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Asymptotical Behaviors for Neumann Boundary Problem with Singular Data

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Abstract In this paper, we will analyze the blow-up behaviors for solutions to the Laplacian equation with exponential Neumann boundary condition. In particular, the boundary value is with a kind of singular data. We show a Brezis–Merle type concentration-compactness theorem, calculate the blow up value at the blow-up point, and give a point-wise estimate for the profile of the solution sequence at the blow-up point.

Keywords Exponential Neumann boundary condition, singular data, blow up analysis, profile of the solution sequence

MR(2010) Subject Classification 35B40, 35J65

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. In the celebrated paper by Brezis and Merle [6], they initiated the study of blow-up analysis for the Liouville equation

$$-\Delta u(x) = V(x)\mathrm{e}^{2u} \quad \text{in } \Omega \tag{1.1}$$

with the energy condition $\int_\Omega V(x)\mathrm{e}^{u} dx < +\infty$. Here $V(x)$ is a nonnegative function. They first showed that any solution to (1.1) belongs to L^∞, and further they analyzed the convergence of a sequence of solutions $\{u_n\}$ to (1.1) and obtained a concentration-compactness result under the uniformly bounded energy condition $\int_\Omega V(x)\mathrm{e}^{u_n} dx < C$. Their results initiate many works on the asymptotic behavior of blow-up solutions. In particular, Yanyan Li and Shafrir [9] showed the quantization of blow-up value at the blow-up point, and Yanyan Li [10] furthermore showed the profile of solution sequences in a neighborhood of a blow-up point provided the oscillation on the boundary of this neighborhood is uniformly bounded.

The corresponding Brezis–Merle type compactness-concentration result and the asymptotic behavior of blow-up solutions to Liouville type equation with singular version

$$-\Delta u(x) = V(x)|x|^{2\alpha}\mathrm{e}^{2u} \quad \text{in } \Omega\backslash\{0\} \tag{1.2}$$

also had been established in [1–4], etc. Here $\alpha > -1$. It turns out that when $\alpha \neq 0$, the problem is more subtler. Since there possibly exist two types of bubbling solutions when a blow-up point occurs at singular point, therefore one needs to analyze the problem deeply to get the results.

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It is well known that Liouville equation (1.1) and (1.2) have a rich background in geometry and physics. In particular, when $\alpha \neq 0$, Eq. (1.2) was studied in the problem of finding a metric on Ω that has a prescribed scalar curvature with a conical singularity at zero, see [13, 14], etc. Beside geometrical interpretations, Eq. (1.1) and Eq. (1.2) is also related to fields of physics and Chern–Simons gauge theory, see [3, 12], etc. They also arise in some problems of combustion and statistical mechanics, see [5, 7] and the reference therein.

The aim of the present paper is to generalize the blow-up analysis for Eq. (1.1) and Eq. (1.2) to the Laplacian equation with exponential Neumann boundary condition and with singular data. In other words, we assume that $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $0 \in \partial\Omega$, and consider the following problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = V(x)|x|^{\alpha}\mathrm{e}^{u} - W(x) & \text{on } \partial\Omega\backslash\{0\}, \end{cases} \tag{1.3}$$

where $\alpha > -1$. When $\alpha = 0$, the problem had been investigated by Guo and Liu in [8]. They proved a Brezis–Merle type concentration-compactness theorem and showed that the all blow-up points of blow-up solutions are on the boundary $\partial\Omega$. They further got the blow-up value, which is $2\pi n$ for $n \in \mathbb{N}$, at a blow-up point for the local problem of (1.3).

In this paper, we study the problem (1.3) with singular data. From [15], we know that a weak solution u of (1.3) satisfies that $u \in C^2(\Omega) \cap C^1(\overline{\Omega}\backslash\{0\})$ and u is continuous at the origin. When $\alpha \geq 0$, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We first prove the following Brezis–Merle type concentration-compactness Theorem:

Theorem 1.1 *Assume that u_n is a sequence of solutions to* (1.3) *satisfying the energy condition*

$$\int_{\partial\Omega} V_n(x)|x|^{\alpha}\mathrm{e}^{u_n}\,dx \leq C, \tag{1.4}$$

where C is a positive number, $\alpha > -1$, $0 < a \leq V_n(x) \leq b$, $V_n(x) \in C^1(\overline{\Omega})$ and $V_n(x) \to V(x)$ uniformly in $\overline{\Omega}$. Moreover we assume $W_n(x) \in C^1(\overline{\Omega})$ and $W_n(x) \to W(x)$ uniformly in $\overline{\Omega}$. Then, there exists a subsequence, denoted still by u_n, satisfying one of the following alternatives:

(i) *u_n is bounded in $L^\infty(\overline{\Omega})$,*

(ii) *$u_n \to -\infty$ uniformly on $\overline{\Omega}$,*

(iii) *there exists a finite blow-up set $S = \{p_1, p_2, \dots, p_m\} \subset \partial\Omega$ such that, for any $1 \leq i \leq m$, there exists $\{x_n\} \subset \partial\Omega$, $x_n \to p_i$, $u_n(x_n) \to +\infty$. Moreover,*

$$u_n(x) \to -\infty \text{ uniformly on compact subsets of } \overline{\Omega}\backslash S,$$

and

$$V_n(x)|x|^{\alpha}\mathrm{e}^{u_n} \rightharpoonup \sum m_i \delta_{p_i}$$

in the sense measure on $\partial\Omega$ with $m_i \geq \min\{\pi(1+\alpha), \pi\}$ for all i and $\alpha > -1$.

Due to the singularity of (1.3), we cannot only use the argument given in [8]. So it is worth mentioning that, if $p = 0$ is a blow-up point, we prove Theorem 1.1 by using the global Green representation formula when $-1 < \alpha \leq 0$ and by using the local Green representation formula and the Pohozaev type identity of equations when $\alpha > 0$.

Next we assume that $\{u_n\}$ is a sequence of blow-up solutions. The important part of the blow-up behaviors of solution sequences is to compute the blow-up value at $p \in S$. The blow-up value at $p \in S$ is defined as

$$m(p) = \lim_{R\to 0}\lim_{n\to\infty}\int_{B_R(p)\cap\partial\Omega} V_n(x)|x|^\alpha \mathrm{e}^{u_n} d\sigma.$$

Then we have the following proposition:

Proposition 1.2 *There exists $G \in W^{1,q}(\Omega)\cap C^2_{\mathrm{loc}}(\overline{\Omega}\backslash S)$ with $\int_\Omega G = 0$ for $1 < q < 2$ such that*

$$u_n - \frac{1}{|\Omega|}\int_\Omega u_n \to G, \tag{1.5}$$

in $C^2_{\mathrm{loc}}(\overline{\Omega}\backslash S)$ and weakly in $W^{1,q}(\Omega)$. Moreover at each blow-up point $p_l \in \partial\Omega\cap S$, there exists $R > 0$ small enough such that $B_R(p_l)\cap S = \{p_l\}$ and

$$G = \frac{1}{\pi}m(p_l)\log\frac{1}{|x-p_l|} + g(x),$$

for $x \in B_R(p_l)\cap\overline{\Omega}\backslash\{p_l\}$ with $g \in C^2(B_R(p_l))$.

Proposition 1.2 and the Pohozaev type identity of equations imply the following theorem:

Theorem 1.3 *If $0 \in S$, then $m(0) = 2\pi(1+\alpha)$. If $p \neq 0$ and $p \in S$, then $m(p) = 2\pi$.*

For purpose of accurate behaviors of solution sequences, it is necessary to show a point-wise estimate for the profile of the solution sequences. Noticing that u_n has uniformly bounded oscillations on compact subsets $\overline{\Omega}\backslash S$ due to (1.5), we assume simply that

$$\begin{cases} -\Delta u_n = 0 & \text{in } B_1^+ \\ \dfrac{\partial u_n}{\partial n} = V_n(x)|x|^\alpha \mathrm{e}^{u_n} - W_n(x) & \text{on } \partial B_1^+\cap\partial\mathbb{R}^2_+\backslash\{0\}, \end{cases} \tag{1.6}$$

with conditions

$$\max u_n - \min u_n \le C, \quad \text{on } \partial B_1^+\cap\mathbb{R}^2_+,$$

and

$$V_n(x)|x|^\alpha \mathrm{e}^{u_n} \rightharpoonup 2\pi(1+\alpha)\delta_0.$$

i.e., 0 is the only blow-up point on $\bar{B}_1^+$. Assume that $\mu_n = u_n(x_n) = \max_{\overline{B_1^+}} u_n(x)$, $x_n = (s_n, t_n) \in \overline{B_1^+}$ and $\lambda_n = \mathrm{e}^{-\frac{\mu_n}{1+\alpha}}$. Then we have $\lambda_n \to 0$ and $x_n \to 0$. Define the scaling functions by

$$\widetilde{u}_n(x) = u_n(\lambda_n x) + (1+\alpha)\ln\lambda_n,$$

for any $x \in B_{\frac{1}{\lambda_n}}\cap\overline{\mathbb{R}^2_+}$. Then $\widetilde{u}_n$ satisfies

$$\begin{cases} -\Delta\widetilde{u}_n = 0, & \text{in } B_{\frac{1}{\lambda_n}}\cap\mathbb{R}^2_+, \\ \dfrac{\partial\widetilde{u}_n}{\partial n} = V_n(\lambda_n x)|x|^\alpha \mathrm{e}^{\widetilde{u}_n} - \lambda_n W_n(\lambda_n x), & \text{on } B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+\backslash\{0\}, \end{cases}$$

with the energy condition

$$\int_{B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+} V_n(\lambda_n x)|x|^\alpha \mathrm{e}^{\widetilde{u}_n} dx \le C.$$

When $\alpha \in (-1, +\infty)$, if we assume $\lim_{n\to\infty} \frac{|x_n|}{\lambda_n} = \Lambda < +\infty$, then we have $\widetilde{u}_n(x) \le 0$ and $\widetilde{u}_n(\frac{x_n}{\lambda_n}) = 0$. Hence by Theorem 1.1, $\{\widetilde{u}_n\}$ admits a subsequence converging to $\widetilde{u}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2_+) \cap C^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+ \backslash \{0\})$ which satisfy

$$\begin{cases} -\Delta \widetilde{u} = 0, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial \widetilde{u}}{\partial n} = V(0)|x|^\alpha \mathrm{e}^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_+ \backslash \{0\}, \end{cases}$$

with the condition $\int_{\partial\mathbb{R}^2_+} V(0)|x|^\alpha \mathrm{e}^{\widetilde{u}} dx \le C$, $\sup_{\overline{\mathbb{R}^2_+}} \widetilde{u}(x) \le C$. By classification results in [15], $\widetilde{u}(x)$ takes the form

$$\widetilde{u}(x) = \ln \frac{\sqrt{8}(\alpha+1)\lambda^{\alpha+1}}{|x^{\alpha+1} - x_0|^2},$$

for some point x_0. Moreover $\int_{\partial\mathbb{R}^2_+} V(0)|x|^\alpha \mathrm{e}^{\widetilde{u}} dx = 2\pi(1+\alpha)$. Our main results is:

Theorem 1.4 *For $\alpha \in (-1, +\infty)$, assume that $\{u_n\}$ satisfies the problem* (1.6) *with its conditions. If $\lim_{n\to\infty} \frac{|x_n|}{\lambda_n} = \Lambda < +\infty$, then there exists two constants $r_0 > 0$ and C independent of n, such that*

$$\left| u_n(x) - \mu_n - \widetilde{u}\left(\frac{1}{\lambda_n} x\right) \right| \le C, \qquad \text{in } B_{r_0} \cap \overline{\mathbb{R}^2_+}. \tag{1.7}$$

Remark 1.5 In Theorem 1.4, we assume $\lim_{n\to\infty} \frac{|x_n|}{\lambda_n} = \Lambda < +\infty$. In fact, we can prove $\lim_{n\to\infty} \frac{|x_n|}{\lambda_n} = \Lambda < +\infty$ when $\alpha \in (-1, 0)$. Now we give a sketch of proof. Suppose that $\lim_{n\to\infty} \frac{|x_n|}{\lambda_n} \to +\infty$ when $\alpha \in (-1, 0)$. Set $\tau_n = \frac{\mathrm{e}^{-u_n(x_n)}}{|x_n|^\alpha} = \lambda_n (\frac{\lambda_n}{|x_n|})^\alpha \to 0$, $n \to \infty$. Letting $\xi_n(x) = u_n(x_n + \tau_n x) - u_n(x_n)$, we see that

$$\begin{cases} -\Delta \xi_n = 0, & \text{in } D_k = \left\{ |x| \le \dfrac{1}{2\tau_n} \right\} \cap \mathbb{R}^2_{-\frac{t_n}{\tau_n}}, \\ \dfrac{\partial \xi_n}{\partial n} = \left| \dfrac{x_n}{|x_n|} + \dfrac{\tau_n}{|x_n|} x \right|^\alpha V_n(x_n + \tau_n x)\mathrm{e}^{\xi_n} - \tau_n W_n(x_n + \tau_n x), & \text{on } \partial\mathbb{R}^2_{-\frac{t_n}{\tau_n}}, \\ \xi_n(0) = \max_{\bar{D}_k} \xi_n = 0. \end{cases}$$

Now we distinguish two cases.

Case 1 $\frac{t_n}{\tau_n} \to +\infty$. Then after passing to a subsequence, ξ_n converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to a function ξ satisfying

$$\begin{cases} -\Delta \xi = 0 & \text{in } \mathbb{R}^2, \\ \xi(x) \le \xi(0) = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

Hence $\xi \equiv 0$. Now we define a function $w_n(x) = \int_{-s_0}^{s_0} \mathrm{e}^{u_n(x+(s,0))} ds, x \in \bar{B}^+_{2s_0}, 0 < s_0 \le \frac{1}{4}$. Then w_n is a subharmonic function and $w_n(x) \le \frac{1}{a} \int_{-s_0}^{s_0} V_n \mathrm{e}^{u_n} ds \le \frac{C}{a}$. On the other hand, we have for $k > \frac{C}{a}$ and n sufficiently large

$$\begin{aligned} w_n(x_n) &= \int_{-s_0}^{s_0} \mathrm{e}^{u_n(x_n+(s,0))} ds \\ &\ge \int_{-k\tau_n}^{k\tau_n} \mathrm{e}^{u_n(x_n+(s,0))} ds \\ &= \int_{-k}^{k} \mathrm{e}^{\xi_n(s,0)} \to \int_{-k}^{k} \mathrm{e}^{\xi(s,0)} > \frac{2C}{a}, \end{aligned}$$

which is a contradiction.

Case 2 $\frac{t_n}{\tau_n} \to t_0 < +\infty$. Then after passing to a subsequence, ξ_n converges in $C^2_{\rm loc}(\mathbb{R}^2_{-t_0}) \cap C^1_{\rm loc}(\overline{\mathbb{R}}^2_{-t_0}\backslash\{0\})$ to a function ξ satisfying

$$\begin{cases} -\Delta\xi = 0 & \text{in } \mathbb{R}^2_{-t_0}, \\ \dfrac{\partial\xi}{\partial n} = V(0)\mathrm{e}^{\xi} & \text{on } \partial\mathbb{R}^2_{-t_0}, \\ \xi(x) \le \xi(0) = 0 & \text{in } \bar{\mathbb{R}}^2_{-t_0}, \end{cases}$$

with the condition

$$\int_{\partial\mathbb{R}^2_{-t_0}} V(0)\mathrm{e}^{\xi} ds \le C, \quad \sup_{\overline{\mathbb{R}}^2_{-t_0}} \xi(x) \le C.$$

By the classification result in [11], we have

$$\int_{\partial\mathbb{R}^2_{-t_0}} V(0)\mathrm{e}^{\xi} ds = 2\pi.$$

However, $\forall\delta > 0$ small,

$$\begin{aligned} \lim_{n\to+\infty} \int_{B_\delta(0)\cap\partial\mathbb{R}^2_+} |x|^\alpha V_n \mathrm{e}^{u_n} dx &\ge \lim_{n\to+\infty} \int_{\{|x-x_n|\le\frac{\delta}{2}\}\cap\partial\mathbb{R}^2_+} |x|^\alpha V_n \mathrm{e}^{u_n} dx \\ &= \lim_{n\to+\infty} \int_{|x|\le\frac{\delta}{2\tau_n}\cap\partial\mathbb{R}^2_{-\frac{t_n}{\tau_n}}} \left|\frac{x_n}{|x_n|} + \frac{\tau_n}{|x_n|}x\right|^\alpha V_n(x_n+\tau_n x)\mathrm{e}^{\xi_n} \\ &\ge \int_{\partial\mathbb{R}^2_{-t_0}} V(0)\mathrm{e}^{\xi} \\ &= 2\pi, \end{aligned}$$

which also is a contradiction.

But for $\alpha \ge 0$, this is an open problem, we will make a further research about this problem.

The proof of Theorem 1.4 follows closely the idea in [1] where they gave the profile of blow-up solutions to mean field equations with singular data. The approach in [1] was designed for $\alpha \ge 0$ and for interior problem. In case of our exponential Neumann boundary and of more general $\alpha \in (-1, +\infty)$, we need some refined calculation on $B_{\frac{1}{\lambda_n}}\backslash B_{(\log\frac{1}{\lambda_n})^k} \cap \overline{\mathbb{R}^2_+}$ for chosen $k > \frac{1}{1+\alpha}$, and need to construct the new barrier functions for the corresponding Neumann boundary problem in $B_{\frac{1}{\lambda_n}}\backslash B_{(\log\frac{1}{\lambda_n})^k} \cap \overline{\mathbb{R}^2_+}$. See the details in Section 3.

2 Brezis–Merle Type Concentration Compactness Theorem

In this section, we would like to prove Theorem 1.1. We begin with a lemma in [8].

Lemma 2.1 ([8]) *Let l be an embedded C^1 curve in $\mathbb{R}^2$. $f \in L^1(l)$. Set $\|f\|_1 = \int_l |f(x)|dx$, and $\rho = \mathrm{diam}\ l$. If we define*

$$\omega(x) = \frac{1}{\pi}\int_l \log\frac{\rho}{|x-y|} f(y)dy,$$

then for every $\delta \in (0,\pi)$ we have

$$\int_l \exp[(\pi-\delta)|\omega(x)|/\|f\|_1]dx \le \frac{C}{\delta}. \tag{2.1}$$

By using Lemma 2.1, we can get the following lemma.

Lemma 2.2 *Set* $f(x) \in L_1(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$*. And we define*

$$\omega(x) = \frac{1}{\pi} \int_{\partial B_r^+(0) \cap \partial \mathbb{R}_+^2} \log \frac{2r}{|x-y|} f(y) dy,$$

then for every $k > 0$ *we have* $\mathrm{e}^{k|\omega|} \in L_1(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$*.*

Proof Let $0 < \epsilon < \frac{1}{k}$. Since $f(x) \in L_1(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$, we can split $f(x)$ as $f(x) = f_1(x) + f_2(x)$ with $\|f_1\|_1 < \epsilon$ and $f_2 \in L^\infty(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$. Write $\omega(x) = \omega_1(x) + \omega_2(x)$ where

$$\omega_i(x) = \frac{1}{\pi} \int_{\partial B_r^+(0) \cap \partial \mathbb{R}_+^2} \log \frac{2r}{|x-y|} f_i(y) dy.$$

Choosing $\delta = \pi - 1$ in Lemma 2.1 we find $\int_{\partial B_r^+(0) \cap \partial \mathbb{R}_+^2} \exp[|\omega_1(x)|/\|f_1\|_1] dx \le C$. This implies that $\mathrm{e}^{k|\omega_1|} \in L_1(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$ for every $k > 0$. Thus the conclusion follows the fact $|\omega| \le |\omega_1| + |\omega_2|$ and $\omega_2 \in L^\infty(\partial B_r^+(0) \cap \partial \mathbb{R}_+^2)$. □

Now we show the small energy regularities by applying Lemma 2.1 to the solutions of (1.3)–(1.4).

Proposition 2.3 *Assume* u_n *is a sequence of solutions of* (1.3)–(1.4)*. Let* $\overline{u}_n = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} u_n$*. If one of the following alternatives is satisfied,*

(i) $x \in \Omega$,

(ii) $x \in \partial \Omega$, $x \ne 0$ *and there exist a neighborhood* B_R *of* x *and a constant* $\beta_1 < \pi$ *such that*

$$\int_{\partial \Omega \cap B_R(x)} V_n(x)|x|^\alpha \mathrm{e}^{u_n} dx \le \beta_1 < \pi, \quad \forall n,$$

(iii) $x = 0 \in \partial \Omega$, $-1 < \alpha < 0$, *and there exist a neighborhood* B_R *of* 0 *and a constant* $\beta_1 < \pi(1+\alpha)$ *such that*

$$\int_{\partial \Omega \cap B_R(0)} V_n(x)|x|^\alpha \mathrm{e}^{u_n} dx \le \beta_2 < \pi(1+\alpha), \quad \forall n,$$

(iv) $x = 0 \in \partial \Omega$, $\alpha \ge 0$, *and there exist a neighborhood* B_R *of* 0 *and a constant* $\beta_3 < \pi$ *such that*

$$\int_{\partial \Omega \cap B_R(0)} V_n(x)|x|^\alpha \mathrm{e}^{u_n} dx \le \beta_3 < \pi, \quad \forall n,$$

then $u_n - \overline{u}_n$ *is bounded near* x*.*

Proof When $\alpha = 0$, it is the smooth case. The results has been shown in [8]. So to prove the proposition, it is sufficient to show (iii) and (iv) for $\alpha \ne 0$.

By using the Green representation, we have

$$u_n(x) = \overline{u}_n + \int_{\partial \Omega} \left(\frac{1}{\pi} \log \frac{\rho}{|x-y|} + R(x,y) \right) (V_n(y)|y|^\alpha \mathrm{e}^{u_n} - W_n(y)) dy. \tag{2.2}$$

Here $R(x,y)$ is the regular part of Green function and $\rho = \operatorname{diam} \Omega$. Denote

$$w_n(x) = \frac{1}{\pi} \int_{\partial \Omega \cap B_R(0)} \log \frac{\rho}{|x-y|} V_n(y)|y|^\alpha \mathrm{e}^{u_n} dy, \tag{2.3}$$

In the case of (iii), by Lemma 2.1 we have for any $\delta \in (0, \pi)$,

$$\int_{\partial \Omega \cap B_R(0)} \exp\left[\frac{\pi - \delta}{\|V_n |x|^\alpha \mathrm{e}^{u_n}\|_1} w_n(y) \right] dy \le \frac{C}{\delta},$$

where $\|V_n|x|^\alpha e^{u_n}\|_1 = \int_{\partial\Omega\cap B_R(0)} V_n(x)|x|^\alpha e^{u_n}dx$. In particular, we can choose δ sufficiently small such that $\frac{\pi-\delta}{\beta_2} = p > \frac{1}{1+\alpha} > 1$. Then we have

$$\int_{\partial\Omega\cap B_R(0)} e^{pw_n} \le C. \tag{2.4}$$

Let $v_n = u_n - w_n$. Then for any $x \in B_{R/2}(0)\cap\bar{\Omega}$ we have

$$|v_n(x) - \overline{u}_n| \le \frac{1}{\pi}\int_{\partial\Omega\setminus B_R(0)} \log\frac{\rho}{R/2}V_n(y)|y|^\alpha e^{u_n}dy + \int_{\partial\Omega} V_n(y)|y|^\alpha e^{u_n}dy + C \le C. \tag{2.5}$$

From (2.4)–(2.5), we obtain

$$\int_{\partial\Omega\cap B_{R/2}(0)} e^{pu_n} \le \int_{\partial\Omega\cap B_{R/2}(0)} e^{p(w_n+\overline{u}_n+C)} \le Ce^{p\overline{u}_n}.$$

Note that when $-1<\alpha<0$, the energy condition (1.4) implies

$$\int_{\partial\Omega} e^{u_n} \le C. \tag{2.6}$$

Hence from Jensen's inequality it holds

$$e^{\overline{u}_n} = e^{\frac{1}{|\partial\Omega|}\int_{\partial\Omega} u_n} \le \frac{1}{|\partial\Omega|}\int_{\partial\Omega} e^{u_n} \le C.$$

Consequently we get

$$\int_{\partial\Omega\cap B_{R/2}(0)} e^{pu_n} \le C.$$

Since $\int_{\partial\Omega\cap B_R(0)} |x|^{\alpha q} \le C$ if $q = \frac{p}{p-1}$, we have for $x \in B_{R/4}(0)\cap\bar{\Omega}$,

$$\begin{aligned}
|u_n(x) - \bar{u}_n(x)| &\le \frac{1}{\pi}\int_{\partial\Omega\cap B_{R/2}(0)} \log\frac{\rho}{|x-y|}V_n(y)|y|^\alpha e^{u_n}dy \\
&\quad + \frac{1}{\pi}\int_{\partial\Omega\setminus B_{R/2}(0)} \log\frac{\rho}{|x-y|}V_n(y)|y|^\alpha e^{u_n}dy + C \\
&\le C\|e^{u_n}\|_{L^p(\partial\Omega\cap B_{R/2}(0))}\Big\||y|^\alpha \log\frac{\rho}{|x-y|}\Big\|_{L^q(\partial\Omega\cap B_{R/2}(0))} \\
&\quad + C\log\frac{\rho}{R/4}\int_{\partial\Omega} V_n(y)|y|^\alpha e^{u_n}dy + C \\
&\le C.
\end{aligned}$$

In the case of (iv), noting that the definition of w_n in (2.3) and using Lemma 2.1 again, we can choose small positive number δ such that $\frac{\pi-\delta}{\beta_3} = p > 1$ and

$$\int_{\partial\Omega\cap B_R(0)} e^{pw_n} \le C. \tag{2.7}$$

Let $v_n = u_n - w_n$ and for $x \in B_{R/2}(0)\cap\bar{\Omega}$ we can also obtain

$$|v_n(x) - \overline{u}_n| \le \frac{1}{\pi}\int_{\partial\Omega\setminus B_R(0)} \log\frac{\rho}{R/2}V_n(y)|y|^\alpha e^{u_n}dy + \int_{\partial\Omega} V_n(y)|y|^\alpha e^{u_n} + C \le C. \tag{2.8}$$

From (2.7)–(2.8), we have

$$\int_{\partial\Omega\cap B_{R/2}(0)} e^{pu_n} \le \int_{\partial\Omega\cap B_{R/2}(0)} e^{p(w_n+\overline{u}_n+C)} \le Ce^{p\overline{u}_n}.$$

Now take t such that $\int_{\partial\Omega}\frac{1}{|x|^{\alpha t}}dx\le C$ and set $s=\frac{t}{t+1}<1$. It follows that

$$\int_{\partial\Omega}\mathrm{e}^{su_n}dx=\int_{\partial\Omega}\mathrm{e}^{su_n}|x|^{\alpha s}|x|^{-\alpha s}dx\le\Big(\int_{\partial\Omega}\mathrm{e}^{u_n}|x|^{\alpha}dx\Big)^{s}\Big(\int_{\partial\Omega}\frac{1}{|x|^{\alpha t}}dx\Big)^{1-s}\le C.$$

From Jensen's inequality

$$\mathrm{e}^{s\overline{u}_n}=\mathrm{e}^{\frac{1}{|\partial\Omega|}\int_{\partial\Omega}su_n}\le\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\mathrm{e}^{su_n}\le C,$$

hence

$$\int_{\partial\Omega\cap B_{R/2}(0)}\mathrm{e}^{pu_n}\le C.$$

Now return to (2.2). For $x\in B_{R/4}(0)\cap\bar{\Omega}$, we have

$$\begin{aligned}|u_n(x)-\bar{u}_n(x)|&\le\frac{1}{\pi}\int_{\partial\Omega\cap B_{R/2}(0)}\log\frac{\rho}{|x-y|}V_n(y)|y|^{\alpha}\mathrm{e}^{u_n}dy\\&\quad+\frac{1}{\pi}\int_{\partial\Omega\backslash B_{R/2}(0)}\log\frac{\rho}{|x-y|}V_n(y)|y|^{\alpha}\mathrm{e}^{u_n}dy+C\\&\le C\|\mathrm{e}^{u_n}\|_{L^p(\partial\Omega\cap B_{\frac{R}{2}}(0))}+C\log\frac{4\rho}{R}\int_{\partial\Omega}V_n(y)|y|^{\alpha}\mathrm{e}^{u_n}dy+C\\&\le C.\end{aligned}$$

Thus we complete the proof of this proposition. □

Remark 2.4 From the proof of Proposition 2.3 we know $\overline{u}_n=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}u_n$ is bounded from above. Define the singular set Σ by

$$\begin{aligned}\Sigma=&\Big\{0\in\partial\Omega\Big|\lim_{R\to0}\lim_{n\to\infty}\int_{\partial\Omega\cap B_R(0)}V_n(x)|x|^{\alpha}\mathrm{e}^{u_n}\ge\pi(1+\alpha)\Big\}\\&\cup\Big\{0\ne x\in\partial\Omega\Big|\lim_{R\to0}\lim_{n\to\infty}\int_{\partial\Omega\cap B_R(x)}V_n(x)|x|^{\alpha}\mathrm{e}^{u_n}\ge\pi\Big\},\end{aligned}$$

when $-1<\alpha<0$, or

$$\Sigma=\Big\{x\in\partial\Omega\Big|\lim_{R\to0}\lim_{n\to\infty}\int_{\partial\Omega\cap B_R(x)}V_n(x)|x|^{\alpha}\mathrm{e}^{u_n}\ge\pi\Big\},$$

when $\alpha\ge0$. Noting that the definition of the blow-up set S in Theorem 1.1 we know that $\Sigma=S$ from Proposition 2.3.

Proof of Theorem 1.1 By Proposition 2.3, $u_n-\overline{u}_n$ is bounded in $L^{\infty}_{\rm loc}(\overline{\Omega}\backslash S)$. If $S=\emptyset$, then (i), (ii) holds depending on whether $\overline{u}_n$ is bounded or $\overline{u}_n\to-\infty$.

If $S\ne\emptyset$, we will show (iii). Suppose $x_0\in S$ and $x_0\ne0$, it follows from the arguments in [8] we know that (iii) holds. Next we suppose $x_0=0\in S$.

The first case is $-1<\alpha<0$. To get the conclusion, it is sufficient to show that $\overline{u}_n\to-\infty$. To this end, let us consider w_n which is defined as in (2.3). Since for each $0<\epsilon<R$ and $x\in\partial\Omega\cap B_R(0)$, we have

$$\begin{aligned}w_n(x)&\ge\frac{1}{\pi}\int_{\partial\Omega\cap B_{\epsilon}(0)}\log\frac{\rho}{|x-y|}V_n(y)|y|^{\alpha}\mathrm{e}^{u_n}dy\\&\ge\frac{1}{\pi}\int_{\partial\Omega\cap B_{\epsilon}(0)}\log\frac{\rho}{|x|+\epsilon}V_n(y)|y|^{\alpha}\mathrm{e}^{u_n}dy.\end{aligned}$$

We obtain from the definition of the singular set that

$$\lim_{n\to\infty} w_n(x) \geq (1+\alpha)\log\frac{1}{|x|} - C, \quad x \in \partial\Omega \cap B_R(0).$$

If $\overline{u}_n$ is bounded, then, by (2.5), $v_n = u_n - w_n$ is bounded in $\overline{\Omega} \cap B_{R/2}(0)$. Hence we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\partial\Omega\cap B_{R/2}(0)} |x|^\alpha \mathrm{e}^{u_n} d\sigma &\geq C \lim_{n\to\infty}\int_{\partial\Omega\cap B_{R/2}(0)} |x|^\alpha \mathrm{e}^{w_n} d\sigma \\ &\geq C \lim_{n\to\infty}\int_{\partial\Omega\cap B_{R/2}(0)} \frac{1}{|x|} d\sigma \\ &= +\infty,\end{aligned}$$

which is a contradiction with the energy condition.

The second case is $\alpha \geq 0$. In this case, it is a little bit subtle. We will use a trick in [3]. We assume 0 is the only blow-up point in $\overline{B_R(0)} \cap \overline{\Omega}$. For simplicity, we assume $\Omega \cap B_R(0) = B_R^+(0)$. And we only need to show that $u_n \to -\infty$ uniformly in compact subsets of $\overline{B_R^+(0)}\backslash\{0\}$. We assume by contradiction that u_n is uniformly bounded in $L^\infty_{\rm loc}(\overline{B_R^+(0)}\backslash\{0\})$. By elliptic estimates and by extracting a subsequence, we may assume that

$$\begin{aligned}&u_n \to \xi \text{ pointwise a.e. and in } C^{1,\delta}_{\rm loc}(\overline{B_R^+(0)}\backslash\{0\}), \quad \text{for some } \delta \in (0,1), \\ &V_n(x)|x|^\alpha \mathrm{e}^{u_n} \to V(x)|x|^\alpha \mathrm{e}^{\xi}, \quad \text{in } C^0_{\rm loc}(\overline{B_R^+(0)}\backslash\{0\}).\end{aligned}$$

Note that by Fatou's Lemma, $V(x)|x|^\alpha \mathrm{e}^\xi \in L^1(\partial B_R^+(0) \cap \partial\mathbb{R}^2_+)$. Then we derive

$$V_n(x)|x|^\alpha \mathrm{e}^{u_n} \rightharpoonup V(x)|x|^\alpha \mathrm{e}^\xi + \beta\delta_0, \tag{2.9}$$

weakly in the sense of measures on $\partial B_R^+(0) \cap \partial\mathbb{R}^2_+$, where

$$\beta = m(0) = \lim_{R\to 0}\lim_{n\to\infty}\int_{\partial B_R^+(0)\cap\partial\mathbb{R}^2_+} V_n(x)|x|^\alpha \mathrm{e}^{u_n} d\sigma.$$

Next we choose r_0 is small enough. Fix $0 < r_0 < R$, and in $\overline{B_{r_0}^+(0)}$ we define

$$\varphi_n(x) = V_n(x)|x|^\alpha \mathrm{e}^{u_n}, \quad \varphi(x) = V(x)|x|^\alpha \mathrm{e}^\xi.$$

By Green's representation formula for u_n in $\overline{B_{r_0}^+(0)}$ and (2.9) to derive that

$$\xi(x) = \frac{\beta}{\pi}\log\frac{1}{|x|} + \phi(x) + \gamma(x), \tag{2.10}$$

with

$$\phi(x) = \frac{1}{\pi}\int_{\partial B_{r_0}^+(0)\cap\partial\mathbb{R}^2_+} \log\frac{1}{|x-y|} V(y)|y|^\alpha \mathrm{e}^\xi dy,$$

and

$$\gamma(x) = \frac{1}{\pi}\int_{\partial B_{r_0}^+(0)\cap\partial\mathbb{R}^2_+} \log\frac{1}{|x-y|}\frac{\partial\xi}{\partial\nu} dy + \frac{1}{\pi}\int_{\partial B_{r_0}^+(0)\cap\partial\mathbb{R}^2_+} \frac{(x-y)\cdot\nu}{|x-y|^2}\xi(y) dy.$$

Clearly,

$$\gamma(x) \in C^1(B_r^+(0)), \quad \text{for every } r \in (0, r_0).$$

For $\phi(x)$, we observe first that $\phi(x)$ is clearly bounded from below on $\overline{B_{r_0}^+(0)}$, i.e.,

$$\phi(x) \geq \frac{1}{\pi}\log\frac{1}{2r_0}\|\varphi\|_{L^1(\partial B_{r_0}^+(0)\cap\partial\mathbb{R}^2_+)}, \quad \forall x \in \overline{B_{r_0}^+(0)}.$$

For r_0 is small enough, by (2.10) we find

$$\varphi(x) = V(x)|x|^\alpha \mathrm{e}^\xi = V(x)\frac{|x|^\alpha}{|x|^{\frac{\beta}{\pi}}}\mathrm{e}^{\phi(x)+\gamma(x)+\beta R(x,0)+\frac{\beta}{\pi}\ln(2r_0)} \geq \frac{C}{|x|^{\frac{\beta}{\pi}-\alpha}}.$$

Thus by the integrability of φ, we see that necessarily

$$\beta < \pi(1+\alpha). \tag{2.11}$$

On the other hand, let us set $s = \frac{\beta}{\pi} - \alpha$. In view of (2.11), we have $s < 1$. Furthermore, we have

$$\varphi(x) = V(x)|x|^\alpha \mathrm{e}^\xi \leq \frac{C}{|x|^s}\mathrm{e}^{\phi(x)}, \quad \text{in } \overline{B^+_{r_0}(0)}.$$

By Lemma 2.2, for every $k > 0$ we have $\mathrm{e}^{k|\phi|} \in L_1(\partial B^+_{r_0}(0) \cap \partial\mathbb{R}^2_+)$, we have by Hölder's inequality to get $\varphi \in L^t(\partial B^+_{r_0}(0) \cap \partial\mathbb{R}^2_+)$ for any $t \in (1, \frac{1}{s})$ if $s > 0$, and $\varphi \in L^t(\partial B^+_{r_0}(0) \cap \partial\mathbb{R}^2_+)$ for any $t > 1$ if $s \leq 0$.

Now we estimate $\nabla\phi(x)$ for $x \in B^+_{r_0}(0)$. First we have

$$\begin{aligned}
|\nabla\phi(x)| &\leq \frac{1}{\pi}\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|}\varphi(y)dy \\
&\quad + \frac{1}{\pi}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|}\varphi(y)dy \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, we fix $t \in (1, \frac{1}{s})$ and choose $\tau > 0$ such that $\frac{\tau t}{t-1} < 1$, and hence we have $0 < \tau < 1 - s$. By Hölder's inequality we obtain

$$\begin{aligned}
I_1 &\leq \frac{1}{\pi}\bigg(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|^{\frac{\tau t}{t-1}}}dy\bigg)^{\frac{t-1}{t}} \\
&\quad \cdot \bigg(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|^{t(1-\tau)}}|\varphi(y)|^t dy\bigg)^{\frac{1}{t}} \\
&\leq \frac{C}{|x|^{1-\tau}}.
\end{aligned}$$

For I_2, since $|x-y| \leq \frac{|x|}{2}$ implies that $|y| \geq \frac{|x|}{2}$, we have

$$\begin{aligned}
I_2 &\leq C\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|}\frac{1}{|y|^s}dy \\
&\leq \frac{C}{|x|^s}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap\partial B^+_{r_0}(0)\cap\partial\mathbb{R}^2_+}\frac{1}{|x-y|}dy \\
&\leq \frac{C}{|x|^{s'}},
\end{aligned}$$

for some s' with $0 < s' < 1$. In conclusion, for all $x \in B^+_{r_0}(0)$ we have

$$|\nabla\phi(x)| \leq \frac{C}{|x|^{1-\tau}} + \frac{C}{|x|^{s'}}, \tag{2.12}$$

for suitable $0 < \tau < 1-s$ and $0 < s' < 1$. At this point we are ready to derive our contradiction by means of a Pohozaev type identity. We multiply all terms in (1.3) by $x \cdot \nabla u_n$ and integrate over $B_r^+(0) \cap \partial\mathbb{R}^2_+$ for any $r \in (0, r_0)$ to get

$$\begin{aligned} & r\int_{\partial B_r^+(0)\cap\mathbb{R}^2_+} \left(\frac{1}{2}|\nabla u_n|^2 - \left|\frac{\partial u_n}{\partial n}\right|^2\right) d\sigma \\ & = -(1+\alpha)\int_{\partial B_r^+(0)\cap\partial\mathbb{R}^2_+} V_n(s,0)|s|^\alpha \mathrm{e}^{u_n} ds + V_n(s,0)|s|^\alpha s\mathrm{e}^{u_n(s,0)} \big|_{s=-r}^{s=r} \\ & \quad - \int_{\partial B_r^+(0)\cap\partial\mathbb{R}^2_+} x\cdot\nabla V_n(x)|s|^\alpha \mathrm{e}^{u_n} ds - \int_{\partial B_r^+(0)\cap\partial\mathbb{R}^2_+} x\cdot\nabla u_n W_n(s,0) ds. \end{aligned} \tag{2.13}$$

Passing to the limit we have

$$\begin{aligned} & r\int_{\partial B_r^+(0)\cap\mathbb{R}^2_+} \left(\frac{1}{2}|\nabla \xi|^2 - \left|\frac{\partial \xi}{\partial n}\right|^2\right) d\sigma \\ & = -(1+\alpha)\int_{\partial B_r^+(0)\cap\partial\mathbb{R}^2_+} V(x)|x|^\alpha \mathrm{e}^{\xi} ds + V(x)|s|^\alpha s\mathrm{e}^{\xi(s,0)} \big|_{s=-r}^{s=r} \\ & \quad - \int_{\partial B_r^+(0)\cap\partial\mathbb{R}^2_+} x\cdot\nabla V(x)|s|^\alpha \mathrm{e}^{\xi} ds - \beta(1+\alpha) + 0_r(1). \end{aligned} \tag{2.14}$$

Set $\eta = \phi + \gamma$. Since $\nabla\xi(x) = -\frac{\beta}{\pi}\frac{x}{|x|^2} + \nabla\eta(x)$ and by (2.12)

$$|\nabla\eta(x)| \le \frac{C}{r^{1-\tau}} + \frac{C}{r^{s'}} + C,$$

with $0 < \tau < 1 - s$ and $0 < s' < 1$, we have

$$\begin{aligned} \Phi_r & := r\int_{\partial B_r^+(0)\cap\mathbb{R}^2_+} \left(\frac{1}{2}|\nabla \xi|^2 - \left|\frac{\partial \xi}{\partial n}\right|^2\right) ds \\ & = r\int_{\partial B_r^+(0)\cap\mathbb{R}^2_+} \frac{1}{2}\left[\frac{\beta^2}{\pi^2|x|^2} - \frac{2\beta x\cdot\nabla\eta}{\pi|x|^2} + |\nabla\eta|^2\right] + \left(\frac{\beta}{\pi|x|} - \frac{x\cdot\nabla\eta}{|x|}\right)^2 ds \\ & = -\frac{1}{2}\left(\frac{\beta}{\pi}\right)^2\pi + r\int_{\partial B_r^+(0)\cap\mathbb{R}^2_+} \frac{\beta x\cdot\nabla\eta}{\pi|x|^2} + \frac{1}{2}|\nabla\eta|^2 - \left(\frac{x\cdot\nabla\eta}{|x|}\right)^2 ds \\ & = -\frac{\beta^2}{2\pi} + o(1), \quad \text{as } r\to 0. \end{aligned} \tag{2.15}$$

Similarly, letting $r \to 0$ on the right side of (2.14) we also can obtain that

$$\Phi_r = -\beta(1+\alpha) + o(1), \quad r \to 0. \tag{2.16}$$

Comparing (2.15) and (2.16), we see that necessarily $\beta = 2\pi(1+\alpha)$, in contradiction with (2.11).

Proof of Proposition 1.2 For any $p > 2$, let $2 > q = \frac{p}{p-1} > 1$. Then we have

$$\|\nabla u_n\|_{L^q(\Omega)} = \sup\left\{\int_\Omega \nabla u_n\nabla\varphi dv \,\middle|\, \forall\varphi\in W^{1,p}(\Omega), \int_\Omega\varphi dv = 0, \|\varphi\|_{W^{1,p}(\Omega)} = 1\right\}.$$

By the Sobolev embedding theorem, we can get $\|\varphi\|_{L^\infty(\Omega)} \le C$. It follows that

$$\left|\int_\Omega \nabla u_n\nabla\varphi dv\right| \le \int_{\partial\Omega}(V_n(x)|x|^\alpha\mathrm{e}^{u_n} + |W_n(x)|)|\varphi| d\sigma \le C.$$

Hence $u_n - \frac{1}{|\Omega|}\int_\Omega u_n$ is uniformly bounded in $W^{1,q}(\Omega)$. Next, we define the Green function G

by

$$\begin{cases} -\Delta G = 0 & \text{in } \Omega, \\ \dfrac{\partial G}{\partial n} = \displaystyle\sum_{p\in\partial\Omega\cap S} m(p)\delta_p - W(x) & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega G = 0. \end{cases}$$

We have for any $\varphi \in C^\infty(\overline{\Omega})$,

$$\int_\Omega \nabla(u_n - G)\nabla\varphi dv = \int_{\partial\Omega} (V_n(x)|x|^\alpha \mathrm{e}^{u_n} - W_n(x) - \Sigma m(p)\delta_p + W(x))\varphi \to 0.$$

Combining the fact that $u_n - \frac{1}{|\Omega|}\int_\Omega u_n$ is uniformly in $W^{1,q}(\Omega)$, we get the conclusion of Proposition 1.2. □

Now we can compute the blow-up value by using the Pohozaev identity and Proposition 1.2.

Proof of Theorem 1.3 First we assume the blow-up point $p = 0$. For sufficiently small $r > 0$, then 0 is the only blow-up point in $B_r(0) \cap \overline{\Omega}$. In view of Pohozaev identity for solutions u_n of (2.13) and Proposition 1.2, we have

$$\begin{aligned} &\lim_{r\to 0}\lim_{n\to\infty} r\int_{\partial B_r^+(0)\cap\mathbb{R}_+^2} \left|\frac{\partial u_n}{\partial n}\right|^2 - \frac{1}{2}|\nabla u_n|^2 \\ &= \lim_{r\to 0}\lim_{n\to\infty} r\int_{\partial B_r^+(0)\cap\mathbb{R}_+^2} \left|\frac{\partial G}{\partial n}\right|^2 - \frac{1}{2}|\nabla G|^2 \\ &= \frac{1}{2\pi}m^2(0). \end{aligned} \tag{2.17}$$

Since $u_n \to -\infty$ uniformly on $\partial B_r^+(0) \cap \mathbb{R}_+^2$, we have

$$\lim_{r\to 0}\lim_{n\to\infty} V_n(s,0)|s|^\alpha s\mathrm{e}^{u_n} |_{-r}^{r} = 0,$$

$$\lim_{r\to 0}\lim_{n\to\infty} \int_{\partial B_r^+(0)\cap\partial\mathbb{R}_+^2} x\cdot\nabla V_n(x)|s|^\alpha \mathrm{e}^{u_n} ds = 0,$$

and

$$\lim_{r\to 0}\lim_{n\to\infty} \int_{\partial B_r^+(0)\cap\partial\mathbb{R}_+^2} x\cdot\nabla u_n W_n(s,0) ds = 0.$$

Letting $r \to 0$ and $n \to \infty$ in (2.13), we get that

$$\frac{1}{2\pi}m^2(0) = (1+\alpha)m(0).$$

It follows that $m(0) = 2\pi(1+\alpha)$. When the blow-up point $p \neq 0$, we can obtain $m(p) = 2\pi$ in a similar way. □

3 Proof of Theorem 1.4

Proof of Theorem 1.4 The proof consists of four steps.

Step 1 From the rescaling functions $\widetilde{u}_n(x) = u_n(\lambda_n x) + (1+\alpha)\ln\lambda_n$, (1.7) is valid in $B_{\lambda_n R} \cap \overline{\mathbb{R}_+^2}$ for any fixed $R > 0$ and for some constants $C > 0$ independent of n. Hence we are left to prove (1.7) are valid on $(B_{r_0}\backslash B_{\lambda_n R}) \cap \overline{\mathbb{R}_+^2}$ for some $r_0 > 0$.

Step 2 It follows from the boundary condition in (1.6) that

$$0 \le u_n(x) - \min_{y\in\partial B_1^+\cap\mathbb{R}_+^2} u_n(y) \le C, \quad \text{on } \partial B_1^+ \cap \mathbb{R}_+^2.$$

Let $w_n(x)$ be the solution of the following problem

$$\begin{cases} -\Delta w_n = 0 & \text{in } B_1^+, \\ w_n = u_n(x) - \min\limits_{y\in\partial B_1^+\cap\mathbb{R}_+^2} u_n(y), & \text{on } \partial B_1^+ \cap \mathbb{R}_+^2, \\ \dfrac{\partial w_n}{\partial n} = 0 & \text{on } \partial B_1^+ \cap \partial\mathbb{R}_+^2. \end{cases}$$

We can apply the maximal principle to obtain that w_n is uniformly bounded in $\overline{B}_1^+$. And the function $v_n = u_n - \min_{y\in\partial B_1^+\cap\mathbb{R}_+^2} u_n(y) - w_n$ satisfies

$$\begin{cases} -\Delta v_n = 0 & \text{in } B_1^+(0), \\ v_n = 0 & \text{on } \partial B_1^+ \cap \mathbb{R}_+^2, \\ \dfrac{\partial v_n}{\partial n} = V_n(x)|x|^\alpha \mathrm{e}^{u_n} - W_n(x) & \text{on } \partial B_1^+ \cap \partial\mathbb{R}_+^2. \end{cases}$$

Then we use the Green representation formula to obtain

$$v_n(x) = \frac{1}{\pi}\int_{\partial B_1^+\cap\partial\mathbb{R}_+^2} \Big(\log\frac{1}{|x-y|}\Big) V_n(y)|y|^\alpha \mathrm{e}^{u_n}dy + R_n(x),$$

where $R_n(x)$ is uniformly bounded function in $\overline{B}_1^+$. Hence we obtain

$$u_n(x) - \min_{y\in\partial B_1^+\cap\mathbb{R}_+^2} u_n(y) = \frac{1}{\pi}\int_{\partial B_1^+\cap\partial\mathbb{R}_+^2} \Big(\log\frac{1}{|x-y|}\Big) V_n(y)|y|^\alpha \mathrm{e}^{u_n}dy + O(1).$$

Here and in the sequel $O(1)$ denotes the uniformly bounded term. We set

$$M_n = \int_{\partial B_1^+\cap\partial\mathbb{R}_+^2} V_n(y)|y|^\alpha \mathrm{e}^{u_n}dy.$$

Recalling the definition of $\widetilde{u}_n(x)$, we get

$$\begin{aligned}\widetilde{u}_n(x) = {} & \frac{1}{\pi}\int_{\{|y|<\frac{1}{\lambda_n}\}\cap\partial\mathbb{R}_+^2} \Big(\log\frac{1}{|x-y|}\Big) V_n(\lambda_n y)|y|^\alpha \mathrm{e}^{\widetilde{u}_n}dy \\ & - \Big[\frac{M_n}{\pi} - (1+\alpha)\Big]\log\lambda_n + \min_{y\in\partial B_1^+\cap\mathbb{R}_+^2} u_n(y) + O(1).\end{aligned}$$

Further from $\widetilde{u}_n(0) = O(1)$ we obtain

$$\begin{aligned}& \frac{1}{\pi}\int_{\{|y|<\frac{1}{\lambda_n}\}\cap\partial\mathbb{R}_+^2} \Big(\log\frac{1}{|y|}\Big) V_n(\lambda_n y)|y|^\alpha \mathrm{e}^{\widetilde{u}_n}dy \\ & = \Big[\frac{M_n}{\pi} - (1+\alpha)\Big]\log\lambda_n - \min_{\partial B_1^+\cap\mathbb{R}_+^2} u_n(y) + O(1).\end{aligned}$$

Putting the above equations together, we get

$$\widetilde{u}_n(x) = \frac{1}{\pi}\int_{\{|y|<\frac{1}{\lambda_n}\}\cap\partial\mathbb{R}_+^2} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha \mathrm{e}^{\widetilde{u}_n}dy + O(1).$$

Claim For small $\delta > 0$ there exist $R = R_\delta > 1$ and $N = N_\delta \in \mathbb{N}$ such that when $x \in B_{\frac{1}{\lambda_n}} \cap \overline{\mathbb{R}^2_+}$ with $|x| > 2R$ and $n > N$ we have

$$\widetilde{u}_n(x) + \frac{M_n}{\pi} \log|x| \le \delta \log|x| + O(1). \tag{3.1}$$

To establish the claim notice that $\lim_{n\to\infty} M_n = 2\pi(1+\alpha)$, therefore for any small $\delta > 0$ and any large n, we can choose R large enough such that

$$\frac{1}{\pi}\int_{\{|y|\le R\}\cap\partial\mathbb{R}^2_+} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \ge \frac{M_n}{\pi} - \frac{\delta}{2(\alpha+2)}.$$

Taking $x \in B_{\frac{1}{\lambda_n}} \cap \overline{\mathbb{R}^2_+}$ with $|x| > 2R$ and decomposing $\widetilde{u}_n$ as

$$\begin{aligned}
\widetilde{u}_n(x) &= \frac{1}{\pi}\int_{\{|y|\le R\}\cap\partial\mathbb{R}^2_+} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\
&\quad + \frac{1}{\pi}\int_{\{R\le|y|\le\frac{|x|}{2}\}\cap\partial\mathbb{R}^2_+} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\
&\quad + \frac{1}{\pi}\int_{B(x,\frac{|x|}{2})\cap\partial\mathbb{R}^2_+} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\
&\quad + \frac{1}{\pi}\int_{\Omega'\cap\partial\mathbb{R}^2_+} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy + O(1) \\
&= I_1 + I_2 + I_3 + I_4 + O(1),
\end{aligned}$$

where $\Omega' = (B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+)\backslash(B_{\frac{|x|}{2}} \cup B(x, \frac{|x|}{2}))$. Notice that $\log\frac{|y|}{|x-y|} \le C$ in $B_{\frac{|x|}{2}} \cup \Omega'$ with a constant $C > 0$, we have that I_2 and I_4 are bounded uniformly with respect to n. On the other hand, recalling that $\widetilde{u}_n(x) \le 0$ and $\frac{|x|}{2} \le |y| \le \frac{3}{2}|x|$ in $B(x, \frac{|x|}{2})$, if we set $D_\alpha = B(x, \frac{|x|}{2}) \cap \{|x-y| < |x|^{-(\alpha+1)}\}$, then we get

$$\begin{aligned}
I_3 &\le \frac{1}{\pi}\int_{D_\alpha\cap\partial\mathbb{R}^2_+} \log\frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\
&\quad + \frac{\alpha+2}{\pi}\log|x| \int_{B(x,\frac{|x|}{2})\cap\partial\mathbb{R}^2_+} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy + O(1) \\
&\le C|x|^\alpha \int_{\{|x-y|\le|x|^{-(\alpha+1)}\}\cap\partial\mathbb{R}^2_+} \log\frac{1}{|x-y|} dy + \frac{\delta}{2}\log|x| + O(1) \\
&\le \frac{\delta}{2}\log|x| + O(1).
\end{aligned}$$

Putting those estimates together, and also noticing that $\frac{1}{2} \le \frac{|x-y|}{|x|} \le \frac{3}{2}$ for $|y| \le R$ and $|x| > 2R$, we find

$$\begin{aligned}
\widetilde{u}_n(x) &\le \frac{1}{\pi}\log\frac{2R}{|x|}\int_{\{|y|\le R\}\cap\partial\mathbb{R}^2_+} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy + \frac{\delta}{2}\log|x| + O(1) \\
&\le -\Big(\frac{M_n}{\pi} - \delta\Big)\log|x| + O(1),
\end{aligned}$$

and (3.1) is established. From (3.1) it follows that

$$e^{\widetilde{u}_n} \le C|x|^{-\frac{M_n}{\pi}+\delta}, \tag{3.2}$$

for $x \in B_{\frac{1}{\lambda_n}} \cap \overline{\mathbb{R}^2_+}$ with $|x| > 2R$. Since $M_n = 2\pi(1+\alpha) + o(1)$, by some computations we can obtain

$$\int_{B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+} |\log|y|||y|^\alpha e^{\widetilde{u}_n} dy \le C. \tag{3.3}$$

Next let us estimate the decay of $\widetilde{u}_n(x)$ and $\nabla \widetilde{u}_n(x)$ at infinity. We choose some k satisfying $k > \frac{1}{1+\alpha}$. Since $\alpha \in (-1, +\infty)$, we have $0 < k < +\infty$. Then we claim for $x \in (B_{\frac{1}{\lambda_n}} \backslash B_{(\log \frac{1}{\lambda_n})^k}) \cap \overline{\mathbb{R}^2_+}$,

$$\Big| \widetilde{u}_n(x) + \frac{M_n}{\pi} \log|x| \Big| \le C, \tag{3.4}$$

$$\Big| \nabla \widetilde{u}_n(x) + \frac{M_n}{\pi} \frac{x}{|x|^2} \Big| \le C\Big(\frac{1}{|x|^{2+\alpha-\delta}} + \frac{1}{|x|^2} \Big). \tag{3.5}$$

To prove the above claim, let us set

$$\widetilde{M}_n(x) = \int_{\{|y| \le \eta_0 |x|\} \cap B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy$$

for any small $\eta_0 > 0$ (can be fixed latter). We can show that

$$\begin{aligned} |M_n - \widetilde{M}_n(x)| &= \int_{B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+ \backslash \{|y| \le \eta_0 |x|\}} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\ &\le \int_{|y| \ge \eta_0 (\log \frac{1}{\lambda_n})^k \cap \partial\mathbb{R}^2_+} V_n(\lambda_n y)|y|^{-\frac{M_n}{\pi}+\delta+\alpha} dy \\ &\le \Big(\log \frac{1}{\lambda_n} \Big)^{-1} \Big(\log \frac{1}{\lambda_n} \Big)^{1-k(1+\alpha+\delta+o(1))} \\ &= o(1) \Big(\log \frac{1}{\lambda_n} \Big)^{-1} \end{aligned} \tag{3.6}$$

for $x \in (B_{\frac{1}{\lambda_n}} \backslash B_{(\log \frac{1}{\lambda_n})^k}) \cap \overline{\mathbb{R}^2_+}$. While by (3.3) we obtain

$$\begin{aligned} \widetilde{u}_n(x) = \frac{1}{\pi} & \int_{B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+ \backslash \{|y| \le \eta_0 |x|\}} \log \frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \\ & + \frac{1}{\pi} \int_{B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+ \cap \{|y| \le \eta_0 |x|\}} \log \frac{1}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy + O(1). \end{aligned}$$

Noting that

$$\begin{aligned} & \Big| \int_{B_{\frac{1}{\lambda_n}} \cap \partial\mathbb{R}^2_+ \backslash \{|y| \le \eta_0 |x|\}} \log \frac{|y|}{|x-y|} V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n} dy \Big| \\ & \le C \int_{\partial\mathbb{R}^2_+ \cap \{|y| \ge \eta_0 (\log \frac{1}{\lambda_n})^k\}} \log|y||y|^\alpha e^{\widetilde{u}_n} dy \\ & = O(1) \Big(\log \frac{1}{\lambda_n} \Big)^{-1}, \end{aligned}$$

hence, it follows from that $(1-\eta_0)|x| \le |x-y| \le (1+\eta_0)|x|$ when $|y| \le \eta_0 |x|$ to get

$$\widetilde{u}_n(x) = -\frac{1}{\pi} \widetilde{M}_n(x) \log|x| + O(1)$$

provided η_0 is small enough. Consequently, by (3.6) we get (3.4).

For (3.5), we use Green representation formula of $\widetilde{u}_n(x)$ (see (3.1)) to obtain

$$\nabla\widetilde{u}_n(x)+\frac{M_n}{\pi}\frac{x}{|x|^2}=\frac{1}{\pi}\int_{B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+}\left[\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\right]V_n(\lambda_n y)|y|^\alpha e^{\widetilde{u}_n}dy+O(1).$$

Set $\Omega_{n,1}=B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+\cap\{|y|\le\frac{|x|}{2}\}$, $\Omega_{n,2}=B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+\cap\{\frac{|x|}{2}\le|y|\le 2|x|\}$, $\Omega_{n,3}=B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+\cap\{|y|\ge 2|x|\}$ for any given $x\in(B_{\frac{1}{\lambda_n}}\backslash B_{(\log\frac{1}{\lambda_n})^k})\cap\overline{\mathbb{R}^2_+}$. Notice that $\frac{1}{|x-y|}\le\frac{1}{|x|-|y|}\le\frac{2}{|x|}$ in $\Omega_{n,1}$ and $\frac{|y|}{|x-y|}\le\frac{|y|}{|y|-|x|}\le 2$ in $\Omega_{n,3}$. Since by the mean value theorem for any $|x|\ge 1$ there holds

$$\left|\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\right|\le\frac{|y|}{|x-y||x|},$$

we obtain from (3.2) that

$$\begin{aligned}
&\frac{1}{\pi}\int_{B_{\frac{1}{\lambda_n}}\cap\partial\mathbb{R}^2_+}\left|\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}|V_n(\lambda_n y)|y|\right|^\alpha e^{\widetilde{u}_n}dy\\
&\le\frac{C}{|x|^2}\int_{\Omega_{n,1}}|y|V_n|y|^\alpha e^{\widetilde{u}_n}dy+C\int_{\Omega_{n,2}}\frac{|y|^\alpha e^{\widetilde{u}_n}}{|x-y|}dy+\frac{C}{|x|}\int_{\Omega_{n,3}}|y|^\alpha e^{\widetilde{u}_n}dy\\
&\le\frac{C}{|x|^2}\int_{\Omega_{n,1}}V_n|y|^{1+\alpha}e^{\widetilde{u}_n}dy+C\int_{\frac{|x|}{2}}^{2|x|}\frac{s^{-2-\alpha+\delta}}{\sqrt{(x_1-s)^2+x_2^2}}ds+\frac{C}{|x|}\int_{2|x|}^{+\infty}s^{-2-\alpha+\delta}ds\\
&\le C\left(\frac{1}{|x|^{2+\alpha-\delta}}+\frac{1}{|x|^2}\right).
\end{aligned}$$

Thus we get (3.5).

Step 3 We want to show that

$$M_n=2\pi(1+\alpha)+O(1)\left(\log\frac{1}{\lambda_n}\right)^{-1}. \tag{3.7}$$

For this purpose, by scaling back to u_n, (3.4)–(3.5) yield

$$u_n(x)=\frac{M_n}{\pi}\log\frac{1}{|x|}+\left[(1+\alpha)-\frac{M_n}{\pi}\right]\log\frac{1}{\lambda_n}+O(1), \tag{3.8}$$

$$\nabla u_n(x)=-\frac{M_n}{\pi}\frac{x}{|x|^2}+O\left(\frac{\lambda_n^{1+\alpha-\delta}}{|x|^{2+\alpha-\delta}}+\frac{\lambda_n}{|x|^2}\right), \tag{3.9}$$

for $x\in(B_1\cap\overline{\mathbb{R}^2_+})\backslash B_{\lambda_n(\log\frac{1}{\lambda_n})^k}$. Now we take $r=\lambda_n(\log\frac{1}{\lambda_n})^{k+1}$ and apply Pohozaev identity (2.13) in B_r. It follows from (3.8)–(3.9) to get

$$r\int_{\partial B_r^+\cap\mathbb{R}^2_+}\left|\frac{\partial u_n}{\partial n}\right|^2-\frac{1}{2}|\nabla u_n|^2=\frac{M_n^2}{2\pi}+O(1)\left(\log\frac{1}{\lambda_n}\right)^{-1},$$

$$\int_{\partial B_r^+\cap\partial\mathbb{R}^2_+}x\cdot\nabla V_n(x)|s|^\alpha e^{u_n}ds=O(1)\left(\log\frac{1}{\lambda_n}\right)^{-1},$$

$$V_n(s,0)|s|^\alpha se^{u_n(s,0)}\,|_{s=-r}^{s=r}=O(1)r^{1+\alpha-\frac{M_n}{\pi}}\left(\frac{1}{\lambda_n}\right)^{1+\alpha-\frac{M_n}{\pi}}=O(1)\left(\log\frac{1}{\lambda_n}\right)^{-1},$$

$$\int_{\partial B_r^+\cap\partial\mathbb{R}^2_+}x\cdot\nabla u_n(x)W_n(s,0)ds=O(1)\left(\log\frac{1}{\lambda_n}\right)^{-1},$$

and

$$\begin{aligned}
&(1+\alpha)M_n - (1+\alpha)\int_{\partial B_r^+\cap\partial\mathbb{R}^2_+} V_n(x)|x|^\alpha \mathrm{e}^{u_n}\,d\sigma\\
&= (1+\alpha)\int_{\partial(B_1^+\backslash B_r^+)\cap\partial\mathbb{R}^2_+} V_n(x)|x|^\alpha \mathrm{e}^{u_n}\,d\sigma\\
&= O(1)r^{1+\alpha-\frac{M_n}{\pi}}\Big(\frac{1}{\lambda_n}\Big)^{1+\alpha-\frac{M_n}{\pi}}\\
&= O(1)\Big(\log\frac{1}{\lambda_n}\Big)^{-1}.
\end{aligned}$$

So we get (3.7).

Step 4 Now we come to prove the local estimate (1.7). From Step 1, we are left to show that

$$|\widetilde{u}_n(x) - \widetilde{u}(x)| \le C \tag{3.10}$$

for $x\in(B_{\frac{1}{\lambda_n}}\backslash B_R)\cap\overline{\mathbb{R}^2_+}$, here R is large enough. Notice that

$$|\widetilde{u}_n(x)-\widetilde{u}(x)| \le |\widetilde{u}_n(x)+2(1+\alpha)\log|x|| + |\widetilde{u}(x)+2(1+\alpha)\log|x||,$$

and with the asymptotic behavior of entire solution

$$|\widetilde{u}(x)+2(1+\alpha)\log|x|| \le C,$$

for $x\in\overline{\mathbb{R}^2_+}\backslash B_R$. So to prove (3.10), it is suffice to prove

$$|\widetilde{u}_n(x)+2(1+\alpha)\log|x|| \le C,$$

for $x\in(B_{\frac{1}{\lambda_n}}\backslash B_R)\cap\overline{\mathbb{R}^2_+}$. For this purpose, by (3.4), (3.7), we have

$$|\widetilde{u}_n(x)+2(1+\alpha)\log|x|| \le \Big|\widetilde{u}_n(x)+\frac{M_n}{\pi}\log|x|\Big| + \Big|\frac{M_n}{\pi}\log|x| - 2(1+\alpha)\log|x|\Big| \le C$$

for $x\in(B_{\frac{1}{\lambda_n}}\backslash B_{(\log\frac{1}{\lambda_n})^k})\cap\overline{\mathbb{R}^2_+}$. Since $\{\widetilde{u}_n(x)\}$ converges to $\widetilde{u}(x)$ in $C^{1,\gamma}_{\rm loc}(\overline{\mathbb{R}^2_+})$ and $\widetilde{u}(x)$ satisfies that $|\widetilde{u}(x)+2(1+\alpha)\log|x||\le C$ for $x\in\overline{\mathbb{R}^2_+}\backslash B_R$, we have

$$|\widetilde{u}_n(x)+2(1+\alpha)\log|x|| \le |\widetilde{u}_n(x)-\widetilde{u}(x)| + |\widetilde{u}(x)+2(1+\alpha)\log|x|| \le C,$$

for $x\in\partial B_R\cap\mathbb{R}^2_+$ and large n and large R. We construct $w_\pm(x)$ as follows:

$$w_\pm(x) = -2(1+\alpha)\log|x| \pm (C_1 - C_2|x|^{-\frac{1}{l}}) \mp \frac{C_3 t}{|x|^{1+p}}$$

for positive constant numbers C_1, C_2 and C_3. Let $0<p<1+\alpha$ and $l>\frac{1}{p}$. Then

$$\Delta w_+(x) = -\frac{C_2}{l^2}|x|^{-\frac{1}{l}-2} + \frac{(1+p)(1-p)C_3 t}{|x|^{3+p}},$$

and

$$\frac{\partial w_+(x)}{\partial t} = -\frac{2(1+\alpha)t}{|x|^2} + \frac{C_2 t}{l|x|^{\frac{1}{l}+2}} + \frac{C_3(1+p)t^2}{|x|^{3+p}} - \frac{C_3}{|x|^{1+p}}.$$

Hence, by a suitable choice of C_1, C_2 and C_3, we have

$$\begin{cases}
-\Delta(\widetilde{u}_n(x)-w_+(x)) \le 0 & \text{in } (B_{(\log\frac{1}{\lambda_n})^k}\backslash B_R)\cap\mathbb{R}^2_+,\\
\dfrac{\partial(\widetilde{u}_n(x)-w_+(x))}{\partial n} \le 0 & \text{on } (B_{(\log\frac{1}{\lambda_n})^k}\backslash B_R)\cap\partial\mathbb{R}^2_+,\\
\widetilde{u}_n(x)-w_+(x) \le 0 & \text{on } \partial(B_{(\log\frac{1}{\lambda_n})^k}\backslash B_R)\cap\mathbb{R}^2_+.
\end{cases}$$

We can apply the maximum principle to conclude

$$\widetilde{u}_n(x) \leq w_+(x),$$

for $(B_{(\log \frac{1}{\lambda_n})^k} \backslash B_R) \cap \overline{\mathbb{R}^2_+}$. By the similar way we also can obtain that

$$w_-(x) \leq \widetilde{u}_n(x),$$

for $(B_{(\log \frac{1}{\lambda_n})^k} \backslash B_R) \cap \overline{\mathbb{R}^2_+}$. Thus we complete the local estimate on u_n. □

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