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Complete Moment Convergence for the Dependent Linear Processes with Random Coefficients

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Abstract In this paper, we investigate the complete moment convergence for dependent linear processes with random coefficients to form $X_t = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{t-j}$, where $\{\varepsilon_n, n \in \mathbb{Z}\}$ is a sequence of END stochastically dominated random variables and $\{A_n, n \in \mathbb{Z}\}$ is a sequence of random varibles. As applications, the convergence rate, Marcinkiewicz–Zygmund strong law and strong law of large numbers for this linear process are established.

Keywords Complete moment convergence, extended negatively dependent random variables, linear processes, random coefficients

MR(2010) Subject Classification 60F15

1 Introduction

Suppose that $\{\varepsilon_n, n \in \mathbb{Z}\}$ is a sequence of independent and identically distributed random variables and $\{a_n, n \in \mathbb{Z}\}$ is a sequence of real numbers. Therefore, $\{X_t, t > 0\}$ is defined as

$$X_t = \sum_{j=-\infty}^{\infty} a_j \varepsilon_{t-j},$$

a linear process or an infinite order moving average process.

Linear processes are one of the most important topics in different applications such as electronic, financial mathematical and time series. The asymptotic behavior of these processes have been considered by many researchers. For example, McLeish [17, 18] verified asymptotic convergence for time series including the linear processes and the checked limit theorems for the dependent random variables of different cases. Ko [10] studied the asymptotic convergence and the limit theorems for the linear processes in Hilbert space. Wang and Wu [27] investigated the central limit theorem for the linear processes generating by the dependent random variable. Philips and Solo [19] established the asymptotic convergence for different cases of linear processes.

A finite collection of random variables $\varepsilon_1, \ldots, \varepsilon_n$ is called extended negatively dependent (END) if there exists a positive constant M independent of n such that both

$$\mathbf{P}(\varepsilon_1 > \epsilon_1, \varepsilon_2 > \epsilon_2, \ldots, \varepsilon_n > \epsilon_n) \le M \prod_{i=1}^{n} \mathbf{P}(\varepsilon_i > \epsilon_i),$$

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and

$$\mathbf{P}(\varepsilon_1 \le \epsilon_1, \varepsilon_2 \le \epsilon_2, \ldots, \varepsilon_n \le \epsilon_n) \le M \prod_{i=1}^{n} \mathbf{P}(\varepsilon_i \le \epsilon_i).$$

An infinity family of random variables $\{\varepsilon_n, n \ge 1\}$ is END, if every finite subfamily is END. In the case of $M = 1$, the notion of END random variables reduces to the well-known notion of so-called negatively dependent (ND, in short) random variables which was introduced by Lehmann [14]. The notion of END seems to be a straightforward generalization of the notion of negative dependence. The extended negative dependence structure is substantially more comprehensive. For the limit theorems of END random variables, one can be referred to other references [21, 23, 28, 31, 36, 37].

The first study in complete moment convergence was performed by Chow [3], where complete moment convergence is a more general expression than complete convergence. Thus, the complete moment convergence is one of the most important problems in the probability theory. Chow [3] obtained the following result.

Theorem 1.1 (Chow [3]) *Supposed that $\{X, X_n, n \ge 1\}$ is a sequence of independent and identically distributed random variables with $\mathbf{E}X = 0$. Let $p \ge 1$, $\alpha > 1/2$, $\alpha p > 1$, $\mathbf{E}|X|^p < \infty$ and $\mathbf{E}[|X| \log(1 + |X|)] < \infty$. Then for all $\epsilon > 0$, we have*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} \mathbf{E}\bigg(\max_{1 \le k \le n} \bigg| \sum_{t=1}^{k} X_t \bigg| - \epsilon n^{\alpha} \bigg)^{+} < \infty.$$

Researchers have extended Theorem 1.1 into the moving average processes. For example, Zhou [38] studied the case of NA random variables. Also Kim and Ko [8] studied the case of φ-mixing random variables. Kim et al. [9] generalized the result of Kim and Ko [8]. Ko et al. [11] investigated the case of $\rho^{\star}$-mixing random variables, while Kim [7] investigated the independent case. Sung [25] extended previous many results such as Kim and Ko [8], Kim et al. [9] and Kim [7] to random variables satisfying some suitable conditions. Yang et al. [33] and Yang and Hu [35] investigated the dependent cases of AANA random variables and pairwise NQD random variables, respectively. For more information about complete moment convergence results, there are multiple studies including Sung [24, 26], Wang et al. [29, 30], Guo [13], Wu et al. [32], Qiu and Chen [20], Yang et al. [34], Guo et al. [5] and so on.

In the following, we review the definition of the linear processes with the random coefficients.

Definition 1.2 *Let $\{\varepsilon_n, n \in \mathbb{Z}\}$ and $\{A_n, n \in \mathbb{Z}\}$ be two sequences of the random variables and*

$$X_t = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{t-j}.$$

Then $\{X_t, t > 0\}$ is a linear process with the random coefficients.

Recently, study on the properties of linear processes with random coefficients has attracted much attention. For example, Saavedre et al. [22] established the estimataion of population spectrum for linear processes with random coefficients. Kulik [12] obtained the limit theorems for moving averages with random coefficients and heavy tailed noise. Hosseini and Nezakati [6] demonstrated the convergence rates in the law of large numbers for END linear processes with random coefficients. The results are as following.

Theorem 1.3 (Hosseini and Nezakati [6]) *Suppose $r > 1$, $1 \le p < 2$, $1 < rp < 2$ and $X_t = \sum_{j=-\infty}^{\infty} A_j \varepsilon_{t-j}$ be a linear process with random coefficients, $\{\varepsilon_n, n \in \mathbb{Z}\}$ is a sequence of* END *random variable with mean zero and stochastically dominated by a nonnegative random variable ε. Moreover, suppose $\{A_n, n \in \mathbb{Z}\}$ be a sequence of* END *random variables with zero mean, $\sum_{j=-\infty}^{\infty} \mathbf{E}|A_j|^p < \infty$, and for some $rp < q \le 2$, $\sum_{j=-\infty}^{\infty} \mathbf{E}|A_j|^q < \infty$. Further, $\{\varepsilon_n, n \in \mathbb{Z}\}$ is independent of $\{A_n, n \in \mathbb{Z}\}$, then for all $\epsilon > 0$, if $\mathbf{E}\varepsilon^{rp} < \infty$,*

$$\sum_{n=1}^{\infty} n^{r-2} \mathbf{P}\left(\left| \sum_{t=1}^{n} X_t \right| > n^{1/p}\epsilon \right) < \infty,$$

and for $r = 1$, if $\mathbf{E}[\varepsilon^p \log(1+\varepsilon)] < \infty$,

$$\sum_{n=1}^{\infty} n^{-1} \mathbf{P}\left(\left| \sum_{t=1}^{n} X_t \right| > n^{1/p}\epsilon \right) < \infty.$$

In this paper, we investigate complete moment convergence and with using that, we prove strong laws and the Marcinkiewicz–Zygmund for dependent linear processes with random coefficients, where the $\{\varepsilon_n, n \in \mathbb{Z}\}$ is a sequence of END stochastically dominated by a nonnegative random variable ε. Some results of this paper are similar to Hosseini and Nezakati [6] . Also, we extend and improve the result of Philips and Solo [19] and Louhichi and Soulier [16] for non-randomly coefficients cases to the case of randomly coefficients. For the details, one can refer to the main results presented in Section 3. Some lemmas are presented in Section 2.

2 Some Lemmas

For the theorems and lemmas that have been presented in the following, we consider C as a positive constant, which may change line by line.

Lemma 2.1 (Sung [24]) *Suppose that $\{X_n, n \ge 1\}$ and $\{Y_n, n \ge 1\}$ be two sequences of random variables. Then for any $n \ge 1$, $q > 1$, $\epsilon > 0$ and $a > 0$, we have*

$$\mathbf{E}\left(\left| \sum_{t=1}^{n} (X_t + Y_t) \right| - \epsilon a \right)^+ \le \left(\frac{1}{\epsilon^q} + \frac{1}{q-1} \right) \frac{1}{a^{q-1}} \mathbf{E}\left| \sum_{t=1}^{n} X_n \right|^q + \mathbf{E}\left| \sum_{t=1}^{n} Y_t \right|,$$

and

$$\mathbf{E}\left(\max_{1\le k\le n} \left| \sum_{t=1}^{k} (X_t + Y_t) \right| - \epsilon a \right)^+ \le \left(\frac{1}{\epsilon^q} + \frac{1}{q-1} \right) \frac{1}{a^{q-1}} \mathbf{E}\left(\max_{1\le k\le n} \left| \sum_{t=1}^{k} X_t \right|^q \right) + \mathbf{E}\left(\max_{1\le k\le n} \left| \sum_{t=1}^{k} Y_t \right| \right).$$

Lemma 2.2 (Adler and Rosalsky [1] and Adler et al. [2]) *Suppose that $\{\varepsilon_n, n \ge 1\}$ is a sequence of random variables stochastically dominated by a nonnegative random variable ε, i.e. $\sup_{n\ge1} \mathbf{P}(|\varepsilon_n| > t) \le C\mathbf{P}(\varepsilon > t)$ for all $t \ge 0$. Then, for all $n \ge 1$, $a > 0$ and $b > 0$, the following conditions hold*

$$\mathbf{E}[|\varepsilon_n|^a \mathbf{I}_{\{|\varepsilon_n| \le b\}}] \le C(\mathbf{E}[\varepsilon^a \mathbf{I}_{\{\varepsilon \le b\}}] + b^a \mathbf{P}(\varepsilon > b)),$$

and

$$\mathbf{E}[|\varepsilon_n|^a \mathbf{I}_{\{|\varepsilon_n| > b\}}] \le C\mathbf{E}[\varepsilon^a \mathbf{I}_{\{\varepsilon > b\}}].$$

Consequently, one has $\mathbf{E}[|\varepsilon_n|^a] \le C\mathbf{E}[\varepsilon^a]$ for all $n \ge 1$.

Lemma 2.3 (Liu [15]) *Let $\{\varepsilon_n, n \geq 1\}$ be a sequence of* END *random variables and $\{f_n(.), n \geq 1\}$ are all nondecreasing (or nonincreasing) functions, then $\{f_n(\varepsilon_n), n \geq 1\}$ are* END *random variables.*

Lemma 2.4 (Ding et al. [4]) *Let $1 < p \leq 2$ and $\{\varepsilon_n, n \geq 1\}$ be a sequence of* END *random variables with some concrete constant $M > 0$. Assume further that $\mathbf{E}\varepsilon_n = 0$ and $\mathbf{E}|\varepsilon_n|^p < \infty$. Then there exists a positive constant $C(M;p)$ depending only on M and p such that*

$$\mathbf{E}\Big|\sum_{i=1}^{n}\varepsilon_i\Big|^p \leq C(M;p)\sum_{i=1}^{n}\mathbf{E}|\varepsilon_i|^p, \tag{2.1}$$

and

$$\mathbf{E}\max_{1<k\leq n}\Big|\sum_{i=1}^{k}\varepsilon_i\Big|^p \leq C(M;p)(\log n)^p\sum_{i=1}^{n}\mathbf{E}|\varepsilon_i|^p. \tag{2.2}$$

Lemma 2.5 (Hosseini and Nezakati [6]) *Suppose that for $1 \leq p \leq 2$, the $\{\varepsilon_n, n \in \mathbb{Z}\}$ be a sequence of* END *random variables with zero mean and $\mathbf{E}|\varepsilon_n|^p < \infty$, the $\{A_n, n \in \mathbb{Z}\}$ be a sequence of random variables with $\mathbf{E}|A_n|^p < \infty$. Further, $\{\varepsilon_n, n \in \mathbb{Z}\}$ is independent of $\{A_n, n \in \mathbb{Z}\}$. Then for fixed $n \geq 1$, if $\sum_{i=-\infty}^{\infty}(\sum_{j=1-i}^{n-i}A_j)\varepsilon_i < \infty$ a.s., then*

$$\mathbf{E}\Big|\sum_{i=-\infty}^{\infty}\Big(\sum_{j=1-i}^{n-i}A_j\Big)\varepsilon_i\Big|^p \leq C(M;p)\sum_{i=-\infty}^{\infty}\mathbf{E}\Big|\sum_{j=1-i}^{n-i}A_j\Big|^p\mathbf{E}|\varepsilon_i|^p.$$

Lemma 2.6 *Suppose that for $1 < p \leq 2$, the $\{\varepsilon_n, n \in \mathbb{Z}\}$ be a sequence of* END *random variables with zero mean and $\mathbf{E}|\varepsilon_n|^p < \infty$, the $\{A_n, n \in \mathbb{Z}\}$ be a sequence of random variables with $\mathbf{E}(\sum_{j=-\infty}^{\infty}|A_j|)^p < \infty$. Further, $\{\varepsilon_n, n \in \mathbb{Z}\}$ is independent of $\{A_n, n \in \mathbb{Z}\}$. Then*

$$\mathbf{E}\max_{1<k\leq n}\Big|\sum_{i=-\infty}^{\infty}\Big(\sum_{j=1-i}^{k-i}A_j\Big)\varepsilon_i\Big|^p \leq C(M;p)(\log n)^p\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|^p.$$

Proof Using of the Hölder's inequality and (2.2), we obtain

$$\begin{aligned}
&\mathbf{E}\max_{1<k\leq n}\Big|\sum_{i=-\infty}^{\infty}\Big(\sum_{j=1-i}^{k-i}A_j\Big)\varepsilon_i\Big|^p\\
&\quad= \mathbf{E}\max_{1<k\leq n}\Big|\sum_{j=-\infty}^{\infty}\Big(\sum_{i=1-j}^{k-j}\varepsilon_i\Big)A_j\Big|^p\\
&\quad\leq \mathbf{E}\Big|\sum_{j=-\infty}^{\infty}|A_j|\max_{1<k\leq n}\Big|\sum_{i=1-j}^{k-j}\varepsilon_i\Big|\Big|^p\\
&\quad= \mathbf{E}\Big|\sum_{j=-\infty}^{\infty}|A_j|^{1-\frac{1}{p}}|A_j|^{\frac{1}{p}}\max_{1<k\leq n}\Big|\sum_{i=1-j}^{k-j}\varepsilon_i\Big|\Big|^p\\
&\quad\leq \mathbf{E}\Big|\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{1-\frac{1}{p}}\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1<k\leq n}\Big|\sum_{i=1-j}^{k-j}\varepsilon_i\Big|^p\Big)^{\frac{1}{p}}\Big|^p\\
&\quad= \sum_{m=1}^{\infty}\mathbf{E}\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^{p-1}\Big(\sum_{j=-\infty}^{\infty}|A_j|\max_{1<k\leq n}\Big|\sum_{i=1-j}^{k-j}\varepsilon_i\Big|^p\Big)\mathbf{I}_{\{m-1\leq\sum_{j=-\infty}^{\infty}|A_j|<m\}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty} m^{p-1}\mathbf{E}[|A_j|\mathbf{I}_{\{m-1\le\sum_{j=-\infty}^{\infty}|A_j|<m\}}]\mathbf{E}\max_{1<k\le n}\Big|\sum_{i=1-j}^{k-j}\varepsilon_i\Big|^p \\
&\le C(M;p)(\log n)^p\sum_{m=1}^{\infty}\sum_{j=-\infty}^{\infty} m^{p-1}\mathbf{E}[|A_j|\mathbf{I}_{\{m-1\le\sum_{j=-\infty}^{\infty}|A_j|<m\}}]\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|^p \\
&\le C(M;p)(\log n)^p\Big(\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|^p\Big)\Big(1+2^p\sum_{m=2}^{\infty}\mathbf{E}\Big[\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^p\mathbf{I}_{\{m-1\le\sum_{j=-\infty}^{\infty}|A_j|<m\}}\Big]\Big) \\
&\le C(M;p)(\log n)^p\Big(\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|^p\Big)\Big(1+2^p\mathbf{E}\Big(\sum_{j=-\infty}^{\infty}|A_j|\Big)^p\Big) \\
&\le C(M;p)(\log n)^p\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|^p.
\end{aligned}
$$

Thus, the proof completes. □

Remark 2.7 In Lemma 2.6, for $p=1$, we have

$$
\mathbf{E}\max_{1<k\le n}\Big|\sum_{i=-\infty}^{\infty}\Big(\sum_{j=1-i}^{k-i}A_j\Big)\varepsilon_i\Big|\le C\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon_i|.
$$

3 Main Results

In the following, we prove the theorem of the complete moment convergence for the linear processes with random coefficients.

Theorem 3.1 *Suppose $\alpha>0$, $1<p<2$ and $X_t=\sum_{j=-\infty}^{\infty}A_j\varepsilon_{t-j}$ be a linear process with random coefficients and the $\{\varepsilon_n, n\in\mathbb{Z}\}$ be a sequence of* END *random variable with mean zero and stochastically dominated by a nonnegative random variable ε with $\mathbf{E}\varepsilon^p<\infty$. Moreover, suppose $\{A_n, n\in\mathbb{Z}\}$ be a sequence of* END *random variables with zero mean,*

$$
\sum_{j=-\infty}^{\infty}\mathbf{E}|A_j|<\infty, \tag{3.1}
$$

and for some $p<q\le 2$,

$$
\sum_{j=-\infty}^{\infty}\mathbf{E}|A_j|^q<\infty. \tag{3.2}
$$

Further, $\{\varepsilon_n, n\in\mathbb{Z}\}$ is independent of $\{A_n, n\in\mathbb{Z}\}$, then for all $\epsilon>0$,

$$
\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}\mathbf{E}\Big(\Big|\sum_{t=1}^{n}X_t\Big|-\epsilon n^{\alpha}\Big)^+<\infty. \tag{3.3}
$$

Proof Since $\mathbf{E}\varepsilon_i=0$, for all $i\in\mathbb{Z}$ and for each $n\ge 1$, we define

$$
\varepsilon_i=\varepsilon'_{ni}-\mathbf{E}\varepsilon'_{ni}+\varepsilon''_{ni}-\mathbf{E}\varepsilon''_{ni}, \tag{3.4}
$$

where

$$
\begin{aligned}
\varepsilon'_{ni}&=\varepsilon_i\mathbf{I}_{\{|\varepsilon_i|\le n^{\alpha}\}}+n^{\alpha}\mathbf{I}_{\{\varepsilon_i>n^{\alpha}\}}-n^{\alpha}\mathbf{I}_{\{\varepsilon_i<-n^{\alpha}\}}\,, \\
\varepsilon''_{ni}&=\varepsilon_i-\varepsilon'_{ni}=(\varepsilon_i-n^{\alpha})\mathbf{I}_{\{\varepsilon_i>n^{\alpha}\}}+(\varepsilon_i+n^{\alpha})\mathbf{I}_{\{\varepsilon_i<-n^{\alpha}\}}\,.
\end{aligned}
$$

Using Lemma 2.3, $\{\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni}, i \in \mathbb{Z}\}$ and $\{\varepsilon''_{ni} - \mathbf{E}\varepsilon''_{ni}, i \in \mathbb{Z}\}$ are END random variables with zero mean. It can be easily seen that

$$|\varepsilon'_{ni}| = |\varepsilon_i|\mathbf{I}_{\{|\varepsilon_i| \le n^\alpha\}} + n^\alpha \mathbf{I}_{\{|\varepsilon_i| > n^\alpha\}}\,, \tag{3.5}$$

$$\begin{aligned} |\varepsilon''_{ni}| &= (\varepsilon_i - n^\alpha)\mathbf{I}_{\{\varepsilon_i > n^\alpha\}} - (\varepsilon_i + n^\alpha)\mathbf{I}_{\{\varepsilon_i < -n^\alpha\}} \\ &\le |\varepsilon_i|\mathbf{I}_{\{|\varepsilon_i| > n^\alpha\}}. \end{aligned} \tag{3.6}$$

Therefore, for each $n \ge 1$, we get

$$\begin{aligned} \sum_{t=1}^{n} X_t &= \sum_{t=1}^{n} \sum_{j=-\infty}^{\infty} A_j \varepsilon_{t-j} \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j \varepsilon_i \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni}) + \sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon''_{ni} - \mathbf{E}\varepsilon''_{ni}). \end{aligned}$$

Hence, by Lemma 2.1 we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbf{E}\left(\left|\sum_{t=1}^{n} X_t\right| - \epsilon n^\alpha\right)^+ \\ &\quad\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \left(\frac{1}{\epsilon^q} + \frac{1}{q-1}\right) \frac{1}{n^{\alpha(q-1)}} \mathbf{E}\left|\sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni})\right|^q \\ &\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbf{E}\left|\sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon''_{ni} - \mathbf{E}\varepsilon''_{ni})\right| \\ &\quad\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \mathbf{E}\left|\sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni})\right|^q \\ &\qquad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \mathbf{E}\left|\sum_{i=-\infty}^{\infty} \sum_{j=1-i}^{n-i} A_j (\varepsilon''_{ni} - \mathbf{E}\varepsilon''_{ni})\right| \\ &\quad= H_1 + H_2. \end{aligned} \tag{3.7}$$

First, to prove H_1, with respect to the Markov's inequality, c_r-inequality, Lemmas 2.2 and 2.5, (3.2) and (3.5), we have

$$\begin{aligned} H_1 &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{i=-\infty}^{\infty} \mathbf{E}\left|\sum_{j=1-i}^{n-i} A_j\right|^q \mathbf{E}|\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni}|^q \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{i=-\infty}^{\infty} \mathbf{E}|\varepsilon'_{ni} - \mathbf{E}\varepsilon'_{ni}|^q \sum_{j=1-i}^{n-i} \mathbf{E}|A_j|^q \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{i=-\infty}^{\infty} \mathbf{E}|\varepsilon'_{ni}|^q \sum_{j=1-i}^{n-i} \mathbf{E}|A_j|^q \\ &\le C \sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2} \sum_{j=-\infty}^{\infty} \sum_{i=1-j}^{n-j} \mathbf{E}|A_j|^q \mathbf{E}[|\varepsilon_i|^q \mathbf{I}_{\{|\varepsilon_i| \le n^\alpha\}}] \end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} \mathbf{E}|A_j|^q \mathbf{P}(|\varepsilon_i| > n^\alpha) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\{\mathbf{E}[\varepsilon^q \mathbf{I}_{\{\varepsilon\leq n^\alpha\}}] + n^{\alpha q}\mathbf{P}(\varepsilon > n^\alpha)\} \sum_{j=-\infty}^{\infty} \mathbf{E}|A_j|^q \\
&+ C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} \mathbf{E}|A_j|^q \mathbf{E}[|\varepsilon_i| \mathbf{I}_{\{|\varepsilon_i|>n^\alpha\}}] \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\mathbf{E}[\varepsilon^q \mathbf{I}_{\{\varepsilon\leq n^\alpha\}}] + C\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbf{P}(\varepsilon > n^\alpha) \\
&+ C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\mathbf{E}[\varepsilon \mathbf{I}_{\{\varepsilon>n^\alpha\}}] \sum_{j=-\infty}^{\infty} \mathbf{E}|A_j|^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\mathbf{E}[\varepsilon^q \mathbf{I}_{\{\varepsilon\leq n^\alpha\}}] + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\mathbf{E}[\varepsilon \mathbf{I}_{\{\varepsilon>n^\alpha\}}] \\
&= H_{11} + H_{12}. \qquad (3.8)
\end{aligned}
$$

For H_{11}, since $p < q$, we have thus

$$
\begin{aligned}
H_{11} &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1} \sum_{i=1}^{n} \mathbf{E}[\varepsilon^q \mathbf{I}_{\{(i-1)^\alpha<\varepsilon\leq i^\alpha\}}] \\
&= C\sum_{i=1}^{\infty} \mathbf{E}[\varepsilon^q \mathbf{I}_{\{(i-1)^\alpha<\varepsilon\leq i^\alpha\}}] \sum_{n=i}^{\infty} n^{\alpha p-\alpha q-1} \\
&\leq C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q}\mathbf{E}[\varepsilon^q \mathbf{I}_{\{(i-1)^\alpha<\varepsilon\leq i^\alpha\}}] \\
&\leq C\mathbf{E}\varepsilon^p < \infty. \qquad (3.9)
\end{aligned}
$$

Next, for H_{12}, with $p > 1$, we obtain

$$
\begin{aligned}
H_{12} &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} \sum_{m=n}^{\infty} \mathbf{E}[\varepsilon \mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}] \\
&= C\sum_{m=1}^{\infty} \mathbf{E}[\varepsilon \mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}] \sum_{n=1}^{m} n^{\alpha p-\alpha-1} \\
&\leq C\sum_{m=1}^{\infty} m^{\alpha p-\alpha}\mathbf{E}[\varepsilon \mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}] \\
&\leq C\mathbf{E}\varepsilon^p < \infty. \qquad (3.10)
\end{aligned}
$$

Thus, the conclusion from (3.9) and (3.10) is $H_1 < \infty$.

For H_2, similar to H_1, we have

$$
\begin{aligned}
H_2 &\leq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \sum_{i=-\infty}^{\infty} \mathbf{E}|\sum_{j=1-i}^{n-i} A_j|\mathbf{E}|\varepsilon''_{ni} - \mathbf{E}\varepsilon''_{ni}| \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j} \mathbf{E}|A_j|\mathbf{E}|\varepsilon''_{ni}|
\end{aligned}
$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j}\mathbf{E}|A_j|\mathbf{E}[|\varepsilon_i|\mathbf{I}_{\{|\varepsilon_i|>n^\alpha\}}]$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon>n^\alpha\}}]\sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j}\mathbf{E}|A_j|$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon>n^\alpha\}}]$$

$$= C\sum_{m=1}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}]\sum_{n=1}^{m} n^{\alpha p-\alpha-1} \tag{3.11}$$

$$\leq C\sum_{m=1}^{\infty} m^{\alpha p-\alpha}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}]\leq C\mathbf{E}\varepsilon^p<\infty. \tag{3.12}$$

Therefore, (3.3) comes out immediately by (3.7)–(3.12). □

Remark 3.2 Let the conditions in Theorem 3.1 hold. Then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbf{P}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|>\epsilon n^\alpha\bigg)<\infty. \tag{3.13}$$

In fact, for all $\epsilon>0$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbf{E}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|-\epsilon n^\alpha\bigg)^+ &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\int_0^\infty \mathbf{P}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|-\epsilon n^\alpha>t\bigg)dt\\ &\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\int_0^{\epsilon n^\alpha} \mathbf{P}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|-\epsilon n^\alpha>t\bigg)dt\\ &\geq \epsilon\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbf{P}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|>2\epsilon n^\alpha\bigg).\end{aligned}$$

Thus (3.3) implies (3.13).

If $\alpha=1/p$, then for all $\epsilon>0$, by using (3.13), we obtain

$$\sum_{n=1}^{\infty} n^{-1}\mathbf{P}\bigg(\bigg|\sum_{t=1}^{n}X_t\bigg|>\epsilon n^{1/p}\bigg)<\infty.$$

Theorem 3.3 *For some $\alpha>0$ and $1<p<q\leq 2$, suppose $X_t=\sum_{j=-\infty}^{\infty}A_j\varepsilon_{t-j}$ be a linear process with random coefficients and the $\{\varepsilon_n, n\in\mathbb{Z}\}$ be a sequence of* END *random variable with mean zero and stochastically dominated by a nonnegative random variable ε with $\mathbf{E}[\varepsilon^p\log^q(1+\varepsilon)]<\infty$. Moreover, suppose $\{A_n, n\in\mathbb{Z}\}$ be a sequence of random variables with*

$$\mathbf{E}\bigg(\sum_{j=-\infty}^{\infty}|A_j|\bigg)^q<\infty. \tag{3.14}$$

Further, $\{\varepsilon_n, n\in\mathbb{Z}\}$ is independent of $\{A_n, n\in\mathbb{Z}\}$, then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbf{E}\bigg(\max_{1\leq k\leq n}\bigg|\sum_{t=1}^{k}X_t\bigg|-\epsilon n^\alpha\bigg)^+<\infty.$$

Proof Similar to the proof of Theorem 3.1, by Lemma 2.1 we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbf{E}\bigg(\max_{1\leq k\leq n}\bigg|\sum_{t=1}^{k}X_t\bigg|-\epsilon n^\alpha\bigg)^+$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\mathbf{E}\bigg(\max_{1\leq k\leq n}\bigg|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i} A_j(\varepsilon'_{ni}-\mathbf{E}\varepsilon'_{ni})\bigg|^q\bigg)$$
$$+\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\mathbf{E}\bigg(\max_{1\leq k\leq n}\bigg|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i} A_j(\varepsilon''_{ni}-\mathbf{E}\varepsilon''_{ni})\bigg|\bigg)$$
$$= H_1^* + H_2^*.$$

To prove H_1^*, with respect to the c_r-inequality, Lemma 2.6 and (3.14), we obtain

$$H_1^* \leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}(\log n)^q \sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon'_{ni}-\mathbf{E}\varepsilon'_{ni}|^q$$
$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}(\log n)^q\mathbf{E}[\varepsilon^q\mathbf{I}_{\{\varepsilon\leq n^\alpha\}}]$$
$$+ C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon> n^\alpha\}}]$$
$$= H_{11}^* + H_{12}^*.$$

For H_{11}^*, we have thus

$$H_{11}^* \leq C\sum_{i=1}^{\infty}\mathbf{E}[\varepsilon^q\mathbf{I}_{\{(i-1)^\alpha<\varepsilon\leq i^\alpha\}}]\sum_{n=i}^{\infty} n^{\alpha p-\alpha q-1}(\log n)^q$$
$$\leq C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q}(\log i)^q\mathbf{E}[\varepsilon^q\mathbf{I}_{\{(i-1)^\alpha<\varepsilon\leq i^\alpha\}}]$$
$$\leq C\mathbf{E}[\varepsilon^p\log^q(1+\varepsilon)]<\infty.$$

Next, for H_{12}^*, with $p>1$, we obtain

$$H_{12}^* \leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}(\log n)^q\sum_{m=n}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}]$$
$$\leq C\sum_{m=1}^{\infty} m^{\alpha p-\alpha}(\log m)^q\mathbf{E}[\varepsilon\mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}]$$
$$\leq C\mathbf{E}[\varepsilon^p\log^q(1+\varepsilon)]<\infty.$$

Finally, for H_2^*, noting that $\sum_{j=-\infty}^{\infty}\mathbf{E}|A_j|<\infty$ by (3.14). Therefore, by using Remark 2.7 we get

$$H_2^* \leq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sup_{j\in\mathbb{Z}}\sum_{i=1-j}^{n-j}\mathbf{E}|\varepsilon''_{ni}-\mathbf{E}\varepsilon''_{ni}|$$
$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon>n^\alpha\}}]$$
$$\leq C\sum_{m=1}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^\alpha<\varepsilon\leq(m+1)^\alpha\}}]\sum_{n=1}^{m} n^{\alpha p-\alpha-1}$$
$$\leq C\mathbf{E}[\varepsilon^p]<\infty.$$

Therefore, the proof completes. □

Remark 3.4 Under the conditions in Theorem 3.3, for every $\epsilon > 0$, it can be concluded that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbf{P}\bigg(\max_{1\le k\le n}\bigg|\sum_{t=1}^{k} X_t\bigg| > \epsilon n^{\alpha}\bigg) < \infty.$$

If $\alpha = 1/p$, then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1}\mathbf{P}\bigg(\max_{1\le k\le n}\bigg|\sum_{t=1}^{k} X_t\bigg| > \epsilon n^{1/p}\bigg) < \infty. \tag{3.15}$$

In the following results, by using (3.15), we show the Marcinkiewicz–Zygmund strong law for linear processes with random coefficients.

Corollary 3.5 *Suppose the conditions in Theorem* 3.3 *hold for* $1 < p < 2$*. Then*

$$\frac{1}{n^{1/p}}\sum_{t=1}^{n} X_t \xrightarrow{a.s.} 0 \quad \textit{as } n \to \infty.$$

Theorem 3.6 *Suppose* $\alpha > 0$ *and* $X_t = \sum_{j=-\infty}^{\infty} A_j\varepsilon_{t-j}$ *be a linear process with random coefficients and the* $\{\varepsilon_n, n \in \mathbb{Z}\}$ *be a sequence of* END *random variable with mean zero and stochastically dominated by a nonnegative random variable* ε *with* $\mathbf{E}[\varepsilon\log(1+\varepsilon)] < \infty$*. Moreover, suppose* $\{A_n, n \in \mathbb{Z}\}$ *be a sequence of* END *random variables with zero mean,*

$$\sum_{j=-\infty}^{\infty} \mathbf{E}|A_j| < \infty,$$

and for some $1 < q \le 2$*,*

$$\sum_{j=-\infty}^{\infty} \mathbf{E}|A_j|^q < \infty.$$

Further, $\{\varepsilon_n, n \in \mathbb{Z}\}$ *is independent of* $\{A_n, n \in \mathbb{Z}\}$*, then for all* $\epsilon > 0$*,*

$$\sum_{n=1}^{\infty} n^{-2}\mathbf{E}\bigg(\bigg|\sum_{t=1}^{n} X_t\bigg| - \epsilon n^{\alpha}\bigg)^{+} < \infty. \tag{3.16}$$

Proof Similar to the proof of Theorem 3.1, from (3.7), for H_{11}, in (3.8), since $1 < q \le 2$, we have thus

$$\begin{aligned}
H_{11} &= C\sum_{n=1}^{\infty} n^{\alpha-\alpha q-1}\sum_{i=1}^{n}\mathbf{E}[\varepsilon^q\mathbf{I}_{\{(i-1)^{\alpha}<\varepsilon\le i^{\alpha}\}}] \\
&= C\sum_{i=1}^{\infty}\mathbf{E}[\varepsilon^q\mathbf{I}_{\{(i-1)^{\alpha}<\varepsilon\le i^{\alpha}\}}]\sum_{n=i}^{\infty} n^{\alpha-\alpha q-1} \\
&\le C\sum_{i=1}^{\infty} i^{\alpha-\alpha q}\mathbf{E}[\varepsilon^q\mathbf{I}_{\{(i-1)^{\alpha}<\varepsilon\le i^{\alpha}\}}] \\
&\le C\mathbf{E}\varepsilon < \infty.
\end{aligned} \tag{3.17}$$

Next, for H_{12}, we obtain

$$\begin{aligned}
H_{12} &= C\sum_{n=1}^{\infty} n^{-1}\sum_{m=n}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^{\alpha}<\varepsilon\le(m+1)^{\alpha}\}}] \\
&= C\sum_{m=1}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^{\alpha}<\varepsilon\le(m+1)^{\alpha}\}}]\sum_{n=1}^{m} n^{-1}
\end{aligned}$$

$$\leq C\sum_{m=1}^{\infty}\log(1+m)\mathbf{E}[\varepsilon\mathbf{I}_{\{m^{\alpha}<\varepsilon\leq(m+1)^{\alpha}\}}]$$
$$\leq C\mathbf{E}[\varepsilon\log(1+\varepsilon)]<\infty. \tag{3.18}$$

Finally, for H_2, in (3.11) with $p=1$, we have

$$H_2\leq C\sum_{m=1}^{\infty}\mathbf{E}[\varepsilon\mathbf{I}_{\{m^{1/p}<\varepsilon\leq(m+1)^{1/p}\}}]\sum_{n=1}^{m}n^{-1}$$
$$\leq C\sum_{m=1}^{\infty}\log(1+m)\mathbf{E}[\varepsilon\mathbf{I}_{\{m^{1/p}<\varepsilon\leq(m+1)^{1/p}\}}]$$
$$=C\mathbf{E}[\varepsilon\log(1+\varepsilon)]<\infty. \tag{3.19}$$

Consequently, by (3.17)–(3.19), we obtain (3.16). Therefore, the proof is completed. □

Remark 3.7 Suppose the conditions in Theorem 3.6 hold. Then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty}n^{\alpha-2}\mathbf{P}\left(\left|\sum_{t=1}^{n}X_t\right|>\epsilon n^{\alpha}\right)<\infty.$$

If $\alpha=1$, then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty}n^{-1}\mathbf{P}\left(\left|\sum_{t=1}^{n}X_t\right|>\epsilon n\right)<\infty.$$

Theorem 3.8 *Suppose $X_t=\sum_{j=-\infty}^{\infty}A_j\varepsilon_{t-j}$ be a linear process with random coefficients and the $\{\varepsilon_n,n\in\mathbb{Z}\}$ be a sequence of* END *random variable with mean zero and stochastically dominated by a nonnegative random variable ε with $\mathbf{E}[\varepsilon\log^3(1+\varepsilon)]<\infty$. Moreover, suppose $\{A_n,n\in\mathbb{Z}\}$ be a sequence of random variables with*

$$\mathbf{E}\left(\sum_{j=-\infty}^{\infty}|A_j|\right)^2<\infty.$$

Further, $\{\varepsilon_n,n\in\mathbb{Z}\}$ is independent of $\{A_n,n\in\mathbb{Z}\}$, then for all $\epsilon>0$,

$$\sum_{n=1}^{\infty}n^{-2}\mathbf{E}\left(\max_{1\leq k\leq n}\left|\sum_{t=1}^{k}X_t\right|-\epsilon n\right)^+<\infty.$$

Proof Similar to the proof of Theorem 3.3, we have thus

$$\sum_{n=1}^{\infty}n^{-2}\mathbf{E}\left(\max_{1\leq k\leq n}\left|\sum_{t=1}^{k}X_t\right|-\epsilon n\right)^+$$
$$\leq C\sum_{n=1}^{\infty}n^{-3}\mathbf{E}\left(\max_{1\leq k\leq n}\left(\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i}A_j(\varepsilon'_{ni}-\mathbf{E}\varepsilon'_{ni})\right)^2\right)$$
$$+\sum_{n=1}^{\infty}n^{-2}\mathbf{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty}\sum_{j=1-i}^{k-i}A_j(\varepsilon''_{ni}-\mathbf{E}\varepsilon''_{ni})\right|\right)$$
$$\leq C\sum_{n=1}^{\infty}n^{-2}\log^2 n\mathbf{E}[\varepsilon^2\mathbf{I}_{\{\varepsilon\leq n\}}]+C\sum_{n=1}^{\infty}n^{-1}\log^2 n\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon>n\}}]+C\sum_{n=1}^{\infty}n^{-1}\mathbf{E}[\varepsilon\mathbf{I}_{\{\varepsilon>n\}}]$$
$$\leq C\sum_{m=1}^{\infty}m^{-1}(\log^2 m+2\log m+2)\mathbf{E}[\varepsilon^2\mathbf{I}_{\{m-1<\varepsilon\leq m\}}]+C\sum_{m=1}^{\infty}\log^3 m\mathbf{E}[\varepsilon\mathbf{I}_{\{m<\varepsilon\leq(m+1)\}}]$$

$$+ C\sum_{m=1}^{\infty}\log m\mathbf{E}[\varepsilon\mathbf{I}_{\{m<\varepsilon\le(m+1)\}}] \le C\mathbf{E}[\varepsilon\log^3(1+\varepsilon)] < \infty.$$

Therefore, the proof is completed. □

Remark 3.9 Suppose the conditions in Theorem 3.8 hold. Then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1}\mathbf{P}\left(\max_{1\le k\le n}\left|\sum_{t=1}^{k} X_t\right| > \epsilon n\right) < \infty. \tag{3.20}$$

In the following results, by using (3.20), we show the strong law of large numbers for linear processes with random coefficients.

Corollary 3.10 *Let the conditions in Theorem* 3.8 *hold. Then*

$$\frac{1}{n}\sum_{t=1}^{n} X_t \xrightarrow{a.s.} 0 \quad \text{as } n\to\infty.$$

Remark 3.11 If $\{A_n = a_n, n\in\mathbb{Z}\}$ is a non-random sequence (the case of constant coefficients), then, with the condition $\sum_{j=-\infty}^{\infty}|a_j| < \infty$, we can get results of Theorem 3.3 and Theorem 3.8 for the linear processes with constant coefficients. These results improve and extend the corresponding results of Philips and Solo [19] and Louhichi and Soulier [16], respectively.

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References

[1] Adler, A., Rosalsky, A.: Some general strong laws for weighted sums of stochastically dominated random variables. *Stoch. Anal. Appl.*, **5**(1), 1–16 (1987)

[2] Adler, A., Rosalsky, A., Taylor, R. L.: Strong laws of large numbers for weighted sums of random elements in normed linear spaces. *Int. J. Math. Math. Sci.*, **12**(3), 507–529 (1989)

[3] Chow, Y. S.: On the rate of moment convergence of sample sums and extremes. *Bull. Inst. Math. Acad. Sin.*, **16**(3), 177–201 (1988)

[4] Ding, Y., Tang, X., Deng, X., et al.: Complete moment convergence for weighted sums of extended negatively dependent random variables. *Filomat*, **31**(14), 4341–4352 (2017)

[5] Guo, M., Zhu, D., Ren, Y.: Complete q th moment convergence of weighted sums for arrays of rowwise negatively associated random variables. *Stochastics*, **87**(2), 257–272 (2015)

[6] Hosseini, S. H., Nezakati, A.: Convergence rates in the law of large numbers for END linear processes with random coefficients, Comm. Statist. Theory Methods, 2018

[7] Kim, H. C.: On the complete moment convergence of moving average processes generated by iid random variables. *Korean Ann. Math.*, **25**(1), 7–17 (2008)

[8] Kim, T. S., Ko, M. H.: Complete moment convergence of moving average processes under dependence assumptions. *Statist. Probab. Lett.*, **78**(7), 839–846 (2008)

[9] Kim, T. S., Ko, M. H., Choi, Y. K.: Complete moment convergence of moving average processes with dependent innovations. *J. Korean Math. Soc.*, **45**(2), 355–365 (2008)

[10] Ko, M. H.: Strong laws of large numbers for linear processes generated by associated random variables in a Hilbert space. *Honam Math. J.*, **30**(4), 703–711 (2008)

[11] Ko, M. H., Kim, T. S., Ryu, D. H.: On the complete moment convergence of moving average processes generated by $\rho^\star$-mixing sequences. *Commun. Korean Math. Soc.*, **23**(4), 597–606 (2008)

[12] Kulik, R.: Limit theorems for moving averages with random coefficients and heavy-tailed noise. *J. Appl. Probab.*, **43**(1), 245–256 (2006)

[13] Le Guo, M.: Equivalent conditions of complete moment convergence of weighted sums for φ-mixing sequence of random variables. *Comm. Statist. Theory Methods*, **43**(10–12), 2527–2539 (2014)

[14] Lehmann, E. L.: Some concepts of dependence. *Ann. Math. Statist.*, 1137–1153 (1966)

[15] Liu, L.: Necessary and sufficient conditions for moderate deviations of dependent random variables with heavy tails. *Sci. China Math.*, **53**(6), 1421–1434 (2010)

[16] Louhichi, S., Soulier, P.: Marcinkiewicz–Zygmund strong laws for infinite variance time series. *Stat. Inference Stoch. Process.*, **3**(1), 31–40 (2000)

[17] McLeish, D. L.: Invariance principles for dependent variables. *Probab. Theory Related Fields*, **32**(3), 165–178 (1975)

[18] McLeish, D. L.: maximal inequality and dependent strong laws. *Ann. Probab.*, **3**(5), 829–839 (1975)

[19] Phillips, P. C., Solo, S.: Asymptotics for linear processes. *Ann. Statist.*, 971–1001 (1992)

[20] Qiu, D., Chen, P.: Complete moment convergence for iid random variables. *Statist. Probab. Lett.*, **91**, 76–82 (2014)

[21] Qiu, D., Chen, P.: Complete moment convergence for product sums of sequence of extended negatively dependent random variables. *J. Inequal. Appl.*, 2015:212, 15 pp., doi:10.1186/s13660-015-0730-4

[22] Saavedra, P., et al.: Estimation of population spectrum for linear processes with random coefficients. *Comput. Statist.*, **23**(1), 79–98 (2008)

[23] Shen, A.: Complete convergence for weighted sums of END random variables and its application to nonparametric regression models. *J. Nonparametr. Stat.*, **28**(4), 702–715 (2016)

[24] Sung, S. H.: Moment inequalities and complete moment convergence. *J. Inequal. Appl.*, Article ID 271265, 14 pp., doi:10.1155/2009/271265

[25] Sung, S. H.: Convergence of moving average processes for dependent random variables. *Comm. Statist. Theory Methods*, **40**(13), 2366–2376 (2011)

[26] Sung, S. H.: Complete q th moment convergence for arrays of random variables. *J. Inequal. Appl.*, 2013:24, 11 pp, doi:10.1186/1029-242X-2013-24

[27] Wang, J., Wu, Q.: Central limit theorem for stationary linear processes generated by linearly negative quadrant-dependent sequence. *J. Inequal. Appl.*, 2012:45, 7 pp., doi:10.1186/1029-242X-2012-45

[28] Wang, S., Wang, X.: Precise large deviations for random sums of END real-valued random variables with consistent variation. *J. Math. Anal. Appl.*, **402**(2), 660–667 (2013)

[29] Wang, X., Shen, A., Li, X.: A note on complete convergence of weighted sums for array of rowwise AANA random variables. *J. Inequal. Appl.*, 2013:359., 13 pp., doi:10.1186/1029-242X-2013-359

[30] Wang, X., Li, X., Hu, S.: Complete convergence of weighted sums for arrays of rowwise φ-mixing random variables. *Appl. Math.*, **59**(5), 589–607 (2014)

[31] Wang, X., et al.: Complete consistency for the estimator of nonparametric regression models based on extended negatively dependent errors. *Statistics*, **49**(2), 396–407 (2015)

[32] Wu, Y. F., Cabrea, M. O., Volodin, A.: Complete convergence and complete moment convergence for arrays of rowwise END random variables. *Glas. Mat.*, **49**(69), 449–468 (2014)

[33] Yang, W., et al.: On complete convergence of moving average process for AANA sequence. Discrete Dyn. Nat. Soc., Article ID315138, 15 pp., doi:10.1155/2012/863931

[34] Yang, W., et al.: The convergence of double-indexed weighted sums of martingale differences and its application. *Abstr. Appl. Anal.*, 2014, Article ID 893906 (2014)

[35] Yang, W., Hu, S.: Complete moment convergence of pairwise NQD random variables. *Stochastics*, **87**(2), 199–208 (2015)

[36] Zhang, G.: Complete convergence for Sungs type weighted sums of END random variables. *J. Inequal. Appl.*, Article ID 383805, 8 pp., doi:10.1155/2010/383805

[37] Zhang, S., Qu, C., Sun, F.: On the Complete Convergence for END Random Variables. *Appl. Math. Sci.*, **12**(4), 161–173 (2018)

[38] Zhou, X.: Complete moment convergence of moving average processes under φ-mixing assumptions. *Statist. Probab. Lett.*, **80**(5–6), 285–292 (2010)