$\mathscr{M}_0\mathcal{S}(M)$ is characterized by the set of all Anosov diffeomorphisms.*

Proof As every Anosov diffeomorphism has the average shadowing property, which implies the $\mathscr{M}_0$-shadowing property, it suffices to check that for any $f \in \mathscr{M}_0\mathcal{S}(M)$, f is Anosov.

It follows from Lemma 3.5 and [8, 10] that f satisfies Axiom A with no-cycles (see Remark 3.1). Then, for spectral decomposition $\Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_l$, there exists a filtration $\emptyset \neq M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_l = M$. We claim that $l = 1$.

In fact, suppose on contrary that $l \geq 2$. For simplicity, let $l = 2$. Take $\varepsilon = \frac{1}{2}\inf\{d(x, y) : x \in M_1, y \in \Lambda_2\} > 0$. From $\Omega(f) = \text{cl}(\text{Per}(f))$, it follows that there exist $p \in \Lambda_1 \cap \text{Per}(f)$ and $q \in \Lambda_2 \cap \text{Per}(f)$. Let $l_1, l_2 \in \mathbb{N}$ be the periods of p and q, respectively and choose $\mathscr{L}_0 = 0$, $L_1 = \mathscr{L}_1 = l_1 \cdot l_2$, $L_n = l_1 \cdot l_2 \cdot 2^{\mathscr{L}_{n-1}}$, and $\mathscr{L}_n = \mathscr{L}_{n-1} + L_n$ for $n \geq 2$. Define a sequence $\xi := \{\xi_n\}_{n=0}^{+\infty}$ as following:

$$\xi_n = \begin{cases} f^n(p), & n \in \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1}), \\ f^n(q), & \text{otherwise}. \end{cases}$$

It is not difficult to check that ξ is a δ-ergodic pseudo-orbit for any $\delta > 0$ and that $\overline{\text{dens}}(\mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})) = \overline{\text{dens}}(\bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})) = 1$. For the above $\varepsilon > 0$, there exists some $z \in M$ such that

$$\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(f^n(z), \xi_n) < \varepsilon\}) > 0.$$

Consider the following two cases:

(1) If $z \in \Lambda_2$, then for any $n \in \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$, $d(f^n(z), \xi_n) = d(f^n(z), f^n(p)) \geq \inf\{d(x, y) : x \in M_1, y \in \Lambda_2\} > \varepsilon$. This implies that

$$\begin{aligned}&\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(f^n(z), \xi_n) < \varepsilon\})\\&= 1 - \overline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(f^n(z), \xi_n) \geq \varepsilon\})\\&= 0,\end{aligned}$$

which is a contradiction;

(2) If $z \notin \Lambda_2$, then there exists a neighborhood U_2 of Λ_2 with $z \notin U_2$. Applying a filtration property yields that there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $f^n(z) \in M_1$. This implies that for any $n \in \mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty} [\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$ with $n \geq N$, $d(f^n(z), \xi_n) = d(f^n(z), f^n(q)) \geq \inf\{d(x,y) : x \in M_1, y \in \Lambda_2\} > \varepsilon$. Then,

$$\begin{aligned}&\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(f^n(z), \xi_n) < \varepsilon\})\\ &\quad = 1 - \overline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(f^n(z), \xi_n) \geq \varepsilon\})\\ &\quad = 0,\end{aligned}$$

which is also a contradiction.

Therefore, $l = 1$, and thus f is Anosov. Actually, $M = W^s(\Lambda_1) \cap W^u(\Lambda_1) = \Lambda_1$. □

4 C^1-stably $\mathscr{M}_0$-shadowing and Dominated Splitting

In this section, the diffeomorphisms with non-trivial transitive sets are studied.

Definition 4.1 ([20]) *For $f \in \text{Diff}^1(M)$ and an invariant set Λ, if there exists a compact neighborhood U of Λ satisfying*

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U),$$

then Λ is called locally maximal or isolated.

Definition 4.2 *For $f \in \text{Diff}^1(M)$ and an invariant set Λ, we call f has a dominated splitting if there exist a continuous Df-invariant splitting $E \oplus F$ of the tangent bundle $T_\Lambda M$, and two constants $C > 0$ and $0 < \lambda < 1$ such that*

$$\|D_x f^n|_{E(x)}\| \cdot \|D_x f^{-n}|_{F(f^n(x))}\| \leq C\lambda^n, \quad \forall x \in \Lambda \text{ and } n \in \mathbb{Z}^+.$$

Definition 4.3 *Let $f \in \text{Diff}^{\,1}(M)$, and Λ be a closed f-invariant subset of M, $\mathcal{U}(f)$ be a C^1 neighborhood of f.*

- *If $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ is locally maximal, where U is a compact neighborhood;*
- *for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{g\in\mathbb{Z}} g^n(U)$ has the $\mathscr{M}_0$-shadowing property,*

then it is said that f has the C^1-stably $\mathscr{M}_0$-shadowing property on Λ.

Definition 4.4 *A set Λ is called transitive if there exists a point $x \in \Lambda$ such that $\omega(x) = \Lambda$.*

Definition 4.5 ([13]) *Let M be a closed manifold with dimension N, $\text{GL}(N)$ be the group of linear isomorphisms of $\mathbb{R}^N$. A sequence $\{\dots, A_{-2}, A_{-1}, A_0, A_1, A_2, \dots\}$ of elements of $\text{GL}(N)$, satisfying $A_{jk+s} = A_s$ for any $j \in \mathbb{Z}$ and $s = 0, 1, \dots, k-1$, denoted by $\mathcal{A} = \{A_0, A_1, \dots, A_{k-1}\}$, $\mathcal{A}$ is said to be a periodic family with period k. Set $M_{\mathcal{A}} := A_{k-1}A_{k-2}\cdots A_1A_0$. A periodic family $\mathcal{A} = \{A_0, A_1, \dots, A_{n-1}\}$ is called to have an l-dominated splitting, if there exists a direct sum $\mathbb{R}^N = E \oplus F$ satisfying*

- *E and F are $M_{\mathcal{A}}$ invariant;*
- *for any $k \in \mathbb{Z}^+$,*

$$\frac{\|A_{k+l-1}\cdots A_{k+1}A_k\|}{m(A_{k+l-1}\cdots A_{k+1}A_k)} \leq \frac{1}{2},$$

where $E_k = A_{k-1}\cdots A_0(E)$, $F_k = A_{k-1}\cdots A_0(F)$, $\|A\| = \sup_{|v|\neq 0} \frac{|Av|}{|v|}$, $m(A) = \inf_{|v|\neq 0} \frac{|Av|}{|v|}$.

Lemma 4.6 ([3]) *Given any $\delta > 0$ and $K > 0$, there exist positive integers $n(\delta, K)$ and $l(\delta, K)$ satisfying: given a periodic family $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$ with period $n \geq n(\delta, K)$ and $\max\{\|A_i\|, \|A_i^{-1}\|\} \leq K$ for any $i = 0, 1, \ldots, n-1$, if $\mathcal{A}$ does not admit any $l(\delta, K)$ dominated splitting, then there exists a periodic family $\mathcal{B} = \{B_0, B_1, \ldots, B_{n-1}\}$ such that*

- $\max\{\|B_i - A_i\|, \|B_i^{-1} - A_i^{-1}\| : i = 0, 1, \ldots, n-1\} < \delta$;
- $\det(M_{\mathcal{A}}) = \det(M_{\mathcal{B}})$;
- *the eigenvalues of $M_{\mathcal{B}}$ are all real and have the same modulus.*

Definition 4.7 ([14]) *For a periodic family $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\}$, if there exists a constant $\delta > 0$ such that any δ perturbation of $\mathcal{A}$ is a sink, that is, for any $\mathcal{B} = \{B_0, B_1, \ldots, B_{n-1}\}$ with $\|B_i - A_i\| < \delta$, the eigenvalues of $M_{\mathcal{B}}$ have moduli less than 1, then $\mathcal{A}$ is said to be a δ-uniformly contracting. Similarly, one can define the δ-uniformly expanding for a periodic family.*

Definition 4.8 ([13]) *Let $f \in \mathrm{Diff}^1(M)$, and $p \in \mathrm{Per}(f)$ with period $\pi(p)$. If $Df^{\pi(p)}(p)$ has a multiplicity one eigenvalue with modulus one and the other eigenvalues have norm strictly less (bigger) than one, then p is called pre-sink (pre-source).*

Lemma 4.9 *Suppose that f has the $\mathscr{M}_0$-stably shadowing property on a closed set Λ. Consider U and $\mathcal{U}(f)$ as in the definition of $\mathscr{M}_0$-stably shadowing property. Then, for any $g \in \mathcal{U}(f)$, g has neither pre-sink nor pre-source with the orbit staying in U.*

Proof Suppose on contrary that there exists $g \in \mathcal{U}(f)$ such that g has a pre-sink with the orbit staying in U. Similar arguments can be applied to show that g has no pre-source.

It follows from the classical Franks' lemma [8, Lemma 1.1] that there exists a small perturbation $g_1 \in \mathcal{U}(f)$ of g such that there exists $\varepsilon_1 > 0$ with the properties $B_{\varepsilon_1}(\mathrm{Orb}(p)) \subset U$ and

$$g_1|_{B_{\varepsilon_1}(g^i(p))} = \exp_{g^{i+1}(p)} \circ D_{g^i(p)} g \circ \exp^{-1}_{g^i(p)}|_{B_{\varepsilon_1}(g^i(p))}, \quad 0 \leq i \leq \pi(p) - 1,$$

where p is a periodic point of g_1 with period $\pi(p)$, and $\mathrm{Orb}(p) = \{g_1^k(p) : k \in \mathbb{Z}\} = \{g_1^k(p) : k = 0, \ldots, \pi(p) - 1\}$.

By the assumption that $D_p g^{\pi(p)}$ has a multiplicity one eigenvalue λ with $|\lambda| = 1$ and other eigenvalues of $D_p g^{\pi(p)}$ have moduli less than 1. Denote by E_p^c and E_p^s the eigenspaces corresponding to λ and other eigenvalues with moduli less than 1, respectively. there exists a direct decomposition of the tangent space $T_pM = E_p^c \oplus E_p^s$. It is evident that $\dim E_p^c = 1$ for $\lambda \in \mathbb{R}$, and $\dim E_p^c = 2$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Now, we study the case that $\dim E_p^c = 1$. Without loss of generality, assume that $\lambda = 1$. Then, there exists a subset $\mathcal{I}_p$ of $B_{\varepsilon_1}(p) \cap \exp_p(E_p^c(\varepsilon_1))$ centered at p such that the restriction of $g_1^{\pi(p)}$ to $\mathcal{I}_p$ is identity. By hypothesis, there exists constant $0 < \varepsilon < \varepsilon_1$ such that for any $0 \leq j < \pi(p)$ and $z \in B_{4\varepsilon}(g_1^j(p))$, there exists a $g_1^{\pi(p)}$ fixed point $x \in g_1^j(\mathcal{I}_p)$ such that $\lim_{n \to +\infty} g^{n\pi(p)}(z) = x$.

Let $\mathscr{L}_0 = 0$ and $L_1 = \mathscr{L}_1 = 2\pi(p)$, and define a sequence of positive integers inductively:

$$L_n = \pi(p) \cdot 2^{\mathscr{L}_{n-1}} \quad \text{and} \quad \mathscr{L}_n = \mathscr{L}_{n-1} + L_n, \ n \geq 2.$$

Fix two distinct points $\xi_-, \xi_+ \in \mathcal{I}_p$ with

$$\max\{d(g_1^i(\xi_-), g_1^i(p)), d(g_1^i(\xi_+), g_1^i(p)) : 0 \leq i < \pi(p)\} < \varepsilon$$

and take a sequence $\xi := \{\xi_n\}_{n=0}^{+\infty}$ as follows:

$$\xi_n = \begin{cases} g_1^n(\xi_+), & n \in \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1}), \\ g_1^n(\xi_-), & \text{otherwise}. \end{cases}$$

From $\overline{\text{dens}}(\mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})) = \overline{\text{dens}}(\bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})) = 1$, it can be verified that ξ is a δ-ergodic pseudo-orbit for any $\delta > 0$.

Let $\varepsilon^* = \min\{\varepsilon, \frac{d(g_1^i(\xi_-), g_1^i(\xi_+))}{4} : 0 \le i < \pi(p)\}$. The uniform continuity of g_1 implies that there exists $0 < \eta < \varepsilon^*$ such that for any $x, y \in M$ with $d(x,y) < \eta$ and any $0 \le i < \pi(p)$, $d(g_1^i(x), g_1^i(y)) < \varepsilon^*$. Since g_1 has the $\mathscr{M}_0$-shadowing property on $\Lambda_{g_1}(U)$, there exists a point $z \in M$ such that ξ is $\mathscr{M}_0$-η-shadowed by z, i.e.,

$$\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(g_1^n(z), \xi_n) < \eta\}) > 0,$$

implying that there exists a time i_0 such that $g_1^{i_0}(z) \in B_\eta(\xi_{i_0}) \subset B_{4\varepsilon}(g_1^{i_0}(p))$. Then, there exists a point $x \in g_1^{i_0}(\mathcal{I}_p)$ such that $\lim_{n\to+\infty} {g_1}^{n\pi(p)}(z) = x$. This implies that

$$\lim_{n\to+\infty} d(g_1^n(z), g_1^n(x)) = 0.$$

Thus,

$$\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(g_1^n(x), \xi_n) < \eta\}) > 0$$

From the choice of η, noting that x is a $g_1^{\pi(p)}$ fixed point, it follows that

• if there exists $i \in \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$ such that $d(g_1^i(x), \xi_i) < \eta$, then $d(g_1^n(x), g_1^n(\xi_+)) < \varepsilon^*$ for all $n \ge 0$, implying that $\{n \in \mathbb{Z}^+ : d(g_1^n(x), \xi_n) < \eta\}) \subset \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$;

• if there exists $i \in \mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$ such that $d(g_1^i(x), \xi_i) < \eta$, then $d(g_1^n(x), g_1^n(\xi_-)) < \varepsilon^*$ for all $n \ge 0$, implying that $\{n \in \mathbb{Z}^+ : d(g_1^n(x), \xi_n) < \eta\}) \subset \mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})$.

Therefore,

$$\begin{aligned} &\underline{\text{dens}}(\{n \in \mathbb{Z}^+ : d(g_1^n(z), \xi_n) < \eta\}) \\ &\quad \le \max\left\{\underline{\text{dens}}\left(\bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})\right), \underline{\text{dens}}\left(\mathbb{Z}^+ \setminus \bigcup_{k=0}^{+\infty}[\mathscr{L}_{2k}, \mathscr{L}_{2k+1})\right)\right\} \\ &\quad = 0, \end{aligned}$$

which is a contradiction.

For the case $\dim E_p^c = 2$. The perturbation g_1 of g such that there exists a disk contained in $B_{\varepsilon_1}(p) \cap \exp_p(E_p^c(\varepsilon_1))$ on which $g_1^{\pi(p)}$ is a rational rotation. As a consequence, this disk consists of $g_1^{\pi(p)}$-invariant circles. The two points ξ_+ and ξ_- can be taken from different circles. By applying similar arguments as above, we could show the conclusion. □

Lemma 4.10 (1) *For a non-trivial transitive set Λ, there exists a sequence of diffeomorphisms $\{g_n\}_{n\in\mathbb{N}}$ and periodic orbits P_n of g_n with period $\pi(P_n) \to \infty$ as $n \to \infty$ satisfying that $P_n \to \Lambda$ and g_n is convergent to f in the C^1 topology.*

(2) *For $p_n \in P_n$, there exists a periodic family*

$$\mathcal{A}_n = \{D_{p_n} g_n, D_{g_n(p_n)} g_n, \dots, D_{g_n^{\pi(p_n)-1}(p_n)} g_n\}.$$

For any $\delta > 0$, there exists an integer $n_0(\delta) > 0$ such that for any $n > n_0(\delta)$, $\mathcal{A}_n$ is neither δ-uniformly contracting nor δ-uniformly expanding.

(3) *For any $\varepsilon > 0$, there exist $n(\varepsilon), l(\varepsilon) > 0$ such that for any $n > n(\varepsilon)$, if P_n does not admit an $l(\varepsilon)$ dominated splitting, then there exists g'_n such that P_n is pre-sink or pre-source with respect to g'_n, where g'_n is ε close to g_n in the C^1 topology, and preserves the orbits of P_n.*

Proof The first statement is derived by [19, Pugh's closing lemma]. The second and third statements are obtained by using the same arguments as in [13, Lemmas 3.4 and 3.5]. □

Lemma 4.11 ([2]) *Suppose that g_n is convergent to f in the C^1 topology, Λ_n is a closed g_n-invariant set, and Λ_n is convergent to Λ in the Hausdorff metric. If Λ_n admits an l-dominated splitting with respect to g_n, then Λ admits an l-dominated splitting with respect to f.*

Theorem 4.12 *Let Λ be a non-trivial transitive set. If f has the $\mathscr{M}_0$-stably shadowing property on Λ, then Λ admits a dominated splitting.*

Proof It can be derived by combing the above lemmas. □

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