Acta Mathematica Sinica, English Series
Sep., 2019, Vol. 35, No. 9, pp. 1491–1510
Published online: August 15, 2019
https://doi.org/10.1007/s10114-019-8149-3
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Moderate Deviations for Stochastic Heat Equation with Rough Dependence in Space

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Abstract In this paper, we establish a moderate deviation principle for the stochastic heat equation driven by a Gaussian noise which is white in time and which has the covariance of a fractional Brownian motion with Hurst parameter $H \in (1/4, 1/2)$ in the space variable. The weak convergence approach plays an important role.

Keywords Stochastic heat equation, fractional Brownian motion, moderate deviations, weak convergence approach

MR(2010) Subject Classification 60F10, 60G15, 60H15

1 Introduction

In this paper, we consider the class of one-dimensional nonlinear stochastic heat equation (SHE in short)

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{\kappa}{2}\frac{\partial^2 u^\varepsilon}{\partial x^2}(t,x) + b(u^\varepsilon(t,x)) + \sqrt{\varepsilon}\sigma(u^\varepsilon(t,x))\dot{W}(t,x), \tag{1.1}$$

with $t \in [0,T], x \in \mathbb{R}$, where $\kappa > 0$ denoting the diffusive constant (in sequel, we will consider $\kappa = 1$ in this paper for simiplicity) and W is a zero-mean Gaussian process with covariance given by

$$E(W(t,x)W(s,y)) = \frac{1}{2}(|x|^{2H} + |y|^{2H} - |x-y|^{2H}) \min\{t,s\}, \tag{1.2}$$

with $t, s \in [0,T], x, y \in \mathbb{R}$, $1/4 < H < 1/2$ and $\varepsilon > 0$.

Since the seminal work of [5] and [18], there has been a lot of interests in the study of stochastic partial differential equations driven by a Brownian motion in time and with spatial homogeneous covariance. Recently, some scholars began to study the SHE (1.1), which is driven by a spatially homogeneous Gaussian noise, which is white in time and behaves in space like a fractional Brownian motion with Hurst index $1/4 < H < 1/2$. For example, when $\sigma(u) = au+b$ is an affine function and the initial condition u_0 is bounded and Hölder continuous of order H, the authors in [1] proved the existence and uniqueness of a mild solution to SHE (1.1). The

Received April 7, 2018, revised September 11, 2018, accepted March 4, 2019
Partially supported by National Natural Science Foundation of China (Grant No. 11771209), Humanities and Social Sciences Foundation of the Ministry of Education of China (Grant No. 18YJCZH101), Natural Science Foundation of Jiangsu Province of China (Grant No. BK20161579), Major Research Plan of Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 18KJA110002), QingLan Project and 333 Talent Training Project of Jiangsu Province (Grant No. BRA2018357)

stochastic integral with respect to the Gaussian noise $\dot{W}$ is defined in the Itô's sense. While in the case of general nonlinear coefficient σ, which has a Lipschitz derivative and satisfies $\sigma(0) = 0$ and $b(x) = 0$, SHE (1.1) has been studied in [11]. They proved the existence and uniqueness of the solution. In [12], the authors studied a class of parabolic Anderson model with the same noise in SHE (1.1). They used the multiple Wiener–Itô integral and a Feynman–Kac formula to study the moment bounds of the solution. While in [4], they studied the stochastic heat equation driven by time fractional Gaussian noise with Hurst parameter $H \in (0, 1/2)$. They established the Feynman–Kac representation of the solution and used this representation to obtain matching lower and upper bounds for the p-th moments of the solution.

On the other hand, the theory of large deviations which was firstly investigated in [7] has been extensively studied recently. The large deviations reveal some important aspects of asymptotic dynamics. Special attention has been paid to studying large deviations principle for stochastic (partial) differential equation (e.g., [2, 3, 6–8, 13, 17]). Like the large deviations, the moderate deviations problems arise in the study of statistical inference naturally. The moderate deviations principle (MDP for short) can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals. Results on MDP for stochastic (partial) differential equation can be found in [14, 15, 19, 20] and references therein.

As $\varepsilon \to 0$, the solutions to SHE (1.1) will tend to the solution of the deterministic equation given by

$$\frac{\partial u^0}{\partial t}(t,x) = \frac{\kappa}{2}\frac{\partial^2 u^0}{\partial x^2}(t,x) + b(u^0(t,x)), \tag{1.3}$$

with $u^0(0,x) = u_0^0(x)$.

In this paper we shall investigate deviations of u^ε given by solution of SHE (1.1) from the deterministic equation u^0 satisfied by Eq. (1.3), as ε decreases to zero, that is, the asymptotic behaviour of $Z^\varepsilon = \{Z^\varepsilon(t,x); t \in [0,T], x \in \mathbb{R}\}$ which is defined by

$$Z^\varepsilon(t,x) = \frac{1}{\sqrt{\varepsilon}\lambda(\varepsilon)}(u^\varepsilon - u^0)(t,x), \quad t \in [0,T],\ x \in \mathbb{R}, \tag{1.4}$$

where $\lambda(\varepsilon)$ is some deviation scale, which influences the asymptotic behavior of the above Z^ε.

We should mention that the case $\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$ provides some large deviations estimates. Under some mild assumptions, in [13], the authors proved that the laws of the solution u^ε to SHE (1.1) satisfies a large deviations principle on the space $X_T^{\frac{1}{2}-H}$ (see Definition 2.3).

If $\lambda(\varepsilon) = 1$, we are in the domain of central limit theorem. In order to fill the gap between the central limit theorem ($\lambda(\varepsilon) = 1$) and large deviations scale ($\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$), in this paper, we will study the moderate deviations, that is when the deviations scale $\lambda(\varepsilon)$ satisfies

$$\lambda(\varepsilon) \to \infty, \quad \sqrt{\varepsilon}\lambda(\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0. \tag{1.5}$$

The moderate deviation principle enables us to refine the estimates obtained through the central limit theorem. It provides the asymptotic behavior of $P(\|u^\varepsilon - u^0\| \geq \delta\sqrt{\varepsilon}\lambda(\varepsilon))$, while the central limit theorem gives asymptotic bounds for $P(\|u^\varepsilon - u^0\| \geq \delta\sqrt{\varepsilon})$. Throughout this paper, we assume that (1.5) is in place.

The rest of this paper is organized as follows. In Section 2, we describe the precise framework we will use later on, such as the noise structure and related space-time function spaces and a criteria for large deviations firstly proved in [3]. In Section 3, the skeleton equation (3.1) is studied. We prove the moderate deviations principle of the solution to SHE (1.1) by the weak convergence approach in Section 4.

2 Preliminaries

2.1 Noise Structure and Related Space-time Function Spaces

Let $\mathcal{D}(\mathbb{R})$ be the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}$. The noise $\dot{W}$ can be represented ([11, 13]) as a zero-mean Gaussian family $W(\varphi) = \{W_t(\varphi), t \in [0,T], \varphi \in \mathcal{D}(\mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, whose covariance structure is given by

$$E(W_t(\varphi)W_s(\phi)) = c_{1,H}(t \wedge s) \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\phi(\xi)}|\xi|^{1-2H} d\xi,$$

where $\mathcal{F}\varphi(\xi)$ and $\mathcal{F}\phi(\xi)$ are the Fourier transform for φ, ϕ and $c_{1,H} = \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)$. Moreover we have the following, by using the fractional derivatives (for example, [13])

$$\begin{aligned} &c_{1,H}(t \wedge s) \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\phi(\xi)}|\xi|^{1-2H} d\xi \\ &= c_H(t \wedge s) \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x))(\phi(x+y) - \phi(x))|y|^{2H-2} dx dy, \end{aligned}$$

with a constant $c_H > 0$.

Let $\mathcal{H}$ denote the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R})$ under the inner product

$$\begin{aligned} \langle \varphi, \phi \rangle_{\mathcal{H}} : &= c_{1,H} \int_{\mathbb{R}} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\phi(\xi)}|\xi|^{1-2H} d\xi \\ &= c_H \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x))(\phi(x+y) - \phi(x))|y|^{2H-2} dx dy. \end{aligned} \tag{2.1}$$

Then the Gaussian family $W(\varphi) = \{W_t(\varphi), t \in [0,T], \varphi \in \mathcal{H}\}$ can be regarded as an $\mathcal{H}$-cylindrical Brownian motion. One can see [11, 13] for the details about the stochastic integral with respect to W.

Now let us recall several classes of function spaces studied in [11]. Let $(B, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$, and let $\beta \in (0,1)$, $\delta \in (0, \infty]$ be fixed numbers. For every function $f : \mathbb{R} \to B$, we introduce the functions $\mathcal{N}_\beta^B f$ and $\mathcal{N}_\beta^{B,(\delta)} f : \mathbb{R} \to [0, \infty]$ defined by

$$\mathcal{N}_\beta^B f(x) = \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}, \tag{2.2}$$

and

$$\mathcal{N}_\beta^{B,(\delta)} f(x) = \left(\int_{|h| \le \delta} \|f(x+h) - f(x)\|^2 |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}. \tag{2.3}$$

When $B = \mathbb{R}$, we abbreviate the notations $\mathcal{N}_\beta^{\mathbb{R}} f = \mathcal{N}_\beta f$ and $\mathcal{N}_\beta^{\mathbb{R},(\delta)} f = \mathcal{N}_\beta^{(\delta)} f$. Note that for $\delta = \infty$, the above two functions defined by (2.2) and (2.3) coincide.

Definition 2.1 *Let $\mathfrak{X}_T^\beta(B)$ be the space of all continuous functions $f : [0,T] \times \mathbb{R} \to B$ such that*

$$\|f\|_{\mathfrak{X}_T^\beta(B)} := \sup_{t \in [0,T], x \in \mathbb{R}} \|f(t,x)\| + \sup_{t \in [0,T], x \in \mathbb{R}} \mathcal{N}_\beta^B f(t,x) < +\infty,$$

where the function $\mathcal{N}_\beta^B f(t,x),(t,x)\in[0,T]\times\mathbb{R}$ *is defined as follows*

$$\mathcal{N}_\beta^B f(t,x)=\bigg(\int_{\mathbb{R}}\|f(t,x+h)-f(t,x)\|^2|h|^{-1-2\beta}dh\bigg)^{\frac{1}{2}}.$$

It was shown in [11] that $\mathfrak{X}_T^\beta(B)$ is a Banach space. Throughout this paper, we write $\mathfrak{X}_T^p$ for $\mathfrak{X}_T^\beta(B)$ with $B=L^p(\Omega),\beta=\frac{1}{2}-H$. For $\theta>0$, define the following semi-norm for f:

$$\|f\|_{\mathfrak{X}_{T,\theta}^p}:=\sup_{t\in[0,T],x\in\mathbb{R}}\mathrm{e}^{-\theta t}\|f(t,x)\|+\sup_{t\in[0,T],x\in\mathbb{R}}\mathrm{e}^{-\theta t}\mathcal{N}_{\frac{1}{2}-H}^{L^p(\Omega)}f(t,x). \tag{2.4}$$

For the uniqueness of the solution to (1.1), we need another space.

Definition 2.2 *$\mathcal{Z}_T^p$ is defined as the space of all random field $f:[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ such that*

$$\|f\|_{\mathcal{Z}_T^p}:=\sup_{t\in[0,T]}\|f(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})}+\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,p}f(t)<\infty, \tag{2.5}$$

where $p\geq 2$ and

$$\mathcal{N}^*_{\frac{1}{2}-H,p}f(t)=\bigg(\int_{\mathbb{R}}\|f(t,\cdot)-f(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})}|h|^{2H-2}dh\bigg)^{\frac{1}{2}}. \tag{2.6}$$

Denote by $C([0,T]\times\mathbb{R})$ the space of all real-valued continuous functions on $[0,T]\times\mathbb{R}$ equipped with the topology of uniform convergence over compact sets. For every $h\in\mathbb{R}$, let τ_h be the translation map in the spatial variable, that is $\tau_h f(t,x)=f(t,x-h)$.

Definition 2.3 *Let X_T^β be the space of all functions $f\in C([0,T]\times\mathbb{R})$ such that*

(1) $(t,x)\mapsto\mathcal{N}_\beta^{(1)}f(t,x)$ *is finite and continuous on* $[0,T]\times\mathbb{R}$.

(2) $\lim_{h\downarrow 0}\sup_{t\in[0,T],x\in[-M,M]}\mathcal{N}_\beta^{(1)}(\tau_h f-f)(t,x)=0$ *for every positive M.*

It turns out that X_T^β is a complete separable metric space equipped with the following topology. A sequence $\{f_n\}$ in X_T^β converges to f in X_T^β if for all $R>0$, the sequences $\{f_n\}$ and $\{\mathcal{N}_\beta^{(1)}(f_n-f)\}$ converge uniformly on $[0,T],x\in[-R,R]$ to f and 0, respectively. We define a metric on X_T^β as follows

$$d_\beta(f,g)=\sum_{n=1}^\infty 2^{-n}\frac{\|f-g\|_{n,\beta}}{1+\|f-g\|_{n,\beta}},$$

where $\|\cdot\|_{n,\beta}$ is the seminorm

$$\|f\|_{n,\beta}:=\sup_{t\in[0,T],x\in[-n,n]}|f(t,x)|+\sup_{t\in[0,T],x\in[-n,n]}\mathcal{N}_\beta^{(1)}f(t,x).$$

We say that u^ε is a mild solution of (1.1) if for all $t\in[0,T]$ and $x\in\mathbb{R}$ we have

$$\begin{aligned}u^\varepsilon(t,x)&=p_t*u_0(x)+\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)b(u^\varepsilon(s,y))dsdy\\&\quad+\sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\sigma(u^\varepsilon(s,y))W(ds,dy),\quad\text{a.s.},\end{aligned} \tag{2.7}$$

where the stochastic integral is understood in the sense of [13, Proposition 2.2] and $p_t*u_0(x)=\int_{\mathbb{R}}p_t(x-y)u_0(y)dy$. The following result has been proved in [11].

Theorem 2.4 *Assume that for* SHE (1.1), *the following conditions hold*:

(1) *The initial condition u_0 is bounded and locally Hölder continuous of order H. Furthermore, for some $p > \frac{6}{4H-1}$, u_0 is in $L^p(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|^2_{L^p(\mathbb{R})} \cdot |h|^{2H-2} dh < \infty.$$

(2) *The coefficient b is Lipschitz continuous with $b(0) = 0$.*

(3) *σ is differentiable, its derivative is Lipschitz continuous and $\sigma(0) = 0$.*

Then there exists a unique solution u^ε to SPDE (1.1) *in $\mathcal{Z}^p_T \cap \mathfrak{X}^p_T$. In addition, the solution has sample paths in $X_T^{1/2-H}$.*

2.2 A Criteria for Large Deviations

Let $\mathcal{H}$ be the Hilbert space introduced in Section 2.1. Define the following space of predictable stochastic processes

$$\mathcal{L}_2 := \left\{ \psi : \Omega \times [0,T] \to \mathcal{H}, \int_0^T \|\psi(s)\|^2_{\mathcal{H}} ds < \infty, \text{ a.s.} - \mathbf{P} \right\}. \tag{2.8}$$

For $N \geq 1$, define

$$\mathcal{H}^N_T = \left\{ f : f \in L^2([0,T];\mathcal{H}), \frac{1}{2}\int_0^T \|f(s)\|^2_{\mathcal{H}} ds \leq N \right\}. \tag{2.9}$$

Moreover $\mathcal{H}^N_T$ will be equipped with the topology of weak convergence in $\mathcal{H}_T := L^2([0,T];\mathcal{H})$.

Define $\mathcal{U}^N$ as follows

$$\mathcal{U}^N = \{ f \in \mathcal{L}_2 : f(\omega) \in \mathcal{H}^N_T, \mathbf{P} - \text{a.s.}\, \omega \}.$$

The following condition will be sufficient to establish a large deviation principle (LDP for short) for a family $\{Z^\varepsilon, \varepsilon > 0\}$ defined by $Z^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}W)$ where $\{\mathcal{G}^\varepsilon\}_{\varepsilon>0}$ is a family of measurable maps from $\mathbb{V} = C([0,T];\mathcal{H}) \subset C([0,T]\times\mathbb{R})$ to $\mathbb{U}$ which is a Polish space.

Condition A There exists a measurable map $\mathcal{G}^0 : \mathbb{V} \to \mathbb{U}$ such that the following two points hold:

(1) For $N \in \mathbb{N}$, let $f_n, f \in \mathcal{H}^N_T$ be such that $f_n \to f$ weakly as $n \to \infty$. Then

$$\mathcal{G}^0\left(\int_0^\cdot f_n(s)ds \right) \to \mathcal{G}^0\left(\int_0^\cdot f(s)ds \right), \quad \text{in } \mathbb{U}. \tag{2.10}$$

(2) For $N \in \mathbb{N}$, let $v^\varepsilon, v \in \mathcal{U}^N$ be such that v^ε converges in distribution to v as $\varepsilon \to 0$. Then

$$\mathcal{G}^\varepsilon\left(W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s)ds \right) \Rightarrow \mathcal{G}^0\left(\int_0^\cdot v(s)ds \right), \quad \text{in } \textit{distribution}. \tag{2.11}$$

The following criteria which was initially established in [3] plays an important role in this paper.

Theorem 2.5 *For $\varepsilon > 0$, let $Z^\varepsilon = \mathcal{G}^\varepsilon(\sqrt{\varepsilon}W)$, and suppose that **Condition A** holds. Define*

$$I(\phi) = \inf_{f \in \mathbb{S}_\phi} \left\{ \frac{1}{2} \int_0^T \|f(s)\|^2_{\mathcal{H}} ds \right\}, \quad \phi \in \mathbb{U},$$

with $\mathbb{S}_\phi = \{ f \in \mathbb{S} = \cup_{N\geq 1}\mathcal{H}^N_T : \phi = \mathcal{G}^0(\int_0^\cdot f(s)ds)\}$, $\phi \in \mathbb{U}$. Then $I(\phi)$ is a rate function on $\mathbb{U}$ and the family $\{Z^\varepsilon, \varepsilon > 0\}$ satisfies a LDP *with rate function I.*

3 Skeleton Equations

In this section, we will study the corresponding skeleton equation associated with SHE (1.1). Let $\{e_k, k \geq 1\}$ be a orthonormal basis of the Hilbert space $\mathcal{H}$. The Gaussian process W admits the following representation

$$W = \sum_{k\geq 1} B_k(t) e_k,$$

where $\{B_k(t), k \geq 1\}$ is a family of independent Brownian motion. The stochastic integral with respect to W can be expressed as

$$\int_0^T \int_{\mathbb{R}} g(s,x) W(ds,dx) = \sum_{k\geq 1} \int_0^T \langle g(s,\cdot), e_k \rangle_{\mathcal{H}} dB_k(s).$$

For $f \in \mathbb{S} = \bigcup_{N\geq 1} \mathcal{H}_T^N$, consider the deterministic evolution equation (called Skeleton equation)

$$\begin{aligned} Z^f(t,x) = p_t * u_0(x) &+ \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}} ds \\ &+ \int_0^t [p_{t-s} * (b'(u^0(s,\star)) Z^f(s,\star))](x) ds, \end{aligned} \tag{3.1}$$

with $t \in [0,T]$ and $x \in \mathbb{R}$. The term $\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}} ds$, in the above equation (3.1) can be written as

$$\sum_{k\geq 1} \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), e_k\rangle_{\mathcal{H}} f_k(s) ds,$$

with $f_k(s) := \langle f(s,\star), e_k\rangle_{\mathcal{H}}, t \in [0,T]$ and $k \geq 1$.

Let us firstly recall a useful lemma which has been proved in [12, 13]. It mainly concerns with the Hölder continuity of the Green function $p_t(\cdot)$.

Lemma 3.1 *The following estimates hold*:

(1) *For any* $0 \leq s \leq t$,

$$\begin{aligned} \int_{\mathbb{R}} |z|^{2H-2} dz \int_{\mathbb{R}^2} &|(p_{t-s}(z+z_1-z_2) - p_{t-s}(z_1-z_2) \\ &- (p_{t-s}(z+z_1) - p_{t-s}(z_1))|^2 |z_2|^{2H-2} dz_1 dz_2 \leq C(t-s)^{-\frac{3}{2}+2H}. \end{aligned}$$

(2) *For any* $s > 0$ *and* $\mu \in (0,1)$,

$$\int_{\mathbb{R}^2} |p_s(z+z_1) - p_s(z_1)|^2 |z|^{-1-2\mu} dz_1 dz \leq C s^{-\frac{1}{2}-\mu}.$$

(3) *For any* $0 \leq s < t \leq T$ *and* $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} |p_{t-s}(x-y)|^2 dy \leq C|t-s|^{-\frac{1}{2}}.$$

Using the similar strategy in the proof of [13, Theorem 4.2], one can prove the following

Proposition 3.2 (Existence of solution to Eq. (3.1)) *Assume the following conditions hold*:

(1) *The initial condition* u_0 *satisfies*

$$\sup_{x\in\mathbb{R}} |u_0(x)| + \sup_{x\in\mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} u_0(x) < \infty, \tag{3.2}$$

(2) *The coefficients* b *and* σ *are differentiable and their derivatives are Lipschitz and* $b(0) = \sigma(0) = 0$. *Furthermore, the derivative* b' *is uniformly bounded.*

Then there exists a unique solution Z^f *to Eq.* (3.1). *Moreover, the solution* Z^f *belongs to the space* $X_T^{\frac{1}{2}-H}$.

Proof We will solve Eq. (3.1) using a successive iteration. Define

$$Z_0^f(t,x) = p_t * u_0(x)$$

and

$$Z_{n+1}^f(t,x) = p_t * u_0(x) + \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}} ds + \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z_n^f(s,\star))](x)ds. \tag{3.3}$$

From the assumption on the initial condition u_0, it follows that $\|Z_0^f(t,\star)\|_\infty < \infty$. First, we will prove a uniform bound for $|Z_{n+1}^f|$. From Eq. (3.3), we have

$$|Z_{n+1}^f(t,x)|^2 \leq C\|u_0\|_\infty^2 + C\int_0^T \|f(s,\star)\|_{\mathcal{H}}^2 ds \int_0^T \|p_{t-s}(x-\star)\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 ds + C\|b'\|_\infty^2 \int_0^t \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 (Z_n^f(s,y))^2 dyds. \tag{3.4}$$

From the proof of upper bound of (4.9) in the proof of [13, Theorem 4.2], one knows that

$$\|p_{t-s}(x-\star)\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 \leq C\|u^0(s,\star)\|_\infty^2 (t-s)^{-1+H} + C\|\mathcal{N}_{\frac{1}{2}-H} u^0(s,\star)\|_\infty^2 (t-s)^{-\frac{1}{2}}. \tag{3.5}$$

On the other hand, one has

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 (Z_n^f(s,y))^2 dyds &\leq \int_0^t \|Z_n^f(s,\star)\|_\infty^2 \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 dyds \\ &\leq \int_0^t (t-s)^{-\frac{1}{2}} \|Z_n^f(s,\star)\|_\infty^2 ds,\end{aligned} \tag{3.6}$$

where we have used the estimate (4.11) in [13].

Combining the inequalities (3.5) and (3.6), one can obtain

$$\begin{aligned}\|Z_{n+1}^f(t,*)\|_\infty^2 \leq C\|u_0\|_\infty^2 &+ C\int_0^t (t-s)^{-1+H} \|u^0(s,\star)\|_\infty^2 ds \\ &+ C\int_0^t (t-s)^{-\frac{1}{2}} \|\mathcal{N}_{\frac{1}{2}-H} u^0(s,\star)\|_\infty^2 ds \\ &+ C\|b'\|_\infty^2 \int_0^t (t-s)^{-\frac{1}{2}} \|Z_n^f(s,\star)\|_\infty^2 ds.\end{aligned} \tag{3.7}$$

Next let us find the upper bounds for $\|u^0(t,\star)\|_\infty^2$ and $\|\mathcal{N}_{\frac{1}{2}-H} u^0(t,\star)\|_\infty^2$. Recall that $u^0(t,x)$ is the solution to Eq. (1.3) which can be written by

$$u^0(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) b(u^0(s,y)) dyds. \tag{3.8}$$

By using Hölder inequality and linear growth condition for the coefficient b, one gets

$$|u^0(t,x)|^2 \leq 2|p_t * u_0(x)|^2 + 2\left|\int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) b(u^0(s,y)) dyds\right|^2$$

$$
\begin{aligned}
&\le 2\|u_0\|_\infty^2 + C\int_0^t (1+\|u^0(s,\star)\|_\infty^2)\int_{\mathbb{R}} |p_{t-s}(x-y)|^2 dyds\\
&\le 2\|u_0\|_\infty^2 + C\int_0^t (1+\|u^0(s,\star)\|_\infty^2)(t-s)^{-\frac{1}{2}} ds. \qquad (3.9)
\end{aligned}
$$

Gronwall's lemma yields that $\|u^0(t,\star)\|_\infty^2$ is bounded by a positive constant. For the second term $\|\mathcal{N}_{\frac{1}{2}-H}u^0(t,\star)\|_\infty^2$, from the definition of the mapping $\mathcal{N}_{\frac{1}{2}-H}u^0(t,x)$, we have

$$
\begin{aligned}
|\mathcal{N}_{\frac{1}{2}-H}u^0(t,x)|^2 &= \int_{\mathbb{R}} |u^0(t,x+z)-u^0(t,x)|^2|z|^{2H-2}dz\\
&= \int_{\mathbb{R}} \bigg| p_t * u_0(x+z) + \int_0^t\int_{\mathbb{R}} p_{t-s}(x+z-y)b(u^0(s,y))dsdy\\
&\quad - p_t * u_0(x) - \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)b(u^0(s,y))dsdy\bigg|^2 |z|^{2H-2}dz\\
&\le 2\int_{\mathbb{R}} |p_t * u_0(x+z) - p_t*u_0(x)|^2|z|^{2H-2}dz\\
&\quad + 2\int_{\mathbb{R}}\bigg|\int_0^t\int_{\mathbb{R}}(p_{t-s}(x+z-y)-p_{t-s}(x-y))b(u^0(s,y))dyds\bigg|^2|z|^{2H-2}dz\\
&\le C + C\int_0^t (1+\|u^0(s,\star)\|_\infty^2)(t-s)^{-1+H}ds\\
&\le C + Ct^H. \qquad (3.10)
\end{aligned}
$$

Then we proved that $\|\mathcal{N}_{\frac{1}{2}-H}u^0(t,\star)\|_\infty^2$ is bounded. Combining Eqs. (3.7), (3.9), (3.10), one can easily obtain that

$$
\|Z_{n+1}^f(t,\star)\|_\infty^2 \le C(1+T^H+\sqrt{T}) + C\int_0^t (t-s)^{-\frac{1}{2}}\|Z_n^f(s,\star)\|_\infty^2 ds. \qquad (3.11)
$$

Next we want to establish a bound for $\mathcal{N}_{\frac{1}{2}-H}Z_{n+1}^f(t,x)$. Recall that $Z_{n+1}^f(t,x)$ satisfies the iteration equation (3.3). Then we have

$$
\begin{aligned}
|\mathcal{N}_{\frac{1}{2}-H}Z_{n+1}^f(t,x)|^2 &\le C|\mathcal{N}_{\frac{1}{2}-H}(p_t*u_0(x))|^2\\
&\quad + C\bigg|\mathcal{N}_{\frac{1}{2}-H}\bigg(\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}}ds\bigg)\bigg|^2\\
&\quad + C\bigg|\mathcal{N}_{\frac{1}{2}-H}\bigg(\int_0^t [p_{t-s}*(b'(u^0(s,\star))Z_n^f(s,\star))](x)ds\bigg)\bigg|^2. \qquad (3.12)
\end{aligned}
$$

The first term $|\mathcal{N}_{\frac{1}{2}-H}(p_t*u_0(x))|^2$ is finite according to (3.10). The second and the third terms in (3.12) can be estimated as follows by using the estimates (4.15)–(4.18) obtained in [13]. Then one can obtain that

$$
\begin{aligned}
&\bigg|\mathcal{N}_{\frac{1}{2}-H}\bigg(\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}}ds\bigg)\bigg|^2\\
&\quad = \int_{\mathbb{R}}\bigg|\int_0^t \langle (p_{t-s}(x+z-\star)-p_{t-s}(x-\star))\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}}ds\bigg|^2\cdot|z|^{2H-2}dz\\
&\quad \le C\int_0^T \|f(s,\star)\|_\infty^2 ds \int_0^t\int_{\mathbb{R}} \|(p_{t-s}(x+z-\star)-p_{t-s}(x-\star))\sigma(u^0(s,\star))\|^2\cdot|z|^{2H-2}dzds\\
&\quad \le C\int_0^t (\|u^0(s,\star)\|_\infty^2(t-s)^{-3/2+2H} + \|\mathcal{N}_{\frac{1}{2}-H}u^0(s,\star)\|_\infty^2(t-s)^{-1+H})ds \qquad (3.13)
\end{aligned}
$$

and

$$\begin{aligned}&\left|\mathcal{N}_{\frac{1}{2}-H}\bigg(\int_0^t [p_{t-s}*(b'(u^0(s,\star))Z_n^f(s,\star))](x)ds\bigg)\right|^2\\&\quad=\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}(p_{t-s}(x+z-y)-p_{t-s}(x-y))b'(\sigma(u^0(s,y)))Z_n^f(s,y)dyds\right|^2\cdot|z|^{2H-2}dz\\&\quad\le C\int_0^t\|Z_n^f(s,\star)\|_\infty^2\int_{\mathbb{R}^2}|p_{t-s}(x+z-y)-p_{t-s}(x-y)|^2\cdot|z|^{2H-2}dydzds\\&\quad\le C\int_0^t\|Z_n^f(s,\star)\|_\infty^2(t-s)^{-1+H}ds. \end{aligned}\tag{3.14}$$

Combining the estimates (3.12), (3.13) and (3.14), we have

$$\begin{aligned}|\mathcal{N}_{\frac{1}{2}-H}Z_{n+1}^f(t,x)|^2\le C+C\int_0^t(&\|u^0(s,\star)\|_\infty^2(t-s)^{-3/2+2H}\\&+\|\mathcal{N}_{\frac{1}{2}-H}u^0(s,\star)\|_\infty^2(t-s)^{-1+H})ds\\&+C\int_0^t\|Z_n^f(s,\star)\|_\infty^2(t-s)^{-1+H}ds.\end{aligned}\tag{3.15}$$

Set

$$A_n(t)=\|Z_n^f(t,\star)\|_\infty^2+\|\mathcal{N}_{\frac{1}{2}-H}Z_n^f(t,\star)\|_\infty^2.$$

According to the estimates (3.11), (3.15), together with the condition $1/4<H<1/2$, we can find a $\kappa_1<1$ such that

$$A_{n+1}(t)\le C+C\int_0^t(t-s)^{-\kappa_1}A_n(s)ds.$$

Then by using a version of Gronwall's lemma (for example, [5, Lemma 9] or [11, Lemma 4.13]), we conclude that

$$\sup_n\sup_{t\in[0,T]}A_n(t)=\sup_n\sup_{t\in[0,T]}\{\|Z_n^f(t,\star)\|_\infty^2+\|\mathcal{N}_{\frac{1}{2}-H}Z_n^f(t,\star)\|_\infty^2\}<+\infty.\tag{3.16}$$

Next we are going to show that $\{Z_n^f,n\ge 0\}$ constitutes a Cauchy sequence in $\mathfrak{X}_T^{\frac{1}{2}-H}$. To this end, we need to bound $\|Z_{n+1}^f(t,x)-Z_n^f(t,x)\|_\infty^2$ and $\|\mathcal{N}_{\frac{1}{2}-H}(Z_{n+1}^f(t,x)-Z_n^f(t,x))\|_\infty^2$. In fact, from (3.3) and (3.6)

$$\begin{aligned}|Z_{n+1}^f(t,x)-Z_n^f(t,x)|^2&=\left|\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)b'(u^0(s,y))(Z_n^f(s,y)-Z_{n-1}^f(s,y))dyds\right|^2\\&\le C\|b'\|_\infty^2\int_0^t(t-s)^{-\frac{1}{2}}\|Z_n^f(s,\star)-Z_{n-1}^f(s,\star)\|_\infty^2ds\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}&|\mathcal{N}_{\frac{1}{2}-H}(Z_{n+1}^f(t,x)-Z_n^f(t,x))|^2\\&\quad=\int_{\mathbb{R}}\bigg|\int_0^t\int_{\mathbb{R}}p_{t-s}(x+z-y)b'(u^0(s,y))(Z_n^f(s,y)-Z_{n-1}^f(s,y))dyds\\&\qquad-\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)b'(u^0(s,y))(Z_n^f(s,y)-Z_{n-1}^f(s,y))dyds\bigg|^2|z|^{2H-2}dz\\&\quad\le C\|b'\|_\infty^2\int_0^t\int_{\mathbb{R}^2}(p_{t-s}(x+z-y)-p_{t-s}(x-y))^2(Z_n^f(s,y)-Z_{n-1}^f(s,y))^2|z|^{2H-2}dzdyds\end{aligned}$$

$$\leq C\|b'\|_\infty^2 \int_0^t (t-s)^{-1+H} \|Z_n^f(s,\star) - Z_{n-1}^f(s,\star)\|_\infty^2 ds. \tag{3.18}$$

Set

$$B_n(t) = \|Z_n^f(t,\star) - Z_{n-1}^f(t,\star)\|_\infty^2 + \|\mathcal{N}_{\frac{1}{2}-H}(Z_n^f(t,\star) - Z_{n-1}^f(t,\star))\|_\infty^2.$$

Combining (3.17) and (3.18), together with the condition $1/4 < H < 1/2$, we can find a $\kappa_2 < 1$ such that

$$\begin{aligned} B_{n+1}(t) &\leq C \int_0^t \|Z_n^f(t,\star) - Z_{n-1}^f(t,\star)\|_\infty^2 ((t-s)^{-1/2} + (t-s)^{-1+H}) ds \\ &\leq C \int_0^t (\|Z_n^f(t,\star) - Z_{n-1}^f(t,\star)\|_\infty^2 + \|\mathcal{N}_{\frac{1}{2}-H}(Z_n^f(t,\star) - Z_{n-1}^f(t,\star))\|_\infty^2) \\ &\quad \cdot ((t-s)^{-1/2} + (t-s)^{-1+H}) ds \\ &\leq C + C \int_0^t (t-s)^{-\kappa_2} B_n(s) ds. \end{aligned} \tag{3.19}$$

Applying Lemma 4.13 in [11], we conclude from (3.19) that $\sum_{n\geq 0} |Z_{n+1}^f(t,x) - Z_n^f(t,x)|^{\frac{2}{p}}$ converges uniformly in $[0,T]$ for all $p \geq 1$. In particular, this implies that the sequence $\{Z_n^f, n \geq 0\}$ is Cauchy in $\mathfrak{X}_T^{\frac{1}{2}-H}$. Denote by Z^f the limit of $\{Z_n^f, n \geq 0\}$. Let n tends to infinity in (3.3), it follows easily that Z^f is the solution of Eq. (3.1). The uniqueness of the solution can be proved by using the standard arguments.

The statement of $Z^f \in X_T^{\frac{1}{2}-H}$ follows from the Hölder continuity of Z^f (see Proposition 3.3 below). □

Proposition 3.3 (Convergence of solution to Eq. (3.1)) *Assume the following conditions hold*:

(1) *The initial condition u_0 is bounded and local Hölder continuous of order H.*

(2) *The coefficients b and σ are differentiable and their derivatives are Lipschitz and $b(0) = \sigma(0) = 0$.*

For $N \in \mathbb{N}$, let $f_n, f \in \mathcal{H}_T^N$ be such that $f_n \to f$ weakly as $n \to \infty$. Let Z^{f_n} denote the solution to Eq. (3.1) *replacing f by f_n. Then*

$$\mathcal{G}^0\bigg(\int_0^\cdot f_n(s)ds\bigg) = Z^{f_n} \to \mathcal{G}^0\bigg(\int_0^\cdot f(s)ds\bigg) = Z^f,$$

as $n \to \infty$ in $X_T^{\frac{1}{2}-H}$.

Proof Recall that for any $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{aligned} Z^{f_n}(t,x) = p_t * u_0(x) &+ \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f_n(s,\star)\rangle_{\mathcal{H}} ds \\ &+ \int_0^t [p_{t-s} * (b'(u^0(s,\star)) Z^{f_n}(s,\star))](x) ds. \end{aligned} \tag{3.20}$$

Since the norm $\{\int_0^T \|f_n(s)\|_{\mathcal{H}}^2 ds, n \geq 1\}$ is bounded by a constant N, invoking similar arguments as in the proof of Proposition 3.2, we can show that

$$\sup_n \sup_{t\in[0,T]} \{\|Z^{f_n}(t,\star)\|_\infty^2 + \|\mathcal{N}_{\frac{1}{2}-H}(Z^{f_n}(t,\star))\|_\infty^2\} < \infty. \tag{3.21}$$

Next, we prove that the family $\{Z^{f_n}, n \geq 1\}$ is equi-Hölder continuous. For $0 \leq t_1 < t_2 \leq T$ and $x \in \mathbb{R}$, by using the property of semigroup of Green function p_t and the Hölder regularity of u_0, one gets

$$\begin{aligned}&|p_{t_2} * u_0(x) - p_{t_1} * u_0(x)|^2\\&= \left| \int_{\mathbb{R}} p_{t_2-t_1}(z) \int_{\mathbb{R}} p_{t_1}(x-2y)(u_0(y-z) - u_0(y))dydz \right|^2\\&\le C \int_{\mathbb{R}} p_{t_2-t_1}(z)|z|^{2H}dz\\&\le C(t_2-t_1)^{2H}.\end{aligned} \tag{3.22}$$

Denote by

$$Z_1^{f_n}(t,x) = \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f_n(s,\star)\rangle_{\mathcal{H}} ds,$$

and

$$Z_2^{f_n}(t,x) = \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z^{f_n}(s,\star))](x)ds.$$

It is sufficient to show that $\{Z_1^{f_n}, n \ge 1\}$ and $\{Z_2^{f_n}, n \ge 1\}$ are Hölder continuous with Hölder exponents being independent of n.

Firstly, for $0 \le t_1 < t_2 \le T$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}&|Z_1^{f_n}(t_2,x) - Z_1^{f_n}(t_1,x)|^2\\&\le C\left| \int_{t_1}^{t_2} \langle p_{t_2-s}(x-\star)\sigma(u^0(s,\star)), f_n(s,\star)\rangle_{\mathcal{H}} ds\right|^2\\&\quad + C\left| \int_0^{t_1} \langle (p_{t_2-s}(x-\star) - p_{t_1-s}(x-\star))\sigma(u^0(s,\star)), f_n(s,\star)\rangle_{\mathcal{H}} ds\right|^2\\&\le C\int_0^T \|f_n(s,\star)\|_{\mathcal{H}}^2 ds \int_{t_1}^{t_2} \|p_{t_2-s}(x-\star)\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 ds\\&\quad + C\int_0^T \|f_n(s,\star)\|_{\mathcal{H}}^2 ds \int_0^{t_1} \|(p_{t_2-s}(x-\star) - p_{t_1-s}(x-\star))\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 ds.\end{aligned} \tag{3.23}$$

By using the similar arguments in the proofs [11, Theorem 4.2 and Theorem 4.3], one obtains, respectively

$$\begin{aligned}&\int_{t_1}^{t_2} \|p_{t_2-s}(x-\star)\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 ds\\&\quad \le C\int_{t_1}^{t_2} (\|u^0(s,\star)\|_\infty^2 (t_2-s)^{-1+H} + \|\mathcal{N}_{\frac{1}{2}-H} u^0(s,\star)\|_\infty^2 (t_2-s)^{-\frac{1}{2}})ds\\&\quad \le C((t_2-t_1)^H + (t_2-t_1)^{\frac{1}{2}}),\end{aligned}$$

and

$$\int_0^{t_1} \|(p_{t_2-s}(x-\star) - p_{t_1-s}(x-\star))\sigma(u^0(s,\star))\|_{\mathcal{H}}^2 ds \le C((t_2-t_1)^H + (t_2-t_1)^{\frac{1}{2}}).$$

Then we see that there exists a constant C (independent of n) such that

$$|Z_1^{f_n}(t_2,x) - Z_1^{f_n}(t_1,x)|^2 \le C|t_2-t_1|^H, \tag{3.24}$$

for all $0 \le t_1, t_2 \le T$ and $x \in \mathbb{R}$. For the second term $Z_2^{f_n}(t,x)$, with $0 \le t_1 < t_2 \le T$ and $x \in \mathbb{R}$, we have

$$|Z_2^{f_n}(t_2,x) - Z_2^{f_n}(t_1,x)|^2$$

$$
\begin{aligned}
&\leq C\left|\int_{t_1}^{t_2}\int_{\mathbb{R}} p_{t_2-s}(x-y)b'(u^0(s,y))Z^{f_n}(s,y)dyds\right|^2 \\
&\quad+\left|\int_0^{t_1}\int_{\mathbb{R}}(p_{t_2-s}(x-y)-p_{t_1-s}(x-y))b'(u^0(s,y))Z^{f_n}(s,y)dyds\right|^2 \\
&\leq C(t_2-t_1)^{\frac{1}{2}},
\end{aligned}
\tag{3.25}
$$

by using some results proved in the proof of [11, Theorem 4.3]. Combing the estimates (3.22), (3.24) and (3.25), we have

$$
|Z^{f_n}(t_2,x)-Z^{f_n}(t_1,x)|^2 \leq C|t_2-t_1|^H, \tag{3.26}
$$

for all $0 \leq t_1, t_2 \leq T$ and $x \in \mathbb{R}$.

Secondly for $x_1, x_2 \in \mathbb{R}$, similarly to the proof of (4.57) in [11], we have

$$
\begin{aligned}
&|Z_1^{f_n}(t,x_1)-Z_1^{f_n}(t,x_2)|^2 \\
&\leq C\left|\int_0^t \langle (p_{t-s}(x_1-\star)-p_{t-s}(x_2-\star))\sigma(u^0(s,\star)), f_n(s,\star)\rangle_{\mathcal{H}} ds\right|^2 \\
&\leq C\int_0^T \|f_n(s,\star)\|_{\mathcal{H}}^2 ds \int_0^t \left\|(p_{t-s}(x_1-\star)-p_{t-s}(x_2-\star))\sigma(u^0(s,\star))\right\|_{\mathcal{H}}^2 ds \\
&\leq C(|x_1-x_2|+|x_1-x_2|^{2H}),
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
&|Z_2^{f_n}(t,x_1)-Z_2^{f_n}(t,x_2)|^2 \\
&\quad\leq C\int_0^t\int_{\mathbb{R}} |(p_{t-s}(x_1-y)-p_{t-s}(x_2-y))b'(u^0(s,y))Z^{f_n}(s,y)|^2 dyds \\
&\quad\leq C|x_1-x_2|.
\end{aligned}
\tag{3.28}
$$

For the term $p_t * u_0(x)$, with $x_1, x_2 \in \mathbb{R}$, we have

$$
|p_t * u_0(x_1)-p_t * u_0(x_2)|^2 \leq C\int_{\mathbb{R}}(p_t(x_1-y)-p_t(x_2-y))^2|u_0(y)|^2 dy \leq C|x_1-x_2|. \tag{3.29}
$$

Collecting the inequalities (3.27), (3.28) and (3.29), we get

$$
|Z^{f_n}(t,x_1)-Z^{f_n}(t,x_2)|^2 \leq C(|x_1-x_2|+|x_1-x_2|^{2H}), \tag{3.30}
$$

for all $x_1, x_2 \in \mathbb{R}$ and $0 \leq t \leq T$.

Recall that

$$
Z^{f_n}(t,x) = p_t * u_0(x) + Z_1^{f_n}(t,x) + Z_2^{f_n}(t,x).
$$

Then it follows from (3.26) and (3.30) that there exists an independent constant $C > 0$ such that

$$
|Z^{f_n}(t_1,x_1)-Z^{f_n}(t_2,x_2)|^2 \leq C(|x_1-x_2|+|x_1-x_2|^{2H}+|t_1-t_2|^H), \tag{3.31}
$$

for all $0 \leq t_1, t_2 \leq T$ and $x_1, x_2 \in \mathbb{R}$. The above uniform estimate along with the Arzela–Ascoli theorem yields that there exists a subsequence $\{n_k, k \geq 1\}$ and a uniform function $Z(t,x)$ such that

$$
\sup_{0\leq t\leq T}\sup_{x\in[-R,R]} |Z^{f_{n_k}}(t,x)-Z(t,x)| \to 0, \tag{3.32}
$$

for every $R > 0$ as $k \to \infty$. First, we will prove that $Z = Z^f$. By the uniqueness of the equation, it is sufficient to show that Z is a solution to (3.1). Applying Fatou lemma and taking into account (3.21) and (3.31), it is easy to see that

$$\sup_{t\in[0,T]} \{\|Z(t,\star)\|_\infty^2 + \|\mathcal{N}_{\frac{1}{2}-H}(Z(t,\star))\|_\infty^2\} < \infty \tag{3.33}$$

and

$$|Z(t_1,x_1) - Z(t_2,x_2)|^2 \le C(|x_1-x_2| + |x_1-x_2|^{2H} + |t_1-t_2|^H), \tag{3.34}$$

for all $0 \le t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{R}$. Recall that

$$\begin{aligned} Z^{f_{n_k}}(t,x) = p_t * u_0(x) &+ \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f_{n_k}(s,\star)\rangle_{\mathcal{H}} ds \\ &+ \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z^{f_{n_k}}(s,\star))](x) ds. \end{aligned} \tag{3.35}$$

To pass the limit in the above equation as $k \to \infty$, it suffices to prove the following

$$\lim_{k\to\infty} \int_0^t \left| \int_{\mathbb{R}} p_{t-s}(x-y) b'(u^0(s,y))(Z^{f_{n_k}}(s,y) - Z(s,y)) dy \right|^2 ds = 0. \tag{3.36}$$

For every $y \in \mathbb{R}$, one has $(Z^{f_{n_k}}(s,y) - Z(s,y))^2$ tends to zero as $k \to \infty$. Applying the dominated convergence theorem, we deduce that

$$\lim_{k\to\infty} \left| \int_{\mathbb{R}} p_{t-s}(x-y) b'(u^0(s,y))(Z^{f_{n_k}}(s,y) - Z(s,y)) dy \right|^2 = 0.$$

Moreover we also have

$$\begin{aligned} &\left| \int_{\mathbb{R}} p_{t-s}(x-y) b'(u^0(s,y))(Z^{f_{n_k}}(s,y) - Z(s,y)) dy \right|^2 \\ &\quad \le C\|b'\|_\infty^2 \int_{\mathbb{R}} |p_{t-s}(x-y)|^2 dy \int_{\mathbb{R}} (Z^{f_{n_k}}(s,y) - Z(s,y))^2 dy \\ &\quad \le C(t-s)^{-\frac{1}{2}}. \end{aligned}$$

Then, by the dominated convergence theorem, we have proved the limit (3.36). Since f_{n_k} converges to f as $k \to \infty$, using (3.36), we have

$$\begin{aligned} &\lim_{k\to\infty} \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z^{f_{n_k}}(s,\star))](x) ds \\ &\quad = \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z^f(s,\star))](x) ds. \end{aligned}$$

Now let $k \to \infty$ in (3.20) to get

$$\begin{aligned} Z(t,x) = p_t * u_0(x) &+ \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), f(s,\star)\rangle_{\mathcal{H}} ds \\ &+ \int_0^t [p_{t-s} * (b'(u^0(s,\star))Z(s,\star))](x) ds, \end{aligned} \tag{3.37}$$

which implies that $Z = Z^f$. To conclude that $Z^{f_n} \to Z^f$ in $X_T^{\frac{1}{2}-H}$, it suffices to show that the family $\{Z^{f_n}, n \ge 1\}$ is relatively compact. According to [11, Proposition 4.18], one only need to check the following three conditions:

(1) $\sup_n |Z^{f_n}(0,0)|$ is finite;

(2) for every $x\in\mathbb{R}$, $\{Z^{f_n}(\cdot,x), n\geq 1\}$ is equi-continuous in time;

(3) for every $R>0$,

$$\lim_{\delta\to 0}\sup_n \sup_{t\in[0,T],x\in[-R,R]}\int_{-\delta}^{\delta}\frac{|Z^{f_n}(t,x+y)-Z^{f_n}(t,x)|^2}{|y|^{2-2H}}dy=0.$$

Actually the first point is clear. The second and third points follow from (3.31). Thus we can conclude the proof of this proposition. □

4 Moderate Deviations Principle

The main aim of this paper is to prove that $Z^\varepsilon(t,x)$ defined by (1.4) satisfies a LDP on the space $X_T^{\frac{1}{2}-H}$, where $\lambda(\varepsilon)$ satisfies (1.5). This special type of LDP is usually called the *moderate deviations principle* of u^ε ([6]). It is stated as follows.

Theorem 4.1 (Moderate deviations principle) *Assuming the conditions in Proposition* 3.3 *hold. Let u^ε be the solution to* SHE (1.1)*, then the law of $Z^\varepsilon(t,x)$ defined by* (1.4) *obeys an large deviations principle in the space $X_T^{\frac{1}{2}-H}$ with speed $\lambda^2(\varepsilon)$ and the good rate function $I(u)$ given by*

$$I(u)=\inf_{f\in S^u}\left\{\frac{1}{2}\int_0^T \|f(s)\|_{\mathcal{H}}^2 ds\right\} \tag{4.1}$$

with

$$S^u=\{f\in L^2([0,T];\mathcal{H}); Z^f=u\},\quad u\in X_T^{\frac{1}{2}-H},$$

where Z^f is the solution to Eq. (3.1)*.*

From (1.4), it is easy to see that $Z^\varepsilon(t,x)$ satisfies the following equation

$$\begin{aligned}Z^\varepsilon(t,x)=&\frac{1}{\lambda(\varepsilon)}\sum_{k\geq 1}\int_0^t\langle p_{t-s}(x-\star)\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^\varepsilon(s,\star)),e_k\rangle_{\mathcal{H}}dB_k(s)\\&+\int_0^t\left\{p_{t-s}*\left[\frac{b(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^\varepsilon(s,\star))-b(u^0(s,\star))}{\sqrt{\varepsilon}\lambda(\varepsilon)}\right]\right\}(x)ds.\end{aligned}\tag{4.2}$$

For any $v\in\mathcal{U}^N$ and $\varepsilon>0$, define the *controlled equation* for $Z^{\varepsilon,v}$ as follows

$$\begin{aligned}Z^{\varepsilon,v}(t,x)=&\,p_t*u_0(x)+\frac{1}{\lambda(\varepsilon)}\sum_{k\geq 1}\int_0^t\langle p_{t-s}(x-\star)\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s,\star)),e_k\rangle_{\mathcal{H}}dB_k(s)\\&+\int_0^t\langle p_{t-s}(x-\star)\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s,\star)),v(s,\star)\rangle_{\mathcal{H}}ds\\&+\int_0^t\left\{p_{t-s}*\left[\frac{b(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v}(s,\star))-b(u^0(s,\star))}{\sqrt{\varepsilon}\lambda(\varepsilon)}\right]\right\}(x)ds.\end{aligned}\tag{4.3}$$

We shall prove the moderate deviations principle satisfied by $u^\varepsilon(t,x)$ given by Theorem 4.1 by using the *weak convergence approach* (for example, [3, 14, 15, 20] and etc). Inspired by [3], let us consider the two conditions given by **Condition A** in Section 2.2 which correspond on the weak convergence approach frame in our setting. Also refer to the weak convergence approach used in [13]. We make the following remarks concerning with the two conditions.

Remark 4.2 Condition (1) (i.e. Eq. (2.10)) says that the level sets of the rate function are compact, and condition (2) (i.e. Eq. (2.11)) is a crucial assumption in the applications of the weak convergence approach and is a statement of weak convergence of the family of random variables $\{Z^{\varepsilon,v^\varepsilon},\varepsilon>0\}$ as $\varepsilon\to 0$.

Remark 4.3 Applying to Theorem 2.5, a verification of condition (1) and condition (2) implies the validity of Theorem 4.1. The first point follows from Proposition 3.3. The second point follows form Proposition 4.5.

One should first show the existence and uniqueness of the stochastic controlled equation given by (4.3). Its proof can be given by following the similar arguments in the proofs of Proposition 3.2. The details are left to the interested readers.

Proposition 4.4 *Assuming the conditions in Proposition* 3.2 *hold, then there exists a unique random field solution* $Z^{\varepsilon,v}=\{Z^{\varepsilon,v}(t,x),t\in[0,T],x\in\mathbb{R}\}$ *to Eq.* (4.3) *such that* $Z^{\varepsilon,v}\in X_T^{\frac12-H}$.

Now we give the weak convergence of the solution to the controlled equation (4.3).

Proposition 4.5 *Assume the following conditions hold*:

(1) *The initial condition* u_0 *is bounded and local Hölder continuous of order* H.

(2) *The coefficients* b *and* σ *are differentiable and their derivatives are Lipschitz and* $b(0)=0$, $\sigma(0)=0$.

For any family $\{v^\varepsilon;\varepsilon>0\}\subset\mathcal{U}^N$ *which converges in distribution as* $\varepsilon\to 0$ *to* $v\in\mathcal{U}^N$, *we have*

$$\lim_{\varepsilon\to 0} Z^{\varepsilon,v^\varepsilon}=Z^v,$$

where Z^v *denotes the solution of Eq.* (3.1) *corresponding to the random variable* v (*instead of a deterministic function* f).

Proof First we claim that for any $p\ge 1$, there exists some $\theta>0$ such that,

$$\sup_{\varepsilon\le 1}\sup_{v\in\mathcal{U}^N}\sup_{t\in[0,T],x\in\mathbb{R}}\|Z^{\varepsilon,v^\varepsilon}(t,x)\|_{\mathfrak{X}^p_{T,\theta}}<\infty, \tag{4.4}$$

where the seminorm $\|\cdot\|_{\mathfrak{X}^p_{T,\theta}}$ is defined by (2.4).

Recall that $Z^{\varepsilon,v^\varepsilon}$ given by (4.3) can be written as

$$Z^{\varepsilon,v^\varepsilon}=p_t*u_0(x)+\Phi_1^\varepsilon(t,x)+\Phi_2^\varepsilon(t,x)+\Phi_3^\varepsilon(t,x),$$

where we denote by

$$\Phi_1^\varepsilon(t,x)=\frac{1}{\lambda(\varepsilon)}\sum_{k\ge 1}\int_0^t\langle p_{t-s}(x-\star)\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\star)),e_k\rangle_{\mathcal{H}}dB_k(s), \tag{4.5}$$

$$\Phi_2^\varepsilon(t,x)=\int_0^t\langle p_{t-s}(x-\star)\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\star)),v^\varepsilon(s,\star)\rangle_{\mathcal{H}}ds, \tag{4.6}$$

and

$$\Phi_3^\varepsilon(t,x)=\int_0^t\left\{p_{t-s}*\left[\frac{b(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)Z^{\varepsilon,v^\varepsilon}(s,\star))-b(u^0(s,\star))}{\sqrt{\varepsilon}\lambda(\varepsilon)}\right]\right\}(x)ds. \tag{4.7}$$

Recall that

$$\|p_t*u_0(x)\|_{\mathfrak{X}^p_{T,\theta}}\le C,$$

then

$$\|Z^{\varepsilon,v^\varepsilon}(t,x)\|_{\mathfrak{X}^p_{T,\theta}} \le C\sum_{i=1}^{3}\Big(\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta t}\|\Phi_i^\varepsilon(t,x)\|_{L^p(\Omega)} + \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta t}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_i^\varepsilon(t,x)\Big).$$

Now let us prove the upper bounds for the $\|\Phi_i^\varepsilon(t,x)\|_{L^p(\Omega)}$ and $\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_i^\varepsilon(t,x)$ with $i=1,2,3$. By using [13, Proposition 3.6], we have that

$$\begin{aligned}
&\mathrm{e}^{-\theta t}\|\Phi_1^\varepsilon(t,x)\|_{L^p(\Omega)}\\
&\quad\le C\varepsilon \sup_{(s,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta s}\|Z^{\varepsilon,v^\varepsilon}\|_{L^p(\Omega)}\bigg(\int_0^t \mathrm{e}^{-2\theta(t-s)}(t-s)^{-1+H}ds\bigg)^{\frac{1}{2}}\\
&\qquad + C\varepsilon \sup_{(s,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta s}\sup_{x\in\mathbb{R}}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}Z^{\varepsilon,v^\varepsilon}(s,x)\bigg(\int_0^t \mathrm{e}^{-2\theta(t-s)}(t-s)^{-1/2}ds\bigg)^{\frac{1}{2}}\\
&\quad\le C\varepsilon\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{H}{2}}+\theta^{-\frac{1}{4}}).
\end{aligned}\tag{4.8}$$

The term $\mathrm{e}^{-\theta t}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_1^\varepsilon(t,x)$ can be estimate as follows. Also by [13, Proposition 3.6] and the definition of the mapping $\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_1^\varepsilon(t,x)$ given by (2.2), we have

$$\begin{aligned}
&\mathrm{e}^{-\theta t}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_1^\varepsilon(t,x)\\
&\quad\le C\varepsilon\lambda^2(\varepsilon) \sup_{(s,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta s}\|Z^{\varepsilon,v^\varepsilon}\|_{L^p(\Omega)}\bigg(\int_0^t \mathrm{e}^{-2\theta(t-s)}(t-s)^{-\frac{3}{2}+2H}ds\bigg)^{\frac{1}{2}}\\
&\qquad + C\varepsilon\lambda^2(\varepsilon) \sup_{(s,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta s}\sup_{x\in\mathbb{R}}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}Z^{\varepsilon,v^\varepsilon}(s,x)\bigg(\int_0^t \mathrm{e}^{-2\theta(t-s)}(t-s)^{-1+H}ds\bigg)^{\frac{1}{2}}\\
&\quad\le C\varepsilon\lambda^2(\varepsilon)\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{H}{2}}+\theta^{\frac{1}{4}-H}).
\end{aligned}\tag{4.9}$$

Combining the above estimates for $\mathrm{e}^{-\theta t}\|\Phi_1^\varepsilon(t,x)\|_{L^p(\Omega)}$ and $\mathrm{e}^{-\theta t}\mathcal{N}^{L^p(\Omega)}_{\frac{1}{2}-H}\Phi_1^\varepsilon(t,x)$ together, we have that

$$\|\Phi_1^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}} \le C\varepsilon\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{H}{2}}+\theta^{-\frac{1}{4}}+\lambda^2(\varepsilon)(\theta^{-\frac{H}{2}}+\theta^{\frac{1}{4}-H})).\tag{4.10}$$

We can also use the similar arguments for $\|\Phi_2^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}}$. Moreover we have

$$\|\Phi_2^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}} \le C\varepsilon\lambda^2(\varepsilon)\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(2\theta^{-\frac{H}{2}}+\theta^{-\frac{1}{4}}+\theta^{\frac{1}{4}-H}).\tag{4.11}$$

Now let us turn to the third term $\|\Phi_3^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}}$. By using the similar arguments for $\|\Phi_1^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}}$ and Lipschitz continuity of b, we have

$$\|\Phi_3^\varepsilon(t,x)\|_{\mathfrak{X}^p_{T,\theta}} \le C\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{1}{4}}+\theta^{-\frac{H}{2}}).\tag{4.12}$$

Then from (4.10), (4.11) and (4.12), we have

$$\begin{aligned}
\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}} &\le C + C\varepsilon\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{H}{2}}+\theta^{-\frac{1}{4}}+\lambda^2(\varepsilon)(\theta^{-\frac{H}{2}}+\theta^{\frac{1}{4}-H}))\\
&\quad + C\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}(\theta^{-\frac{1}{4}}+\theta^{-\frac{H}{2}}).
\end{aligned}\tag{4.13}$$

Now we can choose $\theta>0$ such that

$$C + C\varepsilon(\theta^{-\frac{H}{2}}+\theta^{-\frac{1}{4}}+\lambda^2(\varepsilon)(\theta^{-\frac{H}{2}}+\theta^{\frac{1}{4}-H})) + C(\theta^{-\frac{1}{4}}+\theta^{-\frac{H}{2}}) < 1.$$

Thus from (4.13), we have proved that

$$\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}} < \infty,$$

which is the claim.

In order to prove this proposition, we need a stronger conclusion. For any $\beta < H$ and $p \geq 2$, it holds that

$$\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^{\beta,p}_{T,\theta}} < \infty, \tag{4.14}$$

where

$$\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^{\beta,p}_{T,\theta}} := \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta t}\|Z^{\varepsilon,v^\varepsilon}(t,x)\|_{L^p(\Omega)} + \sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{e}^{-\theta t}\mathcal{N}_\beta^{L^p(\Omega)} Z^{\varepsilon,v^\varepsilon}(t,x).$$

In fact, for $\Phi_i^\varepsilon(t,x), i = 1,2,3$ defined above, by using [13, Proposition 3.6], we have, for large enough θ,

$$\|\Phi_i^\varepsilon\|_{\mathfrak{X}^{\beta,p}_{T,\theta}} \leq C_i(\|\sigma(Z^{\varepsilon,v^\varepsilon})\|_{\mathfrak{X}^p_{T,\theta}} + \|b'(Z^{\varepsilon,v^\varepsilon})\|_{\mathfrak{X}^p_{T,\theta}}) \leq C\|Z^{\varepsilon,v^\varepsilon}\|_{\mathfrak{X}^p_{T,\theta}}.$$

Then (4.14) follows from the above arguments.

In the third step, we need to check the Hölder regularity of $Z^{\varepsilon,v^\varepsilon} = \{Z^{\varepsilon,v^\varepsilon}(t,x), t \in [0,T], x \in \mathbb{R}\}$ with respect to t and x respectively. We also need to use the notation $\Phi_i^\varepsilon(t,x), i = 1,2,3$ defined by (4.5), (4.6) and (4.7).

For any $0 \leq t_1 < t_2 \leq T$ and $x_1, x_2 \in \mathbb{R}$, we can write

$$\begin{aligned} Z^{\varepsilon,v^\varepsilon}(t_2,x_2) - Z^{\varepsilon,v^\varepsilon}(t_1,x_1) &= p_{t_2} * u_0(x_2) - p_{t_1} * u_0(x_1) \\ &\quad + \sum_{i=1}^{3}(\Phi_i^\varepsilon(t_2,x_2) - \Phi_i^\varepsilon(t_1,x_1)). \end{aligned}$$

From (3.22) and (3.29), we have

$$|p_{t_2} * u_0(x_2) - p_{t_1} * u_0(x_1)| \leq C(|t_2 - t_1|^H + |x_2 - x_1|^{\frac{1}{2}}). \tag{4.15}$$

For the first term $\Phi_1^\varepsilon(t_2,x_2) - \Phi_1^\varepsilon(t_1,x_1)$ in the above sum, applying [11, Proposition 3.8], we obtain that

$$\|\Phi_1^\varepsilon(t_2,x_2) - \Phi_1^\varepsilon(t_1,x_1)\|_{L^p(\Omega)} \leq C(|t_2 - t_1|^{\frac{H}{2}} + |x_2 - x_1|^H). \tag{4.16}$$

While the second term $\Phi_2^\varepsilon(t_2,x_2) - \Phi_2^\varepsilon(t_1,x_1)$ in the above sum also satisfies the following

$$\|\Phi_2^\varepsilon(t_2,x_2) - \Phi_2^\varepsilon(t_1,x_1)\|_{L^p(\Omega)} \leq C(|t_2 - t_1|^{\frac{H}{2}} + |x_2 - x_1|^H). \tag{4.17}$$

For the third term $\Phi_3^\varepsilon(t_2,x_2) - \Phi_3^\varepsilon(t_1,x_1)$ in the above sum, we can bound is as follows. In fact, by using Taylor formula, the boundedness of $\|b'\|_\infty$ and Lipschitz continuity of b, we have that

$$\|\Phi_3^\varepsilon(t_2,x_2) - \Phi_3^\varepsilon(t_1,x_1)\|_{L^p(\Omega)} \leq C(|t_2 - t_1|^{\frac{1}{2}} + |t_2 - t_1|^{\frac{H}{2}} + |x_2 - x_1|^{\frac{1}{2}} + |x_2 - x_1|^H), \tag{4.18}$$

with C independent of $\varepsilon > 0$. Then combining the above estimates, we conclude that

$$\|Z^{\varepsilon,v^\varepsilon}(t_2 - x_2) - Z^{\varepsilon,v^\varepsilon}(t_1 - x_1)\|_{L^p(\Omega)} \leq C(|t_2 - t_1|^{\frac{H}{2}} + |x_2 - x_1|^H). \tag{4.19}$$

Using (4.14), (4.19) and [11, Proposition 4.24], we conclude that the law of the family $\{Z^{\varepsilon,v^\varepsilon}, \varepsilon \geq 0\}$ is tight on the space $X_T^{\frac{1}{2}-H}$. Hence, the family $\{(Z^{\varepsilon,v^\varepsilon}, W(\cdot,\cdot), v^\varepsilon); \varepsilon > 0\}$ is tight on the space

$$X_T^{\frac{1}{2}-H} \times C([0,T]\times\mathbb{R}) \times L^2([0,T];\mathcal{H}).$$

Recall that the topology of weak convergence is used for $L^2([0,T];\mathcal{H})$. Choosing a subsequence if necessary, by Skorokhod's embedding theorem, there exists a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{(\mathcal{F})}_t, \overline{P})$ carrying a family of random fields $(\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}, \overline{W}^\varepsilon(\cdot,\cdot), \overline{v}^\varepsilon)$ such that

$$\{(\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}, \overline{W}^\varepsilon(\cdot,\cdot), \overline{v}^\varepsilon); \varepsilon > 0\} = \{(Z^{\varepsilon,v^\varepsilon}, W(\cdot,\cdot), v^\varepsilon); \varepsilon > 0\},$$

in law and $\overline{P}$-almost surely. $\{(\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}, \overline{W}^\varepsilon(\cdot,\cdot), \overline{v}^\varepsilon); \varepsilon > 0\}$ converges to some random fields $(\overline{Z}^{\overline{v}}, \overline{W}(\cdot,\cdot), \overline{v}))$ in the space

$$X_T^{\frac{1}{2}-H} \times C([0,T]\times\mathbb{R}) \times L^2([0,T];\mathcal{H}).$$

Here we should mention that the above $\overline{W}^\varepsilon(\cdot,\cdot)$ is the regularization of the noise $\overline{W}(\cdot,\cdot)$. One can see [11] or [9] for some details about this regularization. In particular, the following stochastic heat equation is held for

$$\begin{aligned}\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(t,x) &= p_t * u_0(x) + \frac{1}{\lambda(\varepsilon)}\sum_{k\geq 1}\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)), e_k\rangle_{\mathcal{H}} dB_k(s)\\ &\quad + \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)), \overline{v}^\varepsilon(s,\star)\rangle_{\mathcal{H}} ds\\ &\quad + \int_0^t \Big\{p_{t-s} * \Big[\frac{b(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)) - b(u^0(s,\star))}{\sqrt{\varepsilon}\lambda(\varepsilon)}\Big]\Big\}(x) ds. \end{aligned} \tag{4.20}$$

Next we want to pass the limit in (4.20) as $\varepsilon \to 0$. First of all, we have

$$\begin{aligned}&\frac{1}{\lambda^2(\varepsilon)}\overline{E}\Big[\Big|\sum_{k\geq 1}\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)), e_k\rangle_{\mathcal{H}} dB_k(s)\Big|^2\Big]\\ &\quad = \frac{1}{\lambda^2(\varepsilon)}\overline{E}\Big[\int_0^t \|p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star))\|_{\mathcal{H}}^2 ds\Big],\end{aligned} \tag{4.21}$$

where $\overline{E}$ stands for the expectation under the probability measure $\overline{P}$. By the Lipschitz continuity of σ it is easy to see that

$$\begin{aligned}&\|p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star))\|_{\mathcal{H}}^2\\ &\quad \leq C\varepsilon\lambda^2(\varepsilon)\int_{\mathbb{R}^2} |p_{t-s}(x-(y+z)) - p_{t-s}(x-y)|^2|\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,y+z)|^2|z|^{2H-2} dydz\\ &\qquad + C\varepsilon\lambda^2(\varepsilon)\int_{\mathbb{R}^2} p_{t-s}^2(x-y)|\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,y+z) - \overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,y)|^2|z|^{2H-2} dydz\\ &\quad \leq C\varepsilon\lambda^2(\varepsilon)[(t-s)^{-1+H} + (t-s)^{-\frac{1}{2}}].\end{aligned}$$

Hence we have the following limit

$$\begin{aligned}&\frac{1}{\lambda^2(\varepsilon)}\overline{E}\Big[\Big|\sum_{k\geq 1}\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)), e_k\rangle_{\mathcal{H}} dB_k(s)\Big|^2\Big]\\ &\quad \leq C\varepsilon\int_0^t [(t-s)^{-1+H} + (t-s)^{-\frac{1}{2}}] ds \to 0, \quad \varepsilon \to 0,\end{aligned} \tag{4.22}$$

where we have used [13, Lemma 4.1] and the fact $\|\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}\|_{\mathfrak{X}_{T,\theta}^2}^2 < \infty$.

Next we will prove

$$\lim_{\varepsilon\to 0}\int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star) + \sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)), \overline{v}^\varepsilon(s,\star)\rangle_{\mathcal{H}} ds$$

$$= \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star)), \overline{v}(s,\star)\rangle_{\mathcal{H}} ds \tag{4.23}$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^t \left\{ p_{t-s} * \left[\frac{b(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)) - b(u^0(s,\star))}{\sqrt{\varepsilon}\lambda(\varepsilon)} \right] \right\}(x) ds \\ &= \int_0^t \{ p_{t-s} * [b'(u^0(s,\star))\overline{Z}^{\overline{v}}(s,\star)](x)\} ds. \end{aligned} \tag{4.24}$$

Since $\overline{v}^\varepsilon \to v$ weakly in $L^2([0,T];\mathcal{H})$ as $\varepsilon \to 0$ and $\int_0^T \|\overline{v}^\varepsilon\|^2 ds \le N$, in order to prove (4.23), it suffices to show that

$$\lim_{\varepsilon\to 0}\int_0^t E[\|p_{t-s}(x-\star)(\sigma(u^0(s,\star)+\sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,\star)) - \sigma(u^0(s,\star)))\|_{\mathcal{H}}^2] ds = 0.$$

And this can be proved by using the estimates (5.22)–(5.31) in [13]. For the limit (4.24), by using Taylor expansion and dominated convergence theorem, we have that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_0^t E\left[\left| \int_{\mathbb{R}} p_{t-s}(x-y) \left(\frac{b(u^0(s,y)+\sqrt{\varepsilon}\lambda(\varepsilon)\overline{Z}^{\varepsilon,\overline{v}^\varepsilon}(s,y)) - b(u^0(s,y))}{\sqrt{\varepsilon}\lambda(\varepsilon)} \right.\right.\right. \\ &\quad \left.\left.\left. - b'(u^0(s,y))\overline{Z}^{\overline{v}}(s,y) \right) dy \right|^2 \right] ds \\ &= 0. \end{aligned}$$

Then the limit (4.24) holds. Now let $\varepsilon \to 0$ in (4.20) and use (4.22), (4.23) and (4.24), we conclude that

$$\begin{aligned} \overline{Z}^{\overline{v}}(t,x) &= p_t * u_0(x) + \int_0^t \langle p_{t-s}(x-\star)\sigma(u^0(s,\star))\overline{v}(s,\star), \overline{v}(s,\star)\rangle_{\mathcal{H}} ds \\ &\quad + \int_0^t \{p_{t-s} * [b'(u^0(s,\star))\overline{Z}^{\overline{v}}(s,\star)]\}(x) ds. \end{aligned} \tag{4.25}$$

Since $\overline{v}^\varepsilon \to \overline{v}$ in distribution and $\overline{v}^\varepsilon$ has the same law as v^ε, $\overline{v}$ must have the same law as that of v. It follows from the uniqueness of the solution of (3.1) that $v(\cdot,\cdot)$, the solution of the (3.1) and $\overline{v}(\cdot,\cdot)$ will have the law. We can finally conclude that

$$Z^{\varepsilon,v^\varepsilon} \to Z^v,$$

in distribution as $\varepsilon \to 0$, which completes the proof of this result. □

Acknowledgements We thank the referees for their time and comments.

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