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Entire Solutions for Nonlocal Dispersal Equations with Bistable Nonlinearity: Asymmetric Case

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Abstract This paper mainly focuses on the entire solutions of a nonlocal dispersal equation with asymmetric kernel and bistable nonlinearity. Compared with symmetric case, the asymmetry of the dispersal kernel function makes more diverse types of entire solutions since it can affect the sign of the wave speeds and the symmetry of the corresponding nonincreasing and nondecreasing traveling waves. We divide the bistable case into two monostable cases by restricting the range of the variable, and obtain some merging-front entire solutions which behave as the coupling of monostable and bistable waves. Before this, we characterize the classification of the wave speeds so that the entire solutions can be constructed more clearly. Especially, we investigate the influence of the asymmetry of the kernel on the minimal and maximal wave speeds.

Keywords Entire solution, traveling wave solution, nonlocal dispersal, asymmetry

MR(2010) Subject Classification 35B08, 35K57

1 Introduction and Main Results

In this paper, we concentrate on some new types of nontrivial entire solutions of the following general nonlocal dispersal equation:

$$u_t(x,t) = (J*u)(x,t) - u(x,t) + f(u(x,t)), \quad (x,t) \in \mathbb{R}^2, \tag{1.1}$$

here $(J*u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]dy$ represents a nonlocal dispersal mechanism. We impose the following assumptions for the kernel function J:

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(J1) $J \in C(\mathbb{R})$, $J(x) \geq 0$, $\int_{\mathbb{R}} J(x)dx = 1$, $\exists\ a < 0 \leq b$ such that $J(a) > 0$ and $J(b) > 0$, $\int_{\mathbb{R}} J(x)|x|dx < \infty$, and $\exists\ \lambda > 0$ such that $\int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda|x|}dx < \infty$.

(J2) J is completely supported.

We separate the above assumptions because some of discussions do not require $(J2)$ in this paper. The most common nonlinear function f in (1.1) usually considered in the literature are such three types: monostable, ignition and bistable. In this paper, we are interested in bistable case, more precisely, f satisfies

(FB) $f \in C^2(\mathbb{R})$, $f(0) = f(1) = 0$, $\exists\rho \in (0,1)$ such that $f|_{(0,\rho)} < 0, f(\rho) = 0$, $f|_{(\rho,1)} > 0$, $f'(0) < 0$, $f'(1) < 0$ and $f'(s) \leq f'(\rho) < 1$ for $s \in (0,1)$.

During the past decades, the equations and systems like (1.1) and variations of it, have been widely introduced to analyze long range effects of the dispersion processes, for example, in materials science [1], phase transitions [2], population dynamics [9], neuronal networks [38], biology [22] and so on. As a special type of solutions and their significant applications in transmission dynamics, traveling wave solutions have been studied extensively, especially their qualitative properties. For the traveling wave solutions of (1.1) with symmetric kernel function J, one can refer to Bates [1], Bates et al. [2], Carr and Chmaj [3], Chen [4], Coville and Dupaigne [9], Pan et al. [23], Schumacher [24] and references therein.

Recently, besides traveling wave solutions, many researchers discovered some new types of entire solutions which behave like the couplings of different traveling wave fronts. These solutions can not only describe the interactions of traveling wave fronts but also characterize new dynamics of diffusion equations. Whether from the viewpoint of dynamical systems or biology or epidemiology, the study of these solutions are interesting and meaningful. About the details, we refer to [5, 12–14, 16, 20, 29, 35] for scalar reaction-diffusion equations with and without delays, [30, 31, 33] for lattice differential equations and systems, [10, 17, 19] for reaction-advection-diffusion equations, [18, 21, 28, 34] for reaction-diffusion systems, and [15, 18, 25, 32, 39] for nonlocal dispersal equations and systems.

Above mentioned researches are all based on the assumption that the kernel function J is symmetric, which corresponds to the situation that the dispersal of individuals or populations are isotropic. However, in the actual environment, the dispersal of individuals or populations can be influenced by many factors (such as wind, sunlight, landscape, food, ...), which correspond to asymmetric or anisotropic dispersal mechanism. Moreover, asymmetry arises widely in applications, for instance, asymmetric effects appear in chemical reactions, flow of solvent in chromatography, population dynamics, movement of insects in wind and predator-prey interaction, where the prey tries to evade the predator. Therefore, it is more practical to study the asymmetric equation (1.1).

So far, when J is asymmetric, the researches on traveling wave solutions and entire solutions of (1.1) are still very little. For traveling wave solutions, one can refer to Coville et al. [8], Yagisita [36] and Sun et al. [26] for monostable case, Yagisita [37] and Coville [7] for bistable and ignition case. However, to the best of our knowledge, there are only two papers to investigate the entire solutions of asymmetric equation (1.1), which are our previous work [27] for monostable nonlinearity and [40] for ignition nonlinearity. Based on the conclusion that the minimal wave speed c^* of nonincreasing traveling waves may be nonpositive (see [8]), in [27] we first proved that the minimal wave speed c^* and $\hat{c}^*$ of nonincreasing and nondecreasing traveling waves can not be negative at the same time, then by combining traveling waves with different speeds, we constructed some new types of entire solutions. In [40], we studied a new entire solution of asymmetric equation (1.1) with ignition nonlinearity. Due to the impact of degeneration, the common method which mainly depends on the Ikehara theorem can not be used to study the asymptotic rates of the traveling wave solutions. We adopted another method based on the construction of appropriate barrier functions to investigate them. Moreover, we gave a special asymmetric kernel function as an example to illustrate our conclusion is reasonable and meaningful.

Consequently, based on the previous work, this paper mainly focuses on those entire solutions which are different with symmetric case. For (1.1), the most common entire solutions are equilibriums, traveling wave solutions, spatial independent solutions and steady-state solutions. However, due to the interactions between these solutions, it might occur many other entire solutions, which behave like traveling fronts or merging fronts as $t \to -\infty$ and so on. In bistable case, (1.1) has no spatial independent solution, but we can divide the bistable equation into two monostable equations by restricting the range of the variable, and further establish some merging-front entire solutions which are generated by the merger of different monostable waves and bistable waves. In addition, compared with symmetric case, the indeterminacy of the symbols of wave speeds and the asymmetry of corresponding nondecreasing and nonincreasing traveling waves will take some difficulties for our research. Therefore we first depict the dependence of wave speeds on the asymmetry of the kernel function, then by constructing different auxiliary functions and super and subsolutions, we establish some new entire solutions.

Before stating the main results, we first introduce the traveling wave solutions of (1.1). From now on, if there is no special explanation, we always assume the nonlinearity f satisfies (FB), and call the traveling wave solutions of (1.1) with bistable nonlinearity as bistable waves, and those with monostable nonlinearity as monostable waves for convenience. The existence of bistable waves of (1.1) and the uniqueness of the wave speed under the condition (J1) have been obtained by Coville [7]. In fact, in [7] the author considered not only the bistable case but also the ignition case. Following from [7], (1.1) possesses a traveling front $\phi(x + ct)$ with speed $c \in \mathbb{R}$ such that

$$\begin{cases} c\phi' = J * \phi - \phi + f(\phi), \\ \phi(-\infty) = 0, \\ \phi(+\infty) = 1, \end{cases} \tag{1.2}$$

where $\phi(\pm\infty)$ denotes the limit of $\phi(\xi)$ as $\xi \to \pm\infty$. We know that if the kernel function J is symmetric, the transformation $x \to -x$ produces a nonincreasing front $\hat{\phi}$ with speed $-c$ from a nondecreasing front ϕ with speed c, since

$$\hat{u}(x,t) := \hat{\phi}(x + (-c)t), \quad \hat{\phi}(\xi) := \phi(-\xi), \quad \xi \in \mathbb{R} \tag{1.3}$$

solves (1.1) whenever $u(x,t) := \phi(x+ct)$ does. But when J is asymmetric, (1.3) no longer gives a solution of (1.1). So we have to obtain the nonincreasing front $\hat{\phi}(x + \hat{c}t)$ with speed $\hat{c} \in \mathbb{R}$ by a similar argument with [7], where $\hat{\phi}$ satisfies

$$\begin{cases} \hat{c}\hat{\phi}' = J * \hat{\phi} - \hat{\phi} + f(\hat{\phi}), \\ \hat{\phi}(-\infty) = 1, \\ \hat{\phi}(+\infty) = 0. \end{cases} \tag{1.4}$$

Clearly, when the kernel function J is asymmetric and f is bistable, the speed $\hat{c}$ of nonincreasing front is not, in general, $-c$, the negative of the speed of nondecreasing front, which implies that there is no symmetry between $\phi(x + ct)$ and $\hat{\phi}(x + \hat{c}t)$.

Motivated by [20], we focus not only on the entire solutions which depict the interactions between bistable waves, but also on other merging-front entire solutions depicting the interactions between monostable waves and bistable waves. Particularly, under the premise of (FB), if we further restrict $f(u)$ to $[0, \rho]$ and $[\rho, 1]$ respectively, (1.1) can be regarded as two monostable equations.

Consider (1.1) on $[0, \rho]$. By taking $v(x, t) = \rho - u(x, t)$, (1.1) reduces to

$$v_t(x, t) = (J * v)(x, t) - v(x, t) + g(v(x, t)), \tag{1.5}$$

where $g(v) = -f(\rho - v)$. From the assumption (FB), we have $g(0) = g(\rho) = 0$, $g(v) > 0$ and $g'(v) \le g'(0)$ for all $v \in (0, \rho)$. Then according to [8] and [26], there exist $\hat{c}_1^* \in \mathbb{R}$ and $\tilde{c}_1^* \in \mathbb{R}$ such that for any $\hat{c}_1 \ge \hat{c}_1^*$ and $\tilde{c}_1 \ge \tilde{c}_1^*$, (1.5) possesses a pair of traveling fronts $\hat{\varphi}_1(x+\hat{c}_1 t)$ and $\tilde{\varphi}_1(x-\tilde{c}_1 t)$ satisfying $\hat{\varphi}_1(-\infty) = 0, \hat{\varphi}_1(+\infty) = \rho$ and $\tilde{\varphi}_1(-\infty) = \rho, \tilde{\varphi}_1(+\infty) = 0$, respectively. Let $c_1 = -\tilde{c}_1, c_1^* = -\tilde{c}_1^*$, $\psi_1(x + c_1 t) = \rho - \tilde{\varphi}_1(x - \tilde{c}_1 t)$ for any $\tilde{c}_1 \ge \tilde{c}_1^*$, and $\hat{\psi}_1(x + \hat{c}_1 t) = \rho - \hat{\varphi}_1(x + \hat{c}_1 t)$ for any $\hat{c}_1 \ge \hat{c}_1^*$. Then simple calculations show that $\psi_1(x + c_1 t)$ and $\hat{\psi}_1(x + \hat{c}_1 t)$ are a pair of nondecreasing and nonincreasing traveling wave solutions of (1.1) with speeds $c_1 \le c_1^*, \hat{c}_1 \ge \hat{c}_1^*$ and satisfying

$$\begin{cases} (J * \psi_1)(x + c_1 t) - \psi_1(x + c_1 t) - c_1\psi_1'(x + c_1 t) + f(\psi_1(x + c_1 t)) = 0, \\ \psi_1(-\infty) = 0, \quad \psi_1(+\infty) = \rho. \end{cases} \tag{1.6}$$

$$\begin{cases} (J * \hat{\psi}_1)(x + \hat{c}_1 t) - \hat{\psi}_1(x + \hat{c}_1 t) - \hat{c}_1\hat{\psi}_1'(x + \hat{c}_1 t) + f(\hat{\psi}_1(x + \hat{c}_1 t)) = 0, \\ \hat{\psi}_1(-\infty) = \rho, \quad \hat{\psi}_1(+\infty) = 0. \end{cases} \tag{1.7}$$

Similarly, on the interval $[\rho, 1]$, by letting $v(x, t) = u(x, t) - \rho$, we obtain that there exist $c_2^* \in \mathbb{R}$ and $\hat{c}_2^* \in \mathbb{R}$ such that for each $c_2 \ge c_2^*$ and $\hat{c}_2 \le \hat{c}_2^*$, (1.1) admits a pair of nondecreasing and nonincreasing traveling wave solutions $\psi_2(x + c_2 t)$ and $\hat{\psi}_2(x + \hat{c}_2 t)$ satisfying $\psi_2(-\infty) = \rho, \psi_2(+\infty) = 1$ and $\hat{\psi}_2(-\infty) = 1, \hat{\psi}_2(+\infty) = \rho$. In analogy to bistable waves, the critical speed c_i^* $(i = 1, 2)$ is not in general $-\hat{c}_i^*$. In other words, the corresponding profiles are not symmetric.

Now we are ready to state our main results. In following Theorems 1.2–1.5, we choose these traveling waves with speeds $c, \hat{c}, c_i, \hat{c}_i$ $(i = 1, 2)$ are not equal to zero, so we first explain that such choices are desirable. We leave the case that at least one of these speeds is equal to zero as a further study.

For monostable wave speeds c_i and $\hat{c}_i$, $i = 1, 2$, since their range are semi-infinity regions, such as $c_1 \le c_1^*$, $\hat{c}_1 \ge \hat{c}_1^*$ and $c_2 \ge c_2^*$, $\hat{c}_2 \le \hat{c}_2^*$, we only need to select those monostable waves with non-zero speeds. For bistable wave speeds c and $\hat{c}$, we will give some easily satisfied restrictive conditions on J and f, to ensure $c \ne 0$ and $\hat{c} \ne 0$.

Next we will constantly use the translation transformation of the convolution operator, precisely, for any $z \in \mathbb{R}^n$, there is $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$, where τ_z is a translation operator defined by $\tau_z f(x) = f(x - z)$. This property can be found in many books of real analysis, such as [11, Chapter 8].

Multiply the first equation of (1.2) and (1.4) by ϕ' and $\hat{\phi}'$ respectively, and then integrate in $\mathbb{R}$, one has

$$c\int_{\mathbb{R}}(\phi'(\xi))^2 d\xi = \int_{\mathbb{R}}\phi'(\xi)\int_{\mathbb{R}}J(y)[\phi(\xi - y) - \phi(\xi)]dyd\xi + \int_0^1 f(u)du, \tag{1.8}$$

and

$$\hat{c}\int_{\mathbb{R}}(\hat{\phi}'(\xi))^2 d\xi = \int_{\mathbb{R}}\hat{\phi}'(\xi)\int_{\mathbb{R}}J(y)[\hat{\phi}(\xi - y) - \hat{\phi}(\xi)]dyd\xi - \int_0^1 f(u)du. \tag{1.9}$$

If J is symmetric, we can prove

$$\int_{\mathbb{R}}\phi'(\xi)\int_{\mathbb{R}}J(y)[\phi(\xi - y) - \phi(\xi)]dyd\xi = 0,$$

and

$$\int_{\mathbb{R}}\hat{\phi}'(\xi)\int_{\mathbb{R}}J(y)[\hat{\phi}(\xi - y) - \hat{\phi}(\xi)]dyd\xi = 0,$$

then

$$c = \frac{\int_0^1 f(u)du}{\int_{\mathbb{R}}(\phi'(\xi))^2 d\xi}, \quad \hat{c} = -\frac{\int_0^1 f(u)du}{\int_{\mathbb{R}}(\hat{\phi}'(\xi))^2 d\xi}, \tag{1.10}$$

which yields the sign of c and $\hat{c}$ is opposite and only depends on the sign of the integral $\int_0^1 f(u)du$. Thus if J is symmetric, as long as $\int_0^1 f(u)du \neq 0$, we obtain the results we desired, namely $c \neq 0$ and $\hat{c} \neq 0$. However, if J is asymmetric, the sign of c and $\hat{c}$ not only depends on the integral $\int_0^1 f(u)du$, but also depends on the properties of the kernel function J.

From [6] and [7], we know $\phi' > 0$, $\hat{\phi}' < 0$ on $\mathbb{R}$ and $\phi'(\xi) \to 0$ and $\hat{\phi}'(\xi) \to 0$ as $\xi \to \pm\infty$. By using the translation transformation of the convolution operator, we compute (1.8) as follows,

$$\begin{aligned}
c\int_{\mathbb{R}}(\phi'(\xi))^2 d\xi &= \int_{-\infty}^{0} J(y)\int_{\mathbb{R}}\phi'(\xi)[\phi(\xi - y) - \phi(\xi)]d\xi dy \\
&\quad + \int_0^{+\infty} J(y)\int_{\mathbb{R}}\phi'(\xi)[\phi(\xi - y) - \phi(\xi)]d\xi dy + \int_0^1 f(u)du \\
&= \int_0^{+\infty} J(-y)\int_{\mathbb{R}}\phi'(\xi)[\phi(\xi + y) - \phi(\xi)]d\xi dy \\
&\quad - \int_0^{+\infty} J(y)\int_{\mathbb{R}}\phi(\xi)[\phi'(\xi - y) - \phi'(\xi)]d\xi dy + \int_0^1 f(u)du \\
&= \int_0^{+\infty} J(-y)y\int_0^1\int_{\mathbb{R}}\phi'(\xi)\phi'(\xi + \theta y)d\xi d\theta dy \\
&\quad + \int_0^{+\infty} J(y)y\int_0^1\int_{\mathbb{R}}\phi(\xi)\phi''(\xi - \theta y)d\xi d\theta dy + \int_0^1 f(u)du \\
&= \int_0^{+\infty} J(-y)y\int_0^1\int_{\mathbb{R}}\phi'(\xi - \theta y)\phi'(\xi)d\xi d\theta dy \\
&\quad - \int_0^{+\infty} J(y)y\int_0^1\int_{\mathbb{R}}\phi'(\xi - \theta y)\phi'(\xi)d\xi d\theta dy + \int_0^1 f(u)du
\end{aligned}$$

$$= \int_0^{+\infty} [J(-y) - J(y)]y \int_0^1 \int_{\mathbb{R}} \phi'(\xi - \theta y)\phi'(\xi) d\xi d\theta dy + \int_0^1 f(u)du. \tag{1.11}$$

Similarly,

$$\hat{c} \int_{\mathbb{R}} (\hat{\phi}'(\xi))^2 d\xi = \int_0^{+\infty} [J(-y) - J(y)]y \int_0^1 \int_{\mathbb{R}} \hat{\phi}'(\xi - \theta y)\hat{\phi}'(\xi) d\xi d\theta dy - \int_0^1 f(u)du. \tag{1.12}$$

(1.11) and (1.12) imply that if $\int_0^1 f(u)du = 0$ and $J(-y) > J(y)$ for $y \in (0, +\infty)$, then $c \neq 0$ and $\hat{c} \neq 0$, more precisely, such J and f make $c > 0$ and $\hat{c} > 0$. Now we give a special example of J and f.

Example 1.1 Let $f(u) = u(1-u)(u - \frac{1}{2})$ on $u \in [0, 1]$, and

$$J(y) = \begin{cases} \dfrac{a}{a^2+1} \mathrm{e}^{-ay}, & y \geq 0, \\ \dfrac{a}{a^2+1} \mathrm{e}^{\frac{1}{a}y}, & y < 0, \end{cases}$$

where $a > 1$. By simple calculations, we can show that J and f given above satisfy $\int_0^1 f(u)du = 0$ and $J(-y) > J(y)$ for $y \in (0, +\infty)$ besides (J1) and (FB). Actually, as long as J and f make the right side of (1.11) and (1.12) not equal to zero, the wave speeds c and $\hat{c}$ will not equal to zero.

Theorem 1.2 *Assume that f satisfies* (FB) *and J satisfies* (J1) (J2). *Let $\psi_i(x + c_i t)$ and $\hat{\psi}_i(x + \hat{c}_i t)$ $(i = 1, 2)$ be the monostable waves of* (1.1) *which are described as the foregoing with $c_1 \leq c_1^*$, $\hat{c}_1 \geq \hat{c}_1^*$ and $c_2 \geq c_2^*$, $\hat{c}_2 \leq \hat{c}_2^*$, respectively. Then* (1.1) *has two entire solutions $U_1(x,t), U_2(x,t) : \mathbb{R}^2 \to [0,1]$ satisfying $\frac{\partial}{\partial x} U_1(x,t) > 0$, $\frac{\partial}{\partial x} U_2(x,t) < 0$,*

$$\lim_{t \to -\infty} \Bigg\{ \sup_{x \leq -\frac{c_1+c_2}{2}t} |U_1(x,t) - \psi_1(x + c_1 t - \omega_1)| + \sup_{x \geq -\frac{c_1+c_2}{2}t} |U_1(x,t) - \psi_2(x + c_2 t + \omega_1)| \Bigg\} = 0, \tag{1.13}$$

and

$$\lim_{t \to -\infty} \Bigg\{ \sup_{x \leq -\frac{\hat{c}_1+\hat{c}_2}{2}t} |U_2(x,t) - \hat{\psi}_2(x + \hat{c}_2 t - \omega_2)| + \sup_{x \geq -\frac{\hat{c}_1+\hat{c}_2}{2}t} |U_2(x,t) - \hat{\psi}_1(x + \hat{c}_1 t + \omega_2)| \Bigg\} = 0, \tag{1.14}$$

for some constants ω_1 and ω_2. Furthermore, according to the size of the critical speeds c_1^*, c_2^*, $\hat{c}_1^*$ and $\hat{c}_2^*$ (see Lemma 2.5), the following asymptotics hold.

(i) *When $c_2 > c_1 > 0$, the entire solution $U_1(x,t)$ behaves as two monostable waves $\psi_2(x + c_2 t)$ and $\psi_1(x + c_1 t)$ propagating in the same direction from right to left of x-axis as $t \to -\infty$, and ψ_2 will catch up ψ_1 finally.*

(ii) *When $0 > c_2 > c_1$, $U_1(x,t)$ behaves as two monostable waves ψ_1 and ψ_2 propagating in the same direction from left to right of x-axis as $t \to -\infty$, and ψ_1 will catch up ψ_2 finally.*

(iii) *When $c_2 > 0 > c_1$, $U_1(x,t)$ behaves as two monostable waves ψ_1 and ψ_2 coming from the opposite sides of x-axis as $t \to -\infty$, and ψ_1 and ψ_2 may emerge finally.*

(iv) *The similar corresponding properties hold for $U_2(x,t)$.*

Theorem 1.3 *Let all assumptions of Theorem* 1.2 *be satisfied and* $\phi(x+ct), \hat{\phi}(x+\hat{c}t)$ *be the bistable waves of* (1.1) *which satisfy* (1.2) *and* (1.4), *respectively. If* $c > \hat{c}_1$ *and* $\hat{c} > c_2$, *then for some constants* ω_3 *and* ω_4, *there exist two entire solutions* $V_1(x,t), V_2(x,t) : \mathbb{R}^2 \to [0,1]$ *of* (1.1) *satisfying*

$$\lim_{t\to-\infty}\Big\{\sup_{x\le-\frac{c+\hat{c}_1}{2}t}|V_1(x,t)-\hat{\psi}_1(x+\hat{c}_1t-\omega_3)| + \sup_{x\ge-\frac{c+\hat{c}_1}{2}t}|V_1(x,t)-\phi(x+ct+\omega_3)|\Big\}=0, \tag{1.15}$$

and

$$\lim_{t\to-\infty}\Big\{\sup_{x\le-\frac{\hat{c}+c_2}{2}t}|V_2(x,t)-\psi_2(x+c_2t-\omega_4)| + \sup_{x\ge-\frac{\hat{c}+c_2}{2}t}|V_2(x,t)-\hat{\phi}(x+\hat{c}t+\omega_4)|\Big\}=0. \tag{1.16}$$

In addition, $V_1(x,t)$ *and* $V_2(x,t)$ *have analogous properties as* (i)–(iii) *of Theorem* 1.2 *for* U_1 *and* U_2 *depending on the size of wave speeds.*

Theorem 1.4 *Assume* (J1)–(J2) *and* (FB) *hold. Let* $\phi(x+ct)$ *and* $\hat{\phi}(x+\hat{c}t)$ *be the bistable waves of* (1.1). *Suppose* $c \neq \hat{c}$, *then for some constants* ω_5 *and* ω_6, (1.1) *possesses an entire solution* $W(x,t) : \mathbb{R}^2 \to [0,1]$ *such that the following statements hold.*

(i) *When* $c > \hat{c}$,

$$\lim_{t\to-\infty}\Big\{\sup_{x\le-\frac{c+\hat{c}}{2}t}|W(x,t)-\hat{\phi}(x+\hat{c}t-\omega_5)| + \sup_{x\ge-\frac{c+\hat{c}}{2}t}|W(x,t)-\phi(x+ct+\omega_5)|\Big\}=0. \tag{1.17}$$

Moreover, $W(x,t)$ *has the following properties*:

(a) *If* $c > 0 > \hat{c}$, *then* $\frac{\partial W}{\partial t} > 0$; *However, if* $c > \hat{c} > 0$ *or* $0 > c > \hat{c}$, $W(x,t)$ *has no monotonicity with respect to* t;

(b) $\lim_{t\to-\infty} W(x,t) = 0$ *locally in* $x \in \mathbb{R}$;

(c) $\lim_{t\to+\infty} |W(x,t)-1| = 0$ *for* $(x,t) \in \mathbb{R}^2$.

(ii) *When* $\hat{c} > c$,

$$\lim_{t\to-\infty}\Big\{\sup_{x\le-\frac{c+\hat{c}}{2}t}|W(x,t)-\phi(x+ct-\omega_6)| + \sup_{x\ge-\frac{c+\hat{c}}{2}t}|W(x,t)-\hat{\phi}(x+\hat{c}t+\omega_6)|\Big\}=0. \tag{1.18}$$

Correspondingly,

(a) *If* $\hat{c} > 0 > c$, *then* $\frac{\partial W}{\partial t} < 0$; *if* $\hat{c} > c > 0$ *or* $0 > \hat{c} > c$, $W(x,t)$ *has no monotonicity with respect to* t;

(b) $\lim_{t\to-\infty} W(x,t) = 1$ *locally in* $x \in \mathbb{R}$;

(c) $\lim_{t\to+\infty} W(x,t) = 0$ *for* $(x,t) \in \mathbb{R}^2$.

Theorem 1.5 *Let* $U_i(x,t)$, $V_i(x,t)$ $(i=1,2)$ *and* $W(x,t)$ *be the entire solutions of* (1.1) *that are established in Theorems* 1.2–1.4. *In order to simplify the statement, we denote all of them by* $u(x,t)$. *Then there exist some positive constants* L_1 *and* L_2 *such that*

$$|u(x+\eta,t)-u(x,t)| \le L_1\eta, \qquad \left|\frac{\partial u(x+\eta,t)}{\partial t} - \frac{\partial u(x,t)}{\partial t}\right| \le L_2\eta$$

for any $(x,t) \in \mathbb{R}^2$ *and* $\eta \in (0,+\infty)$.

Remark 1.6 Combining Lemma 2.5 with Theorems 1.2–1.4, we find out that the asymmetry of the kernel function has great influence on the sign, range and size of the wave speeds, which further makes the types and properties of entire solutions more diverse.

2 Preliminaries

In the first part of this section, we give some valuable lemmas refer to the existence and a priori estimate of solutions of the Cauchy problem of (1.1) since our main results are proved by studying the solving sequence of Cauchy problem with suitable initial value. Afterwards, we consider the sign and size of the speeds of monostable and bistable waves because they play an important role in constructing different types of entire solutions.

2.1 Cauchy Problem

Definition 2.1 *Let $\tau < T$ be any two real constants. A function $\overline{U}(x,t)$ is called a supersolution of* (1.1) *on $(x,t) \in \mathbb{R} \times [\tau, T)$, if $\overline{U}(x,t) \in C^{0,1}(\mathbb{R} \times [\tau, T), \mathbb{R})$ and satisfies*

$$\overline{U}_t(x,t) \geq (J * \overline{U})(x,t) - \overline{U}(x,t) + f(\overline{U}(x,t)), \quad \forall (x,t) \in \mathbb{R} \times [\tau, T). \tag{2.1}$$

Furthermore, if for any $\tau < T$, $\overline{U}(x,t)$ is a supersolution of (1.1) *on $(x,t) \in \mathbb{R} \times [\tau, T)$, then $\overline{U}(x,t)$ is called a supersolution of* (1.1) *on $(x,t) \in \mathbb{R} \times (-\infty, T)$. Similarly, a subsolution $\underline{U}(x,t)$ can be defined by replacing $\overline{U}(x,t)$ by $\underline{U}(x,t)$ and reversing the inequality* (2.1).

Lemma 2.2 *Assume* (J1) *and* (FB) *hold. Then the following statements hold for the Cauchy problem of* (1.1):

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = (J * u)(x,t) - u(x,t) + f(u(x,t)), & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

(i) *For any $u_0(x) \in C(\mathbb{R}, [0,1])$,* (2.2) *admits a unique solution $u(x,t;u_0) \in C(\mathbb{R} \times [0,\infty), [0,1])$.*

(ii) *Suppose that $\overline{U}(x,t)$ and $\underline{U}(x,t)$ are a pair of supersolution and subsolution of* (1.1) *on $\mathbb{R} \times [0, +\infty)$ with $\underline{U}(x,0) \leq \overline{U}(x,0)$ and $0 \leq \underline{U}(x,t), \overline{U}(x,t) \leq 1$ for $(x,t) \in \mathbb{R} \times [0,+\infty)$, then $0 \leq \underline{U}(x,t) \leq \overline{U}(x,t) \leq 1$ for all $(x,t) \in \mathbb{R} \times [0,+\infty)$.*

Lemma 2.3 *Assume* (J1)–(J2) *and* (FB) *hold. Let $u(x,t)$ be a solution of* (2.2) *with $u_0(x) \in [0,1]$, then there exists a constant $M_1 > 0$ independent of x, t and $u_0(x)$, such that*

$$|u_t(x,t)|, \ |u_{tt}(x,t)| \leq M_1 \quad \text{for any } x \in \mathbb{R}, \quad t > 0.$$

In addition, if there exists a positive constant L_0 such that

$$|u_0(x+\eta) - u_0(x)| \leq L_0 \eta, \tag{2.3}$$

then for any $x \in \mathbb{R}$, $t > 0$ and $\eta > 0$, we have

$$|u(x+\eta,t) - u(x,t)| \leq M_2 \eta, \quad \left| \frac{\partial u}{\partial t}(x+\eta,t) - \frac{\partial u}{\partial t}(x,t) \right| \leq M_2 \eta, \tag{2.4}$$

where $M_2 > 0$ is some constant independent of u_0 and η.

Remark 2.4 The proofs of Lemmas 2.2 and 2.3 can be found in many references, such as [15, 27] and the references therein. Here what must be illustrated is that in order to prove

Lemma 2.2, we need some proper maximum principles and comparison principles for asymmetric kernel function established by Coville [6], that is why we assume $\exists\ a \leq 0 \leq b, a \neq b$ such that $J(a) > 0$ and $J(b) > 0$ in assumption (J1).

2.2 Classification of the Traveling Wave Speeds

In this subsection, we mainly analyze the sign and size of the critical speeds of monostable waves. Recalling [8, 26], by simple calculations and transformations, we get

$$c_1^* = \hat{c}_2^* := \sup_{\lambda>0} \frac{1}{\lambda}\Big(- \int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda x} dx + 1 - f'(\rho)\Big), \tag{2.5}$$

$$\hat{c}_1^* = c_2^* := \inf_{\lambda>0} \frac{1}{\lambda}\Big(\int_{\mathbb{R}} J(-x)\mathrm{e}^{\lambda x} dx - 1 + f'(\rho)\Big). \tag{2.6}$$

Let $h(\lambda) = \frac{1}{\lambda}(- \int_{\mathbb{R}} J(x)\mathrm{e}^{\lambda x} dx + 1 - f'(\rho))$. Since $J(x) \not\equiv 0$ in $\mathbb{R}^+$, we have

$$h(\lambda) \to -\infty \quad \text{when } \lambda \to -\infty,$$
$$h(\lambda) \to -\infty \quad \text{when } \lambda \to 0.$$

Note that $h(\lambda)$ is continuous, thus there exists $\lambda_1^* > 0$ such that $c_1^* = \hat{c}_2^* = \max_{\lambda>0} h(\lambda) = h(\lambda_1^*)$ and $h'(\lambda_1^*) = 0$. Note

$$h'(\lambda) = \frac{-\lambda \int_{\mathbb{R}} xJ(x)\mathrm{e}^{\lambda x}dx - \lambda h(\lambda)}{\lambda^2},$$

we have $h(\lambda_1^*) = - \int_R xJ(x)\mathrm{e}^{\lambda_1^* x}dx$ which implies

$$\begin{aligned} c_1^* = \hat{c}_2^* = h(\lambda_1^*) &= - \int_{-\infty}^{0} xJ(x)\mathrm{e}^{\lambda_1^* x}dx - \int_0^{+\infty} xJ(x)\mathrm{e}^{\lambda_1^* x}dx \\ &< - \int_{-\infty}^{0} xJ(x)dx - \int_0^{+\infty} xJ(x)dx \\ &= \int_0^{+\infty} x[J(-x) - J(x)]dx, \end{aligned} \tag{2.7}$$

then if $J(-x) \leq J(x)$ and $J(-x) \not\equiv J(x)$ in $\mathbb{R}^+$, $c_1^* = \hat{c}_2^* < 0$. If $J(-x) = J(x)$ in $\mathbb{R}^+$, $c_1^* = \hat{c}_2^* < 0$, which is consistent with the case that J is symmetric. If $J(-x) \geq J(x)$ and $J(-x) \not\equiv J(x)$ in $\mathbb{R}^+$, the sign of c_1^* and $\hat{c}_2^*$ should be further study.

Similarly, let $g(\lambda) = \frac{1}{\lambda}(\int_{\mathbb{R}} J(-x)\mathrm{e}^{\lambda x}dx - 1 + f'(\rho))$. Since $J(x) \not\equiv 0$ in $\mathbb{R}^-$, we get $\hat{c}_1^* = c_2^* = \min_{\lambda>0} g(\lambda) = g(\lambda_2^*)$ for some $\lambda_2^* > 0$, and

$$\begin{aligned} \hat{c}_1^* = c_2^* = g(\lambda_2^*) &= \int_{-\infty}^{0} xJ(-x)\mathrm{e}^{\lambda_2^* x}dx + \int_0^{+\infty} xJ(-x)\mathrm{e}^{\lambda_2^* x}dx \\ &> \int_0^{+\infty} x[J(-x) - J(x)]dx \\ &= \int_{\mathbb{R}} xJ(-x)dx, \end{aligned} \tag{2.8}$$

which implies that if $J(-x) \geq J(x)$ and $J(-x) \not\equiv J(x)$ in $\mathbb{R}^+$, $\hat{c}_1^* = c_2^* > 0$. If $J(-x) \equiv J(x)$ in $\mathbb{R}^+$, $\hat{c}_1^* = c_2^* > 0$, which is consistent with the case that J is symmetric. If $J(-x) \leq J(x)$ and $J(-x) \not\equiv J(x)$ in $\mathbb{R}^+$, the sign of $\hat{c}_1^*$ and c_2^* should be further study.

Moreover, (2.7) and (2.8) show that

$$\hat{c}_1^* - c_1^* > \int_0^{+\infty} x[J(-x) - J(x)]dx - \int_0^{+\infty} x[J(-x) - J(x)]dx = 0,$$

So we obtain that

$$\hat{c}_1^* = c_2^* > c_1^* = \hat{c}_2^*.$$

Actually, take $J(x) = \frac{1}{\varepsilon}P(\frac{x}{\varepsilon})$ with $\varepsilon > 0$, where $P(x)$ is a general mollification function with support $x \in [-1, 1]$. Then for any smooth function $u(x)$, by Taylor's formula, we have the following approximation

$$(J * u)(x) - u(x) = \varepsilon^2 \alpha u''(x) + \varepsilon \beta u'(x) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \tag{2.9}$$

where $\alpha = \frac{1}{2}\int_{\mathbb{R}} P(-z)z^2 dz$, $\beta = \int_{\mathbb{R}} P(-z)zdz$. When J is symmetric, it is obvious that $\beta = 0$, for general asymmetric kernel J, the monostable wave problem of (1.1) in the interval $[\rho, 1]$ is approximate to the following problem

$$\tilde{\alpha} u'' - (c - \tilde{\beta})u' + f(u) = 0, \quad u(-\infty) = \rho, \quad u(+\infty) = 1$$

for some $\tilde{\alpha} \geq 0$ and $\tilde{\beta} \in \mathbb{R}$, thus the minimal speed is $c^* + \tilde{\beta}$, where $c^* > 0$ and $\tilde{\beta}$ depends on $\int_{\mathbb{R}} J(-z)zdz$ which is related to the asymmetry of J.

Based on the above discussion, we classify the sign and size of the critical speeds as follows.

Lemma 2.5 *As for the size of critical speeds $c_i^* \in \mathbb{R}$ and $\hat{c}_i^* \in \mathbb{R}$, there are only four possibilities*:

(i) $\hat{c}_1^* = c_2^* > c_1^* = \hat{c}_2^* \geq 0$;

(ii) $\hat{c}_1^* = c_2^* > 0 \geq c_1^* = \hat{c}_2^*$;

(iii) $\hat{c}_1^* = c_2^* \geq 0 > c_1^* = \hat{c}_2^*$;

(iv) $0 \geq \hat{c}_1^* = c_2^* > c_1^* = \hat{c}_2^*$.

3 Existence and Properties of Entire Solutions

3.1 Asymptotic Behaviors of Traveling Wave Solutions

Since we will frequent use the asymptotic behaviors and estimates of the monostable and bistable waves to prove the existence and properties of the entire solutions, we present the results as follows. The proofs are similar to those in [7, 8, 26], so we omit them.

For $i = 1, 2$, let $\psi_i(x + c_i t)$ and $\hat{\psi}_i(x + \hat{c}_i t)$ be the monostable waves stated in Section 1. Define characteristic functions as follows:

$$\begin{aligned}
\Gamma_i(\lambda) &= \int_{\mathbb{R}} J((-1)^{i-1}z)\mathrm{e}^{\lambda z}dz - 1 + (-1)^{i-1}c_i\lambda + f'(\alpha_i), \\
\Gamma_i(\mu) &= \int_{\mathbb{R}} J((-1)^{i-1}z)\mathrm{e}^{\mu z}dz - 1 + (-1)^{i-1}c_i\mu + f'(\beta_i), \\
\widehat{\Gamma}_i(\hat{\lambda}) &= \int_{\mathbb{R}} J((-1)^{i}z)\mathrm{e}^{\hat{\lambda} z}dz - 1 + (-1)^{i}\hat{c}_i\hat{\lambda} + f'(\alpha_i), \\
\widehat{\Gamma}_i(\hat{\mu}) &= \int_{\mathbb{R}} J((-1)^{i}z)\mathrm{e}^{\hat{\mu} z}dz - 1 + (-1)^{i}\hat{c}_i\hat{\mu} + f'(\beta_i),
\end{aligned}$$

where $\lambda, \mu, \hat{\lambda}, \hat{\mu} \in \mathbb{C}$, $(\alpha_1, \beta_1) = (0, \rho), (\alpha_2, \beta_2) = (\rho, 1)$ and $i = 1, 2$. Then the following lemma holds.

Lemma 3.1 *Assume that J satisfies* (J1) *and f satisfies* (FB). *Then $\psi_i(x+c_i t)$ and $\hat{\psi}_i(x+\hat{c}_i t)$ satisfying*:

(i) *For* $c_1 < c_1^*$ *and* $c_2 > c_2^*$,

$$\lim_{z\to-\infty}(\psi_i(z)-\alpha_i)\mathrm{e}^{-\lambda_i z}=A_i,\quad \lim_{z\to-\infty}\psi_i'(z)\mathrm{e}^{-\lambda_i z}=A_i\lambda_i,$$

For $c_1 = c_1^*$ *and* $c_2 = c_2^*$,

$$\lim_{z\to-\infty}(\psi_i(z)-\alpha_i)|z|^{-1}\mathrm{e}^{-\lambda_i z}=A_i,\quad \lim_{z\to-\infty}\psi_i'(z)|z|^{-1}\mathrm{e}^{-\lambda_i z}=A_i\lambda_i.$$

(ii) *For* $c_1 \le c_1^*$ *and* $c_2 \ge c_2^*$,

$$\lim_{z\to+\infty}(\beta_i-\psi_i(z))\mathrm{e}^{-\mu_i z}=B_i,\quad \lim_{z\to+\infty}\psi_i'(z)\mathrm{e}^{-\mu_i z}=-B_i\mu_i.$$

(iii) *For* $\hat{c}_1 > \hat{c}_1^*$ *and* $\hat{c}_2 < \hat{c}_2^*$,

$$\lim_{z\to+\infty}(\hat{\psi}_i(z)-\alpha_i)\mathrm{e}^{-\hat{\lambda}_i z}=\widehat{A}_i,\quad \lim_{z\to+\infty}\hat{\psi}_i'(z)\mathrm{e}^{-\hat{\lambda}_i z}=\widehat{A}_i\hat{\lambda}_i.$$

For $\hat{c}_1 = \hat{c}_1^*$ *and* $\hat{c}_2 = \hat{c}_2^*$,

$$\lim_{z\to+\infty}(\hat{\psi}_i(z)-\alpha_i)|z|^{-1}\mathrm{e}^{-\hat{\lambda}_i z}=\widehat{A}_i,\quad \lim_{z\to+\infty}\hat{\psi}_i'(z)|z|^{-1}\mathrm{e}^{-\hat{\lambda}_i z}=\widehat{A}_i\hat{\lambda}_i.$$

(iv) *For* $\hat{c}_1 \ge \hat{c}_1^*$ *and* $\hat{c}_2 \le \hat{c}_2^*$,

$$\lim_{z\to-\infty}(\beta_i-\hat{\psi}_i(z))\mathrm{e}^{-\hat{\mu}_i z}=\widehat{B}_i,\quad \lim_{z\to-\infty}\hat{\psi}_i'(z)\mathrm{e}^{-\hat{\mu}_i z}=-\widehat{B}_i\hat{\mu}_i.$$

Here $i = 1, 2$, $-\lambda_1 < 0$ *and* $\lambda_2 > 0$ *are the largest negative root of* $\Gamma_1(\lambda)$ *and the smallest positive root of* $\Gamma_2(\lambda)$, *respectively.* $-\mu_1 > 0$ *and* $\mu_2 < 0$ *are the smallest positive root of* $\Gamma_1(\mu)$ *and the largest negative root of* $\Gamma_2(\mu)$, *respectively.* $\hat{\lambda}_1 < 0$ *and* $-\hat{\lambda}_2 > 0$ *are the largest negative root of* $\widehat{\Gamma}_1(\hat{\lambda})$ *and the smallest positive root of* $\widehat{\Gamma}_2(\hat{\lambda})$, *respectively.* $\hat{\mu}_1 > 0$ *and* $-\hat{\mu}_2 < 0$ *are the smallest positive root of* $\widehat{\Gamma}_1(\hat{\mu})$ *and the largest negative root of* $\widehat{\Gamma}_2(\hat{\mu})$, *respectively.* $A_i, B_i, \widehat{A}_i$ *and* $\widehat{B}_i$ *are some positive constants.*

For bistable waves $\phi(x + ct)$ and $\hat{\phi}(x + \hat{c}t)$ of (1.1), define four complex functions

$$\begin{aligned}
\Delta_1(\lambda) &= \int_{\mathbb{R}} J(-z)\mathrm{e}^{\lambda z}dz - 1 - c\lambda + f'(0),\\
\Delta_2(\lambda) &= \int_{\mathbb{R}} J(-z)\mathrm{e}^{\lambda z}dz - 1 - c\lambda + f'(1),\\
\widehat{\Delta}_1(\mu) &= \int_{\mathbb{R}} J(z)\mathrm{e}^{\mu z}dz - 1 + \hat{c}\mu + f'(0),\\
\widehat{\Delta}_2(\mu) &= \int_{\mathbb{R}} J(z)\mathrm{e}^{\mu z}dz - 1 + \hat{c}\mu + f'(1),
\end{aligned}$$

where $\lambda, \mu \in \mathbb{C}$. Simple computations show that $\Delta_1(0) = \widehat{\Delta}_1(0) = f'(0) < 0$, $\Delta_2(0) = \widehat{\Delta}_2(0) = f'(1) < 0$, and

$$\begin{aligned}
\frac{\partial}{\partial\lambda^2}\Delta_i(\lambda) &= \int_{\mathbb{R}} J(-z)z^2\mathrm{e}^{\lambda z}dz > 0,\\
\frac{\partial}{\partial\mu^2}\widehat{\Delta}_i(\mu) &= \int_{\mathbb{R}} J(z)z^2\mathrm{e}^{\mu z}dz > 0,
\end{aligned}$$

with $i = 1, 2$. Moreover, the assumption (J1) yields that $J(z) \not\equiv 0$ in $\mathbb{R}^-$, then

$$\Delta_1(\lambda) = \int_{-\infty}^{0} J(-z)\mathrm{e}^{\lambda z}dz + \int_{0}^{+\infty} J(-z)\mathrm{e}^{\lambda z}dz - 1 - c\lambda + f'(0) \to +\infty \quad \text{as } \lambda \to +\infty.$$

Similarly,

$$\begin{aligned} &\Delta_1(\lambda) \to +\infty \quad \text{as } \lambda \to -\infty, \quad \Delta_2(\lambda) \to +\infty \quad \text{as } \lambda \to \pm\infty, \\ &\widehat{\Delta}_1(\mu) \to +\infty \quad \text{as } \mu \to \pm\infty, \quad \widehat{\Delta}_2(\mu) \to +\infty \quad \text{as } \mu \to \pm\infty. \end{aligned}$$

Then we obtain the following lemmas.

Lemma 3.2 *Assume J and f satisfy* (J1) *and* (FB) *respectively, then the equation $\Delta_i(\lambda) = 0$ ($\widehat{\Delta}_i(\mu) = 0$) ($i = 1, 2$) has two real roots $\lambda_{i1} < 0$ ($\mu_{i1} < 0$) and $\lambda_{i2} > 0$ ($\mu_{i2} > 0$) such that*

$$\Delta_i(\lambda)\ (\widehat{\Delta}_i(\mu)) \begin{cases} > 0 & \text{for } \lambda < \lambda_{i1}\ (\mu < \mu_{i1}), \\ < 0 & \text{for } \lambda \in (\lambda_{i1}, \lambda_{i2})\ (\mu \in (\mu_{i1}, \mu_{i2})), \\ > 0 & \text{for } \lambda > \lambda_{i2}\ (\mu > \mu_{i2}). \end{cases} \tag{3.1}$$

Lemma 3.3 *Assume J satisfies* (J1) *and f satisfies* (FB). *Let $\phi(x + ct)$ and $\hat{\phi}(x + \hat{c}t)$ be the nondecreasing and nonincreasing solutions of* (1.2) *and* (1.4)*, respectively. Then there exist some positive constants $A, B, \widehat{A}$ and $\widehat{B}$ such that*

(i) $\lim_{\xi\to-\infty} \phi(\xi)\mathrm{e}^{-\lambda_{12}\xi} = A$, $\lim_{\xi\to-\infty} \phi'(\xi)\mathrm{e}^{-\lambda_{12}\xi} = A\lambda_{12}$;
$\lim_{\xi\to+\infty}(1 - \phi(\xi))\mathrm{e}^{-\lambda_{21}\xi} = B$, $\lim_{\xi\to+\infty} \phi'(\xi)\mathrm{e}^{-\lambda_{21}\xi} = -B\lambda_{21}$.

(ii) $\lim_{\xi\to+\infty} \hat{\phi}(\xi)\mathrm{e}^{-\mu_{11}\xi} = \widehat{A}$, $\lim_{\xi\to+\infty} \hat{\phi}'(\xi)\mathrm{e}^{-\mu_{11}\xi} = \widehat{A}\mu_{11}$;
$\lim_{\xi\to-\infty}(1 - \hat{\phi}(\xi))\mathrm{e}^{-\mu_{22}\xi} = \widehat{B}$, $\lim_{\xi\to-\infty} \hat{\phi}'(\xi)\mathrm{e}^{-\mu_{22}\xi} = -\widehat{B}\mu_{22}$.

Then the following estimate lemma can be derived from Lemmas 3.1 and 3.3.

Lemma 3.4 *Assume J satisfies* (J1) *and* (J2)*, then there exist some positive constants $C_0, C_1, C_2, \tau_1, \tau_2$ and δ which depend on $c_1, c_2, \hat{c}_1, \hat{c}_2, c, \hat{c}$, such that*

(i) *For $\xi \le a_0$,*

$$|\psi_i'(\xi)|,\ |\hat{\psi}_i'(\xi)|,\ |\phi'(\xi)|,\ |\hat{\phi}'(\xi)| \le C_0\mathrm{e}^{\tau_1\xi}, \tag{3.2}$$

$$C_1\mathrm{e}^{\tau_1\xi} \le |\phi(\xi)|, \quad |1 - \hat{\phi}(\xi)|,\ |\psi_i(\xi) - \alpha_i|,\ |\beta_i - \hat{\psi}_i(\xi)| \le C_2\mathrm{e}^{\tau_1\xi}, \tag{3.3}$$

$$\frac{|\phi'(\xi)|}{|\phi(\xi)|},\ \frac{|\hat{\phi}'(\xi)|}{|1 - \hat{\phi}(\xi)|},\ \frac{|\psi_i'(\xi)|}{|\psi_i(\xi) - \alpha_i|},\ \frac{|\hat{\psi}_i'(\xi)|}{|\beta_i - \hat{\psi}_i(\xi)|} \ge \delta. \tag{3.4}$$

(ii) *For $\xi \ge -a_0$,*

$$|\psi_i'(\xi)|,\ |\hat{\psi}_i'(\xi)|,\ |\phi'(\xi)|,\ |\hat{\phi}'(\xi)| \le C_0\mathrm{e}^{-\tau_2\xi}, \tag{3.5}$$

$$C_1\mathrm{e}^{-\tau_2\xi} \le |1 - \phi(\xi)|, \quad |\hat{\phi}(\xi)|,\ |\beta_i - \psi_i(\xi)|,\ |\hat{\psi}_i(\xi) - \alpha_i| \le C_2\mathrm{e}^{-\tau_2\xi}, \tag{3.6}$$

$$\frac{|\phi'(\xi)|}{|1 - \phi(\xi)|},\ \frac{|\hat{\phi}'(\xi)|}{|\hat{\phi}(\xi)|},\ \frac{|\psi_i'(\xi)|}{|\beta_i - \psi_i(\xi)|},\ \frac{|\hat{\psi}_i'(\xi)|}{|\hat{\psi}_i(\xi) - \alpha_i|} \ge \delta, \tag{3.7}$$

where a_0 is the radius of the support set of kernel J, $(\alpha_1, \beta_1) = (0, \rho)$, $\alpha_2, \beta_2 = (\rho, 1)$ and $i = 1, 2$.

3.2 Proof of Theorem 1.2

Before starting the main proofs, we consider the following ordinary differential equations which play an elementary role in the construction of super and subsolutions later.

$$\begin{cases} p_1'(t) = c_0 - N\mathrm{e}^{\sigma p_1(t)}, \quad t < 0, \\ p_1(0) < \min\left\{0, \dfrac{1}{\sigma}\ln\dfrac{c_0}{2N}\right\}, \end{cases} \qquad \begin{cases} p_2'(t) = c_0 + N\mathrm{e}^{\sigma p_2(t)}, \quad t < 0, \\ p_2(0) < 0, \end{cases} \tag{3.8}$$

where c_0, N, σ are positive constants and the initial value $p_1(0) \le p_2(0)$. (3.8) can be explicitly solved as

$$p_1(t) = p_1(0) + c_0 t - \frac{1}{\sigma} \ln \left\{ 1 - \frac{N}{c_0} e^{\sigma p_1(0)} (1 - e^{c_0 \sigma t}) \right\},$$
$$p_2(t) = p_2(0) + c_0 t - \frac{1}{\sigma} \ln \left\{ 1 + \frac{N}{c_0} e^{\sigma p_2(0)} (1 - e^{c_0 \sigma t}) \right\}.$$

If we further assume $p_2(0) = p_1(0) - \frac{1}{\sigma} \ln(1 - \frac{2N}{c_0} e^{\sigma p_1(0)})$, then

$$p_1(t) - c_0 t - \omega_1 = -\frac{1}{\sigma} \ln \left\{ 1 - \frac{r_1}{1 + r_1} e^{c_0 \sigma t} \right\}, \quad r_1 = -\frac{N}{c_0} e^{\sigma p_1(0)}, \tag{3.9}$$
$$p_2(t) - c_0 t - \omega_1 = -\frac{1}{\sigma} \ln \left\{ 1 - \frac{r_2}{1 + r_2} e^{c_0 \sigma t} \right\}, \quad r_2 = \frac{N}{c_0} e^{\sigma p_2(0)}, \tag{3.10}$$

where

$$\omega_1 := p_1(0) - \frac{1}{\sigma} \ln \left\{ 1 - \frac{N}{c_0} e^{\sigma p_1(0)} \right\} = p_2(0) - \frac{1}{\sigma} \ln \left\{ 1 + \frac{N}{c_0} e^{\sigma p_2(0)} \right\}.$$

Furthermore,

$$\lim_{t \to -\infty} (p_2(t) - p_1(t)) = 0.$$

In addition, there exists a positive constant R_0 such that

$$0 < p_2(t) - p_1(t) \le R_0 e^{c_0 \sigma t}, \quad t \le 0, \tag{3.11}$$

and

$$-\frac{R_0}{2} e^{c_0 \sigma t} \le p_1(t) - c_0 t - \omega_1 < 0 < p_2(t) - c_0 t - \omega_1 \le \frac{R_0}{2} e^{c_0 \sigma t}, \quad t \le 0. \tag{3.12}$$

Now define an auxiliary function as follows:

$$P(x, y) = \frac{(1 - \rho) x y}{x(y - \rho) + \rho(1 - y)}, \quad (x, y) \in D_1 := \{[0, \rho] \times [\rho, 1]\} \setminus \{(0, 1)\}. \tag{3.13}$$

Rewrite $P(x, y)$ as

$$\begin{aligned} P(x, y) &= x + x(y - \rho) \left\{ \frac{1 - x}{x(y - \rho) + \rho(1 - y)} \right\} \\ &= y + (x - \rho)(y - 1) \left\{ \frac{-y}{x(y - \rho) + \rho(1 - y)} \right\}, \quad \text{for } (x, y) \in D_1. \end{aligned}$$

Simple calculations imply the following lemma.

Lemma 3.5 *The functions $P(x, y)$ defined by* (3.13) *satisfying*

$$P_x(x, \rho) = P_y(\rho, y) = 1, \quad P_x(x, 1) = P_y(0, y) = 0, \quad (x, y) \in D_1,$$

and

$$P_{xx}(x, \rho) = P_{xx}(x, 1) = P_{yy}(0, y) = P_{yy}(\rho, y) = 0, \quad (x, y) \in D_1.$$

Moreover, there exist functions $\widetilde{P}_{11j}, \widetilde{P}_{22j} \in C^1(D_1), j = 1, 2$ such that

$$P_{xx}(x, y) = (y - \rho) \widetilde{P}_{111}(x, y) = (y - 1) \tilde{P}_{112}(x, y),$$
$$P_{yy}(x, y) = x \widetilde{P}_{221}(x, y) = (x - \rho) \widetilde{P}_{222}(x, y), \quad (x, y) \in D_1.$$

Now combining above discussions on $p_i(t)$ with Lemmas 3.4 and 3.5, we are ready to construct a pair of supersolution and subsolution of (1.1).

Lemma 3.6 *Let all assumptions of Theorem* 1.2 *be satisfied. Set* $\bar{c} = (c_1 + c_2)/2$ *and* $c_0 = (c_2 - c_1)/2$. *Let* $(p_1(t), c_0)$ *and* $(p_2(t), c_0)$ *be the solutions of* (3.8). *Then the functions defined by*

$$\begin{cases} \overline{U}(x,t) := P(\psi_1(x + \bar{c}t - p_1(t)), \psi_2(x + \bar{c}t + p_2(t))), \\ \underline{U}(x,t) := P(\psi_1(x + \bar{c}t - p_2(t)), \psi_2(x + \bar{c}t + p_1(t))), \end{cases}$$

are a pair of supersolution and subsolution of (1.1) *for* $t \le 0$. *Moreover, there are*

$$\underline{U}(x,t) \le \overline{U}(x,t), \quad \sup_{x\in\mathbb{R}}(\overline{U}(x,t) - \underline{U}(x,t)) \le C\mathrm{e}^{c_0\sigma t}, \quad t \le 0 \tag{3.14}$$

for some positive constant C *and* σ *as in* (3.8).

Proof From Lemma 2.5, we know $c_2 \ge c_2^* > c_1^* \ge c_1$, thus $c_0 > 0$. Denote

$$\mathcal{F}(u) = u_t - (J * u - u) - f(u). \tag{3.15}$$

To prove this lemma, it suffices to show that

$$\mathcal{F}(\overline{U}) \ge 0 \quad \text{and} \quad \mathcal{F}(\underline{U}) \le 0, \quad \text{for } (x,t) \in \mathbb{R} \times (-\infty, 0].$$

By using the above prepared results, direct calculations give that

$$\begin{aligned} \mathcal{F}(\overline{U}) &= P_x\psi_1'(-p_1' + \bar{c}) + P_y\psi_2'(p_2' + \bar{c}) - f(P) - (J * P - P) \\ &= P_x\psi_1'(-p_1' + \bar{c} - c_1) + P_y\psi_2'(p_2' + \bar{c} - c_2) + P_x f(\psi_1) + P_y f(\psi_2) - f(P) \\ &\quad + P_x(J * \psi_1 - \psi_1) + P_y(J * \psi_2 - \psi_2) - (J * P - P) \\ &= P_x\psi_1' N\mathrm{e}^{\sigma p_1} + P_y\psi_2' N\mathrm{e}^{\sigma p_2} - F(\psi_1, \psi_2) - H(\psi_1, \psi_2), \end{aligned} \tag{3.16}$$

where $P = P(\psi_1, \psi_2)$, and

$$\begin{aligned} F(\psi_1, \psi_2) &= f(P) - P_x f(\psi_1) - P_y f(\psi_2), \\ H(\psi_1, \psi_2) &= (J * P - P) - P_x(J * \psi_1 - \psi_1) - P_y(J * \psi_2 - \psi_2). \end{aligned}$$

By virtue of (3.11), we have $\mathrm{e}^{\sigma p_2(t)} \ge \mathrm{e}^{\sigma p_1(t)}$ for $t \le 0$. Then it follows from (3.16) that

$$\mathcal{F}(\overline{U}) \ge A(\psi_1, \psi_2)[N\mathrm{e}^{\sigma p_1(t)} - G(\psi_1, \psi_2)], \tag{3.17}$$

where

$$A(\psi_1, \psi_2) := P_x\psi_1' + P_y\psi_2', \quad G(\psi_1, \psi_2) := \frac{F(\psi_1, \psi_2) + H(\psi_1, \psi_2)}{A(\psi_1, \psi_2)}.$$

Indeed, following from (3.13), we have

$$P_x(x,y) = \frac{\rho(1-\rho)y(1-y)}{[x(y-\rho) + \rho(1-y)]^2}, \quad P_y(x,y) = \frac{\rho(1-\rho)x(1-x)}{[x(y-\rho) + \rho(1-y)]^2}.$$

Since $0 < \psi_1 < \rho$, $\rho < \psi_2 < 1$ and $\psi_i' > 0$ $(i = 1, 2)$ for all $(x,t) \in \mathbb{R}^2$, we have $A(\psi_1, \psi_2) > 0$ for $(x,t) \in \mathbb{R} \times (-\infty, 0]$.

Next we verify $\mathcal{F}(\overline{U}(x,t)) \ge 0$ for $(x,t) \in \mathbb{R} \times (-\infty, 0]$. The remainder of the proof is divided into three steps.

Step 1 We give some estimates for the function $P(\psi_1(x+\bar{c}t-p_1(t)),\psi_2(x+\bar{c}t+p_2(t)))$. If $p_2(0)\ll -1$, then $p_2(t)$ can be small enough. And it follows from (3.3) of Lemma 3.4 that

$$0<\psi_2(x+\bar{c}t+p_2(t))-\rho\le C_2\mathrm{e}^{\tau_1(x+\bar{c}t+p_2)}\le C_2\mathrm{e}^{\tau_1 p_2}\le\frac{1-\rho}{2}\quad\text{for } x+\bar{c}t\le 0,\quad t\le 0. \tag{3.18}$$

Thus, there exists a constant $\gamma_1>0$ such that

$$\begin{aligned}P_x(\psi_1,\psi_2)&=\frac{\rho(1-\rho)\psi_2(1-\psi_2)}{[\psi_1(\psi_2-\rho)+\rho(1-\psi_2)]^2}\\&\ge\frac{\rho^2(1-\rho)(1-\psi_2)}{[2\rho(1-\rho)]^2}\ge\gamma_1\quad\text{for } x+\bar{c}t\le 0,\quad t\le 0.\end{aligned} \tag{3.19}$$

By a similar argument, if $p_1(0)\ll -1$, (3.6) shows that

$$\begin{aligned}0<\rho-\psi_1(x+\bar{c}t-p_1(t))&\le C_2\mathrm{e}^{-\tau_2(x+\bar{c}t-p_1(t))}\\&\le C_2\mathrm{e}^{\tau_2 p_1(t)}\\&\le\frac{\rho}{2}\quad\text{for } x+\bar{c}t\ge 0,\quad t\le 0.\end{aligned} \tag{3.20}$$

Therefore, there exists $\gamma_2>0$ such that

$$P_y(\psi_1,\psi_2)=\frac{\rho(1-\rho)\psi_1(1-\psi_1)}{[\psi_1(\psi_2-\rho)+\rho(1-\psi_2)]^2}\ge\frac{\rho(1-\rho)^2\psi_1}{[2\rho(1-\rho)]^2}\ge\gamma_2 \tag{3.21}$$

for $x+\bar{c}t\ge 0$ and $t\le 0$.

Next, we estimate the second derivative of $P(\psi_1,\psi_2)$.

$$P_{xx}(\psi_1,\psi_2)=(\psi_2-\rho)(\psi_2-1)\frac{2\rho(1-\rho)\psi_2}{[\psi_1(\psi_2-\rho)+\rho(1-\psi_2)]^3}, \tag{3.22}$$

$$P_{xy}(\psi_1,\psi_2)=\rho(1-\rho)\frac{(2\rho-1)\psi_1\psi_2+\rho(1-\psi_1-\psi_2)}{[\psi_1(\psi_2-\rho)+\rho(1-\psi_2)]^3}, \tag{3.23}$$

$$P_{yy}(\psi_1,\psi_2)=\psi_1(\psi_1-\rho)\frac{2\rho(1-\rho)(\psi_1-1)}{[\psi_1(\psi_2-\rho)+\rho(1-\psi_2)]^3}. \tag{3.24}$$

From (3.20), we have $\psi_1(x+\bar{c}t-p_1(t))\ge\rho/2$ for $x+\bar{c}t\ge 0$ and $t\le 0$. Thus

$$\begin{aligned}\psi_1(\psi_2-\rho)+\rho(1-\psi_2)&\ge\frac{\rho}{2}[\psi_2(x+\bar{c}t+p_2(t))-\rho]+\rho[1-\psi_2(x+\bar{c}t+p_2(t))]\\&=\frac{\rho}{2}[2-\rho-\psi_2(x+\bar{c}t+p_2(t))]\\&\ge\frac{\rho(1-\rho)}{2}\end{aligned}$$

for $x+\bar{c}t\ge 0$ and $t\le 0$. Similarly, from (3.18), we can obtain

$$\psi_1(\psi_2-\rho)+\rho(1-\psi_2)\ge\frac{\rho(1-\rho)}{2}\quad\text{for } x+\bar{c}t\le 0,\ t\le 0.$$

Thus, there exists a constant $\widetilde{C}$ such that

$$\begin{aligned}&|P_{xx}(\psi_1,\psi_2)|,\quad|P_{xy}(\psi_1,\psi_2)|,\quad|P_{yy}(\psi_1,\psi_2)|\le\widetilde{C}\\&\text{uniformly in } (x,t)\in\mathbb{R}\times(-\infty,0].\end{aligned} \tag{3.25}$$

Step 2 In this step, we prove the following estimations:

$$\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\le L_1\mathrm{e}^{\tau_1 p_2(t)}\quad\text{for } x+\bar{c}t\le 0,\ t\le 0, \tag{3.26}$$

$$\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} \le L_1 \mathrm{e}^{\tau_2 p_1(t)} \quad \text{for } x+\bar{c}t \ge 0,\ t \le 0, \tag{3.27}$$

for some constant $L_1 > 0$. From the definition of $P(\psi_1,\psi_2)$ and Lemma 3.5, we have

$$F(\psi_1,\rho) = f(P(\psi_1,\rho)) - P_x(\psi_1,\rho)f(\psi_1) - P_y(\psi_1,\rho)f(\rho) = f(\psi_1) - f(\psi_1) = 0.$$

Analogously, one has

$$F(\psi_1,1) = F(\psi_1,\rho) = F(0,\psi_2) = F(\rho,\psi_2) = 0.$$

Thus, there exist functions $F_1, F_2, F_3 \in C(D_1)$ such that for $x+\bar{c}t \le p_1(t)$, we have the following expression

$$F(\psi_1,\psi_2) = \psi_1(\psi_2-\rho)F_1(\psi_1,\psi_2).$$

Similarly, we have

$$F(\psi_1,\psi_2) = (\psi_1-\rho)(\psi_2-1)F_2(\psi_1,\psi_2)$$

for $x+\bar{c}t \ge -p_2(t)$, and

$$F(\psi_1,\psi_2) = (\psi_1-\rho)(\psi_2-\rho)F_3(\psi_1,\psi_2)$$

for $p_1(t) \le x+\bar{c}t \le -p_2(t)$, where $\psi_1 = \psi_1(x+\bar{c}t-p_1(t))$ and $\psi_2 = \psi_2(x+\bar{c}t+p_2(t))$. It is easy to see that there exists a positive constant C_3 such that $|(F_1,F_2,F_3)(\psi_1,\psi_2)| \le C_3$.

Next we consider two cases: $x+\bar{c}t \in (-\infty, p_1(t)] \cup [-p_2(t), +\infty)$ and $x+\bar{c}t \in [p_1(t), -p_2(t)]$, respectively.

Case I $x+\bar{c}t \in (-\infty, p_1(t)] \cup [-p_2(t), +\infty)$. By using Lemma 3.4, (3.19) and the above prepared results, for $x+\bar{c}t \le p_1(t)$ and $t \le 0$, we have

$$\begin{aligned}\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} &= \frac{\psi_1(\psi_2-\rho)F_1(\psi_1,\psi_2)}{P_x\psi_1' + P_y\psi_2'} \\ &\le \frac{(\psi_2-\rho)|F_1(\psi_1,\psi_2)|}{P_x(\psi_1'/\psi_1)} \\ &\le \frac{C_2C_3\mathrm{e}^{\tau_1(x+\bar{c}t+p_2(t))}}{\gamma_1\delta} \\ &\le L_2\mathrm{e}^{\tau_1 p_2(t)}\end{aligned} \tag{3.28}$$

with some constant $L_2 > 0$. Similarly, we can prove that there exists a constant $L_3 > 0$ such that

$$\begin{aligned}\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} &= \frac{(\psi_1-\rho)(\psi_2-1)F_2(\psi_1,\psi_2)}{P_x\psi_1' + P_y\psi_2'} \\ &\le L_3\mathrm{e}^{\tau_2 p_1(t)} \quad \text{for } x+\bar{c}t \ge -p_2(t),\ t \le 0.\end{aligned} \tag{3.29}$$

Case II $z \in [p_1(t), -p_2(t)]$. Firstly, for $p_1(t) \le x+\bar{c}t \le 0$ and $t \le 0$, there is

$$\begin{aligned}\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} &= \frac{(\psi_1-\rho)(\psi_2-\rho)F_3(\psi_1,\psi_2)}{P_x\psi_1' + P_y\psi_2'} \\ &\le \frac{|\psi_2-\rho||F_3|}{P_x\psi_1'/(\rho-\psi_1)} \\ &\le L_4\mathrm{e}^{\tau_1 p_2(t)}.\end{aligned} \tag{3.30}$$

For $0 \le x + \bar{c}t \le -p_2(t)$ and $t \le 0$, we also have

$$\frac{F(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} \le L_5 \mathrm{e}^{\tau_2 p_1(t)}. \tag{3.31}$$

Then taking $L_1 = \max\{L_i, i = 2,3,4,5\}$ and combining (3.28)–(3.31), we conclude that (3.26) and (3.27) hold.

Step 3 Next, we prove that there exists a positive constant $\widetilde{L}_1$ such that

$$\frac{H(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} \le \widetilde{L}_1 \mathrm{e}^{\tau_1 p_2(t)} \quad \text{for } x + \bar{c}t \le 0,\ t \le 0, \tag{3.32}$$

$$\frac{H(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} \le \widetilde{L}_1 \mathrm{e}^{\tau_2 p_1(t)} \quad \text{for } x + \bar{c}t \ge 0,\ t \le 0. \tag{3.33}$$

In order to simplify the complex expressions, we denote

$$\tilde{\psi}_1(\theta) = \psi_1(x + \bar{c}t - p_1(t) - \theta r) \quad \text{and} \quad \tilde{\psi}_2(\theta) = \psi_2(x + \bar{c}t + p_2(t) - \theta r),$$

where $\theta \in [0,1]$ and $r \in \mathrm{supp}(J)$. Note that

$$\begin{aligned}
&H(\psi_1(x+\bar{c}t-p_1(t)), \psi_2(x+\bar{c}t+p_2(t)))\\
&\quad= \int_{\mathbb{R}} J(r)[P(\tilde{\psi}_1(1),\tilde{\psi}_2(1)) - P(\tilde{\psi}_1(0),\tilde{\psi}_2(0))]dr\\
&\qquad - P_x \int_{\mathbb{R}} J(r)[\tilde{\psi}_1(1) - \tilde{\psi}_1(0)]dr - P_y \int_{\mathbb{R}} J(r)[\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]dr\\
&\quad= \int_{\mathbb{R}} J(r) P_x(\theta_1\tilde{\psi}_1(1) + (1-\theta_1)\tilde{\psi}_1(0), \tilde{\psi}_2(1))[\tilde{\psi}_1(1) - \tilde{\psi}_1(0)]dr\\
&\qquad + \int_{\mathbb{R}} J(r) P_y(\tilde{\psi}_1(0), \theta_2\tilde{\psi}_2(1) + (1-\theta_2)\tilde{\psi}_2(0))[\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]dr\\
&\qquad - \int_{\mathbb{R}} J(r) P_x(\tilde{\psi}_1(0), \tilde{\psi}_2(0))[\tilde{\psi}_1(1) - \tilde{\psi}_1(0)]dr\\
&\qquad - \int_{\mathbb{R}} J(r) P_y(\tilde{\psi}_1(0), \tilde{\psi}_2(0))[\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]dr\\
&\quad= \int_{\mathbb{R}} J(r)\{[P_x(\theta_1\tilde{\psi}_1(1) + (1-\theta_1)\tilde{\psi}_1(0), \tilde{\psi}_2(1)) - P_x(\tilde{\psi}_1(0), \tilde{\psi}_2(0))][\tilde{\psi}_1(1) - \tilde{\psi}_1(0)]\\
&\qquad + [P_y(\tilde{\psi}_1(0), \theta_2\tilde{\psi}_2(1) + (1-\theta_2)\tilde{\psi}_2(0)) - P_y(\tilde{\psi}_1(0), \tilde{\psi}_2(0))][\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]\}dr\\
&\quad= \int_{\mathbb{R}} J(r)\{P_{xx}(\theta_3\theta_1\tilde{\psi}_1(1) + (1-\theta_3\theta_1)\tilde{\psi}_1(0), \tilde{\psi}_2(1))\theta_1[\tilde{\psi}_1(1) - \tilde{\psi}_1(0)]^2\\
&\qquad + P_{xy}(\tilde{\psi}_1(0), \theta_4\tilde{\psi}_2(1) + (1-\theta_4)\tilde{\psi}_2(0))[\tilde{\psi}_1(1) - \tilde{\psi}_1(0)][\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]\\
&\qquad + P_{yy}(\tilde{\psi}_1(0), \theta_5\theta_2\tilde{\psi}_2(1) + (1-\theta_5\theta_2)\tilde{\psi}_2(0))\theta_2[\tilde{\psi}_2(1) - \tilde{\psi}_2(0)]^2\}dr,
\end{aligned}$$

where $\theta_i \in (0,1)\,(i = 1,\dots,5)$. According to (3.22)–(3.25), there exists a positive constant $\widehat{C}$ such that

$$\begin{aligned}
&|P_{xx}(\theta_3\tilde{\psi}_1(1) + (1-\theta_3)\tilde{\psi}_1(0), \tilde{\psi}_2(1))| \le \widehat{C}(\tilde{\psi}_2(1) - \rho),\\
&|P_{yy}(\tilde{\psi}_1(0), \theta_5\tilde{\psi}_2(1) + (1-\theta_5)\tilde{\psi}_2(0))| \le \widehat{C}\tilde{\psi}_1(0),\\
&|P_{xy}(\tilde{\psi}_1(0), \theta_4\tilde{\psi}_2(1) + (1-\theta_4)\tilde{\psi}_2(0))| \le \widehat{C}.
\end{aligned}$$

Therefore, we have

$$\frac{H(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} \le \widehat{C}\int_{\mathbb{R}} J(r)\left\{\frac{[\tilde{\psi}_1(1)-\tilde{\psi}_1(0)]^2[\tilde{\psi}_2(1)-\rho]}{P_x\psi_1'(x+\bar{c}t-p_1)+P_y\psi_2'(x+\bar{c}t+p_2)} + \frac{[\tilde{\psi}_1(1)-\tilde{\psi}_1(0)][\tilde{\psi}_2(1)-\tilde{\psi}_2(0)]+[\tilde{\psi}_2(1)-\tilde{\psi}_2(0)]^2\tilde{\psi}_1(0)}{P_x\psi_1'(x+\bar{c}t-p_1)+P_y\psi_2'(x+\bar{c}t+p_2)}\right\}dr.$$

Let

$$\begin{aligned}
B(\psi_1,\psi_2) &= r^2[\psi_1'(x+\bar{c}t-p_1-\theta_6 r)]^2[\psi_2(x+\bar{c}t+p_2-r)-\rho],\\
C(\psi_1,\psi_2) &= r^2\psi_1'(x+\bar{c}t-p_1-\theta_7 r)\psi_2'(x+\bar{c}t+p_2-\theta_8 r),\\
D(\psi_1,\psi_2) &= r^2[\psi_2'(x+\bar{c}t+p_2-\theta_9 r)]^2\psi_1(x+\bar{c}t-p_1),
\end{aligned}$$

where $\theta_i\in(0,1)\,(i=6,\dots,9)$ and $r\in[-a_0,a_0]$, a_0 is defined as in Lemma 3.4.

For $t\le 0$ and $x+\bar{c}t\le p_1(t)<0$, we have $x+\bar{c}t-p_1(t)-\theta_6 r\le a_0$ and $x+\bar{c}t+p_2(t)-r\le a_0$. Then by (3.2), (3.3), (3.4) and (3.19), we get

$$\begin{aligned}
\frac{B(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} &\le \frac{r^2[\psi_1'(x+\bar{c}t-p_1-\theta_6 r)]^2[\psi_2(x+\bar{c}t+p_2-r)-\rho]}{P_x\psi_1'(x+\bar{c}t-p_1)}\\
&\le \frac{a_0^2C_0^2\mathrm{e}^{2\tau_1(x+\bar{c}t-p_1-\theta_6 r)}}{\gamma_1\delta\psi_1(x+\bar{c}t-p_1)}[\psi_2(x+\bar{c}t+p_2-r)-\rho]\\
&\le \frac{a_0^2C_0^2\mathrm{e}^{2\tau_1(x+\bar{c}t-p_1-\theta_6 r)}}{\gamma_1\delta C_1\mathrm{e}^{\tau_1(x+\bar{c}t-p_1)}}C_2\mathrm{e}^{\tau_1(x+\bar{c}t+p_2-r)}\\
&\le \widetilde{L}_2\mathrm{e}^{\tau_1 p_2(t)}
\end{aligned}$$

for some constant $\widetilde{L}_2>0$. Similarly, for $t\le 0$ and $p_1(t)\le x+\bar{c}t\le 0$, there also exists a constant $\widetilde{L}_3>0$ such that

$$\frac{B(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\le \widetilde{L}_3\mathrm{e}^{\tau_1 p_2(t)}.$$

By a similar argument as above, we obtain that there exist positive constants $\widetilde{L}_4$ and $\widetilde{L}_5$ such that

$$\begin{aligned}
&\frac{C(\psi_1,\psi_2)}{A(\psi_1,\psi_2)},\quad \frac{D(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\le \widetilde{L}_4\mathrm{e}^{\tau_1 p_2(t)} \quad \text{for } x+\bar{c}t\le 0,\ t\le 0,\\
&\frac{B(\psi_1,\psi_2)}{A(\psi_1,\psi_2)},\quad \frac{C(\psi_1,\psi_2)}{A(\psi_1,\psi_2)},\quad \frac{D(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\le \widetilde{L}_5\mathrm{e}^{\tau_2 p_1(t)} \quad \text{for } x+\bar{c}t\ge 0,\ t\le 0.
\end{aligned}$$

Then by setting $\widetilde{L}_1=\widehat{C}\max\{3\widetilde{L}_5,\max\{\widetilde{L}_2+\widetilde{L}_3\}+2\widetilde{L}_4\}$, we get

$$\begin{aligned}
\frac{H(\psi_1,\psi_2)}{A(\psi_1,\psi_2)} &\le \widehat{C}\int_{\mathbb{R}} J(r)\left\{\frac{B(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}+\frac{C(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}+\frac{D(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\right\}dr\\
&\le \widetilde{L}_1\mathrm{e}^{\tau_1 p_2(t)} \quad \text{for } x+\bar{c}t\le 0,\ t\le 0,
\end{aligned}$$

and

$$\frac{H(\psi_1,\psi_2)}{A(\psi_1,\psi_2)}\le \widetilde{L}_1\mathrm{e}^{\tau_2 p_1(t)} \quad \text{for } x+\bar{c}t\ge 0,\ t\le 0.$$

Choosing $0<\tau_3<\min\{\tau_1,(p_2(0)\tau_1)/p_1(0)\}$ and letting $g(t):=\tau_1 p_2(t)-\tau_3 p_1(t)$, it follows from (3.8) that $g(0)=\tau_1 p_2(0)-\tau_3 p_1(0)<0$ and

$$g'(t)=c_0(\tau_1-\tau_3)+N(\tau_1\mathrm{e}^{\sigma p_2}+\tau_3\mathrm{e}^{\sigma p_1})>0,$$

which implies that $\tau_1 p_2(t) \le \tau_3 p_1(t) < 0$ for $t \le 0$. Applying (3.26), (3.27), (3.32) and (3.33), letting $N \ge L_1 + \widetilde{L}_1$ and $\sigma \le \min\{\tau_2, \tau_3\}$, we have

$$\begin{aligned}\mathcal{F}(\overline{U}) &\ge A(\psi_1,\psi_2)[N\mathrm{e}^{\sigma p_1(t)} - (L_1 + \widetilde{L}_1)\mathrm{e}^{\tau_1 p_2(t)}]\\ &\ge A(\psi_1,\psi_2)[N\mathrm{e}^{\sigma p_1(t)} - (L_1 + \widetilde{L}_1)\mathrm{e}^{\tau_3 p_1(t)}]\\ &\ge 0\end{aligned}$$

uniformly in $(x + \bar{c}t, t) \in (-\infty, 0] \times (-\infty, 0]$. And

$$\mathcal{F}(\overline{U}) \ge A(\psi_1,\psi_2)[N\mathrm{e}^{\sigma p_1(t)} - (L_1 + \widetilde{L}_1)\mathrm{e}^{\tau_2 p_1(t)}] \ge 0$$

uniformly in $(x + \bar{c}t, t) \in [0, +\infty) \times (-\infty, 0]$. Thus, $\mathcal{F}(\overline{U}) \ge 0$ for all $(x,t) \in \mathbb{R} \times (-\infty, 0]$, and $\overline{U}$ is a supersolution of (1.1) for $t \le 0$. By a similar argument, we can prove $\mathcal{F}(\underline{U}) \le 0$ for all $(x,t) \in \mathbb{R} \times (-\infty, 0]$.

Finally, we show (3.14). From the definition of $\overline{U}(x,t)$ and $\underline{U}(x,t)$, one has

$$\begin{aligned}\overline{U}(x,t) - \underline{U}(x,t) &= P_x(\theta_1\psi_1(x+\bar{c}t-p_1) + (1-\theta_1)\psi_1(x+\bar{c}t-p_2), \psi_2(x+\bar{c}t+p_2))\\ &\quad\cdot [\psi_1(x+\bar{c}t-p_1) - \psi_1(x+\bar{c}t-p_2)]\\ &\quad + P_y(\psi_1(x+\bar{c}t-p_2), \theta_2\psi_2(x+\bar{c}t+p_2) + (1-\theta_2)\psi_2(x+\bar{c}t+p_1))\\ &\quad\cdot [\psi_2(x+\bar{c}t+p_2) - \psi_2(x+\bar{c}t+p_1)].\end{aligned}$$

Since $P_x \ge 0$ and $P_y \ge 0$ on D_1 and $\psi_i' > 0\,(i = 1, 2)$, from (3.11) we know that $\psi_1(x + \bar{c}t - p_1) - \psi_1(x + \bar{c}t - p_2) \ge 0$ and $\psi_2(x + \bar{c}t + p_2) - \psi_2(x + \bar{c}t + p_1) \ge 0$. Consequently, we have $\overline{U}(x,t) \ge \underline{U}(x,t)$ and

$$\sup_{x\in\mathbb{R}} \big(\overline{U}(x,t) - \underline{U}(x,t)\big) \le |P_x||\psi_1'|(p_2(t) - p_1(t)) + |P_y||\psi_2'|(p_2(t) - p_1(t)) \le C\mathrm{e}^{c_0\sigma t}.$$

The proof is complete. □

Proof of Theorem 1.2 Now we are ready to prove the existence of entire solutions of (1.1) described as in Theorem 1.2. We only prove the existence and properties of $U_1(x,t)$ since those for $U_2(x,t)$ can be proved by similar processes. Given any $n \in \mathbb{N}$, consider the following Cauchy problem

$$\begin{cases} u_t^n(x,t) = (J * u^n - u^n)(x,t) + f(u^n(x,t)), \quad x \in \mathbb{R}, \quad t > -n,\\ u^n(x,-n) := \underline{U}(x,-n) = P(\psi_1(x - \bar{c}n - p_2(-n)),\\ \psi_2(x - \bar{c}n + p_1(-n))), \quad x \in \mathbb{R}. \end{cases} \tag{3.34}$$

Lemma 2.2 shows that (3.34) has a unique solution $u^n(x,t)$ which satisfies $0 \le u^n(x,t) \le 1$ for $(x,t) \in \mathbb{R} \times [-n, +\infty)$. In addition, by Lemmas 2.2 and 3.6, we have

$$0 \le \underline{U}(x,t) \le u^n(x,t) \le u^{n+1}(x,t) \le \min\{1, \overline{U}(x,t)\} \quad \text{for } x \in \mathbb{R}, -(n+1) \le t < 0. \tag{3.35}$$

In other words, the solution sequence $\{u^n(x,t)\}_{n\in\mathbb{N}}$ is bounded and non-decreasing about n for any $(x,t) \in \mathbb{R} \times (-n, +\infty)$.

Moreover, note that $\psi_1(x + c_1 t)$ and $\psi_2(x + c_2 t)$ are monotone monostable waves of (1.1) which satisfy (1.6), and $c_1 \le c_1^*$, $c_2 \ge c_2^*$ with $c_1, c_2 \ne 0$. Lemma 2.5 implies three cases for the size of c_1 and c_2: (1) $c_2 > c_1 > 0$; (2) $c_2 > 0 > c_1$; (3) $0 > c_2 > c_1$. In either case, we always have $0 < \psi_1' \le \frac{2\rho + M_1}{|c_1|}$ and $0 < \psi_2' \le \frac{2 + M_2}{|c_2|}$, where $M_1 = \max_{s\in[0,\rho]} |f(s)|$ and

$M_2 = \max_{s\in[\rho,1]}|f(s)|$. In addition, it is easy to see that there exists a constant $M_3 > 0$ such that $|P_x(x,y)| \le M_3$ for any $(x,y) \in D_1$. Therefore, the initial functions $u^n(x,-n) = P(\psi_1(x-\bar{c}n - p_2(-n)), \psi_2(x-\bar{c}n + p_1(-n)))$ of (3.34) satisfy (2.3) in Lemma 2.3. Thus, Lemma 2.3 implies that the solutions $u^n(x,t)$ of (3.34) and $\frac{\partial}{\partial t}u^n(x,t)$ are globally Lipschitz in x. Then by using Arzela–Ascoli Theorem and the diagonal extraction process, there exists a subsequence $\{u^{n_i}\}_{i\in N}$ of $\{u^n\}_{n\in N}$ such that $u^{n_i}(x,t)$ converge uniformly to a function $U_1(x,t)$ in $\mathcal{T}$, which means for any compact set $S \subset \mathbb{R}^2$, the sequences $u^{n_i}(x,t)$ and $\frac{\partial}{\partial t}u^{n_i}(x,t)$ converge uniformly in $(x,t) \in S$ to $U_1(x,t)$ and $\frac{\partial}{\partial t}U_1(x,t)$ as $i \to \infty$, respectively. Obviously, $U_1(x,t)$ is an entire solution of (1.1). And (3.35) implies that

$$\underline{U}(x,t) \le U_1(x,t) \le \overline{U}(x,t) \quad \text{for any } x \in \mathbb{R} \text{ and } t \le 0. \tag{3.36}$$

Next, we study the asymptotic behavior (1.13). Assume $x \le -\bar{c}t = -\frac{c_1+c_2}{2}t$, then from (3.36) we have

$$\begin{aligned} |U_1(x,t) - \psi_1(x+c_1t-\omega_1)| &\le |\overline{U}(x,t) - \underline{U}(x,t)| + |\underline{U}(x,t) - \psi_1(x+\bar{c}t - p_2(t))| \\ &\quad + |\psi_1(x+\bar{c}t-p_2(t)) - \psi_1(x+c_1t-\omega_1)| \\ &\le C\mathrm{e}^{c_0\sigma t} + \frac{R_0}{2}\mathrm{e}^{c_0\sigma t}\sup_{\xi\in\mathbb{R}}\psi_1'(\xi) \\ &\quad + |\underline{U}(x,t) - \psi_1(x+\bar{c}t-p_2(t))|. \end{aligned} \tag{3.37}$$

The last inequality is ensured by (3.14) and (3.12). Now we estimate the last term of (3.37).

$$\begin{aligned} &|\underline{U}(x,t) - \psi_1(x+\bar{c}t-p_2(t))| \\ &\quad = |P(\psi_1(x+\bar{c}t-p_2(t)), \psi_2(x+\bar{c}t+p_1(t))) - \psi_1(x+\bar{c}t-p_2(t))| \\ &\quad = \frac{\psi_1(x+\bar{c}t-p_2(t))[1-\psi_1(x+\bar{c}t-p_2(t))][\psi_2(x+\bar{c}t+p_1(t))-\rho]}{\psi_1(x+\bar{c}t-p_2(t))[\psi_2(x+\bar{c}t+p_1(t))-\rho]+\rho[1-\psi_2(x+\bar{c}t+p_1(t))]}. \end{aligned}$$

Note that for $x \le -\bar{c}t$ and $t \ll -1$,

$$\psi_1(x+\bar{c}t-p_2(t))[\psi_2(x+\bar{c}t+p_1(t))-\rho]+\rho[1-\psi_2(x+\bar{c}t+p_1(t))] \ge \rho(1-\psi_2(0)).$$

Since $\rho \le \psi_2(x+\bar{c}t+p_1(t)) \le \psi_2(p_1(t)) \to \rho$ as $t \to -\infty$ for $x \le -\bar{c}t$, we have

$$\lim_{t\to-\infty}\sup_{x\le-\bar{c}t}|\underline{U}(x,t) - \psi_1(x+\bar{c}t-p_2(t))| = 0.$$

Therefore from (3.37),

$$\lim_{t\to-\infty}\sup_{x\le-\bar{c}t}|U_1(x,t) - \psi_1(x+c_1t-\omega_1)| = 0.$$

Similarly, we can prove that

$$\lim_{t\to-\infty}\sup_{x\ge-\bar{c}t}|U_1(x,t) - \psi_2(x+c_2t+\omega_1)| = 0.$$

Thus (1.13) holds. (1.14) can be proved similarly and (i)–(iv) are straightforward.

3.3 Proofs of Theorems 1.3 and 1.4

To prove Theorems 1.3 and 1.4, we recommend another auxiliary function:

$$Q(x,y) = \frac{\rho(x+y)-(1+\rho)xy}{\rho-xy}, \quad (x,y) \in D_2 := \{[0,\rho]\times[\rho,1]\} \setminus \{(\rho,1)\}. \tag{3.38}$$

Simple calculations show that $Q(x,y)$ has the similar properties with $P(x,y)$. And by using analogous arguments with the proof of Lemma 3.6, we can get the following two lemmas.

Lemma 3.7 *Let all assumptions of Theorem* 1.3 *be satisfied. Set* $\bar{c}=(c+\hat{c}_1)/2$, $c_0=(c-\hat{c}_1)/2$, *and* $(p_i(t),c_0)$ $(i=1,2)$ *be the solutions of* (3.8). *If* $c>\hat{c}_1$, *then the functions defined by*

$$\begin{cases}\overline{V}(x,t):=Q(\hat{\psi}_1(x+\bar{c}t-p_2(t)),\phi(x+\bar{c}t+p_2(t)))\\ \underline{V}(x,t):=Q(\hat{\psi}_1(x+\bar{c}t-p_1(t)),\phi(x+\bar{c}t+p_1(t)))\end{cases}$$

are a pair of supersolution and subsolution of (1.1) *for* $(x,t)\in\mathbb{R}\times(-\infty,0]$. *Moreover,* (3.14) *holds for* $\overline{V}(x,t)$ *and* $\underline{V}(x,t)$.

Lemma 3.8 *Let all assumptions of Theorem* 1.4 *be satisfied and* $c>\hat{c}$. *Set* $\bar{c}=\frac{c+\hat{c}}{2}$, $c_0=\frac{c-\hat{c}}{2}$ *and* $(p_i(t),c_0)$ $(i=1,2)$ *be the solutions of* (3.8). *Then the functions defined by*

$$\begin{cases}\overline{W}(x,t):=Q(\phi(x+\bar{c}t-p_2(t)),\hat{\phi}(x+\bar{c}t+p_2(t)))\\ \underline{W}(x,t):=Q(\phi(x+\bar{c}t-p_1(t)),\hat{\phi}(x+\bar{c}t+p_1(t)))\end{cases}$$

constitute a pair of supersolution and subsolution of (1.1) *for* $(x,t)\in\mathbb{R}\times(-\infty,0]$. *In addition,* (3.14) *also holds for* $\overline{W}(x,t)$ *and* $\underline{W}(x,t)$.

Afterwards, following the similar processes with Theorem 1.2, we obtain the conclusions of Theorems 1.3 and 1.4. In particular, for the case $\hat{c}>c$ in Theorem 1.4, we prove it just by exchanging the roles of (ϕ,c) and $(\hat{\phi},\hat{c})$. In addition, since the proof of Theorem 1.5 is standard and common by using Lemma 2.3, we omit the details here.

Remark 3.9 In Theorems 1.2 and 1.3, we do not get the asymptotic properties of the entire solutions as $t\to+\infty$ since the subsolutions we constructed here just hold for $t<0$. However, in Theorem 1.4, we obtained the asymptotic behavior and monotonicity of $W(x,t)$ with respect to t, because under the conditions of Theorem 1.4, the function

$$\underline{W}(x,t):=\max\{\phi(x+ct+\theta_1),\hat{\phi}(x+ct+\theta_2)\}$$

is also a subsolution of (1.1) which holds for $t\in\mathbb{R}$.

4 The Effects of Asymmetry of J on the Wave Speeds

For a general kernel function, to consider the influence of the asymmetry of the kernel is very difficult, so we take J as a translation of the Gauss kernel, i.e. $J(x)=\frac{1}{\sqrt{4\pi r}}\mathrm{e}^{-\frac{(x-a)^2}{4r}}$, where a is some constant (not necessarily positive). Obviously, $J\in C^\infty$, $\int_{\mathbb{R}}J(x)dx=1$ and $J(x)\neq J(-x)$. Section 2.2 shows that there exists a positive constant λ_2^* such that

$$\begin{aligned}
c_2^*=\hat{c}_1^*&=\frac{\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi r}}\mathrm{e}^{-\frac{(-x-a)^2}{4r}}\mathrm{e}^{\lambda_2^*x}dx-1+f'(\rho)}{\lambda_2^*}.\\
\frac{dc_2^*}{da}&=\frac{1}{\lambda_2^*\sqrt{4\pi r}}\int_{\mathbb{R}}\frac{2(-x-a)}{4r}\mathrm{e}^{\lambda_2^*x-\frac{(-x-a)^2}{4r}}dx\\
&=-\frac{1}{2r\lambda_2^*\sqrt{4\pi r}}\int_{\mathbb{R}}(x+a)\mathrm{e}^{\lambda_2^*(x+a)-\frac{(x+a)^2}{4r}}\mathrm{e}^{-a\lambda_2^*}dx\\
&=-\frac{1}{2r\lambda_2^*\sqrt{4\pi r}}\mathrm{e}^{-a\lambda_2^*}\int_{\mathbb{R}}y\mathrm{e}^{\lambda_2^*y-\frac{y^2}{4r}}dy
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2r\lambda_2^*\sqrt{4\pi r}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}}\int_{\mathbb{R}} ye^{-\frac{(y-2r\lambda_2^*)^2}{4r}}dy \\
&= -\frac{1}{2r\lambda_2^*\sqrt{4\pi r}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}}\int_{\mathbb{R}} ze^{-\frac{z^2}{4r}}dz - \frac{1}{2r\lambda_2^*\sqrt{4\pi r}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}}\int_{\mathbb{R}} 2r\lambda_2^* e^{-\frac{z^2}{4r}}dz \\
&= -\frac{1}{\lambda_2^*\sqrt{4\pi r}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}}\int_0^{+\infty} e^{-s}ds - \frac{1}{\sqrt{\pi}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}}\int_{\mathbb{R}} e^{-t^2}dt \\
&= -\frac{1}{\lambda_2^*\sqrt{4\pi r}}e^{-a\lambda_2^*}e^{r\lambda_2^{*2}} - e^{-a\lambda_2^*}e^{r\lambda_2^{*2}} \\
&< 0.
\end{aligned}
$$

In above calculations, we frequently use variable substitution, such as $y = x + a$, $z = y - 2r\lambda_2^*$, $s = (\frac{z}{\sqrt{4r}})^2$ and $t = \frac{z}{\sqrt{4r}}$, and the special integral $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}$. Thus we obtain that c_2^* is decreased with respect to a, which means that the symmetry axis of the kernel J is more right ($a >> 1$), the smaller the minimal wave speed is, and the symmetry of the kernel J is more left ($a << -1$), the larger the minimal wave speed is.

Similarly, we obtain

$$\frac{d\hat{c}_2^*}{da} = \frac{dc_1^*}{da} < 0$$

which implies that $\hat{c}_2^*$ is also decreased with respect to a, namely, the symmetry axis of the kernel J is more right ($a >> 1$), the smaller the maximal wave speed is, and the symmetry of the kernel J is more left ($a << -1$), the larger the maximal wave speed is. When $a = 0$, it is obviously that $c_2^* > 0$ and $\hat{c}_2^* < 0$. Combining with the above derivative relations, it it natural to know that the case of $c_2^* < 0$ and $\hat{c}_2^* > 0$ is impossible.

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