Acta Mathematica Sinica, English Series
Jun., 2019, Vol. 35, No. 6, pp. 1074–1084
Published online: May 15, 2019
https://doi.org/10.1007/s10114-019-8199-6
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Extrapolation for the L^p Dirichlet Problem in Lipschitz Domains

Zhongwei Shen
Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506*, USA*
E-mail: *zshen2@uky.edu*

Dedicated to Carlos E. Kenig on the Occasion of His 65th Birthday

Abstract Let $\mathcal{L}$ be a second-order linear elliptic operator with complex coefficients. It is shown that if the L^p Dirichlet problem for the elliptic system $\mathcal{L}(u) = 0$ in a fixed Lipschitz domain Ω in $\mathbb{R}^d$ is solvable for some $1 < p = p_0 < \frac{2(d-1)}{d-2}$, then it is solvable for all p satisfying

$$p_0 < p < \frac{2(d-1)}{d-2} + \varepsilon.$$

The proof is based on a real-variable argument. It only requires that local solutions of $\mathcal{L}(u) = 0$ satisfy a boundary Cacciopoli inequality.

Keywords Dirichlet problem, Lipschitz domain, extrapolation

MR(2010) Subject Classification 35J57

1 Introduction

In this paper we consider the L^p Dirichlet problem for an $m \times m$ second-order elliptic system,

$$\begin{cases} \mathcal{L}(u) = 0 & \text{in } \Omega, \\ u = f \in L^p(\partial\Omega; \mathbb{C}^m) & \text{on } \partial\Omega, \\ N(u) \in L^p(\partial\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$ and $N(u)$ denotes the (modified) nontangential maximal function of u. The operator $\mathcal{L}$ in (1.1) is a second-order linear elliptic operator with complex coefficients. It may contain lower order terms and needs not to be in divergence form. Instead we shall impose the following condition.

Let $r_0 = \mathrm{diam}(\Omega)$. There exist constants $\kappa > 0$ and $c_0 > 0$ such that the boundary Cacciopoli inequality

$$\int_{B(x_0,r)\cap\Omega} |\nabla u|^2 \, dx \le \frac{\kappa}{r^2} \int_{B(x_0,2r)\cap\Omega} |u|^2 \, dx \tag{1.2}$$

holds, whenever $x_0 \in \partial\Omega$, $0 < r < c_0 r_0$, and $u \in W^{1,2}(B(x_0, 2r) \cap \Omega; \mathbb{C}^m)$ is a weak solution to $\mathcal{L}(u) = 0$ in $B(x_0, 2r) \cap \Omega$ with $u = 0$ on $B(x_0, 2r) \cap \partial\Omega$.

Received May 10, 2018, accepted October 26, 2018
Supported in part by NSF (Grant No. DMS-1600520)

Theorem 1.1 *Let Ω be a (fixed) bounded Lipschitz domain in $\mathbb{R}^d$ and $1 < p_0 < \frac{2(d-1)}{d-2}$. Let $\mathcal{L}$ be a second-order linear elliptic operator satisfying the condition* (1.2)*. Assume that for any $f \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m)$, there exists a weak solution $u \in W^{1,2}(\Omega;\mathbb{C}^m)$ to $\mathcal{L}(u) = 0$ in Ω such that $u = f$ on $\partial\Omega$ in the sense of trace, and $\|N(u)\|_{L^{p_0}(\partial\Omega)} \le C_0\|f\|_{L^{p_0}(\partial\Omega)}$. Then the weak solution u satisfies the L^p estimate*

$$\|N(u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)} \tag{1.3}$$

for any p satisfying

$$p_0 < p < \frac{2(d-1)}{d-2} + \varepsilon, \tag{1.4}$$

where $\varepsilon > 0$ depends only on d, m, p_0, κ, c_0, C_0 and the Lipschitz character of Ω. The constant C in (1.3) *depends on d, m, p_0, p, κ, c_0, C_0 and the Lipschitz character of Ω.*

We remark that in the scalar case $m = 1$ with real coefficients, the maximum principle $\|u\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\partial\Omega)}$ holds for weak solutions of $\mathcal{L}(u) = 0$ in Ω. It follows by interpolation that if the estimate (1.3) holds for $p = p_0$, then it holds for any $p_0 < p \le \infty$. However, it is known that the maximum principle or its weak version $\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^\infty(\partial\Omega)}$ is not available in Lipschitz domains for elliptic systems or scalar elliptic equations with complex coefficients. Theorem 1.1 provides a partial solution to this problem.

The analogous of Theorem 1.1 also holds if Ω is the region above a Lipschitz graph,

$$\Omega = \{(x', x_d) \in \mathbb{R}^d : \ x_d > \psi(x')\}, \tag{1.5}$$

where $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with $\|\nabla\psi\|_\infty \le M$.

Theorem 1.2 *Let Ω be a (fixed) graph domain in $\mathbb{R}^d$, given by* (1.5)*, and $1 < p_0 < \frac{2(d-1)}{d-2}$. Let $\mathcal{L}$ be a second-order linear elliptic operator satisfying the condition* (1.2) *with $r_0 = \infty$. Assume that for any $f \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m)$, there exists a weak solution $u \in W^{1,2}_{\text{loc}}(\overline{\Omega};\mathbb{C}^m)$ to $\mathcal{L}(u) = 0$ in Ω such that $u = f$ on $\partial\Omega$ in the sense of trace, and $\|N(u)\|_{L^{p_0}(\partial\Omega)} \le C_0\|f\|_{L^{p_0}(\partial\Omega)}$. Then the weak solution u satisfies the estimate* (1.3) *for any p satisfying* (1.4)*, where $\varepsilon > 0$ depends only on d, m, p_0, κ, C_0 and M. The constant C in* (1.3) *depends on d, m, p_0, p, κ, C_0 and M.*

Remark 1.3 Regarding the boundary Cacciopoli inequality (1.2) in a graph domain Ω, consider the elliptic operator

$$(\mathcal{L}(u))^\alpha = -\frac{\partial}{\partial x_i}\left\{a_{ij}^{\alpha\beta}(x)\frac{\partial u^\beta}{\partial x_j}\right\} + b_j^{\alpha\beta}(x)\frac{\partial u^\beta}{\partial x_j}, \tag{1.6}$$

where $1 \le \alpha, \beta \le m$ and $1 \le i, j \le d$ (the repeated indices are summed). Assume that the coefficients $a_{ij}^{\alpha\beta}(x)$ are complex-valued bounded functions satisfying $\|a_{ij}^{\alpha\beta}\|_\infty \le \mu^{-1}$ and the ellipticity condition

$$\mathrm{Re}(a_{ij}^{\alpha\beta}(x)\xi_j^\beta\overline{\xi_i^\alpha}) \ge \mu|\xi|^2 \tag{1.7}$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{C}^{m\times d}$, where $\mu > 0$. Also assume that there exists some $\nu > 0$ such that

$$\delta(x)|b_j^{\alpha\beta}(x)| \le \nu \tag{1.8}$$

for any $x \in \Omega$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Then there exists a constant $\nu_0 > 0$, depending only on d, m, μ and M, such that if $\nu \le \nu_0$, the Cacciopoli inequality (1.2) holds for any $0 < r < \infty$. This may be proved by using Hardy's inequality. In the case of a bounded Lipschitz domain, one only needs to assume (1.8) with $\nu \le \nu_0$ for x sufficiently close to $\partial\Omega$ ($\delta(x) \le c_0 r_0$).

Remark 1.4 Let $d \ge 3$. If the Dirichlet problem (1.1) is solvable for $p = p_0 = \frac{2(d-1)}{d-2}$, our argument gives the solvability for $p_0 < p < p_0 + \varepsilon$.

The L^p boundary value problems for second-order elliptic equations and systems in Lipschitz domains have been studied extensively. We refer the reader to [1, 2, 6, 11–14, 16] for references. In particular, the L^2 Dirichlet problem is solvable for elliptic systems with real constant coefficients satisfying the Legendre–Hadamard condition and the symmetry condition [5, 7, 8, 10]. It is also known that under the same assumption, the L^p Dirichlet problem is solvable for $2 - \varepsilon < p \le \infty$ if $d = 3$ [4], and for $2 - \varepsilon < p < \frac{2(d-1)}{d-3} + \varepsilon$ if $d \ge 4$ [16]. More recent work in this area focuses on operators with complex coefficients or real coefficients without the symmetry condition [1, 2, 11–13].

As in [16], the proof of Theorems 1.1 and 1.2 is based on a real-variable method, which may be regarded as a dual version of the celebrated Calderón–Zygmund Lemma. The method was originated in [3] and was further developed in [15–17]. It reduces the L^p estimate (1.3) to the reverse Hölder inequality,

$$\left(\fint_{B(x_0,r)\cap\partial\Omega} |N(u)|^q \, d\sigma \right)^{1/q} \le C \left(\fint_{B(x_0,2r)\cap\partial\Omega} |N(u)|^{p_0} \, d\sigma \right)^{1/p_0} \tag{1.9}$$

for $q = \frac{2(d-1)}{d-2}$ (for any $2 < q < \infty$, if $d = 2$), where $x_0 \in \partial\Omega$, u is a weak solution to $\mathcal{L}(u) = 0$ in Ω with $u = 0$ in $B(x_0, 3r) \cap \partial\Omega$. To prove (1.9), we replace $N(u)$ by $N^r(u)$, a localized nontangential maximal function at height r (see Section 2 for definition), and use the observation

$$\int_{B(x_0,r)\cap\partial\Omega} |N^r(u)|^q \, d\sigma \le C \int_{B(x_0,2r)\cap\Omega} |u(y)|^q \delta(y)^{-1} \, dy. \tag{1.10}$$

The right-hand side of (1.10) is then handled by using Sobolev inequality and Hardy's inequality,

$$\int_{B(x_0,2r)\cap\Omega} \frac{|u(y)|^2}{\delta(y)^2} \, dy \le C \int_{B(x_0,2r)\cap\Omega} |\nabla u|^2 \, dy. \tag{1.11}$$

The exponent $q = \frac{2(d-1)}{d-2}$ arises in the use of Sobolev inequality

$$\|u\|_{L^{2(q-1)}(B(x_0,2r)\cap\Omega)} \le C\|\nabla u\|_{L^2(B(x_0,2r)\cap\Omega)}. \tag{1.12}$$

It may be worthy to point out that q is also the exponent in the boundary Sobolev inequality $\|u\|_{L^q(\partial\Omega)} \le C\|u\|_{H^{1/2}(\partial\Omega)}$.

2 Reverse Hölder Inequalities

Throughout this section we assume that Ω is the region above a Lipschitz graph in $\mathbb{R}^d$, given by (1.5) with $\|\nabla\psi\|_\infty \le M$. A nontangential approach region at $z \in \partial\Omega$ is given by

$$\Gamma_a(z) = \{x \in \Omega : \ |x - z| < a\,\delta(x)\}, \tag{2.1}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and $a > 1 + 2M$. We also need a truncated version

$$\Gamma_a^h(z) = \{x \in \Omega : \ |x - z| < a\,\delta(x) \text{ and } \delta(x) < h\}, \tag{2.2}$$

where $h > 0$. For $u \in L^2_{\mathrm{loc}}(\Omega)$, the modified nontangential maximal function of u is defined by

$$N_a(u)(z) = \sup\left\{ \left(\fint_{B(x,(1/4)\delta(x))} |u|^2 \right)^{1/2} : \ x \in \Gamma_a(z) \right\} \tag{2.3}$$

for each $z \in \partial\Omega$. Similarly, we introduce

$$N_a^h(u)(z) = \sup\left\{ \left(\fint_{B(x,(1/4)\delta(x))} |u|^2 \right)^{1/2} : \ x \in \Gamma_a^h(z) \right\}. \tag{2.4}$$

The definitions of $N_a(u)$ and $N_a^h(u)$ are same if Ω is a bounded Lipschitz domain. We will drop the subscript a if there is no confusion.

Lemma 2.1 *Let $2 \le q < \infty$. Then*

$$N_a^h(u)(z) \le C\left(\int_{\Gamma_{2a}^{2h}(z)} |u(y)|^q \delta(y)^{-d}\, dy \right)^{1/q} \tag{2.5}$$

for any $z \in \partial\Omega$, where C depends only on d and q.

Proof Fix $x \in \Gamma_a^h(z)$. Let $y \in B(x,(1/4)\delta(x))$. Note that

$$\delta(y) \le \delta(x) + |x-y| < (5/4)\delta(x).$$

Since $\delta(x) \le \delta(y) + |x-y| < \delta(y) + (1/4)\delta(x)$, we obtain $(3/4)\delta(x) < \delta(y)$. It follows that

$$\begin{aligned} |y-z| &\le |x-z| + |x-y| < (a+(1/4))\delta(x) \\ &\le (4/3)(a+(1/4))\delta(y) \\ &\le 2a\delta(y), \end{aligned}$$

where we have used the fact $a > 1$. Also observe that $\delta(y) < (5/4)\delta(x) < (5/4)h$. Thus we have proved that $B(x,(1/4)\delta(x)) \subset \Gamma_{2a}^{2h}(z)$. This, together with Hölder's inequality, gives

$$\begin{aligned} \left(\fint_{B(x,(1/4)\delta(x))} |u(y)|^2\, dy \right)^{1/2} &\le \left(\fint_{B(x,(1/4)\delta(x))} |u(y)|^q\, dy \right)^{1/q} \\ &\le C\left(\int_{\Gamma_{2a}^{2h}(z)} |u(y)|^q \delta(y)^{-d}\, dy \right)^{1/q}, \end{aligned}$$

where C depends only on d and q. The inequality (2.5) now follows by definition. □

Assume that $\psi(0) = 0$. For $r > 0$, define

$$\begin{aligned} D_r &= \{(x', x_d) \in \mathbb{R}^d : \ |x'| < r \text{ and } \psi(x') < x_d < 2(M+1)r\}, \\ \Delta_r &= \{(x', \psi(x')) \in \mathbb{R}^d : \ |x'| < r\}. \end{aligned} \tag{2.6}$$

Lemma 2.2 *Suppose that $u \in H^1(D_r)$ and $u = 0$ on Δ_r. Then*

$$\int_{D_r} \frac{|u(x)|^2}{\widetilde{\delta}(x)^2}\, dx \le 4\int_{D_r} |\nabla u|^2\, dx, \tag{2.7}$$

where $\widetilde{\delta}(x) = |x_d - \psi(x')|$.

Proof Using $u(x', \psi(x')) = 0$ and Fubini's Theorem, we obtain

$$\begin{aligned} \int_{D_r} \frac{|u(x)|^2}{\widetilde{\delta}(x)^2}\, dx &= \int_{|x'|<r} \int_{\psi(x')}^{2(1+M)r} \frac{|u(x', x_d)|^2}{|x_d - \psi(x')|^2}\, dx_d dx' \\ &\le 4\int_{|x'|<r} \int_{\psi(x')}^{2(1+M)r} \left| \frac{\partial u}{\partial x_d} \right|^2 dx_d dx' \\ &\le 4\int_{D_r} |\nabla u|^2\, dx, \end{aligned}$$

where we have used the Hardy inequality (see e.g. [18, p. 272]) for the first inequality. □

The following lemma is one of the main steps in our argument.

Lemma 2.3 *Let $u \in H^1(B(0,6kr)\cap\Omega;\mathbb{C}^m)$ be a weak solution to $\mathcal{L}(u)=0$ in $B(0,6kr)\cap\Omega$ with $u=0$ on $B(0,6kr)\cap\partial\Omega$ for some $0<r<\infty$, where $k=10a(M+2)$. Assume that*

$$\int_{B(0,kr)\cap\Omega} |\nabla u|^2\,dx \le \frac{C_0}{r^2}\int_{B(0,2kr)\cap\Omega} |u|^2\,dx. \tag{2.8}$$

Then

$$\left(\fint_{\Delta_r} |N_a^r(u)|^q\,d\sigma\right)^{1/q} \le C\left(\fint_{\Delta_{2kr}} |N_a^{4kr}(u)|^2\,d\sigma\right)^{1/2}, \tag{2.9}$$

where $q=\frac{2(d-1)}{d-2}$ for $d\ge 3$ and $2<q<\infty$ for $d=2$. The constant C depends only on d, m, M, C_0, and q (if $d=2$).

Proof We give the proof for the case $d\ge 3$. With minor modification, the same argument works for $d=2$. It follows from (2.5) and Fubini's Theorem that

$$\begin{aligned}\int_{\Delta_r} |N_a^r(u)|^q\,d\sigma &\le C\int_{\Delta_r}\int_{\Gamma_{2a}^{2r}(z)} |u(y)|^q\delta(y)^{-d}\,dyd\sigma(z)\\ &\le C\int_E |u(y)|^q\delta(y)^{-1}\,dy,\end{aligned}$$

where

$$E=\bigcup_{z\in\Delta_r}\Gamma_{2a}^{2r}(z).$$

Note that if $y\in E$, then $y\in\Gamma_{2a}^{2r}(z)$ for some $z\in\Delta_r$. Hence,

$$\begin{aligned}|y| &\le |y-z|+|z| < 2a\delta(y)+(1+M)r\\ &\le (4a+1+M)r\\ &\le 5ar,\end{aligned}$$

where we have used the fact $a\ge 1+M$. This shows that $|y'|\le 5ar$ and $|y_d|<5ar$. As a result, we obtain $E\subset D_{5ar}$. Thus,

$$\begin{aligned}\int_{\Delta_r} |N_a^r(u)|^q\,d\sigma &\le C\int_{D_{5ar}} |u(y)|^q\delta(y)^{-1}\,dy\\ &\le C\left(\int_{D_{5ar}} |u|^{2(q-1)}\,dy\right)^{1/2}\left(\int_{D_{5ar}}\frac{|u(y)|^2}{\delta(y)^2}\,dy\right)^{1/2},\end{aligned} \tag{2.10}$$

where we have used the Cauchy inequality for the last step.

To bound the right-hand side of (2.10), we first note that

$$\frac{1}{\sqrt{2}(M+1)}|x_d-\psi(x')| \le \delta(x) \le |x_d-\psi(x')|.$$

In view of Lemma 2.2 we obtain

$$\int_{D_{5ar}}\frac{|u(y)|^2}{\delta(y)^2}\,dy \le C\int_{D_{5ar}} |\nabla u(y)|^2\,dy, \tag{2.11}$$

where C depends only on M. Recall that $q=\frac{2(d-1)}{d-2}$. Thus $2(q-1)=\frac{2d}{d-2}$. Since $u=0$ on Δ_{5ar}, we may apply the Sobolev inequality to obtain

$$\left(\int_{D_{5ar}} |u|^{2(q-1)}\,dy\right)^{1/(2(q-1))} \le C\left(\int_{D_{5ar}} |\nabla u|^2\,dy\right)^{1/2}. \tag{2.12}$$

This, together with (2.10) and (2.11), leads to

$$\begin{aligned}\int_{\Delta_r} |N_a^r(u)|^q \, d\sigma &\le C\Big(\int_{D_{5ar}} |\nabla u|^2 \, dy\Big)^{q/2} \\ &\le C\Big(\int_{B(0,10a(M+2)r)\cap\Omega} |\nabla u|^2 \, dy\Big)^{q/2}, \end{aligned} \tag{2.13}$$

where we have used the observation $D_{5ar} \subset B(0, 10a(M+2)r)$ for the last step. Hence,

$$\begin{aligned}\Big(\fint_{\Delta_r} |N_a^r(u)|^q \, d\sigma\Big)^{1/q} &\le C r^{\frac{d}{2}-\frac{d-1}{q}} \Big(\fint_{B(0,10a(M+2)r)\cap\Omega} |\nabla u|^2 \, dy\Big)^{1/2} \\ &\le C\Big(\fint_{B(0,20a(M+2)r)\cap\Omega} |u|^2 \, dy\Big)^{1/2}, \end{aligned} \tag{2.14}$$

where we have used the assumption (2.8) for the last step.

Finally, we note that if $|x-y| < (1/5)\delta(y)$, then $|x-y| < (1/4)\delta(x)$. Thus, by Fubini's Theorem,

$$\begin{aligned}\fint_{B(0,R)\cap\Omega} |u|^2 \, dx &\le C \fint_{B(0,R)\cap\Omega} \Big(\fint_{B(x,(1/4)\delta(x))} |u|^2 \, dy\Big) dx \\ &\le C \fint_{\Delta_R} |N_a^{2R}(u)|^2 \, d\sigma\end{aligned}$$

for any $R > 0$. This, together with (2.14), yields the reverse Hölder inequality (2.9). □

We are now ready to prove the main result of this section.

Theorem 2.4 *Let $u \in H^1(B(0,9kR)\cap\Omega; \mathbb{C}^m)$ be a weak solution to $\mathcal{L}(u) = 0$ in $B(0,9kR)\cap\Omega$ with $u = 0$ on $B(0,9kR)\cap\partial\Omega$ for some $0 < R < \infty$, where $k = 10a(M+2)$. Assume that*

$$\int_{B(z,r)\cap\Omega} |\nabla u|^2 \, dx \le \frac{C_0}{r^2} \int_{B(z,2r)\cap\Omega} |u|^2 \, dx \tag{2.15}$$

for any $0 < r < 3kR$ and any $z \in B(0,3kR)\cap\partial\Omega$. Then for any $0 < r < R$,

$$\Big(\fint_{\Delta_r} |N_a^{4kR}(u)|^q \, d\sigma\Big)^{1/q} \le C \fint_{\Delta_{2r}} N_a^{4kR}(u) \, d\sigma, \tag{2.16}$$

where $q = \frac{2(d-1)}{d-2}$ for $d \ge 3$ and $2 < q < \infty$ for $d = 2$. The constant C depends only on d, m, M, C_0, and q (if $d = 2$).

Proof We first show that for any $0 < r < R$,

$$\Big(\fint_{\Delta_r} |N_a^{4kR}(u)|^q \, d\sigma\Big)^{1/q} \le C\Big(\fint_{\Delta_{2kr}} |N_a^{4kR}(u)|^2 \, d\sigma\Big)^{1/2}. \tag{2.17}$$

Let $z \in \Delta_r$ and $x \in \Gamma_a^{4kR}(z)$. If $\delta(y) < r$, we have

$$\Big(\fint_{B(x,(1/4)\delta(x))} |u|^2\Big)^{1/2} \le N_a^r(u)(z).$$

Suppose $\delta(x) > r$. It follows by a simple geometric observation that there exists a constant $c_0 \in (0,1)$, depending only on d, M and a, such that

$$|\{y \in \Delta_{\Delta_{2kr}} : \ x \in \Gamma_a^{4kR}(y)\}| \ge c_0 r^{d-1}.$$

This implies that

$$\left(⨍_{B(x,(1/4)\delta(x))} |u|^2 \right)^{1/2} \le C ⨍_{\Delta_{2kr}} N_a^{4kR}(u)\, d\sigma.$$

Hence, for any $z \in \Delta_r$,

$$N_a^{4kR}(u)(z) \le N_a^r(u)(z) + C ⨍_{\Delta_{2kr}} N_a^{4kR}(u)\, d\sigma, \tag{2.18}$$

which, together with (2.9), gives (2.17).

The fact that (2.17) implies (2.16) follows from a convexity argument, found in [9]. For $z = (z', \psi(z')) \in \partial\Omega$ and $r > 0$, define the surface ball $\Delta_r(z)$ on $\partial\Omega$ by

$$\Delta_r(z) = \{(x', \psi(x')) \in \mathbb{R}^d : \ |x' - z'| < r\}. \tag{2.19}$$

Note that $\Delta_r = \Delta_r(0)$. By translation the inequality (2.17) continues to hold if Δ_r and Δ_{2kr} are replaced by $\Delta_r(z)$ and $\Delta_{2kr}(z)$, respectively. Let $0 < s < t < 1$. We may cover Δ_{sr} by a finite number of surface balls $\{\Delta_{c(t-s)r}(z_\ell)\}$ with the property $\Delta_{2kc(t-s)r}(z_\ell) \subset \Delta_{tr}$. Note that

$$\begin{aligned} ⨍_{\Delta_{sr}} |N_a^{4kR}(u)|^q\, d\sigma &\le C s^{1-d}(t-s)^{d-1} \sum_\ell ⨍_{\Delta_{c(t-s)r}(z_\ell)} |N_a^{4kR}(u)|^q\, d\sigma \\ &\le C s^{1-d}(t-s)^{d-1} \sum_\ell \left(⨍_{\Delta_{2kc(t-s)r}(z_\ell)} |N_a^{4kR}(u)|^2\, d\sigma \right)^{q/2} \\ &\le C s^{1-d}(t-s)^{d-1} \left(\sum_\ell ⨍_{\Delta_{2kc(t-s)r}(z_\ell)} |N_a^{4kR}(u)|^2\, d\sigma \right)^{q/2} \\ &\le C s^{1-d}(t-s)^{d-1} t^{\frac{q}{2}(d-1)} (t-s)^{-\frac{q}{2}(d-1)} \left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|^2\, d\sigma \right)^{q/2}. \end{aligned}$$

It follows that for any $0 < s < t < 1$,

$$\left(⨍_{\Delta_{sr}} |N_a^{4kR}(u)|^q\, d\sigma \right)^{1/q} \le C s^{\frac{1-d}{q}} t^{\frac{d-1}{2}} (t-s)^{(d-1)(\frac{1}{q}-\frac{1}{2})} \left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|^2\, d\sigma \right)^{1/2}. \tag{2.20}$$

Write $\frac{1}{2} = \frac{\theta}{q} + \frac{\theta}{1}$, where $\theta \in (0, 1)$. By Hölder's inequality,

$$\left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|^2\, d\sigma \right)^{1/2} \le \left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|^q\, d\sigma \right)^{(1-\theta)/q} \left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|\, d\sigma \right)^{\theta}. \tag{2.21}$$

Let

$$I(t) = \left(⨍_{\Delta_{tr}} |N_a^{4kR}(u)|^q\, d\sigma \right)^{1/q} \Big/ ⨍_{\Delta_r} N_a^{4kR}(u)\, d\sigma.$$

By (2.20) and (2.21) we obtain

$$I(s) \le C s^{\frac{1-d}{q}} t^{(d-1)(\frac{1}{2}-\theta)} (t-s)^{(d-1)(\frac{1}{q}-\frac{1}{2})} [I(t)]^{1-\theta}.$$

Hence,

$$\log I(s) \le \log(C s^{\frac{1-d}{q}} t^{(d-1)(\frac{1}{2}-\theta)} (t-s)^{(d-1)(\frac{1}{q}-\frac{1}{2})}) + (1-\theta) \log I(t).$$

Let $s = t^b$, where $b > 1$ is chosen so that $b^{-1} > 1 - \theta$. We integrate the inequality above in t with respect to $t^{-1}dt$ over the interval $(1/2, 1)$. This gives

$$\frac{1}{b} \int_{(1/2)^b}^1 \log I(t) \frac{dt}{t} \le C + (1-\theta) \int_{1/2}^1 \log I(t)\, \frac{dt}{t}.$$

It follows that

$$\left(\frac{1}{b}-\theta\right)\int_{1/2}^{1}\log I(t)\frac{dt}{t}\le C.$$

Since $I(t)\ge cI(1/2)$ for $t\in(1/2,1)$, we obtain $I(1/2)\le C$, which gives (2.16).

Remark 2.5 By translation the inequality (2.16) continues to hold if Δ_r and Δ_{2r} are replaced by surface balls $\Delta_r(z)$ and $\Delta_{2r}(z)$, respectively, where $z\in\Delta_R$. In the case $d\ge 3$, (2.16) in fact holds for some $\overline{q}=\frac{2(d-1)}{d-2}+\varepsilon$, where $\varepsilon>0$ depends only on d, m, M and C_0. This follows from the well known self-improving property of the reverse Hölder inequality.

3 Proof of Theorem 1.2

Throughout this section we assume that Ω is a graph domain, given by (1.5), with $\psi(0)=0$ and $\|\nabla\psi\|_\infty\le M$. Consider the map $\Phi:\partial\Omega\to\mathbb{R}^{d-1}$, defined by $\Phi(x',\psi(x'))=x'$. We say $Q\subset\partial\Omega$ is a surface cube of $\partial\Omega$ if $\Phi(Q)$ is a cube of $\mathbb{R}^{d-1}$ (with sides parallel to the coordinate planes). A dilation of Q is defined by $\alpha Q=\Phi^{-1}(\alpha\Phi(Q))$. We call $z\in Q$ the center of Q if $\Phi(z)$ is the center of $\Phi(Q)$. Similarly, the side length of Q is defined to be the side length of $\Phi(Q)$.

Proofs of Theorems 1.1 and 1.2 are based on a real variable argument.

Theorem 3.1 *Let $F\in L^{p_0}(2Q_0)$ for some surface cube Q_0 of $\partial\Omega$ and $1\le p_0<\infty$. Let $p_1>p_0$ and $f\in L^p(2Q_0)$ for some $p_0<p<p_1$. Suppose that for each surface cube $Q\subset Q_0$ with $|Q|\le\beta|Q_0|$, there exist two integrable functions F_Q and R_Q such that*

$$|F|\le|F_Q|+|R_Q|\quad\text{on } 2Q, \tag{3.1}$$

$$\left(\fint_{2Q}|R_Q|^{p_1}\,d\sigma\right)^{1/p_1}\le C_1\left\{\left(\fint_{\alpha Q}|F|^{p_0}\,d\sigma\right)^{1/p_0}+\sup_{2Q_0\supset Q'\supset Q}\left(\fint_{Q'}|f|^{p_0}\,d\sigma\right)^{1/p_0}\right\}, \tag{3.2}$$

$$\left(\fint_{2Q}|F_Q|^{p_0}\,d\sigma\right)^{1/p_0}\le C_2\sup_{2Q_0\supset Q'\supset Q}\left(\fint_{Q'}|f|^{p_0}\,d\sigma\right)^{1/p_0}, \tag{3.3}$$

where $C_1,C_2>0$ and $0<\beta<1<\alpha$. Then

$$\left(\fint_{Q_0}|F|^p\,d\sigma\right)^{1/p}\le C\left(\fint_{2Q_0}|F|^{p_0}\,d\sigma\right)^{1/p_0}+C\left(\fint_{2Q_0}|f|^p\,d\sigma\right)^{1/p}, \tag{3.4}$$

where $C>0$ depends at most on d, M, p_0, p_1, p, C_1, C_2, α and β.

Proof This theorem with $p_0=1$ was formulated and proved in [17, Theorem 3.2 and Remark 3.3]. Its proof was inspired by a paper of Caffarelli and Peral [3]. The case $p_0>1$ follows readily from the case $p_0=1$ by considering the functions $|F|^{p_0}$ and $|f|^{p_0}$. □

Assume $d\ge 3$. To prove Theorem 1.2, we fix $f\in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m)$. By the assumption of the theorem, there exists a weak solution $u\in H^1_{\text{loc}}(\Omega;\mathbb{C}^m)$ to the elliptic system $\mathcal{L}(u)=0$ in Ω such that $u=f$ on $\partial\Omega$ in the sense of trace and $\|N(u)\|_{L^{p_0}(\partial\Omega)}\le C_0\|f\|_{L^{p_0}(\partial\Omega)}$, where $1<p_0<\frac{2(d-1)}{d-2}$. We need to show that $\|N(u)\|_{L^p(\partial\Omega)}\le C\|f\|_{L^p(\partial\Omega)}$ for $p_0<p<\frac{2(d-1)}{d-2}+\varepsilon$, where $\varepsilon>0$ depends only on d, m, p_0, p, M and C_0.

To this end we fix $Q_0=Q(0,R)$, a surface cube centered at the origin with side length R. Let $Q=Q(z,r)\subset Q_0$ be a surface cube centered at z with side length $r\le\beta R$, where $\beta\in(0,1)$ is sufficiently small. Let $g=\varphi f$, where φ is a smooth cut-off function such that $0\le\varphi\le 1$, $\varphi=1$ in $\Delta_{\gamma^2 r}(z)$, and $\varphi=0$ in $\partial\Omega\setminus\Delta_{2\gamma^2 r}(z)$, where

$$2Q\subset\Delta_{\gamma r}\subset\Delta_{2\gamma^2 r}(z)\subset 2Q_0$$

and $\gamma=\gamma(M)>1$ is large. By the assumption there exists a weak solution v to $\mathcal{L}(v)=0$ in Ω such that $v=\varphi f$ on $\partial\Omega$ and

$$\|N(v)\|_{L^{p_0}(\partial\Omega)}\le C_0\|\varphi f\|_{L^{p_0}(\partial\Omega)}. \tag{3.5}$$

Let $w=u-v$ and define

$$F=N(u),\quad F_Q=N(v)\quad\text{and}\quad R_Q=N(w). \tag{3.6}$$

Using $N(u)\le N(v)+N(w)$, we obtain (3.1). To verify (3.3), we use the estimate (3.5) to obtain

$$\begin{aligned}\Big(\fint_{2Q}|F_Q|^{p_0}\,d\sigma\Big)^{1/p_0}&\le C\Big(\frac{1}{|Q|}\int_{\partial\Omega}|N(v)|^{p_0}\,d\sigma\Big)^{1/p_0}\\&\le C\Big(\frac{1}{|Q|}\int_{\Delta_{2\gamma^2r}(z)}|f|^{p_0}\,d\sigma\Big)^{1/p_0}\\&\le C\sup_{2Q_0\supset Q'\supset Q}\Big(\fint_{Q'}|f|^{p_0}\,d\sigma\Big)^{1/p_0}.\end{aligned} \tag{3.7}$$

To verify (3.2), we use Theorem 2.4. Observe that $\mathcal{L}(w)=0$ in Ω and $w=0$ on $\Delta_{\gamma^2r}(z)$. By choosing $\gamma=\gamma(M)>1$ sufficiently large, it follows from (2.16) as well as Remark 2.5 that

$$\Big(\fint_{\Delta_{\gamma r}(z)}|N^{4k\gamma r}(w)|^{\overline{q}}\,d\sigma\Big)^{1/\overline{q}}\le C\fint_{\Delta_{2\gamma r}(z)}N^{4k\gamma r}(w)\,d\sigma, \tag{3.8}$$

where $\overline{q}=\frac{2(d-1)}{d-2}+\varepsilon$ and $\varepsilon>0$ depends only on d, m, M and C_0. Note that for any $y\in\Delta_{\gamma r}(z)$,

$$N(w)(y)\le N^{4k\gamma r}(w)(y)+C\fint_{\Delta_{2\gamma r}(z)}N(w)\,d\sigma \tag{3.9}$$

(see the proof of (2.18)). This, together with (3.8), yields

$$\Big(\fint_{\Delta_{\gamma r}(z)}|N(w)|^{\overline{q}}\,d\sigma\Big)^{1/\overline{q}}\le C\fint_{\Delta_{2\gamma r}(z)}N(w)\,d\sigma. \tag{3.10}$$

Hence,

$$\begin{aligned}\Big(\fint_{2Q}|R_Q|^{\overline{q}}\,d\sigma\Big)^{1/\overline{q}}&\le C\Big(\fint_{\Delta_{\gamma r}(z)}|N(w)|^{\overline{q}}\,d\sigma\Big)^{1/\overline{q}}\\&\le C\Big(\fint_{\Delta_{2\gamma r}(z)}|N(w)|^{p_0}\,d\sigma\Big)^{1/p_0}\\&\le C\Big(\fint_{\Delta_{2\gamma r}(z)}|N(u)|^{p_0}\,d\sigma\Big)^{1/p_0}+C\Big(\fint_{\Delta_{2\gamma r}(z)}|N(v)|^{p_0}\,d\sigma\Big)^{1/p_0}\\&\le C\Big(\fint_{\alpha Q}|F|^{p_0}\,d\sigma\Big)^{1/p_0}+C\sup_{2Q_0\supset Q'\supset Q}\Big(\fint_{Q'}|f|^{p_0}\,d\sigma\Big)^{1/p_0},\end{aligned} \tag{3.11}$$

where $\alpha Q\supset\Delta_{2\gamma r}(z)$ and we have used (3.5) for the last inequality.

To summarize, we have verified the conditions in Theorem 3.1. As a result, we may conclude that

$$\Big(\fint_{Q_0}|N(u)|^{p}\,d\sigma\Big)^{1/p}\le C\Big(\fint_{2Q_0}|N(u)|^{p_0}\,d\sigma\Big)^{1/p_0}+C\Big(\fint_{2Q_0}|f|^{p}\,d\sigma\Big)^{1/p} \tag{3.12}$$

for any $p_0 < p < \frac{2(d-1)}{d-2} + \varepsilon$. It follows that

$$\left(\int_{Q_0} |N(u)|^p \, d\sigma \right)^{1/p} \le C|Q_0|^{(d-1)(\frac{1}{p}-\frac{1}{p_0})} \left(\int_{2Q_0} |N(u)|^{p_0} \, d\sigma \right)^{1/p_0} + C\left(\int_{2Q_0} |f|^p \, d\sigma \right)^{1/p}$$
$$\le C|Q_0|^{(d-1)(\frac{1}{p}-\frac{1}{p_0})} \|f\|_{L^{p_0}(\partial\Omega)} + C\|f\|_{L^p(\partial\Omega)}.$$

By letting the side length of Q_0 go to infinity in the inequalities above, we obtain the desired estimate $\|N(u)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}$.

Finally, note that if $d = 2$, the same argument yields the estimate (1.3) for $p_0 < p < \infty$.

4 Proof of Theorem 1.1

Theorem 1.1 follows from the proof of Theorem 1.2 by a simple localization technique. Fix $z \in \partial\Omega$. Let $r_0 = \operatorname{diam}(\Omega)$ and $r = c_0 r_0$, where $c_0 > 0$ is sufficiently small such that

$$B(z, r) \cap \Omega = B(z, r) \cap \{(x', x_d) : x_d > \psi(x')\}$$

in a new coordinate system, obtained from the standard system through translation and rotation. It follows from the estimate (3.12) that

$$\begin{aligned} \left(\int_{B(z,c_1 r)\cap\partial\Omega} |N(u)|^p \, d\sigma \right)^{1/p} &\le C r_0^{(d-1)(\frac{1}{p}-\frac{1}{p_0})} \|N(u)\|_{L^{p_0}(\partial\Omega)} + C\|f\|_{L^p(\partial\Omega)} \\ &\le C r_0^{(d-1)(\frac{1}{p}-\frac{1}{p_0})} \|f\|_{L^{p_0}(\partial\Omega)} + C\|f\|_{L^p(\partial\Omega)} \\ &\le C\|f\|_{L^p(\partial\Omega)}, \end{aligned} \tag{4.1}$$

where $c_1 = c_1(\Omega) > 0$ is small and we have used Hölder's inequality as well as the fact $|\partial\Omega| \le C r_0^{d-1}$ for the last step. By covering $\partial\Omega$ with a finite number of balls $\{B(z_\ell, c_1 r)\}$ we obtain the estimate (1.3).

References

[1] Alfonseca, M. A., Auscher, P., Axelsson, A., et al.: Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients. *Adv. Math.*, **226**(5), 4533–4606 (2011)

[2] Auscher, P., Axelsson, A.: Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I. *Invent. Math.*, **184**(1), 47–115 (2011)

[3] Caffarelli, L., Peral, I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.*, **51**, 1–21 (1998)

[4] Dahlberg, B., Kenig, C.: L^p estimates for the three-dimensional system of elastostatics on Lipschitz domains, Lecture Notes in Pure and Applied Mathematics (Cora Sadoesky, ed.), vol. 122, Dekker, 1990, pp. 631–634

[5] Dahlberg, B., Kenig, C., Verchota, G.: Boundary value problems for the system of elastostatics in Lipschitz domains. *Duke Math. J.*, **57**(3), 795–818 (1988)

[6] Dindos, M., Pipher, J., Rule, D.: Boundary value problems for second-order elliptic operators satisfying a Carleson condition. *Comm. Pure Appl. Math.*, **70**(7), 1316–1365 (2017)

[7] Fabes, E.: Layer potential methods for boundary value problems on Lipschitz domains. *Lecture Notes in Math.*, **1344**, 55–80 (1988)

[8] Fabes, E., Kenig, C., Verchota, G.: The Dirichlet problem for the Stokes system on Lipschitz domains. *Duke Math. J.*, **57**(3), 769–793 (1988)

[9] Fefferman, C., Stein, E.: H^p spaces of several variables. *Acta Math.*, **129**(3–4), 137–193 (1972)

[10] Gao, W.: Layer potentials and boundary value problems for elliptic systems in Lipschitz domains. *J. Funct. Anal.*, **95**, 377–399 (1991)

[11] Hofmann, S., Kenig, C., Mayboroda, S., et al.: The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients. *Math. Ann.*, **361**(3–4), 863–907 (2015)

[12] Hofmann, S., Kenig, C., Mayboroda, S., et al.: Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators. *J. Amer. Math. Soc.*, **28**(2), 483–529 (2015)

[13] Hofmann, S., Mayboroda, S., McIntosh, A.: Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces. *Ann. Sci. Ec. Norm. Super.* (4), **44**(5), 723–800 (2011)

[14] Kenig, C.: Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Regional Conference Series in Math., vol. 83, AMS, Providence, RI, 1994

[15] Shen, Z.: Bounds of Riesz transforms on L^p spaces for second order elliptic operators. *Ann. Inst. Fourier* (*Grenoble*), **55**, 173–197 (2005)

[16] Shen, Z.: Necessary and sufficient conditions for the solvability of the L^p Dirichlet problem on Lipschitz domains. *Math. Ann.*, **336**(3), 697–724 (2006)

[17] Shen, Z.: The L^p boundary value problems on Lipschitz domains. *Adv. Math.*, **216**, 212–254 (2007)

[18] Stein, E.: Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, 1970