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Finite p-groups All of Whose Minimal Nonabelian Subgroups are Nonmetacyclic of Order p^3

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Abstract Assume p is an odd prime. We investigate finite p-groups all of whose minimal nonabelian subgroups are of order p^3. Let $\mathcal{P}_1$-groups denote the p-groups all of whose minimal nonabelian subgroups are nonmetacyclic of order p^3. In this paper, the $\mathcal{P}_1$-groups are classified, and as a by-product, we prove the Hughes' conjecture is true for the $\mathcal{P}_1$-groups.

Keywords Finite p-groups, a minimal nonabelian subgroup, the Hughes subgroup, p-groups of maximal class

MR(2010) Subject Classification 20D15, 20D25

1 Introduction

A finite group G is said to be minimal nonabelian if G is nonabelian but all its proper subgroups are abelian. Obviously, every finite nonabelian group contains a minimal nonabelian subgroup. In particular, every finite nonabelian p-group, by [4, Proposition 10.28], can be generated by its minimal nonabelian subgroups. Hence the behavior of minimal nonabelian subgroups deeply influence the structure of finite nonabelian p-groups, see for example [3, 15, 17–19]. Berkovich and Janko in their long paper [6] introduced a more general concept than that of minimal nonabelian p-groups. A finite nonabelian p-group is called an $\mathcal{A}_t$-group, $t \in \mathbb{N}$, if it has a nonabelian subgroup of index p^{t-1} but all its subgroups of index p^t are abelian. In other words, an $\mathcal{A}_t$-group is a finite nonabelian p-group whose every nonabelian subgroup of index p^{t-1} is minimal nonabelian. Obviously, $\mathcal{A}_1$-groups are minimal nonabelian p-groups. Given a nonabelian p-group G, there is a $t \in \mathbb{N}$ such that G is an $\mathcal{A}_t$-group. Hence the study of finite nonabelian p-groups can be regarded as that of $\mathcal{A}_t$-groups for some $t \in \mathbb{N}$.

$\mathcal{A}_1$-groups were classified by Rédei in [13]. $\mathcal{A}_2$-groups were classified by Kazarin in his unpublished thesis, but his exposition did not give a full proof. Berkovich and Janko gave an elementary treatment, see [6, Sections 3–5]. However, they did not solve the isomorphism problem for some classes of groups. Zhang et al. in [20] gave a new proof of the classification of $\mathcal{A}_2$-groups and gave all non-isomorphic types of the $\mathcal{A}_2$-groups. Subsequently, Zhang et al. in [21] classified $\mathcal{A}_3$-groups. We observed that $\mathcal{A}_2$-groups are the p-groups all of whose $\mathcal{A}_1$-subgroups are of index p. $\mathcal{A}_3$-groups are the p-groups all of whose $\mathcal{A}_1$-subgroups are of index p or p^2. In other words, the $\mathcal{A}_1$-subgroups of $\mathcal{A}_2$-, $\mathcal{A}_3$-groups are of large order. Moreover, An

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and Qu et al. in a series of papers [1, 2, 10–12] classified finite p-groups with an $\mathcal{A}_1$-subgroup of index p. In this paper, we consider an opposite question, that is, what can be said about finite p-groups all of whose $\mathcal{A}_1$-subgroups are of the smallest order? Obviously, the smallest order of $\mathcal{A}_1$-subgroups is p^3. Moreover, Berkovich and Janko proposed the following:

Problem ([5, Problem 920]) Classify the p-groups all of whose $\mathcal{A}_1$-subgroups are of order p^3.

For $p = 2$, the problem was solved by Janko in [9]. For odd prime p, the problem is open. We investigate this problem. For convenience, we use $\mathcal{P}_1$-groups to denote the p-groups all of whose $\mathcal{A}_1$-subgroups are nonmetacyclic of order p^3, $\mathcal{P}_2$-groups the p-groups all of whose $\mathcal{A}_1$-subgroups are metacyclic of order p^3 and $\mathcal{P}_3$-groups the p-groups all of whose $\mathcal{A}_1$-subgroups are of order p^3, respectively. In this paper, some properties of $\mathcal{P}_i$-groups are given, and $\mathcal{P}_1$-groups are classified. It turns out that the structure of $\mathcal{P}_1$-groups is closely related to the Hughes subgroup, and the Hughes' conjecture is true for $\mathcal{P}_1$-groups. By the way, Zhao et al. in [22] classified finite p-groups with exactly one $\mathcal{A}_1$-subgroup of given structure of order p^3.

For a finite p-group G, we use $M \lessdot G$ to denote M is a maximal subgroup of G, $G \in \mathcal{A}_t$ to denote G is an $\mathcal{A}_t$-group and $G \in \mathcal{P}_i$ to denote G is a $\mathcal{P}_i$-group. For a nilpotent group G, let

$$G = G_1 > G_2 > \cdots > G_{c+1} = 1$$

denote the lower central series of G, where $G_i = [G_{i-1}, G]$ and $c = c(G)$, and

$$1 = Z_0(G) \leq Z_1(G) \leq \cdots \leq Z_c(G) = G$$

the upper central series of G, where $Z_i(G)/Z_{i-1}(G) = Z(G/Z_{i-1}(G))$.

Assume G is a group, $A \leq G$, $B \leq G$ and $K = A \cap B$. G is called a center product of A and B if $G = AB$ and $[A, B] = 1$, denoted by $A *_K B$.

2 Preliminaries

In the following, we list some known results which are often used.

Theorem 2.1 ([16, Lemma 2.2]) *Let G be a finite p-group. Then the following statements are equivalent.*

(1) *G is minimal nonabelian.*

(2) *$d(G) = 2$ and $|G'| = p$.*

(3) *$d(G) = 2$ and $Z(G) = \Phi(G)$.*

We use $\mathrm{M}_p(m, n)$ to denote the p-groups

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle, \quad \text{where } m \geq 2,$$

and $\mathrm{M}_p(m, n, 1)$ to denote the p-groups

$$\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where $m \geq n$, and $m + n \geq 3$ for $p = 2$. It is not difficult to know that $\mathrm{M}_p(m, n)$ is metacyclic, and $\mathrm{M}_p(m, n, 1)$ is nonmetacyclic. We can give a presentation of minimal nonabelian p-groups as follows.

Theorem 2.2 (Rédei [13]) *Let G be a minimal nonabelian p-group. Then G is one of Q_8, $\mathrm{M}_p(m, n)$ and $\mathrm{M}_p(m, n, 1)$.*

Let G be a group, p odd prime and $H_p(G)$ the subgroup of G generated by the elements of order different from p. $H_p(G)$ is called a Hughes subgroup of G.

Lemma 2.3 ([7]) *Let G be a finite p-group with $H_p(G) \neq 1$. Then $|G : H_p(G)| \leq p$ if G satisfies one of the following conditions*:

(a) G_i/G_{i+1} *is cyclic for all* $i \geq 2$,

(b) G *is metabelian.*

Proposition 2.4 *Assume $\mathcal{P}$ is a set consisting of all p-groups with a fixed property. Then the following statements are equivalent.*

(1) *If $G \in \mathcal{P}$, then $G/N \in \mathcal{P}$, where $N \trianglelefteq G$ with $|N| = p$.*

(2) *If $G \in \mathcal{P}$, then $G/N \in \mathcal{P}$, where $N \trianglelefteq G$.*

Proof (2) $\Rightarrow$ (1): It is obvious.

(1) $\Rightarrow$ (2): Assume $N \trianglelefteq G$ and $|N| > p$ without loss of generality. We use induction on $|N|$. Let K be a minimal normal subgroup of G contained in N. Then $|K| = p$. Thus $G/K \in \mathcal{P}$ by (1). Since $|N/K| < |N|$, $(G/K)/(N/K) \in \mathcal{P}$ by induction hypothesis. Hence $G/N \in \mathcal{P}$. □

3 Some Properties of $\mathcal{P}_i$-groups

In the following, we always assume p is an odd prime.

Lemma 3.1 *Assume $G \in \mathcal{P}_i$. Then*

(1) *Subgroups of G are $\mathcal{P}_i$-groups for $i = 1, 2, 3$.*

(2) *Quotient groups of G are $\mathcal{P}_i$-groups for $i = 1, 3$.*

Proof (1) It is obvious.

(2) Assume $N \trianglelefteq G$ and $\overline{G} = G/N$. By Proposition 2.4 we may assume $|N| = p$. We will prove $\overline{G} \in \mathcal{P}_i$ for $i = 1, 3$ by a counterexample.

If $\overline{G} \notin \mathcal{P}_i$, then there exists an $\mathcal{A}_1$-subgroup $\overline{H}$ of $\overline{G}$ such that $\overline{H} \notin \mathcal{P}_i$. It follows by Theorem 2.1 that $d(\overline{H}) = 2$ and $|\overline{H}'| = p$. Let $\overline{H} = \langle \overline{a}, \overline{b} \rangle$ and $N = \langle x \rangle$. Then $d(H) \leq 3$. We claim that

$$d(H) = 2, \quad H = \langle a, b \rangle \quad \text{and} \quad |H'| = p^2.$$

If $d(H) \neq 2$, then $d(H) = 3$ and $H = \langle a, b, x \rangle = \langle a, b \rangle \times \langle x \rangle$. It follows by (1) that $\langle a, b \rangle \in \mathcal{P}_i$. On the other hand, it is easy to verify that $\langle a, b \rangle \cong \overline{H}$. This contradicts $\overline{H} \notin \mathcal{P}_i$. So $d(H) = 2$. Since $x \in Z(G)$ and H is nonabelian, $H = \langle a, b \rangle$. Notice that $|\overline{H}| \geq p^3$. So $|H| \geq p^4$. Since $d(H) = 2$ and $H \in \mathcal{P}_i$, $|H'| \neq p$ by Theorem 2.1. On the other hand, since $\overline{H} \in \mathcal{A}_1$, $|\overline{H}'| = p$ by Theorem 2.1. It follows that $p \leq |H'| \leq p^2$. Hence $|H'| = p^2$.

In the following, we will prove the conclusion for $i = 1$ and 3 respectively.

Case 1 $i = 3$

We claim that AN is an abelian subgroup of index p of H if $\overline{A} \lessdot \overline{H}$ and $d(\overline{A}) = 2$. If not, then AN is an nonabelian subgroup of index p of H. Since $N \leq Z(G)$, A is nonabelian. Since $\overline{H} \in \mathcal{A}_1$, $\overline{A}$ is abelian. Moreover, $N = A' < A$. Hence $AN = A$ and $d(A) = d(\overline{A}) = 2$. Since $|A'| = |N| = p$, $A \in \mathcal{A}_1$ by Theorem 2.1. On the other hand, it follows from $\overline{H} \notin \mathcal{P}_3$ that $|\overline{H}| > p^3$. Hence $|\overline{A}| > p^2$. So $|A| > p^3$. This contradicts G is a $\mathcal{P}_3$-group.

Now we discuss the possible case of $\overline{H}$. Theorem 2.2 tells us that $\overline{H} \cong \mathrm{M}_p(m, n)$ or $\mathrm{M}_p(m, n, 1)$.

If $\overline{H} \cong \mathrm{M}_p(m,n)$, then $\Phi(\overline{H}) = \langle \overline{a}^p, \overline{b}^p \rangle$. Let $\overline{M}_1 = \langle \overline{a}^p, \overline{b} \rangle$ and $\overline{M}_2 = \langle \overline{a}, \overline{b}^p \rangle$. Then $\overline{M}_1 \lessdot \overline{H}$ and $\overline{M}_2 \lessdot \overline{H}$. By the claim above paragraph we get M_1N and M_2N are abelian subgroups of index p of H. It follows that $|H'| \le p$. This contradicts $|H'| = p^2$.

If $\overline{H} \cong \mathrm{M}_p(m,1,1)$, then let $\overline{M}_1 = \langle \bar{a}, \bar{c} \rangle$ and $\overline{M}_2 = \langle \overline{ba}, \bar{c} \rangle$. We have $\overline{M}_1 \lessdot \overline{H}$ and $\overline{M}_2 \lessdot \overline{H}$. By a same argument as that of above paragraph, a contradiction occurs.

If $\overline{H} \cong \mathrm{M}_p(m,n,1)$, where $n > 1$, then assume $c = [a,b]$ without loss of generality. If $[a,c] = [b,c] = 1$, then $H_3 = 1$ and $H_2 = \langle c \rangle \cong C_{p^2}$. Since $|\overline{H}'| = p$ and $|H'| = p^2$, $N \le H'$. Hence $\langle c^p \rangle = N$. It follows from $H_3 = 1$ that $[a^p, b] = [a,b]^p = c^p$. So $\langle a^p, b \rangle' = \langle c^p \rangle = N$. Moreover, $\langle a^p, b \rangle \in \mathcal{A}_1$. On the other hand, $|\langle a^p, b \rangle| = o(\overline{a^p})o(\overline{b})|N| > p^3$. This contradicts G is a $\mathcal{P}_3$-group. This implies that either $[a,c]$ or $[b,c]$ is not identity. Assume $[a,c] \ne 1$ without loss of generality. Then $\langle a, c \rangle' = N$. It follows by Theorem 2.1 that $\langle a, c \rangle \in \mathcal{A}_1$. On the other hand, $|\langle a, c \rangle| = o(\overline{a})o(\overline{c})|N| > p^3$. This contradicts G is a $\mathcal{P}_3$-group again. To sum up, $\overline{G} \in \mathcal{P}_3$.

Case 2 $i = 1$

Since $\mathcal{P}_1$-group is a $\mathcal{P}_3$-group, $G \in \mathcal{P}_3$. By the argument of Case 1 we get $\overline{G} \in \mathcal{P}_3$. Hence $|\overline{H}| = p^3$, and so $|H| = p^4$. If $\overline{G} \notin \mathcal{P}_1$, then $\overline{H} \cong \mathrm{M}_p(2,1)$. For convenience assume

$$\overline{H} = \langle \bar{a}, \bar{b} \mid \bar{a}^{p^2} = \bar{1}, \bar{b}^p = \bar{1}, [\bar{a}, \bar{b}] = \bar{a}^p, [\bar{a}^p, \bar{b}] = \bar{1} \rangle.$$

If H has a cyclic subgroup of index p, then, since $p > 2$, $H \cong \mathrm{M}_p(3,1)$. This contradicts $G \in \mathcal{P}_1$. So $a^{p^2} = 1$. If $H_3 = 1$, then $H' = \langle [a,b] \rangle = \langle a^p x^t \rangle$, where t is an integer. Moreover, $|H'| = p$. This contradicts $|H'| = p^2$. So $H_3 \ne 1$. Hence H is a p-group of maximal class and $N = Z(H)$. Obviously, $\exp(H/Z(H)) = \exp(\overline{H}) \ne p$. On the other hand, $\exp(H/Z(H)) = p$ by [4, Lemma 9.3]. This is a contradiction. So $\overline{G} \in \mathcal{P}_1$. □

There exists a $\mathcal{P}_2$-group whose a quotient group is not a $\mathcal{P}_2$-group.

Example 3.2 Let $G = \langle a, b \mid a^{p^2} = b^p = c^p = 1, [a,b] = c, [c,a] = a^p, [c,b] = 1 \rangle$, where $p \ge 5$. Then $G \in \mathcal{P}_2$ and $G/Z(G) \notin \mathcal{P}_2$.

Proof It is not difficult to verify that G is a p-group of maximal class of order p^4. So $G \notin \mathcal{A}_1$. Let $K \le G$ and $K \in \mathcal{A}_1$. Then $|K| = p^3$. By a simple calculation we have K is one of $\langle a, c \rangle$ and $\langle ab^i, c \rangle$, where $i = 1, 2, \dots, p-1$. Notice that $G' = \langle a^p, c \rangle$. Thus G is metabelian. By a simple calculation we get $\langle a, c \rangle \cong \langle ab^i, c \rangle \cong \mathrm{M}_p(2,1)$. Thus $G \in \mathcal{P}_2$. On the other hand, since G is of maximal class, $Z(G) = G_3 = \langle a^p \rangle$. It is not difficult to verify

$$G/Z(G) = \langle \bar{a}, \bar{b} \mid \bar{a}^p = \bar{b}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle.$$

So $G/Z(G) \cong \mathrm{M}_p(1,1,1)$. That is, $G/Z(G) \in \mathcal{P}_1$. □

Lemma 3.3 *Assume G is a finite nonabelian p-group and $G \in \mathcal{P}_3$, $z \in Z(G)$ and $o(z) = p^2$. If H is an $\mathcal{A}_1$-subgroup of G, then $H' = \langle z^p \rangle$.*

Proof By Theorem 2.1 we may assume $H = \langle x, y \rangle$. Let $K = \langle xz, y \rangle$. Then $K' = H'$. Thus $K \in \mathcal{A}_1$ by Theorem 2.1. Since $G \in \mathcal{P}_3$, $|H| = |K| = p^3$. It follows by Theorem 2.2 that $\mho_1(H) \le H'$ and $\mho_1(K) \le K'$. Thus

$$x^p z^p = (xz)^p \in \mho_1(K) \le K' = H' \quad \text{and} \quad x^p \le \mho_1(H) \le H'.$$

So $z^p \in H'$. Since $|\langle z^p \rangle| = p = |H'|$, $H' = \langle z^p \rangle$. □

Lemma 3.4 *Assume G is a finite nonabelian p-group and $G \in \mathcal{P}_i$. Then*

(1) $|\mho_1(Z(G))| = 1$ *for* $i = 1, 2$.

(2) $|\mho_1(Z(G))| \le p$ *for* $i = 3$.

Proof (1) It suffices to show $\exp(Z(G)) = p$. If not, then there exists $x \in Z(G)$ such that $o(x) = p^2$.

If $i = 1$, then there exists $H \le G$ such that $H \cong \mathrm{M}_p(1,1,1)$. By Theorem 2.2 we may assume

$$H = \langle a, b \mid a^p = b^p = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle.$$

Let $K = \langle ax, b \rangle$. Then $K' = H'$. By Theorem 2.1, we get $K \in \mathcal{A}_1$. Since $G \in \mathcal{P}_1$, $K \cong \mathrm{M}_p(1,1,1)$. Thus $o(ax) = p$. On the other hand, $o(ax) = o(x) = p^2$ by hypothesis. A contradiction.

If $i = 2$, then there exists $H \le G$ such that $H \cong \mathrm{M}_p(2,1)$. By Theorem 2.2 we may assume

$$H = \langle a, b \mid a^{p^2} = 1, b^p = 1, [a,b] = a^p \rangle.$$

It follows by Lemma 3.3 that $\langle a^p \rangle = H' = \langle x^p \rangle$. Assume $a^p = x^p$ without loss of generality. Then $(ax^{-1})^p = 1$. Let $K = \langle ax^{-1}, b \rangle$. Then $K \cong \mathrm{M}_p(1,1,1)$. This contradicts $G \in \mathcal{P}_2$.

(2) Since $G \in \mathcal{P}_3$, there exists $K \le G$ such that $K \in \mathcal{A}_1$ and $|K| = p^3$. Assume $K = \langle a, b \rangle$. Take $1 \ne x \in Z(G)$. Let $H = \langle ax, b \rangle$. Then $H' = K'$. It follows by Theorem 2.1 that $H \in \mathcal{A}_1$. Since $G \in \mathcal{P}_3$, $|H| = p^3$. If $o(x) \ge p^3$, then $o(ax) \ge p^3$. Hence $H = \langle ax \rangle$. This contradicts $H \in \mathcal{A}_1$. So $o(x) \le p^2$. It follows that $\exp(Z(G)) \le p^2$.

If $\exp(Z(G)) = p$, then $|\mho_1(Z(G))| = 1$. If $\exp(Z(G)) = p^2$, then assume $o(x) = p^2$. Since $G \in \mathcal{P}_3$, $H' = \langle x^p \rangle$ by Lemma 3.3. Hence $\mho_1(Z(G)) = H'$. Moreover, $|\mho_1(Z(G))| = |H'| = p$. □

Lemma 3.5 *Assume G is a nonabelian p-group and $G \in \mathcal{P}_3$. Then $\Phi(G) = G'$.*

Proof It suffices to show $x^p \in G'$ for all $x \in G$. If $x \in Z(G)$, then, since $G \in \mathcal{P}_3$, $o(x) \le p^2$ by Lemma 3.4 (2). If $o(x) = p^2$, then $\langle x^p \rangle \le G'$ by Lemma 3.3. If $x \notin Z(G)$, then there exists $y \in G$ such that $[x,y] \ne 1$. Let $L = \langle x, y \rangle$ and $\overline{L} = L/\mho_1(L')L_3$. It is not difficult to see that $|\overline{L}'| = p$. Thus $\overline{L} \in \mathcal{A}_1$ by Theorem 2.1. Since $G \in \mathcal{P}_3$, $L \in \mathcal{P}_3$ by Lemma 3.1 (1). It follows by Lemma 3.1 (2) that $\overline{L} \in \mathcal{P}_3$. Then $|\overline{L}| = p^3$. Thus $\bar{x}^p \in \overline{L}'$. Since $\mho_1(L')L_3 \le L'$, $x^p \in L' \le G'$. □

Proposition 3.6 *Assume G is a finite p-group. Then G is a $\mathcal{P}_i$-group if and only if all two-generator subgroups of G are $\mathcal{P}_i$-groups, where $i = 1, 2, 3$.*

Proof $\Longrightarrow$: It follows by Lemma 3.1 (1).

$\Longleftarrow$: Let $K \le G$ and $K \in \mathcal{A}_1$. Then $d(K) = 2$ by Theorem 2.1. If $K \in \mathcal{P}_1$, then $K \cong \mathrm{M}_p(1,1,1)$. Hence $G \in \mathcal{P}_1$. Similarly, if $K \in \mathcal{P}_2$ or $K \in \mathcal{P}_3$, respectively, then so do G. □

Theorem 3.7 *Assume G is a finite nonabelian p-group and p odd prime. Then $G \in \mathcal{P}_1$ if and only if nonabelian subgroups of G are generated by elements of order p.*

Proof $\Longrightarrow$: Since $G \in \mathcal{P}_1$, all $\mathcal{A}_1$-subgroups of G are isomorphic to $\mathrm{M}_p(1,1,1)$ by Theorem 2.2. [4, Proposition 10.28] tells us that G is generated by its $\mathcal{A}_1$-subgroups. Hence nonabelian subgroups of G are generated by elements of order p.

$\Longleftarrow$: Let $K \le G$ and $K \in \mathcal{A}_1$. Then, by hypothesis, K is generated by elements of order p. On the other hand, $d(K) = 2$ and $|K'| = p$ by Theorem 2.1. Thus $c(K) = 2 < p$. Hence

K is regular. It follows by [4, Theorem 7.2 (b)] that $\exp(K) = p$, and so $K \cong \mathrm{M}_p(1,1,1)$ by Theorem 2.2. That is, $G \in \mathcal{P}_1$. □

Hughes had conjectured in [8] that $|G : H_p(G)| = p$ if $G > H_p(G) > 1$. We will prove the structure of $\mathcal{P}_1$-groups is closely related to the Hughes subgroup, and the Hughes's conjecture is true for $\mathcal{P}_1$-groups.

Theorem 3.8 *Assume G is a finite nonabelian p-group and $\exp(G) > p > 2$. Then $G \in \mathcal{P}_1$ if and only if $H_p(G)$ is abelian subgroup of index p of G. In particular, $\mathcal{P}_1$-groups satisfy the Hughes conjecture.*

Proof $\Longleftarrow$: Let $K \leq G$ and $K \in \mathcal{A}_1$. It is enough to show $K \cong \mathrm{M}_p(1,1,1)$. This needs only to prove K can be generated by its elements of order p by Theorem 2.2. By Theorem 2.1 we may assume $K = \langle x, y\rangle$. Since $H_p(G)$ is abelian subgroup of index p of G, $K \nleq H_p(G)$. Assume $x \notin H_p(G)$ without loss of generality. If $y \notin H_p(G)$, then $o(x) = o(y) = p$. If $y \in H_p(G)$, then $xy \notin H_p(G)$ and $K = \langle x, xy\rangle$. In any case, we have K can be generated by its elements of order p.

$\Longrightarrow$: For convenience, the Hughes subgroup $H_p(G)$ is denoted by H. Take $x \in H$ with $o(x) > p$ and $y \in Z_2(H)$. Let $K = \langle x, y\rangle$. Then K is abelian or of class 2. If K is nonabelian, then $K \in \mathcal{P}_1$ by Lemma 3.1 (1). Hence all $\mathcal{A}_1$-subgroups of K are isomorphic to $\mathrm{M}_p(1,1,1)$. It follows by [4, Proposition 10.28] that K is generated by its elements of order p. Since $p > 2$, K is regular. Hence $\exp(K) = p$ by [4, Theorem 7.2 (b)]. This contradicts $o(x) > p$. Hence K is abelian. Thus $x \in C_G(Z_2(H))$. Moreover, $H \leq C_G(Z_2(H))$. It follows that $H = C_H(Z_2(H))$. So $Z_2(H) = Z(H)$. Hence $Z(H) = H$. That is, H is abelian.

Notice that G is nonabelian. We have $|G : H| \neq 1$. If $|G : H| > p$, then there exists $K \leq G$ such that $H < K$ and $|K : H| = p^2$. Hence K/H is abelian of order p^2. So $K' \leq H$. Hence K is metabelian. Since G is nonabelian and $\exp(G) > p$, $H > 1$. It follows by Lemma 2.3 that $|K : H| = p$. This contradicts $|K : H| = p^2$. So $|G : H| = p$. □

The following Theorems 3.9 and 3.10 describe the rough structure of $\mathcal{P}_3$-groups.

Theorem 3.9 *Assume G is a finite nonabelian p-group, and $G = H \times C_p^n$. If $H \in \mathcal{P}_i$ for $i = 1, 2, 3$, then $G \in \mathcal{P}_i$.*

Proof We use induction on n. If $n = 1$, then $G = H \times C_p$. Assume $G = H \times \langle x\rangle$, where $o(x) = p$. Let $K \leq G$ and $K \in \mathcal{A}_1$. It follows by Theorem 2.1 that $d(K) = 2$. Assume $K = \langle a, b\rangle$. Then $a = h_1x^i$ and $b = h_2x^j$, where $h_1, h_2 \in H$, $1 \leq i, j \leq p-1$. By a simple calculation we get $\langle h_1, h_2\rangle' = \langle a, b\rangle'$. It follows by Theorem 2.1 that $\langle h_1, h_2\rangle \in \mathcal{A}_1$. Since $H \in \mathcal{P}_i$, $\langle h_1, h_2\rangle \cong \mathrm{M}_p(2,1)$ or $\mathrm{M}_p(1,1,1)$.

If $\langle h_1, h_2\rangle \cong \mathrm{M}_p(2,1)$, then

$$o(a) = o(h_1) = p^2, \quad o(b) = o(h_2) = p \quad \text{and} \quad [a,b] = [h_1,h_2] = h_1^p = a^p.$$

Hence $\langle a, b\rangle$ is isomorphic to a quotient group of $\langle h_1, h_2\rangle$ by [14, 2.2.1]. Since $|\langle a, b\rangle| \geq p^3$, $\langle a, b\rangle \cong \langle h_1, h_2\rangle$. In the same way, if $\langle h_1, h_2\rangle \cong \mathrm{M}_p(1,1,1)$, then $\langle a, b\rangle \cong \langle h_1, h_2\rangle$. So the conclusion is true for $n = 1$.

Assume the conclusion is true for $n = k$. If $n = k+1$, then $G = H \times C_p^{k+1}$. Let $L = H \times C_p^k$. By induction hypothesis, $L \in \mathcal{P}_i$. Thus $G = L \times C_p \in \mathcal{P}_i$. □

Theorem 3.10 *Assume G is a finite nonabelian p-group. If $G \in \mathcal{P}_3$, then there exists a subgroup H of G satisfying $H \in \mathcal{P}_3$ and $Z(H) \le \Phi(H)$ such that $G = H \times C_p^k$ or $G = (H *_N C_{p^2}) \times C_p^k$, where $|N| = p$.*

Proof Assume G is a counterexample of the smallest order. Then $Z(G) \nleq \Phi(G)$. Let $x \in Z(G) \setminus \Phi(G)$. Then there exists $M \lessdot G$ such that $x \notin M$. Hence $G = \langle M, x\rangle = M\langle x\rangle$. It follows by Lemma 3.1 (1) that $M \in \mathcal{P}_3$. Since $|M| < |G|$, there exists $L \le M$ with $L \in \mathcal{P}_3$ and $Z(L) \le \Phi(L)$ such that $M = L \times C_p^m$ or $(L *_N C_{p^2}) \times C_p^m$, where $|N| = p$.

Since $G \in \mathcal{P}_3$ and $x \in Z(G)$, $o(x) \le p^2$ by Lemma 3.4 (2). If $o(x) = p$, then $G = M \times \langle x\rangle$. Thus $G = L \times C_p^{m+1}$ or $(L *_N C_{p^2}) \times C_p^{m+1}$, where $Z(L) \le \Phi(L)$ and $|N| = p$. This contradicts G is a counterexample.

If $o(x) = p^2$, then $G = M *_N \langle x\rangle$, where $N = \langle x^p\rangle$. By Lemma 3.4 (2), $|\mho_1(Z(G))| \le p$. It follows that $\mho_1(Z(G)) = \langle x^p\rangle$. On the other hand, $|\mho_1(Z(M))| \le p$ by Lemma 3.4 (2). If $|\mho_1(Z(M))| = p$, then $\mho_1(Z(M)) = \langle x^p\rangle$. Hence, there exists an element y of order p^2 of $Z(M)$ such that $\mho_1(Z(M)) = \langle y^p\rangle$. Thus $x^p = y^{jp}$, where $(j, p) = 1$. Let $x' = xy^{-j}$. Then $o(x') = p$ and $x' \in Z(G)\setminus\Phi(G)$. This reduces to the case mentioned above. Hence $|\mho_1(Z(M))| = 1$. That is, $\exp(Z(M)) = p$. Thus $M = L\times C_p^m$. Hence $G = (L\times C_p^m) *_N \langle x\rangle \cong (L\times C_p^m) *_N C_{p^2}$. We will prove $N \le L$. If not, then $L\langle x\rangle = L \times \langle x\rangle$. Obviously, L is nonabelian. Thus there exists $H \le L$ such that $H \cong \mathrm{M}_p(1,1,1)$ or $\mathrm{M}_p(2,1)$. It is easy to verify that $\mathrm{M}_p(1,1,1) \times C_{p^2}$ has a subgroup which is isomorphic to $\mathrm{M}_p(2,1,1)$, and $\mathrm{M}_p(2,1) \times C_{p^2}$ has a subgroup which is isomorphic to $\mathrm{M}_p(2,2)$. This contradicts $G \in \mathcal{P}_3$. So $N \le L$. It follows that $G = (L *_N C_{p^2}) \times C_p^m$. This contradicts G is a counterexample. □

The group $(H *_N C_{p^2}) \times C_p^k$ in Theorem 3.10 is not necessarily a $\mathcal{P}_3$-group.

Example 3.11 Let $H = S \times T$, where $S \cong T \cong \mathrm{M}_p(1,1,1)$, $K = \langle x\rangle$ and $\langle x^p\rangle = T'$. Then $H \in \mathcal{P}_3$ with $Z(H) \le \Phi(H)$, and $G = (H *_{T'} K) \times C_p^k$ is not a $\mathcal{P}_3$-group.

Proof Since $\exp(H) = p$, $H \in \mathcal{P}_1$. Thus $H \in \mathcal{P}_3$. Obviously, $Z(H) \le \Phi(H)$. Assume without loss of generality $S = \langle a_1, b_1\rangle$ and $T = \langle a_2, b_2\rangle$. Since $S \cap K \le (H \cap K) \cap S = T' \cap S = 1$, $SK = S \times K$. It follows that $\langle a_1x, b_1\rangle \cong \mathrm{M}_p(2,1,1)$. Hence $G = (H *_N K) \times C_p^k$ is not a $\mathcal{P}_3$-group. □

4 A Classification of $\mathcal{P}_1$-groups

Let G be a group of maximal class of order p^n. For convenience, in this section a group of maximal class of order p^2 means an elementary abelian group. In the following, we give a more intensive description about the structure of $\mathcal{P}_1$-groups.

Lemma 4.1 *Assume G is a finite nonabelian p-group with $d(G) = 2$ and $G \in \mathcal{P}_1$. If G has an abelian subgroup H of index p, then G is a group of maximal class, and all elements of $G\setminus H$ are of order p.*

Proof Since $G \in \mathcal{P}_1$, $G' = \Phi(G)$ by Lemma 3.5. It follows from $d(G) = 2$ that $|G/G'| = p^2$. Since G has an abelian subgroup H of index p, G is of maximal class by [4, §1, Exercise 4].

Let $a \in G\setminus H$, then $G = \langle a\rangle H$. Since $d(G) = 2$, there exists $b \in H$ such that $G = \langle a, b\rangle$. If $|G| = p^3$, then it follows from $G \in \mathcal{P}_1$ that $G \cong \mathrm{M}_p(1,1,1)$. Thus $o(a) = p$. Assume $|G| = p^n \ge p^4$. Let $K = \langle a, [b,(n-3)a]\rangle$. Notice that $G_{n-1} = \langle [b,(n-2)a], G_n\rangle$. Since G is of maximal class, $G_n = 1$. Thus $K' = G_{n-1}$, and $|G_{n-1}| = p$. It follows by Theorem 2.1 that $K \in \mathcal{A}_1$. Since $G \in \mathcal{P}_1$, $K \cong \mathrm{M}_p(1,1,1)$. Thus, $o(a) = p$. $\square$

Lemma 4.2 *Assume $G \in \mathcal{P}_1$. If G has an abelian subgroup H of index p, and all elements of $G\backslash H$ are of order p, then $G = H \rtimes \langle a\rangle$, a semidirect product of H by $\langle a\rangle$, where $H = B_1 \times B_2 \times \cdots \times B_n$ and $o(a) = p$. Moreover, for all $1 \le i \le n$, $B_i\langle a\rangle$ is a group of maximal class with an abelian subgroup B_i of index p, and all elements of $B_i\langle a\rangle\backslash B_i$ are of order p.*

Proof Obviously, there exists $a \in G\backslash H$ such that $o(a) = p$. Thus $G = H \rtimes \langle a\rangle$, a semidirect product of H by $\langle a\rangle$. Since all elements of $G\backslash H$ are of order p, all elements of $B_i\langle a\rangle\backslash B_i$ are of order p.

It suffices to show that $H = B_1 \times B_2 \times \cdots \times B_n$ and for all $1 \le i \le n$, $B_i\langle a\rangle$ is a group of maximal class with an abelian subgroup B_i of index p. We prove by induction on $|G|$.

It is clear that the conclusion holds for $|G| \le p^2$. Assume $|G| \ge p^3$. Take $N \le H$ such that $N \trianglelefteq G$ and $|N| = p$. Let $\overline{G} = G/N$. Then, by Lemma 3.1 (2), $\overline{G} \in \mathcal{P}_1$. Obviously, $\overline{H}$ is an abelian subgroup of index p of $\overline{G}$. By induction hypothesis,

$$\overline{H} = \overline{B_1} \times \overline{B_2} \times \cdots \times \overline{B_n},$$

and $\overline{B_i}\langle\overline{a}\rangle$ is a group of maximal class with an abelian subgroup $\overline{B_i}$ of index p for all $1 \le i \le n$.

Since $\overline{B_i}\langle\overline{a}\rangle$ is of maximal class, $\overline{B_i}\langle\overline{a}\rangle$ is generated by two elements. Assume $\overline{B_i}\langle\overline{a}\rangle = \langle\overline{b_i}, \overline{a}\rangle$ without loss of generality. Let $\langle\overline{b_i}, \overline{a}\rangle = A_i/N = \overline{A_i}$. Then $A_i = \langle b_i, a\rangle N$. By modular law we have

$$\overline{A_i} = (\overline{A_i} \cap \overline{H})\langle\overline{a}\rangle.$$

Since $\overline{B_i} \le \overline{A_i} \cap \overline{H}$ and $\overline{A_i} = \overline{B_i}\langle\overline{a}\rangle$, $\overline{B_i} = \overline{A_i} \cap \overline{H}$. Thus $B_i = A_i \cap H$ for all $1 \le i \le n$.

Case 1 There exists i such that $N \not\le \langle b_i, a\rangle$.

In this case, $A_i = \langle b_i, a\rangle \times N$. By modular law we have

$$B_i = A_i \cap H = (\langle b_i, a\rangle \cap H)N.$$

Let

$$H_1 = B_1 \cdots B_{i-1}B_{i+1} \cdots B_n.$$

Since $N \le H_1$,

$$H = B_iH_1 = (\langle b_i, a\rangle \cap H)NH_1 = (\langle b_i, a\rangle \cap H)H_1.$$

Since $\overline{B_i} \cap \overline{H_1} = 1$, $B_i \cap H_1 = N$. Thus

$$(\langle b_i, a\rangle \cap H) \cap H_1 = (\langle b_i, a\rangle \cap H) \cap B_i \cap H_1 = (\langle b_i, a\rangle \cap H) \cap N = 1.$$

It follows that $H = (\langle b_i, a\rangle \cap H) \times H_1$. Let $D_i = \langle b_i, a\rangle \cap H$.

Since $H_1\langle a\rangle \in \mathcal{P}_1$ and $|H_1\langle a\rangle| < |G|$, by induction hypothesis, without loss of generality assume

$$H_1 = D_1 \times \cdots \times D_{i-1} \times D_{i+1} \times \cdots \times D_n,$$

and $D_j\langle a\rangle$ is a group of maximal class with an abelian subgroup D_j of index p for all $1 \le j \le n$ and $j \ne i$. Thus $H = H_1 \times D_i = D_1 \times D_2 \times \cdots \times D_n$.

Now we prove $D_i\langle a\rangle$ is a group of maximal class with an abelian subgroup D_i of index p. Obviously, $D_i\langle a\rangle = \langle b_i, a\rangle$. If $\langle b_i, a\rangle$ is nonabelian, then, by Lemma 4.1, $\langle b_i, a\rangle$ is a group of maximal class. If $\langle b_i, a\rangle$ is abelian, then, since $a, ab_i \in G\backslash H$, $o(a) = o(ab_i) = p$ by hypothesis. It follows that $\langle b_i, a\rangle = \langle ab_i, a\rangle \cong \mathrm{C}_p^2$. Thus $\langle b_i, a\rangle$ is of maximal class.

Case 2 $N \le \langle b_i, a\rangle$ for all $1 \le i \le n$.

In this case, $A_i = \langle b_i, a\rangle$ and $|A_i| \ge p^3$. If A_i is abelian, then $|A_i| = p^2$ by a same argument as that of paragraph above. This is a contradiction. Hence A_i is nonabelian. Thus A_i is a group of maximal class by Lemma 4.1. Let $|B_i| = p^{m_i}$. Since B_i is an abelian subgroup of index p of A_i, $m_i \ge 2$. Without loss of generality assume that m_1 is minimum among m_i, where $1 \le i \le n$.

Since $A_i = \langle b_i, a\rangle$ is a group of maximal class of order p^{m_i+1}, $N = Z(A_i) = (A_i)_{m_i}$. It follows that $N = \langle [b_i, (m_i - 1)a]\rangle$ for all $1 \le i \le n$. Without loss of generality assume

$$[b_1, (m_1 - 1)a] = [b_2, (m_2 - 1)a].$$

Let $b = [b_2, (m_2 - m_1)a]$. Then $[b_1 b^{-1}, (m_1 - 1)a] = 1$.

Let $A = \langle b_1 b^{-1}, a\rangle$. If A is nonabelian, then, by Lemma 4.1, A is of maximal class. Since $A_{m_1} = 1$, $c(A) \le m_1 - 1$. It follows that $|A| \le p^{m_1}$. If $A = \langle b_1 b^{-1}, a\rangle$ is abelian, then, since $a, b_1 b^{-1} a \in G\backslash H$, $o(a) = o(b_1 b^{-1} a) = p$ by hypothesis. Hence $A \cong C_p^2$. It follows that $|A| = p^2 \le p^{m_1}$. In any case, we have $|A| \le p^{m_1}$.

Let $B = A \cap H$. Then $|B| \le p^{m_1 - 1} < |B_1|$. Let $H_1 = B_2 B_3 \cdots B_n$. Then $G = (BH_1)\langle a\rangle$. It follows that $BH_1 = H$. Since $B_1 H_1 = H$ and $B_1 \cap H_1 \le N$, $B \cap H_1 = 1$ by comparing the order. Thus $H = B \times H_1$. The conclusion follows by induction hypothesis. □

Lemma 4.3 *Let G be a group, N a normal subgroup of G. Then for any $a \in G, b \in N$,*

$$(ab)^k = a^k \cdot f(a, b, k), \quad \text{where } f(a, b, k) = b^{a^{k-1}} \cdot b^{a^{k-2}} \cdots b^a \cdot b.$$

In particular, if N is abelian and $b_1, b_2 \in N$, then $f(a, b_1 b_2, k) = f(a, b_1, k) f(a, b_2, k)$.

Proof It follows by a simple calculation. □

Lemma 4.4 *Assume $G = H \rtimes \langle a\rangle$ with $\exp(G) > p$, where $H = B_1 \times B_2 \times \cdots \times B_n$ is abelian and $o(a) = p$. If B_i is an abelian subgroup of index p of $\langle B_i, a\rangle$ and all elements of $\langle B_i, a\rangle \setminus B_i$ are of order p for all $1 \le i \le n$, then $H = H_p(G)$. In particular, G is an $\mathcal{P}_1$-group.*

Proof Assume $g^p = 1$ for every $g \in G \setminus H$. Then $H \ge H_p(G)$. On the other hand, since H is abelian, we may assume

$$H = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_t\rangle, \quad \text{where } o(a_1) \ge o(a_2) \ge \cdots \ge o(a_t).$$

Since $\exp(G) > p$, $\exp(H) > p$. It follows that $o(a_1) > p$. Obviously,

$$H = \langle a_1, a_1 a_2, \ldots, a_1 a_t\rangle.$$

This means H can be generated by its elements of order $> p$. It follows that $H \le H_p(G)$. Thus $H = H_p(G)$. Since $H_p(G)$ is an abelian subgroup of index p of G, by Theorem 3.8, we get G is an $\mathcal{P}_1$-group.

Now it suffices to show that $g^p = 1$ for every $g \in G \setminus H$. Let $g = a^s b_1 b_2 \cdots b_n$, where $b_i \in B_i$ and $(s, p) = 1$. Since H is an abelian normal subgroup of G and $o(a) = p$, we get by Lemma 4.3

$$g^p = (a^s b_1 b_2 \cdots b_n)^p = a^{sp} \cdot f(a^s, b_1 b_2 \cdots b_n, p)$$

$$= f(a^s, b_1 b_2 \cdots b_n, p) = f(a^s, b_1, p) f(a^s, b_2, p) \cdots f(a^s, b_n, p).$$

We will prove $f(a^s, b_i, p) = 1$ for all $1 \le i \le n$. Since B_i is a subgroup of index p of $\langle B_i, a\rangle$, $B_i \trianglelefteq \langle B_i, a\rangle$. We get by Lemma 4.3 that

$$(a^s b_i)^p = a^{sp} f(a^s, b_i, p) = f(a^s, b_i, p).$$

Obviously, $a^s b_i \in \langle B_i, a\rangle \setminus B_i$. Thus $(a^s b_i)^p = 1$. That is, $f(a^s, b_i, p) = 1$. It follows that $g^p = 1$. □

Now, by Theorem 3.8, Lemmas 4.2 and 4.4, we have the following:

Theorem 4.5 *Assume G is a finite nonabelian p-group and p an odd prime. Then $G \in \mathcal{P}_1$ if and only if G is one of the following groups*:

(1) *nonabelian groups with* $\exp(G) = p$;

(2) $G = H_p(G) \rtimes \langle a\rangle$, *a semidirect product of $H_p(G)$ by $\langle a\rangle$, where $H_p(G) = B_1 \times B_2 \times \cdots \times B_n$ is an abelian subgroup of index p and $o(a) = p$. Moreover, for all $1 \le i \le n$, $B_i\langle a\rangle$ is a group of maximal class with an abelian subgroup B_i of index p, and all elements of $B_i\langle a\rangle \backslash B_i$ are of order p.*

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