Acta Mathematica Sinica, English Series
Mar., 2019, Vol. 35, No. 3, pp. 427–437
Published online: October 30, 2018
https://doi.org/10.1007/s10114-018-7519-6
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

2-Local Automorphisms on Basic Classical Lie Superalgebras

Li YU Ying WANG[1)]
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024*, P. R. China*
E-mail: 791742592*@mail.dlut.edu.cn* *wangying@dlut.edu.cn*

Hai Xian CHEN
School of Mathematical Sciences, Shanxi University, Taiyuan 030006*, P. R. China*
E-mail: 290804797*@qq.com*

Ji Zhu NAN
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024*, P. R. China*
E-mail: *jznan@*163*.com*

Abstract Let G be a basic classical Lie superalgebra except $A(n,n)$ and $D(2,1,\alpha)$ over the complex number field $\mathbb{C}$. Using existence of a non-degenerate invariant bilinear form and root space decomposition, we prove that every 2-local automorphism on G is an automorphism. Furthermore, we give an example of a 2-local automorphism which is not an automorphism on a subalgebra of Lie superalgebra spl$(3,3)$.

Keywords Basic classical Lie superalgebra, 2-local automorphism, automorphism

MR(2010) Subject Classification 17A36, 17B20, 17B40

1 Introduction

In 1997, Šemrl introduced the concepts of 2-local automorphism and 2-local derivation on associative algebras in [13]. Let A be an associative algebra, a map $\Phi : A \to A$ is called a 2-local automorphism if for $\forall x, y \in A$, there is an automorphism $\Phi_{x,y} : A \to A$ such that $\Phi(x) = \Phi_{x,y}(x)$ and $\Phi(y) = \Phi_{x,y}(y)$. A map $T : A \to A$ is called a 2-local derivation if for $\forall x, y \in A$, there is a derivation $\delta_{x,y} : A \to A$ such that $T(x) = \delta_{x,y}(x)$ and $T(y) = \delta_{x,y}(y)$. 2-Local automorphisms and 2-local derivations are close to automorphisms and derivations. P. Šemrl studied 2-local automorphisms on the algebra B(H) of all bounded linear operators on the infinite-dimensional separable Hilbert space H and proved that every 2-local automorphism on B(H) is an automorphism [13]. Later, the mathematicians considered the similar problems on semi-finite von Neumann algebras and von Neumann algebras and they obtained the same conclusions on these algebras (see [1, 2, 5, 6]). Afterwards, the similar problems were extended for non-associative algebras, in particular, Lie algebras and Lie superalgebras.

Received December 5, 2017, accepted August 3, 2018
The second author is supported by the National Natural Science Foundation of China (Grant No. 11471090)
1) Corresponding author

Recently in [4], the authors proved that every 2-local derivation on a semi-simple Lie algebra is a derivation and gave an example of 2-local derivations which are not a derivation on finite-dimensional nilpotent Lie algebras with dimension larger than two. In [8], Chen and Wang initiated the study of 2-local automorphisms on finite-dimensional simple Lie algebras. They proved that every 2-local automorphism is an automorphism on a simple Lie algebra of type $A(l), l \geq 1$, $D(l), l \geq 4$, $E_k(k = 6, 7, 8)$ over an algebraically closed field of characteristic zero. In [3], authors obtained the same result for finite-dimensional semi-simple Lie algebras and gave example of nilpotent Lie algebras which admit 2-local automorphisms which are not automorphisms. In the present paper, we consider the same problem for Lie superalgebras.

In 1975, Kac in [11] presented the classification of finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero: classical Lie superalgebras and the Lie superalgebras of Cartan type. The classical Lie superalgebras include basic classical Lie superalgebras and two strange series $P(n)$ and $Q(n)$. The properties of basic classical Lie superalgebras are similar to those of semi-simple Lie algebras. The notions of a 2-local superderivation and a 2-local automorphism for Lie superalgebras are defined similar to the associative case. In [9, 16], authors proved that every local superderivation (2-local superderivation) on basic classical Lie superalgebras except $A(n, n)$ over the complex numbers field $\mathbb{C}$ is a superderivation. Furthermore, they gave examples of Lie superalgebras with local superderivations (2-local superderivations) which are not superderivations. The present paper is devoted to the study of 2-local automorphisms of basic classical Lie superalgebras.

The Killing forms of classical Lie superalgebras are different from those of semi-simple Lie algebras. The Killing forms of classical Lie superalgebras are non-degenerate or equal to zero. If a basic classical Lie superalgebra has non-degenerate Killing form, then it is isomorphic to a simple Lie algebra or is isomorphic to one of $A(m, n)(m \neq n), B(m, n), D(m, n)(m - n \neq 1), C(n), G(3), F(4)$. In Section 3, we describe the 2-local automorphisms on these basic classical Lie superalgebras. Since the Killing form of Lie superalgebra $D(n + 1, n)$ is zero, we study its 2-local automorphisms in Section 4 by trace form $b(x, y) = \mathrm{str}(xy)$. The case of 2-local automorphisms of classical Lie superalgebras $A(n, n)$, $D(2, 1, \alpha)$, $P(n)$ and $Q(n)$ are still an unsolved problem. In Section 5, we give an example of a 2-local automorphism which is not an automorphism on a subalgebra of Lie superalgebra $\mathrm{spl}(3, 3)$.

We denote by $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{Z}_2$ the sets of all integers, complex numbers and residue classes modulo 2, respectively. The Lie superalgebras in this paper are all finite-dimensional over the complex numbers field $\mathbb{C}$.

2 Preliminaries

In this section, we recall the definition and significant properties of basic classical Lie superalgebras.

Definition 2.1 ([12]) *Let $b(,)$ be a bilinear form on finite-dimensional Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$.*

(1) *If $\forall x \in G_\alpha, y \in G_\beta, \alpha, \beta \in \mathbb{Z}_2$, $b(x, y) = (-1)^{\alpha\beta} b(y, x)$, then $b(,)$ is called supersymmetric.*

(2) *If $\forall x, y, z \in G$, $b([x, y], z) = b(x, [y, z])$, then $b(,)$ is called invariant.*

(3) *If* $\{x \mid b(x,y)=0, \forall y\in G\}=\{0\}$, *then* $b(,)$ *is called non-degenerate.*

(4) *If* $b(G_{\bar 0},G_{\bar 1})=0$, *then* $b(,)$ *is called even; if* $b(G_{\bar i},G_{\bar i})=0$, $i\in\mathbb{Z}_2$, *then* $b(,)$ *is called odd.*

Definition 2.2 ([12]) *A finite-dimensional Lie superalgebra* $G=G_{\bar 0}\oplus G_{\bar 1}$ *is called a basic classical Lie superalgebra if*

(1) G *is a simple Lie superalgebra;*

(2) $G_{\bar 0}$ *is a reductive Lie algebra;*

(3) *there exists a non-degenerate even supersymmetric invariant bilinear form on* G.

Proposition 2.3 ([12]) *Let* G *be a finite-dimensional basic classical Lie superalgebra. Then either* G *is a simple Lie algebra or* G *is isomorphic to one of the following algebras:*

$A(m,n)=\mathrm{spl}(m+1,n+1), m\neq n, m,n\geq 0,$

$A(n,n)=\mathrm{spl}(n+1,n+1), n\geq 1,$

$B(m,n)=\mathrm{osp}(2m+1,2n), m\geq 0,\ n\geq 1,$

$C(n)=\mathrm{osp}(2,2n-2), n\geq 2,$

$D(m,n)=\mathrm{osp}(2m,2n), m\geq 2,\ n\geq 1,$

$D(2,1;a), a\neq 0,-1,$

$G(3),$

$F(4).$

Proposition 2.4 ([12]) *Let* G *be a simple Lie superalgebra.*

(1) *Any invariant bilinear form on* G *either is non-degenerate or equals to zero.*

(2) *Any invariant bilinear form on* G *is super-symmetric.*

(3) *Any two non zero invariant bilinear forms on* G *are proportional.*

(4) *The invariant bilinear forms on* G *are either all even or all odd.*

Definition 2.5 ([12]) *Let* G *be a finite-dimensional simple Lie superalgebra. For* $g\in G$, *if* $g=\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\mathrm{gl}(m,n)$, *we define the supertrace of* g, $\mathrm{str}(g)$, *by* $\mathrm{str}(g)=\mathrm{tr}(A)-\mathrm{tr}(D)$ *and define the Killing form of* G *by* $\kappa(x,y)=\mathrm{str}(\mathrm{ad}x\mathrm{ad}y), \forall x,y\in G$.

Let G be a basic classical Lie superalgebra and H_0 be the Cartan subalgebra of $G_{\bar 0}$. Set

$$G^{\alpha}=\{x\in G \mid [h_0,x]=\alpha(h_0)x, \forall h_0\in H_0\},$$

then $\Delta=\{\alpha\in H_0^* \mid \alpha\neq 0, G^{\alpha}\neq 0\}$ is the set of roots of G. Evidently, $\Delta=\Delta_{\bar 0}\cup\Delta_{\bar 1}$ where $\Delta_{\bar 0}$ is the root system of $G_{\bar 0}$, $\Delta_{\bar 1}$ is the weight system of $G_{\bar 0}$ on $G_{\bar 1}$. Since the action of H_0 on any finite-dimensional simple $G_{\bar 0}$-module is diagonalizable, there is a root space decomposition

$$G=H\oplus\left(\bigoplus_{\alpha\in\Delta}G^{\alpha}\right),$$

where $H=G^0$ is the centralizer of H_0 in G.

Proposition 2.6 ([12]) *Let* G *be a basic classical Lie superalgebra. Then* $\dim G^{\alpha}=1$ *and* $H=H_0$.

We can represent the root vector of the root α by $e_{\alpha}\neq 0$ for $\alpha\in\Delta$.

Definition 2.7 *Let* G *be a finite-dimensional Lie superalgebra. A map* $\Psi: G\to G$ *is called a 2-local automorphism if for* $\forall x,y\in G$, *there exists an automorphism* $\Phi_{x,y}: G\to G$ *such that* $\Phi_{x,y}(x)=\Psi(x)$ *and* $\Phi_{x,y}(y)=\Psi(y)$.

Definition 2.8 ([10, 14]) *Let G be a finite-dimensional complex simple Lie superalgebra and Ω be the component connected of the identity of a Lie group, with Lie algebra $G_{\bar{0}}$. We call Ω the inner automorphisms group of G.* $\mathrm{Out}(G) = \mathrm{Aut}(G)/\Omega$ *is called the outer automorphisms group.*

3 Basic Classical Lie Superalgebras with Non-degenerate Killing Forms

We know that the Killing form of a simple Lie superalgebra G either is non-degenerate or equals to zero. If $G = A(m, n)$, then $\kappa(x, y) = (m - n)\mathrm{str}(xy)$. Thus the Killing form of $A(m, m)$ equals to zero. If a basic classical Lie superalgebra G has the non-degenerate invariant Killing form, then it is isomorphic to a simple Lie algebra or is isomorphic to one of $A(m, n)(m \neq n), B(m, n), D(m, n)(m - n \neq 1), C(n), G(3), F(4)$ in [15]. In this section, G denotes one of these algebras. The main result is Theorem 3.1.

Theorem 3.1 *If G is a basic classical Lie superalgebra with non-degenerate Killing form, then every 2-local automorphism on G is an automorphism.*

The proof of Theorem 3.1 consists of several lemmas.

Lemma 3.2 *There exists an element $d \in H$ such that $\alpha(d) \neq \beta(d)$ for $\forall \alpha, \beta \in \Delta \cup \{0\}$ and $\alpha \neq \beta$. d is called a strongly regular element of G.*

Proof The proof is similar to Lemma 2.2 of [7]. Since the restriction to H_0 of the Killing form is non-degenerate, there exists a bijection $H_0^* \to H_0$, $\phi \mapsto h_\phi$ satisfying $\phi(h) = \kappa(h_\phi, h)$ for $\forall h \in H_0$ and we may transfer the Killing form to H_0^* by $(\gamma, \delta) = \kappa(h_\gamma, h_\delta)$. Let

$$D = \{\gamma - \delta \mid \gamma, \delta \in \Delta \cup \{0\}, \gamma \neq \delta\}.$$

Obviously, D is a finite set. For $\alpha \in D$, there exists a hyperplane

$$P_\alpha = \{\gamma \in H_0^* \mid (\gamma, \alpha) = 0\}.$$

Then we have

$$\mathrm{dim} P_\alpha = \mathrm{dim} H_0^* - 1.$$

It is generally acknowledged truth that the union of finitely many hyperplanes can't exhaust H_0^*. So we have

$$\gamma_0 \in H_0^* - \bigcup_{\alpha \in D} P_\alpha,$$

i.e.,

$$(\gamma_0, \alpha) \neq 0, \quad \forall \alpha \in D.$$

There exists corresponding $h_{\gamma_0} \in H_0$, such that

$$\alpha(h_{\gamma_0}) = \kappa(h_{\gamma_0}, h_\alpha) = (\gamma_0, \alpha) \neq 0, \quad \alpha \in D.$$

By the definition of D, we know that $d = h_{\gamma_0}$ is a strongly regular element. □

Lemma 3.3 *If d is a strongly regular element of G, then $\{x \in G \mid [x, d] = 0\} = H$.*

Proof $d \in H$ implies that $H \subseteq C_G(d)$. Conversely, let $y \in C_G(d)$, then y can be written as

$$y = \sum_{\beta \in \Delta} a_\beta e_\beta + h, \quad h \in H.$$

We obtain $[y,d] = -\sum_{\beta\in\Delta} a_\beta\beta(d)e_\beta = 0$. By Lemma 3.2, we know that $\beta(d) \neq 0$. Thus $a_\beta = 0, y \in H$. □

Lemma 3.4 *The Killing form is* $\mathrm{Aut}(G)$*-invariant.*

Proof Recall the definition of the Killing form on G, $\kappa(x,y) = \mathrm{str}(\mathrm{ad}x\mathrm{ad}y)$. Let $\Phi \in \mathrm{Aut}(G)$. Because $\deg(\Phi) = \bar{0}$, we can choose homogeneous elements as the basis of G. The matrix of Φ in the basis is a block diagonal matrix, i.e.,

$$\Phi = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

It is obvious that A, D are invertible matrices. It is easy to see that

$$\mathrm{ad}\Phi(x) = \Phi \circ \mathrm{ad}x \circ \Phi^{-1}.$$

Then

$$\begin{aligned} \kappa(\Phi(x),\Phi(y)) &= \mathrm{str}(\mathrm{ad}\Phi(x)\mathrm{ad}\Phi(y)) \\ &= \mathrm{str}(\Phi\mathrm{ad}x\Phi^{-1}\Phi\mathrm{ad}y\Phi^{-1}) \\ &= \mathrm{str}(\Phi\mathrm{ad}x\mathrm{ad}y\Phi^{-1}). \end{aligned} \tag{3.1}$$

Let $\left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix}\right)$ represent the matrix corresponding to $\mathrm{ad}x\mathrm{ad}y$. Then

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} AMA^{-1} & AND^{-1} \\ DCA^{-1} & DQD^{-1} \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathrm{str}(\Phi\mathrm{ad}x\mathrm{ad}y\Phi^{-1}) &= \mathrm{tr}(AMA^{-1}) - \mathrm{tr}(DQD^{-1}) = \mathrm{tr}(M) - \mathrm{tr}(Q) \\ &= \mathrm{str}(\mathrm{ad}x\mathrm{ad}y) = \kappa(x,y). \end{aligned}$$

By (3.1), we have $\kappa(\Phi(x),\Phi(y)) = \kappa(x,y)$. □

Lemma 3.5 *Let* $\Phi \in \mathrm{Aut}(G)$ *and* d *be a strongly regular element of* G *such that* $\Phi(d) = d$.

(a) *If* $\alpha \in \Delta$, *then* $\Phi(e_\alpha) = c_\alpha e_\alpha, c_\alpha \in \mathbb{C}^*$.

(b) *If* $h \in H$, *then* $\Phi(h) = h$.

Proof (a) For $\alpha \in \Delta$, according to the root space decomposition, we can represent the element $\Phi(e_\alpha)$ in the form

$$\Phi(e_\alpha) = h + \sum_{\beta\in\Delta} c_\beta e_\beta, \tag{3.2}$$

where $h \in H$, $c_\beta \in \mathbb{C}$. By $\Phi(d) = d$, we have

$$[d,\Phi(e_\alpha)] = [\Phi(d),\Phi(e_\alpha)] = \Phi([d,e_\alpha]) = \Phi(\alpha(d)e_\alpha) = \alpha(d)\Phi(e_\alpha),$$

i.e.,

$$[d,\Phi(e_\alpha)] = \alpha(d)\Phi(e_\alpha). \tag{3.3}$$

Plug (3.2) into equation (3.3), we have

$$\sum c_\beta\beta(d)e_\beta = \alpha(d)h + \sum c_\beta\alpha(d)e_\beta. \tag{3.4}$$

Since d is a strongly regular element of G, we have $\alpha(d) \neq 0, \beta(d) \neq \alpha(d)$ for any $\alpha \neq \beta$. Thus by the equality (3.4), we obtain $h = 0, c_\beta = 0, \forall \beta \neq \alpha, \alpha, \beta \in \Delta$. So $\Phi(e_\alpha) = c_\alpha e_\alpha$, $c_\alpha \neq 0$.

(b) For arbitrary $h \in H$, by $[d, \Phi(h)] = [\Phi(d), \Phi(h)] = \Phi([d, h]) = \Phi(0) = 0$ and Lemma 3.3, we have $\Phi(h) \in H$. Thus for $\alpha \in \Delta$,

$$[\Phi(h), \Phi(e_\alpha)] = [\Phi(h), c_\alpha e_\alpha] = c_\alpha \alpha(\Phi(h)) e_\alpha.$$

On the other hand, we have

$$[\Phi(h), \Phi(e_\alpha)] = \Phi([h, e_\alpha]) = \Phi(\alpha(h) e_\alpha) = \alpha(h) c_\alpha e_\alpha.$$

Thus

$$c_\alpha \alpha(\Phi(h)) = \alpha(h) c_\alpha$$

which implies that

$$\alpha(\Phi(h)) = \alpha(h), \quad \forall \alpha \in \Delta.$$

Therefore $\Phi(h) = h$. □

Lemma 3.6 *Let Ψ be a 2-local automorphism on G and d be a strongly regular element of G such that $\Psi(d) = d$. Then*

(a) $\Psi(h) = h$ *for* $\forall h \in H$,

(b) $\Psi(e_\alpha) = c_\alpha e_\alpha, c_\alpha \in \mathbb{C}^*$ *for* $\forall \alpha \in \Delta$.

Proof (a) Since Ψ is a 2-local automorphism on G, for arbitrary $h \in H$, there exists an automorphism $\Phi_{h,d}$ such that

$$\Phi_{h,d}(h) = \Psi(h), \quad \Phi_{h,d}(d) = \Psi(d).$$

Therefore

$$\Phi_{h,d}(d) = \Psi(d) = d.$$

By Lemma 3.5, we have

$$\Psi(h) = \Phi_{h,d}(h) = h.$$

(b) For arbitrary $\alpha \in \Delta$, we can take an automorphism Φ_{d,e_α} such that

$$\Phi_{d,e_\alpha}(d) = \Psi(d), \quad \Phi_{d,e_\alpha}(e_\alpha) = \Psi(e_\alpha).$$

By Lemma 3.5

$$\Phi_{d,e_\alpha}(e_\alpha) = c_\alpha e_\alpha.$$

Thus $\Psi(e_\alpha) = c_\alpha e_\alpha$. □

Lemma 3.7 *Let Ψ be a 2-local automorphism on G and d be a strongly regular element such that $\Psi(d) = d$. Then Ψ is linear.*

Proof Because Killing form is Aut(G)-invariant, for any $x, y \in G$, we have

$$\kappa(\Psi(x), \Psi(y)) = \kappa(\Phi_{x,y}(x), \Phi_{x,y}(y)) = \kappa(x, y).$$

For any $x, y, z \in G$, we have

$$\begin{aligned}\kappa(\Psi(x+y), \Psi(z)) &= \kappa(x+y, z) = \kappa(x, z) + \kappa(y, z) \\ &= \kappa(\Psi(x), \Psi(z)) + \kappa(\Psi(y), \Psi(z)) \\ &= \kappa(\Psi(x) + \Psi(y), \Psi(z)),\end{aligned}$$

i.e.,

$$\kappa(\Psi(x+y),\Psi(z))=\kappa(\Psi(x)+\Psi(y),\Psi(z)).$$

Furthermore

$$\kappa(\Psi(x+y)-\Psi(x)-\Psi(y),\Psi(z))=0.$$

Set $z=e_\alpha$ and $z=h\in H$, respectively, then by Lemma 3.6, we have

$$\kappa(\Psi(x+y)-\Psi(x)-\Psi(y),e_\alpha)=0$$

and

$$\kappa(\Psi(x+y)-\Psi(x)-\Psi(y),h)=0.$$

Thus for all $\omega\in G$, we have

$$\kappa(\Psi(x+y)-\Psi(x)-\Psi(y),\omega)=0.$$

Since the Killing form is non-degenerate on G, we obtain

$$\Psi(x+y)=\Psi(x)+\Psi(y),\quad \forall x,y\in G.$$

Finally,

$$\Psi(\lambda x)=\Phi_{\lambda x,x}(\lambda x)=\lambda\Phi_{\lambda x,x}(x)=\lambda\Psi(x).$$

Thus Ψ is linear. □

Next, we prove Theorem 3.1 in this section.

Proof Let Ψ be a 2-local automorphism on G and d be a strongly regular element. Put $q=\sum_{\alpha\in\Delta}e_\alpha$. By the definition of 2-local automorphisms, there exists an automorphism $\Phi_{d,q}$ such that

$$\Phi_{d,q}(d)=\Psi(d),\quad \Phi_{d,q}(q)=\Psi(q).$$

Set $\Psi'=\Phi_{d,q}^{-1}\circ\Psi$, then Ψ' is a 2-local automorphism such that

$$\Psi'(d)=d,\quad \Psi'(q)=q.$$

By Lemma 3.7, we obtain that Ψ' is linear. Take into account Lemma 3.6, we have

$$\Psi'(q)=\Psi'\Big(\sum e_\alpha\Big)=\sum\Psi'(e_\alpha)=\sum c_\alpha e_\alpha. \tag{3.5}$$

On the other hand,

$$\Psi'(q)=q=\sum e_\alpha. \tag{3.6}$$

Comparing the equality (3.5) with (3.6), we have $c_\alpha=1$ for all $\alpha\in\Delta$. We can get $\Psi'(x)=x,\forall x\in G$, i.e.,

$$\Psi'=\mathrm{Id}|_G.$$

Thus $\Psi=\Phi_{d,q}$ is an automorphism. □

4 The Lie Superalgebra $D(n+1,n)$

We know that the Killing form of $D(n+1,n)$ is zero. Taking $b(x,y)=\mathrm{str}(xy)$ as a non-degenerate even invariant bilinear form on $D(n+1,n)$, we prove that $b(,)$ is $\mathrm{Aut}(D(n+1,n)$-invariant.

Let GL_n, SO_n, SP_n denote the general linear groups, special orthogonal groups and symplectic groups, respectively.

$$\mathrm{gl}(m,n):=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A\in\mathbb{C}^{m\times m}, D\in\mathbb{C}^{n\times n}, B\in\mathbb{C}^{m\times n}, C\in\mathbb{C}^{n\times m}\right\}.$$

Let $\mathrm{gl}(n)$ denote the general linear Lie algebra. There is a natural homorphism $\mathrm{Ad}:\mathrm{GL}_n\to\mathrm{Aut}(\mathrm{gl}(n))$, where $\mathrm{Ad}X: A\mapsto XAX^{-1}$, $X\in\mathrm{GL}_n, A\in\mathrm{gl}(n)$.

For $(X,Y)\in\mathrm{GL}_m\times\mathrm{GL}_n$, we define $\mathrm{Ad}(X,Y):\mathrm{gl}(m,n)\to\mathrm{gl}(m,n)$ by

$$\mathrm{Ad}(X,Y)(G)=\mathrm{diag}(X,Y)\cdot G\cdot\mathrm{diag}(X,Y)^{-1},\quad G\in\mathrm{gl}(m,n).$$

This yields a group homorphism $\mathrm{Ad}:\mathrm{GL}_m\times\mathrm{GL}_n\to\mathrm{Aut}(\mathrm{gl}(m,n))$.

By Table 4.1 of [10], for $D(n+1,n)=\mathrm{osp}(2(n+1),2n)$, we know $\Omega\cong\mathrm{SO}_{2(n+1)}\times\mathrm{SP}_{2n}/u_2$, where $\ker\mathrm{Ad}=u_2=(-I_{2n+2},-I_{2n})$. This show that every inner automorphism is in the form $\mathrm{Ad}(X,Y)$ where X,Y are invertible matrices.

Let $B_{2k,n}=\mathrm{diag}(I_{2k},J_n)$, where $J_n=\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0\end{smallmatrix}\right)$. By Theorem 1 of [14], we know $\mathrm{Out}(G)\cong\mathrm{Ad}(J_{n+1,n})$ for $G=D(n+1,n)$, where $J_{n+1,n}\in\mathrm{gl}(2n+2,2n)$ with $\det(J_{n+1,n})=-1$, $J_{n+1,n}^2=I_{2(n+1)+2n}$, $J_{n+1,n}B_{2(n+1),n}J_{k,n}=B_{2(n+1),n}$. By simple calculation one can verify that $J_{n+1,n}$ is a block diagonal matrix. Let $J_{n+1,n}=\left(\begin{smallmatrix} X & O \\ O & Y\end{smallmatrix}\right)$, $X\in GL_{2n+2}, Y\in GL_{2n}$. For $A\in D(n+1,n)$, we have

$$\mathrm{Ad}(J_{n+1,n})(A)=(J_{n+1,n})A(J_{n+1,n})^{-1}=\mathrm{Ad}(X,Y)(A).$$

Therefore every automorphism on $D(n+1,n)$ is the form $\mathrm{Ad}(X,Y)$ where X, Y are invertible matrices.

Lemma 4.1 *$b(,)$ is $\mathrm{Aut}(D(n+1,n))$-invariant.*

Proof For $\Phi\in\mathrm{Aut}(D(n+1,n))$, we know that Φ is $\mathrm{Ad}(X,Y)$ for some invertible matrices X and Y. Set

$$x=\begin{pmatrix} A & B \\ C & D \end{pmatrix},\quad \tilde{x}=\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}\in D(n+1,n).$$

Then

$$\begin{aligned} b(x,\tilde{x})=\mathrm{str}(x\tilde{x}) &= \mathrm{tr}(A\tilde{A}+B\tilde{C})-\mathrm{tr}(C\tilde{B}+D\tilde{D}) \\ &= \mathrm{tr}(X(A\tilde{A}+B\tilde{C})X^{-1})-\mathrm{tr}(Y(C\tilde{B}+D\tilde{D})Y^{-1}) \\ &= \mathrm{Str}(\mathrm{diag}(X,Y)\cdot x\tilde{x}\cdot\mathrm{diag}(X,Y)^{-1})=\mathrm{Str}(\mathrm{Ad}(X,Y)(x)\cdot\mathrm{Ad}(X,Y)(\tilde{x})) \\ &= b(\Phi(x),\Phi(\tilde{x})). \end{aligned}$$

Therefore $b(,)$ on $D(n+1,n)$ is $\mathrm{Aut}(D(n+1,n))$-invariant. □

Lemma 4.2 *There exists a strongly regular element on $D(n+1,n)$.*

Proof The proof is similar to that of Lemma 3.2. □

Replacing Killing form with Trace form, we can prove the following Theorem 4.3 by the same method of Theorem 3.1.

Theorem 4.3 *Every 2-local automorphism on $D(n+1,n)$ is an automorphism.*

Corollary 4.4 *If G is a basic classical Lie superalgebras except $A(n,n)$ and $D(2,1,\alpha)$ over $\mathbb{C}$, then every 2-local automorphism on G is an automorphism.*

Proof This is the result of Theorem 3.1, Theorem 4.3 and References [3, 8]. □

5 A 2-local Automorphism on a Subalgebra of spl(3, 3)

In [3], the authors give an example of a 2-local automorphism which is not an automorphism of nilpotent Lie algebras. Referring to their method, we construct a non-nilpotent Lie superalgebra with its 2-local automorphism which is not an automorphism.

Suppose that G is a Lie superalgebra over $\mathbb{C}$. Let $Z(G)$ and $[G,G]$ denote the center and derived algebra of G, respectively. Let $\delta : G \to G$ be a linear map of degree $\alpha (\alpha \in \mathbb{Z}_2)$ such that $\delta|_{[G,G]} = 0$ and $\delta(G) \subseteq Z(G)$. Then δ is a superderivation.

Let S be a subalgebra of spl$(3,3)$ consisting of the following elements:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ d_1 & 0 & d_2 & 0 & 0 & d_4 \\ d_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_6 & d_7 & 0 \\ 0 & 0 & 0 & d_8 & -d_6 & 0 \\ d_5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $d_i \in \mathbb{C}, i = 1, \dots, 8$. We can obtain that $[S,S] = \mathbb{C}E_{21} + \mathbb{C}(E_{44} - E_{55}) + \mathbb{C}E_{45} + \mathbb{C}E_{54}$, $Z(S) = \mathbb{C}E_{21}$, where E_{ij} is a 6×6 matrix with 1 in the (i,j) position and 0 elsewhere. There exists a decomposition of S as follows:

$$S = [S,S] + \mathbb{C}E_{23} + \mathbb{C}E_{31} + \mathbb{C}E_{26} + \mathbb{C}E_{61}.$$

We define a function f on $\mathbb{C}^2$.

$$f(k_1,k_2) = \begin{cases} \dfrac{k_1^2}{k_2}, & \text{if } k_2 \neq 0, \\ 0, & \text{if } k_2 = 0, \end{cases}$$

where $k_1, k_2 \in \mathbb{C}$. Define a map T on S by

$$T(x) = f(k_1,k_2)E_{21}, \quad \text{for } x = x_1 + k_1E_{23} + k_2E_{31} + k_3E_{26} + k_4E_{61},$$

where $x_1 \in [S,S]$, $k_1, k_2, k_3, k_4 \in \mathbb{C}$.

Let

$$\Psi(x) = x + T(x), \quad x \in S.$$

It is obvious that Ψ is not an automorphism since it is not linear. In following sections we prove that Ψ is a 2-local automorphism.

Define a map δ on S by

$$\delta(x) = (ak_1 + bk_2)E_{21}, \quad \text{for } x = x_1 + k_1E_{23} + k_2E_{31} + k_3E_{26} + k_4E_{61},$$

where $x_1 \in [S,S]$, $k_1, k_2, k_3, k_4 \in \mathbb{C}$. δ is a superderivation of degree $\bar{0}$ since $\delta|_{[S,S]} = 0$ and $\delta(S) \subseteq Z(S)$.

Let $x = x_1 + k_1E_{23} + k_2E_{31} + k_3E_{26} + k_4E_{61}$ and $y = x_1' + k_1'E_{23} + k_2'E_{31} + k_3'E_{26} + k_4'E_{61}$ be the elements of S. Since the function f is homogeneous, the system of linear equations

$$\begin{cases} k_1a + k_2b = f(k_1, k_2), \\ k_1'a + k_2'b = f(k_1', k_2') \end{cases}$$

in variables a and b has a solution. Namely, there exists $\delta_{x,y}$ such that

$$T(x) = \delta_{x,y}(x), \quad T(y) = \delta_{x,y}(y).$$

Therefore, $\forall x, y \in S$ there exists a superderivation $\delta_{x,y}$ such that

$$\Psi(x) = x + T(x) = x + \delta_{x,y}(x), \quad \Psi(y) = y + T(y) = y + \delta_{x,y}(y).$$

Suppose $\phi_{x,y} = \mathrm{Id}|_S + \delta_{x,y}$, then $\phi_{x,y}$ is a linear map. For $\forall x', y' \in S$, $[\phi_{x,y}(x'), \phi_{x,y}(y')] = [x', y'] = \phi_{x,y}([x', y'])$. It is obvious that $\phi_{x,y}$ is bijective and $\deg(\phi_{x,y}) = \bar{0}$. Therefore $\phi_{x,y}$ is an automorphism. Namely, there exists an automorphism $\phi_{x,y}$ such that

$$\Psi(x) = \phi_{x,y}(x), \quad \Psi(y) = \phi_{x,y}(y).$$

So Ψ is a 2-local automorphism on S but it's not an automorphism.

Acknowledgements We thank the referees for their time and comments.

References

[1] Ayupov, Sh., Arzikulov, F: 2-Local derivations on semi-finite von Neumann algebras. *Glasg. Math. J.*, **56**(1), 9–12 (2014)

[2] Ayupov, Sh., Kudaybergenov, K.: 2-Local derivations on von Neumann algebras. *Positivity*, **19**(3), 445–455 (2015)

[3] Ayupov, Sh., Kudaybergenov, K.: 2-Local automorphisms on finite-dimensional Lie algebras. *Linear Algebra Appl.*, **507**, 121–131 (2016)

[4] Ayupov, Sh., Kudaybergenov, K., Rakhimov, I.: 2-Local derivations on finite-dimensional Lie algebras. *Linear Algebra Appl.*, **474**, 1–11 (2015)

[5] Burgos, M., Fernández-Polo, F., Garcés, J., et al.: 2-Local triple homomorphisms on von Neumann algebras and JBW*-triples. *J. Math. Anal. Appl.*, **426**(1), 43–63 (2015)

[6] Chen, L., Huang, L., Lu, F.: 2-Local Lie isomorphism of operator algebras on Banach spaces. *Studia Math.*, **229**(1), 1–11 (2015)

[7] Chen, Z., Wang, D.: Nonlinear maps satisfying derivability on standard parabolic subalgebras of finite-dimensional simple Lie algebras. *Linear Multilinear Algebra*, **59**(3), 261–270 (2011)

[8] Chen, Z., Wang, D.: 2-Local automorphisms of finite-dimensional simple Lie algebras. *Linear Algebra Appl.*, **486**, 335–344 (2015)

[9] Chen, H., Wang, Y., Nan, J.: Local superderivations on the basic classical Lie superalgebras. *Algebra Colloq.*, **24**(4), 673–684 (2018)

[10] Grantcharov, D., Pianzola, A.: Automorphisms and twisted loop algebras of finite-dimensional simple Lie superalgebras. *Int. Math. Res. Not.*, **73**, 3937–3962 (2004)

[11] Kac, V. G.: Lie superalgebras. *Adv. Math.*, **26**(1), 8–96 (1977)

[12] Musson, I.: Lie Superalgebras and Enveloping Algebras. Graduate Studies in Mathematics, 131. Providence, RI, 2012

[13] Šemrl, P.: Local automorphisms and derivations on B(H). *Proc. Amer. Math. Soc.*, **125**(9), 2677–2680 (1997)

[14] Serganova, V.: Automorphisms of simple Lie superalgebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, **48**(3), 585–598 (1984)

[15] Sun, H., Han, Q.: Lie algebras and Lie superalgebras and their applications in Physics (in Chinese), Peking University, Beijing, 1995

[16] Wang, Y., Chen, H., Nan, J.: 2-Local superderivations on basic classical Lie superalgebras. *J. Math. Res. Appl.*, **37**(5), 527–534 (2017)