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Quantitative Absolute Continuity of Harmonic Measure and the Dirichlet Problem: A Survey of Recent Progress

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Dedicated to Prof. Carlos Kenig on the occasion of his 65th birthday

Abstract It is a well-known folklore result that quantitative, scale invariant absolute continuity (more precisely, the weak-A_∞ property) of harmonic measure with respect to surface measure, on the boundary of an open set $\Omega \subset \mathbb{R}^{n+1}$ with Ahlfors–David regular boundary, is equivalent to the solvability of the Dirichlet problem in Ω, with data in $L^p(\partial\Omega)$ for some $p < \infty$. Drawing an analogy to the famous Wiener criterion, which characterizes the domains in which the classical Dirichlet problem, with continuous boundary data, can be solved, one may seek to characterize the open sets for which L^p solvability holds, thus allowing for singular boundary data.

It has been known for some time that absolute continuity of harmonic measure is closely tied to rectifiability properties of $\partial\Omega$, but also that rectifiability alone is not sufficient to guarantee absolute continuity. In this note, we survey recent progress in this area, culminating in a geometric characterization of the weak-A_∞ property, and hence of solvability of the L^p Dirichlet problem for some finite p. This characterization, obtained under rather optimal background hypotheses, follows from a combination of the present author's joint work with Martell, and the work of Azzam, Mourgoglou and Tolsa.

Keywords Harmonic measure, Poisson kernel, uniform rectifiability, weak local John condition, big pieces of chord-arc domains, Carleson measures

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A classical result of F. Riesz and M. Riesz [44] states that for a simply connected domain Ω in the complex plane, rectifiability of $\partial\Omega$ implies that harmonic measure for Ω is absolutely continuous with respect to arclength measure on the boundary. A quantitative version of this theorem was later proved by Lavrentiev [38]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones [15] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion. They also present a counter-example to show that the result of [44] may fail in the absence of some connectivity hypothesis (e.g., simple connectedness).

The present survey is concerned with quantitative (and higher dimensional) versions of the result of [44], in which quantitative scale invariant versions of absolute continuity are tied to

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quantitative rectifiability of $\partial\Omega$, necessarily (by virtue of the counter-example of [15]) in the presence of some scale invariant connectivity property. Specifically, we shall discuss recent work of Azzam [8], of the present author jointly with Martell [29, 30], and of Azzam, Mourgoglou and Tolsa [12], which in combination yield geometric characterizations of the open sets for which quantitative absolute continuity holds (either with doubling [8], or without [29, 30] and [12]). The quantitative version of the result of [44], in turn, is known to be equivalent to the solvability of the Dirichlet problem with boundary data in L^p, for some $p < \infty$; we shall return to this point below.

We remark that there have been other recent related developments, concerning qualitative (as opposed to quantitative) absolute continuity [4, 5, 10]; elliptic operators with variable coefficients [32, 33]; and substitute boundary estimates for solutions, which still hold in the absence of connectivity, even though absolute continuity may fail [9, 16, 25, 31]. We do not discuss these issues here. Instead we shall focus on quantitative absolute continuity results dealing only with the classical harmonic measure associated to the Laplace operator $\mathcal{L} := \sum_{i=1}^{n+1}(\partial x_i)^2$, in open sets $\Omega \subset \mathbb{R}^{n+1}$.

Let us begin by making these notions precise. In the sequel, surface measure on the boundary of an open set $\Omega \subset \mathbb{R}^{n+1}$ is denoted by $\sigma := \mathcal{H}^n\lfloor_{\partial\Omega}$, the restriction of n-dimensional Hausdorff measure to $\partial\Omega$. We shall assume that $\partial\Omega$ is (n-dimensional) Ahlfors–David Regular (ADR), i.e., that there is a uniform constant C such that

$$\frac{1}{C}\, r^n \leq \sigma(\Delta(x,r)) \leq C\, r^n, \quad \forall r \in (0, \mathrm{diam}(\partial\Omega)),\ x \in \partial\Omega, \tag{1}$$

where $\mathrm{diam}(\partial\Omega)$ may be infinite. Here, $\Delta(x,r) := \partial\Omega \cap B(x,r)$ is the *surface ball* of radius r. The ADR hypothesis is in the nature of best possible for the results that we shall discuss; we shall return to this point below.

Quantitative absolute continuity is expressed by the A_∞ property, or more generally, the weak-A_∞ property. In the sequel, given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a point $X \in \Omega$, we let ω^X denote harmonic measure for Ω, with pole at X. If we wish to specify the domain explicitly, we shall write ω^X_Ω.

Definition 1 (Local A_∞ and local weak-A_∞) *We say that harmonic measure ω is locally in A_∞ (resp., locally in weak-A_∞) on $\partial\Omega$, if there are uniform positive constants C and s such that for every ball $B = B(x,r)$ centered on $\partial\Omega$, with radius $r < \mathrm{diam}(\partial\Omega)/4$, and associated surface ball $\Delta = B \cap \partial\Omega$,*

$$\omega^X(E) \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^s \omega^X(\Delta), \quad \forall\, X \in \Omega \setminus 4B, \forall \text{ Borel } E \subset \Delta, \tag{2}$$

or, respectively, that

$$\omega^X(E) \leq C\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^s \omega^X(2\Delta), \quad \forall\, X \in \Omega \setminus 4B, \forall \text{ Borel } E \subset \Delta. \tag{3}$$

Thus, weak-A_∞ is essentially A_∞ without the doubling property. Observe that if we fix a surface ball $\Delta_0 = B_0 \cap \partial\Omega$, and a point $X \in \Omega \setminus 4B_0$, then the local A_∞ (resp., local weak-A_∞) property states that the inequality in (2) (resp., (3)) holds for the same fixed X, for all

$\Delta = B \cap \partial\Omega$, with $B \subset B_0$. In this case we say that $\omega^X \in A_\infty(\Delta_0)$ (resp., $\omega^X \in$ weak-$A_\infty(\Delta_0)$).

We recall that, as is well known, the condition $\omega^X \in$ weak-$A_\infty(\Delta_0)$ is equivalent to the property that $\omega^X \ll \sigma$ in Δ_0, and that for some $q > 1$, the Poisson kernel $k^X := d\omega^X/d\sigma$ satisfies the weak reverse Hölder estimate

$$\left(\fint_{\Delta} (k^X)^q d\sigma\right)^{1/q} \lesssim \fint_{2\Delta} k^X \, d\sigma \approx \frac{\omega^X(2\Delta)}{\sigma(\Delta)}, \quad \Delta = B \cap \partial\Omega,\, 2B \subseteq B_0 \tag{4}$$

(with B, B_0 centered on $\partial\Omega$). We shall refer to the inequality in (4) as a "weak-RH_q" estimate, and we shall say that $k^X \in$ weak-$RH_q(\Delta_0)$ if k^X satisfies (4).

Before proceeding, let us introduce some further terminology.

Definition 2 (Corkscrew condition) *Following* [35]*, we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the Corkscrew condition if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a Corkscrew point relative to Δ. We note that we may allow $r < C\operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c. In order to emphasize that $B(X_\Delta, cr) \subset \Omega$, we shall sometimes refer to this property as the interior Corkscrew condition.*

Definition 3 (Harnack Chains, and the Harnack Chain condition [35]) *Given two points $X, X' \in \Omega$, and a pair of numbers $M, N \geq 1$, an (M, N)-Harnack Chain connecting X to X', is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \emptyset$ and $M^{-1}\operatorname{diam}(B_k) \leq \operatorname{dist}(B_k, \partial\Omega) \leq M\operatorname{diam}(B_k)$. We say that Ω satisfies the Harnack Chain condition if there is a uniform constant M such that for any two points $X, X' \in \Omega$, there is an (M, N)-Harnack Chain connecting them, with N depending only on the ratio $|X - X'|/(\min(\delta(X), \delta(X')))$.*

Definition 4 (NTA) *Again following* [35]*, we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ is* NTA *(Non-tangentially accessible) if it satisfies the Harnack Chain condition, and if both Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ satisfy the Corkscrew condition.*

A domain which satisfies the Harnack Chain condition and only an interior (but not necessarily exterior) Corkscrew condition is known as a **uniform** (aka **1-sided NTA**) domain. A well known folkloric result is that uniformity of a domain Ω can equivalently be characterized by the property that there is some finite M such that any two points $Y, X \in \Omega$ may be connected by an *M-cigar path* (or just *cigar path*, when the constant M is implictly understood); that is, a rectifiable curve $\gamma = \gamma(Y, X)$, with length $\ell(\gamma) \leq M|X - Y|$, whose subarcs $\gamma(Y, Z)$ and $\gamma(Z, X)$ satisfy

$$\min\big\{\ell\big(\gamma(Y, Z)\big), \ell\big(\gamma(Z, X)\big)\big\} \leq M \operatorname{dist}(Z, \partial\Omega), \quad \forall\, Z \in \gamma. \tag{5}$$

A **semi-uniform** domain Ω is one for which each point $X \in \Omega$ can be connected by a cigar path to every $y \in \partial\Omega$ (but not necessarily to another point $Y \in \Omega$), with uniform control of the constant M. Thus, for example, the unit disk centered at the origin, with the slit $-1/2 \leq x \leq 1/2, y = 0$ removed, is semi-uniform, but not uniform.

We recall also that Ω is a **John** domain if there is a point $X_0 \in \Omega$ (the *John center*), and a constant $M < \infty$, such that each $X \in \Omega$ may be joined to X_0 by a rectifiable curve

$\gamma = \gamma(X, X_0)$, with

$$\ell(\gamma(X,Y)) \leq M \operatorname{dist}(Y, \partial\Omega), \quad \forall\, Y \in \gamma.$$

Definition 5 (CAD) *We say that a connected open set* $\Omega \subset \mathbb{R}^{n+1}$ *is a* CAD (*Chord-arc domain*)*, if it is* NTA*, and if* $\partial\Omega$ *is* ADR (*see* (1) *above*)*.*

Definition 6 (UR) (aka *uniformly rectifiable*) *An n-dimensional* ADR (*hence closed*) *set* $E \subset \mathbb{R}^{n+1}$ *is* UR *if and only if it contains "Big Pieces of Lipschitz Images" of* $\mathbb{R}^n$ *("*BPLI*"). This means that there are positive constants* c_1 *and* C_1*, such that for each* $x \in E$ *and each* $r \in (0, \operatorname{diam}(E))$*, there is a Lipschitz mapping* $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$*, with Lipschitz constant no larger than* C_1*, such that*

$$H^n(E \cap B(x,r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})) \geq c_1 r^n.$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [23]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2 boundedness of certain special singular integral operators (the "Riesz Transforms"), suffices to characterize uniform rectifiability (see [40] for the case $n = 1$, and [43] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a uniform domain), but that are totally non-rectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [24, Chapter 1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher co-dimensions) [23, 24].

A variant of the "Big Pieces" approximation used in the previous definition, is the following.

Definition 7 ("Interior Big Pieces") *Given a class* $\mathbf{A}$ *of domains, defined by certain quantitative properties* (*which we refer to as the* $\mathbf{A}$*-constants*)*, we say that an open set* $\Omega \subset \mathbb{R}^{n+1}$ *with* ADR *boundary has Interior Big Pieces of domains of class* $\mathbf{A}$ (IBPA) *if there exist positive constants* η *and* C*, such that for every* $X \in \Omega$*, with* $\delta(X) := \operatorname{diam}(X, \partial\Omega) < \operatorname{diam}(\partial\Omega)$*, there is a subdomain* $\Omega_X \subset \Omega$ *satisfying*

- $X \in \Omega_X \in \mathbf{A}$.
- $\operatorname{dist}(X, \partial\Omega_X) \geq \eta\delta(X)$.
- $\operatorname{diam}(\Omega_X) \leq C\delta(X)$.
- $\sigma(\partial\Omega_X \cap \Delta_X) \geq \eta\sigma(\Delta_X) \approx \eta\delta(X)^n$*, where* $\Delta_X := B(X, 10\delta(X)) \cap \partial\Omega$.
- *The* $\mathbf{A}$*-constants of the subdomains* Ω_X *are uniform in* X*.*

Thus, "interior big pieces" entails having ample, scale invariant overlap with the boundary of a subdomain in the given class, locally, on $\partial\Omega$.

Let us now briefly review the history of developments in the quantitative setting. The first quantitative absolute continuity result was obtained by Lavrentiev [38], who in effect showed that for chord-arc domains in the plane, harmonic measure is A_∞ with respect to σ. This result is remarkable, since its publication in 1936 precedes the introduction of the class A_∞ [19, 42] by more than 35 years.

Progress in higher dimensions came later. The fundamental result is due to Dahlberg [20], and states that harmonic measure belongs to the class A_∞ in the sense of Definition 1, on the boundary of a Lipschitz domain. The result of Dahlberg was extended to the class of Chord-arc domains, that is, NTA domains with ADR boundaries, by David and Jerison [22], and independently by Semmes [45]. The Chord-arc hypothesis was weakened to that of a two-sided Corkscrew condition (Definition 2) by Bennewitz and Lewis [14], who then drew the conclusion that harmonic measure is locally in weak-A_∞; the latter condition is the best conclusion that can be obtained under the weakened geometric conditions considered in [14].

Let us briefly outline the strategy used in [22], and a refinement of it appearing in [14], as these will be relevant to us in the sequel. The starting point is an idea that, to the present author's knowledge, first appears in [35] in the context of BMO_1 domains (that is, domains whose boundaries coincide locally with the graph of a function whose gradient belongs to BMO), and which can be formalized as the following general principle.

Proposition 1 *Let Ω be an open set with* ADR *boundary, and let* $\mathbf{A}$ *denote the class of domains with* ADR *boundaries for which harmonic measure belongs locally to A_∞, in the sense of Definition* 1 *above. Suppose further that Ω has Interior Big Pieces of* $\mathbf{A}$*, in the sense of Definition* 7*. Then there are constants $\varepsilon \in (0,1)$ and $c \in (0,1)$ such that for each $X \in \Omega$, with* $\delta(X) < \operatorname{diam}(\partial\Omega)$*, if E is a Borel subset of Δ_X, then*

$$\sigma(E) \geq (1-\varepsilon)\sigma(\Delta_X) \implies \omega^X(E) \geq c. \tag{6}$$

Here, $\Delta_X = B(X, 10\delta(X)) \cap \partial\Omega$, as in Definition 7.

Proof The proof, following [35], is quite simple. Let $\varepsilon = \eta/2$, where η is the constant in Definition 7. Fix $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, and let $E \subset \Delta_X$ satisfy the first inequality in (6). By Definition 7, there is a subdomain Ω_X of class $\mathbf{A}$, containing X, such that $\sigma(\partial\Omega_X \cap \Delta_X) \geq \eta\sigma(\Delta_X)$, and therefore

$$\sigma(E \cap \partial\Omega_X) \geq \frac{\eta}{2}\sigma(\Delta_X) \approx \eta\delta(X)^n \approx \eta\,\mathcal{H}^n(\partial\Omega_X),$$

where we have used that both $\partial\Omega$ and $\partial\Omega_X$ are ADR. Since $\omega^X_{\Omega_X}$ is A_∞ with respect to surface measure $\sigma_{\partial\Omega_X} := \mathcal{H}^n\lfloor_{\partial\Omega_X}$, we have that

$$\omega^X_{\Omega_X}(E \cap \partial\Omega_X) \geq c(\eta)\omega^X_{\Omega_X}(\partial\Omega_X) = c(\eta),$$

where we have used that $\omega^X_{\Omega_X}$ is a probability measure on $\partial\Omega_X$ (and we have also used that A_∞ is an equivalence relationship, so that the analogue of (2) holds on $\partial\Omega_X$, but with the roles of $\omega^X_{\Omega_X}$ and $\sigma_{\partial\Omega_X}$ reversed). The right-hand inequality in (6) now follows immediately by the maximum principle. □

The method of proof in [22] utilized the principle elucidated in Proposition 1 as follows. First, the authors show that a CAD (or even an open set with ADR boundary, satisfying both an interior and exterior Corkscrew condition[1)]) has Interior Big Pieces of Lipschitz domains.

1) In fact, even the latter condition may be relaxed somewhat; we refer the reader to [22] for details.

By the result of Dahlberg, Lipschitz domains belong to the class **A** of Proposition 1, so that the implication (6) holds. Observe that for a given pole X, (6) entails a condition like A_∞ (since $\omega^X(\Delta_X) \approx 1$), but at only one scale. To obtain A_∞, one needs to change scales, replacing Δ_X by any surface ball $\Delta \subset \Delta_X$, or, equivalently, one needs to change the pole, while working at a given scale. To prove the requisite pole change formula, it suffices to have two ingredients: (1) "CFMS" type estimates relating the Green function and harmonic measure; and (2) the boundary Harnack principle (called by some authors the "comparison" principle), which allows one to control the ratio of Green's functions with two different poles. In [35], these two properties were shown to hold in NTA domains (in particular, in chord-arc domains), hence the pole change formula is valid, and thus $\omega \in A_\infty$.

On the other hand, the last step just described uses connectivity in an essential way: Aikawa has shown that the boundary Harnack principle (plus a John condition) characterizes the class of uniform domains with ADR (or even CDC[2)]) boundaries; see [1, 2]. In particular, then, for domains with ADR boundary, the boundary Harnack principle requires the Harnack chain condition.

Nonetheless, it turns out that the strong connectivity entailed by the Harnack chain condition can be relaxed, at the expense of replacing A_∞ by weak-A_∞. The following result is due to Bennewitz and Lewis [14].

Theorem 1 ([14]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with* ADR *boundary. Suppose that there are constants $\varepsilon \in (0,1)$ and $c \in (0,1)$ such that* (6) *holds for each $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, when E is a Borel subset of Δ_X. Then harmonic measure is locally in weak-A_∞ in the sense of Definition* 1.

This result is not stated explicitly in this form in [14]: rather, it follows from a combination of [14, Lemma 2.2] and its proof, and [14, Lemma 3.1].

The method of proof in [14] is quite interesting, and in some sense amounts to proving a sort of weak version of the pole change formula, which includes an error term. This error term in turn leads to the conclusion that harmonic measure ω is in weak-A_∞, rather than A_∞, but as mentioned above, this conclusion is best possible, since the results of [14] apply to domains in which the doubling property fails for ω. In particular, in [14], the authors observe that their methods apply to an open set Ω with ADR boundary, satisfying a 2-sided (i.e., both interior and exterior) Corkscrew condition: indeed, such Ω have Interior Big Pieces of Lipschitz domains, by the geometric result of [22], and therefore the principle of Proposition 1 applies, using once again Dahlberg's theorem [20].

Before turning to recent developments, let us record the fact that the weak-A_∞ property of harmonic measure is equivalent to solvability of the L^p Dirichlet problem for the Laplacian $\mathcal{L}$. We state this problem as follows:

$$(\mathrm{D})_p \quad \begin{cases} \mathcal{L}u = 0 \quad \text{in } \Omega, \\ N_*(u) \in L^p(\partial\Omega), \\ u|_{\partial\Omega} = f \in L^p(\partial\Omega). \end{cases}$$

2) "Capacity Density Condition": a quantitative version of Wiener regularity.

Here, for a continuous u defined on Ω, the non-tangential maximal function of u is given by

$$N_* u(x) := \sup_{Y \in \Upsilon(x)} |u(Y)|, \tag{7}$$

where the (possibly disconnected) non-tangential approach region $\Upsilon(x)$, with vertex at $x \in \partial\Omega$, is defined as

$$\Upsilon(x) := \bigcup_{I \in \mathcal{W}(x)} I, \tag{8}$$

and in turn, for some fixed collection $\mathcal{W}$ of standard Whitney cubes covering Ω, given $x \in \partial\Omega$, we set

$$\mathcal{W}(x) := \{I \in \mathcal{W} : \operatorname{dist}(x, I) \leq 100\operatorname{diam}(I), \text{ with } \operatorname{diam}(I) < 10\operatorname{diam}(\partial\Omega)\}. \tag{9}$$

The statement "$u|_{\partial\Omega} = f$" is understood in the sense of non-tangential convergence almost everywhere, i.e.,

$$\lim_{Y \to x,\, Y \in \Upsilon(x)} u(Y) = f(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega.$$

We say that $(\mathrm{D})_p$ is *solvable* if for all $f \in C_c(\partial\Omega)$, the harmonic measure solution defined by $u(X) = \int_{\partial\Omega} f d\omega^X$ satisfies $(\mathrm{D})_p$. We note that in particular, the ADR condition implies that every point on $\partial\Omega$ is Wiener regular (see, e.g., [26] or [28]), so the harmonic measure solution is continuous all the way to the boundary.

We have the following equivalence.

Proposition 2 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with* ADR *boundary. Then harmonic measure is locally in weak-A_∞ in the sense of Definition* 1*, if and only if there is a $p \in (1, \infty)$ for which the problem $(\mathrm{D})_p$ is solvable, with the uniform estimate*

$$\|N_*(u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \quad \forall f \in C_c(\partial\Omega). \tag{10}$$

Let us observe that by density of $C_c(\partial\Omega)$ in $L^p(\partial\Omega)$, $1 < p < \infty$, one may extend by continuity to construct, for all $f \in L^p(\partial\Omega)$, a solution u satisfying (10) and $(\mathrm{D})_p$.

It is perhaps worthwhile to comment on the matter of non-tangential convergence to the data. Given estimate (10), and continuity up to the boundary of harmonic measure solutions with data in $C_c(\partial\Omega)$, one obtains non-tangential convergence σ-a.e., for the solution with data $f \in L^p(\partial\Omega)$, by standard techniques from the maximal estimate (10) (e.g., as in the proof of Lebesgue's differentiation theorem using the Hardy–Littlewood maximal theorem). On the other hand, in principle this non-tangential convergence may hold vacuously, since we have no guarantee that in the present generality, the non-tangential approach region $\Upsilon(x)$ is non-empty in every neighborhood of x. For non-tangential convergence to hold in a non-vacuous manner would seem to require that the Whitney boxes in the definition of $\mathcal{W}(x)$ (see (9)) exist at infinitely many scales, for a.e. $x \in \partial\Omega$; e.g., the interior Corkscrew condition would be more than enough to guarantee this property.

Proposition 2 is well known in more restrictive classes of domains (say, Lipschitz domains, or even chord-arc domains), see [36, Chapter 2]. In the present generality, the result is folkloric, although the proof of the direction weak-A_∞ implies solvability of $(\mathrm{D})_p$ is given in [27, Proposition 4.5]. We shall present the short proof of the converse direction here.

We recall that $\Delta_X := B(X, 10\delta(X)) \cap \partial\Omega$, and that by an estimate of Bourgain [17] (alternatively, by capacitary estimates), since $\partial\Omega$ is ADR we have

$$\omega^X(\Delta_X) \approx 1, \tag{11}$$

where the implicit constants depend only on n and ADR.

$(\mathrm{D})_p$ *with estimate* (10) *implies weak-*A_∞. We follow the standard argument, as in [36]. By Theorem 1, it is enough to show that there are constants $\varepsilon \in (0,1)$ and $c \in (0,1)$ such that (6) holds for each $X \in \Omega$, with $\delta(X) < \operatorname{diam}(\partial\Omega)$, when E is a Borel subset of Δ_X.

To this end, we fix a point $X \in \Omega$. Let $f \in C_c(\Delta_X)$, with $\|f\|_{L^p(\partial\Omega)} \le 1$, and let u be the harmonic measure solution with this data f. We may assume that $f \ge 0$, and thus also $u \ge 0$. Let $\hat{x} \in \partial\Omega$ be a touching point for X, i.e., $\delta(X) = |X - \hat{x}|$. Set $r := \delta(X)/1000$. Let $I \in \mathcal{W}$ be a Whitney cube containing X, and note that $I \in \mathcal{W}(z)$, for every $z \in \Delta := \Delta(\hat{x}, r) = B(\hat{x}, r) \cap \partial\Omega$, by definition of $\mathcal{W}(z)$. Consequently,

$$u(Y) \lesssim \left(\fint_\Delta (N_* u)^p d\sigma \right)^{1/p}, \quad \forall Y \in I.$$

Thus, by the mean value property of harmonic functions, we have

$$u(X) \lesssim \left(\fint_I (u(Y))^p dY ds \right)^{1/p} \lesssim \left(\fint_\Delta (N_* u)^p \, d\sigma \right)^{1/p} \lesssim \delta(X)^{-n/p},$$

where in the last step we have applied the lower ADR estimate to Δ, and used (10) and the fact that $\|f\|_p \le 1$. Taking a supremum over all non-negative $f \in C_c(\Delta_X)$ such that $\|f\|_p \le 1$, we obtain by Riesz representation that

$$\left(\int_{\Delta_X} (k^X)^q d\sigma \right)^{1/q} \lesssim \delta(X)^{-(n)/p}, \tag{12}$$

with $q = p/(p-1)$.

We now claim that the latter estimate implies (6), for suitable $\varepsilon, c \in (0,1)$, in which case we are done. To prove this claim, note first that by ADR,

$$\sigma(\Delta_X) \approx \delta(X)^n. \tag{13}$$

Let $E \subset \Delta_X$ satisfy the left-hand estimate in (6), for $\varepsilon > 0$ to be chosen, and set $A := \Delta_X \setminus E$, so that

$$\sigma(A) \le \varepsilon \sigma(\Delta_X). \tag{14}$$

Then

$$\begin{aligned} \omega^X(A) &\le \sigma(A)^{1/p} \left(\iint_{\Delta_X} (k^X)^q d\sigma \right)^{1/q} \\ &\lesssim \sigma(A)^{1/p} \delta(X)^{-(n)/p} \\ &\lesssim \varepsilon^{1/p} \\ &\approx \varepsilon^{1/p} \omega^X(\Delta_X), \end{aligned}$$

where in the last three steps we have used (12)–(14), and then (11). Taking compliments, and using (11) once again, for ε small enough we obtain (6). $\square$

We remark that the previous proof does not depend on harmonicity, per se: for solutions of a uniformly elliptic divergence form operator, one simply uses Moser's local boundedness in place of the mean value property.

We now turn to quite recent developments, which have led to geometric characterizations of those domains for which harmonic measure is A_∞, or is weak-A_∞, with respect to surface measure. In each case, there is a background hypothesis that the boundary be ADR (see (1) above). This hypothesis is rather optimal, as we shall describe below.

The first result is a sort of free boundary result for Poisson kernels, which says that the weak-A_∞ property implies uniform rectifiability of the boundary.

Theorem 2 ([29]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying an interior Corkscrew condition, with* ADR *boundary, and suppose that harmonic measure for Ω belongs locally to weak-A_∞ in the sense of Definition* 1. *Then $\partial\Omega$ is uniformly rectifiable.*

This result appeared first in the present author's unpublished joint work with Martell [29]; the published version, which treats also analogous results in the p-harmonic setting, appears in [28]. An alternative proof, based on the deep characterization of UR sets via Riesz transforms in [43], is presented in the work of Mourgoglou and Tolsa [41].

We remark that the interior Corkscrew hypothesis is in the nature of best possible: in [30], we present a counter-example, based on the 4-corners Cantor set construction of Garnett, to show that the conclusion may fail in the absence of interior Corkscrews. We summarize this construction at the end of the paper.

We further remark that Theorem 2 may be viewed as a sort of large constant version of a result of Kenig and Toro [37][3)], which says that in the presence of a Reifenberg flatness condition and Ahlfors–David regularity, $\log k \in \mathrm{VMO}$ implies that $\nu \in \mathrm{VMO}$, where k is the Poisson kernel with pole at some fixed point, and ν is the unit normal to the boundary. Moreover, given the same background hypotheses, the condition that $\nu \in \mathrm{VMO}$ is equivalent to a uniform rectifiability condition with vanishing trace, thus $\log k \in \mathrm{VMO}$ implies vanishing UR. On the other hand, our large constant version "almost" says "$\log k \in \mathrm{BMO}$ implies UR", given interior Corkscrews and ADR boundary. Indeed, it is well known that the A_∞ condition (to be clear: not weak-A_∞) implies that $\log k \in \mathrm{BMO}$, while if $\log k \in \mathrm{BMO}$ with small norm, then $k \in A_\infty$.

The proof in [29] and [28] is a refinement of an argument of Lewis and Vogel [39], who treated the case that (up to a normalization), the Poisson kernel $k = d\omega/d\sigma \approx 1$. Let us describe, very briefly and roughly, the ideas underlying the proof. Uniform rectifiability of the boundary may be characterized in terms of some appropriate flatness, modulo a Carleson packing condition for the boundary "dyadic cubes" (see [21] or [18]). Flatness, in turn, using an idea going back to Alt and Caffarelli [6][4)], can be deduced from local small oscillations of ∇G, the gradient of the Green function with some fixed pole (fairly far away). Let us explain heuristically this idea of [6]: suppose for the sake of illustration that ∇G has not just small, but zero oscillation, i.e., ∇G is constant, so that G is linear. For some fixed $Y \in \Omega$, we let $\hat{y} \in \partial\Omega$ be a "touching point" for Y, i.e., $|Y - \hat{y}| = \delta(Y)$. In particular, the (open) touching ball $B(Y, \delta(Y))$ lies inside

3) which is itself an endpoint version of the results of [6] and of [34]
4) This idea is also exploited in [37].

Ω. By translation and rotation, we may suppose that $\hat{y} = 0$, and that Y lies vertically above $\hat{y}$, i.e., in $\mathbb{R}^{n+1}$, after rescaling, $Y = e_{n+1}$ (as usual e_{n+1} denotes the unit basis vector in the X_{n+1} direction). Since $\hat{y} = 0 \in \partial\Omega$, and since $\partial\Omega$ is the zero set of G, which we have assumed to be linear, we find therefore that $G(X) = \vec{\alpha} \cdot X$, for some constant vector $\vec{\alpha}$. Observe that $\vec{\alpha}$ is parallel to e_{n+1}: indeed, if not, then there exists $X \in B(Y, \delta(Y))$ such that $\vec{\alpha} \cdot X < 0$, which contradicts the positivity of G. But then, for some constant $a > 0$, $G(X) = aX_{n+1}$, so its zero set $\partial\Omega$ is the hyperplane $\{X_{n+1} = 0\}$, and thus flat. Of course, in reality, all of this must be quantified, and occurs modulo errors. Moreover, the bad boundary cubes where small oscillation of ∇G fails nearby, must be packed; one does this by proving local square function bounds for $\nabla^2 G$, using the weak-A_∞ property of ω (this is not too hard, given known techniques).

Concerning the A_∞ property, the following characterization is due to Azzam [8], in combination with Proposition 1, Theorems 1 and 2.

Theorem 3 ([8]) *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with* ADR *boundary. Then the following are equivalent*:

(1) Ω *is a semi-uniform domain with uniformly rectifiable* (UR) *boundary.*

(2) Ω *is a semi-uniform domain with* ADR *boundary and very big pieces of chord-arc subdomains* (VBPCAS): *for every ball B centered on $\partial\Omega$ and $\varepsilon > 0$, there is a* CAD $\Omega_B \subset B \cap \Omega$ (*with chord-arc constants depending on ε, but uniform with respect to B*), *so that*

$$\sigma((\partial\Omega \cap B \setminus \partial\Omega_B) < \varepsilon\sigma((\partial\Omega \cap B). \tag{15}$$

(3) *Harmonic measure is A_∞ with respect to σ, in the sense of Definition* 1.

Some discussion of this result, and of the author's approach in [8], is in order. It was shown by Aikawa and Hirata in [3], that for a domain satisfying a John condition and the Capacity Density Condition (in particular, for a domain with an ADR boundary), semi-uniformity characterizes the doubling property of harmonic measure. Azzam first proves a preliminary result, interesting in its own right, which improves the result of [3] by removing the John domain assumption; that is, he shows that in the presence of CDC, semi-uniformity is equivalent to an appropriate version of the doubling property of harmonic measure, with no further background hypotheses. Combining this result with Theorem 2, one obtains the implication (3) implies (1) in Theorem 3: by Theorem 2, the A_∞ property of harmonic measure gives that $\partial\Omega$ is UR[5)], and, since A_∞ measures are doubling, Azzam's refinement of the result of [3] yields semi-uniformity. It also yields, in conjunction with Proposition 1, and Theorem 1, the implication (2) implies (3): the last two ingredient give that harmonic measure is in weak-A_∞, which in turn self-improves to A_∞, since semi-uniformity implies doubling. We observe that for this step, the full strength of the "Very Big Piece" approximation in (2) is not needed: a standard interior big piece approximation (i.e., (15) with just one fixed $\varepsilon < 1$) would suffice.

Finally, the main new step in the proof of Theorem 3 is to show that (1) implies (2). Briefly,

5) The alert reader may object that in Theorem 2, one assumes an interior Corkscrew condition; in fact, the latter condition holds automatically in domains (with ADR boundaries) whose harmonic measure is doubling, and of course, A_∞ entails doubling.

the (very rough) idea is to start with a bilateral version of the Corona approximation of UR sets by Lipschitz graphs with small constant (see [31] for the bilateral version, [23] for the original unilateral version), and then to use semi-uniformity to connect Lipschitz subdomains at different scales to build a CAD that touches $\partial\Omega$ in a (locally) ample way. In practice, of course, this is somewhat delicate.

We now turn to the matter of characterizing the weak-A_∞ property. This is important because, as noted above in Proposition 2, at least for domains with ADR boundaries, the weak-A_∞ property of harmonic measure is equivalent to L^p solvability of the Dirichlet problem, with non-tangential maximal function estimates and non-tangential convergence of the solution to its boundary values, σ-a.e. on $\partial\Omega$. In addition, in light of the example of [15], it has been an important open problem to determine the minimal connectivity assumption, which, in conjunction with uniform rectifiability of the boundary, yields quantitative absolute continuity of harmonic measure with respect to surface measure. Let us now describe the connectivity hypothesis, milder than semi-uniformity, which answers the latter question. We observe that the connectivity condition in [8], i.e., semi-uniformity, is tied to the doubling property of harmonic measure, and not to absolute continuity.

The connectivity condition which turns out to be tied to absolute continuity says, roughly speaking, that from each point $X \in \Omega$, there is local non-tangential access to an ample portion of a surface ball at a scale on the order of $\delta(X) := \operatorname{dist}(X, \partial\Omega)$. Let us make this a bit more precise. A **carrot path** (aka *non-tangential path*) or more precisely, a **λ-carrot path**, joining a point $X \in \Omega$, and a point $y \in \partial\Omega$, is a rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, such that for some $\lambda \in (0,1)$ and for all $Z \in \gamma$,

$$\lambda\, \ell(\gamma(y,Z)) \leq \delta(Z). \tag{16}$$

For $X \in \Omega$, as above set

$$\Delta_X := B(X, 10\delta(X)) \cap \partial\Omega.$$

Definition 8 (Weak local John condition) *We say that Ω satisfies a weak local John condition if there are constants $\lambda \in (0,1)$ and $\theta \in (0,1]$, such that every point $X \in \Omega$ may be joined by a λ-carrot path to each y in a "θ-ample piece" of Δ_X, i.e., to each y in a Borel subset $F \subset \Delta_X$, with $\sigma(F) \geq \theta\sigma(\Delta_X)$.*

We remark that the terminology *weak local semi-uniformity* would probably be equally appropriate.

The weak local John condition is strictly weaker than semi-uniformity: for example, the unit disk centered a the origin, with either the cross $\{-1/2 \leq x \leq 1/2, y = 0\} \cup \{-1/2 \leq y \leq 1/2, x = 0\}$ removed, or with the slit $0 \leq x \leq 1, y = 0$ removed, satisfies the weak local John condition, although semi-uniformity fails in each case.

The geometric characterization of quantitative absolute continuity of harmonic measure, and of the L^p solvability of the Dirichlet problem, is as follows.

Theorem 4 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying an interior Corkscrew condition, and suppose that $\partial\Omega$ is Ahlfors–David regular* (ADR). *Then the following are equivalent*:

(1) $\partial\Omega$ *is uniformly rectifiable* (UR) *and* Ω *satisfies the weak local John condition.*

(2) Ω *satisfies an Interior Big Pieces of Chord-Arc Domains* (IBPCAD) *condition.*

(3) *Harmonic measure* ω *is locally in weak-*A_∞ *with respect to surface measure* σ *on* $\partial\Omega$*, in the sense of Definition* 1.

(4) *The* L^p *Dirichlet problem is solvable for some* $p \in (1,\infty)$.

This theorem is a combination of very recent work of the present author jointly with Martell [30], and of Azzam, Mourgoglou and Tosa [12], along with previously described results. There are two new ingredients: that (1) implies (2) is the main result of [30], and that (3) implies the weak local John condition is the main result of [12]. Otherwise, the remaining implications were known: (2) implies (3) follows immediately from Proposition 1 and Theorem 1, using that chord arc domains belong to the class **A** of Proposition 1, by [22] or [45]; and that (3) implies uniform rectifiability is Theorem 2 (it is here that one uses the interior Corkscrew condition). Finally, that (3) is equivalent to (4) is Proposition 2.

Let us discuss briefly the new ingredients, namely that (1) implies (2) [30], and that (3) implies the weak local John condition [12]. As a matter of general principle, we note that the difference between Theorem 4 and Theorem 3, is that in the former, the doubling property of harmonic measure is lacking, and this causes nontrivial difficulties: connectivity in the doubling case is both stronger, and easier to put one's hands on.

The proof of (1) implies (2) is based on a bilateral Corona type approximation of UR sets by (boundaries of) chord arc domains, established in [31]. To turn this into an "interior big piece" approximation entails finding carrot paths to connect chord arc domains at different scales; this is done via a two parameter induction argument, in which one bootstraps both the parameter θ in Definition 8, starting at $\theta = 1$, and working downwards to any given $\theta_0 > 1$, as well as the Carleson packing constant in the bilateral Corona type approximation, starting with constant 0 and working upward. It is the fact that θ may be strictly less than 1, that seems to require this elaborate approach. In the case $\theta = 1$, something along the lines of Azzam's arguments in [8] would be enough: indeed, when $\theta = 1$, it is not too hard to build carrot paths connecting the chord arc domains at different scales. We remark that this part of the argument does not require the interior Corkscrew condition.

To prove that (3) implies the weak local John condition, entails building the required carrot paths by hand. We sketch a few of the ideas in very rough terms. Fix a dyadic boundary cube Q_0, which by scale invariance we shall take to have diameter 1, hence $\sigma(Q_0) \approx 1$, by ADR, and suppose that there is an appropriate Corkscrew point X_0 relative to Q_0 such $\omega^{X_0}(Q_0) \approx 1$. The weak-$A_\infty$ condition yields an ample stopping time regime (or "tree") of subcubes of Q_0, on which the averages $\omega^{X_0}(Q)/\sigma(Q)$ are uniformly bounded above and below; here "ampleness" means that at least a fixed σ-portion of Q_0 is covered by arbitrarily small cubes in the tree. In turn, a weak version of "CFMS" type estimates ensures that for each cube Q in the tree, there is a Corkscrew point X_Q relative to Q where the Green function satisfies $G(X_0, X_Q) \approx \operatorname{diam}(Q)$. One then uses the Alt–Caffarelli–Friedman monotonicity formula [7] to show that X_Q may be connected to another Corkscrew point X', further from the boundary (thus, relative to a somewhat larger cube), at which $G(X_0, X') \gtrsim \delta(X')$. But this is only part of the battle: one would like eventually to trace a path back to X_0, but Whitney regions relative to a given cube may have multiple components, and this also causes connectivity problems. In practice, this is all quite delicate.

Sharpness of background hypotheses. Finally, we discuss some counter–examples that show that the background hypotheses in Theorem 4 (namely, ADR and interior Corkscrew condition) are in the nature of best possible. In the first two examples, Ω is a domain satisfying an interior Corkscrew condition, such that $\partial\Omega$ satisfies exactly one (but not both) of the upper or the lower ADR bounds, and for which harmonic measure ω fails to be weak-A_∞ with respect to surface measure σ on $\partial\Omega$. In this setting, in which full ADR fails, there is no established notion of uniform rectifiability, but in each case, the domain will enjoy some substitute property which would imply uniform rectifiability of the boundary in the presence of full ADR.

In the last example, we construct an open set Ω with ADR boundary, and for which $\omega \in$ weak-A_∞ with respect to surface measure, but for which the interior Corkscrew condition fails, and $\partial\Omega$ is not uniformly rectifiable.

Failure of the upper ADR *bound.* In [11], the authors construct an example of a Reifenberg flat domain $\Omega \subset \mathbb{R}^{n+1}$ for which surface measure $\sigma = H^n\lfloor_{\partial\Omega}$ is locally finite on $\partial\Omega$, but for which the upper ADR bound

$$\sigma(\Delta(x,r)) \leq Cr^n \tag{17}$$

fails, and for which harmonic measure ω is not absolutely continuous with respect to σ. Note that the hypothesis of Reifenberg flatness implies in particular that Ω and $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ are both NTA domains, hence both enjoy the Corkscrew condition, so by the relative isoperimetric inequality, the lower ADR bound

$$\sigma(\Delta(x,r)) \geq cr^n \tag{18}$$

holds. Thus, it is the failure of (17) which causes the failure of absolute continuity: in the presence of (17), the results of [22] apply, and one has that $\omega \in A_\infty(\sigma)$, and that $\partial\Omega$ satisfies a "big pieces of Lipschitz graphs" condition (see [22] for a precise statement), and hence is uniformly rectifiable. We note that by a result of Badger [13], a version of the Lipschitz approximation result of [22] still holds for NTA domains with locally finite surface measure, even in the absence of the upper ADR condition.

Failure of the lower ADR *bound.* In [5, Example 5.5], the authors give an example of a domain satisfying the interior Corkscrew condition, whose boundary is rectifiable (indeed, it is contained in a countable union of hyperplanes), and satisfies the upper ADR condition (17), but not the lower ADR condition (18), but for which surface measure σ fails to be absolutely continuous with respect to harmonic measure, and in fact, for which the non-degeneracy condition

$$E \subset \Delta_X := B(X, 10\delta(X)) \cap \partial\Omega, \quad \sigma(E) \geq (1-\eta)\sigma(\Delta_X) \implies \omega^X(E) \geq c \tag{19}$$

fails to hold uniformly for $X \in \Omega$, for any fixed positive η and c, and therefore ω cannot be weak-A_∞ with respect to σ. We note that in the presence of the full ADR condition, if $\partial\Omega$ were contained in a countable union of hyperplanes (as it is in the example), then in particular it would satisfy the "BAUP" condition of [24], and thus would be uniformly rectifiable [24, Theorem I.2.18, p. 36].

Failure of the interior Corkscrew condition. The example is based on the construction of Garnett's 4-corners Cantor set $\mathcal{C} \subset \mathbb{R}^2$ (see, e.g., [24, Chapter 1]). Let I_0 be a unit square positioned with lower left corner at the origin in the plane, and in general for each $k = 0, 1, 2, \ldots$, we let I_k be the unit square positioned with lower left corner at the point $(2k, 0)$ on the x-axis. Set $\Omega_0 := I_0$. Let Ω_1 be the first stage of the 4-corners construction, i.e., a union of four squares of side length $1/4$, positioned in the corners of the unit square I_1, and similarly, for each k, let Ω_k be the k-th stage of the 4-corners construction, positioned inside I_k. Note that $\operatorname{dist}(\Omega_k, \Omega_{k+1}) = 1$ for every k. Set $\Omega := \bigcup_k \Omega_k$. It is easy to check that $\partial\Omega$ is ADR, and that the non-degeneracy condition (19) holds in Ω for some uniform positive η and c, and thus by the criterion of [14], $\omega \in$ weak-$A_\infty(\sigma)$. On the other hand, the interior Corkscrew condition clearly fails to hold in Ω (it holds only for decreasingly small scales as k increases), and certainly $\partial\Omega$ cannot be uniformly rectifiable: indeed, if it were, then $\partial\Omega_k$ would be UR, with uniform constants, for each k, and this would imply that $\mathcal{C}$ itself was UR, whereas in fact, as is well known, it is totally non-rectifiable. One can produce a similar set in 3 dimensions by simply taking the cylinder $\Omega' = \Omega \times [0, 1]$. Details are left to the interested reader.

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