Acta Mathematica Sinica, English Series
Aug., 2018, Vol. 34, No. 8, pp. 1259–1268
Published online: April 20, 2018
https://doi.org/10.1007/s10114-018-7409-y
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

On Compact Hermitian Manifolds with Flat Gauduchon Connections

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In Memory of Professor Qikeng Lu (1927–2015)

Abstract Given a Hermitian manifold (M^n, g), the Gauduchon connections are the one parameter family of Hermitian connections joining the Chern connection and the Bismut connection. We will call $\nabla^s = (1-\frac{s}{2})\nabla^c + \frac{s}{2}\nabla^b$ the s-Gauduchon connection of M, where ∇^c and ∇^b are respectively the Chern and Bismut connections. It is natural to ask when a compact Hermitian manifold could admit a flat s-Gauduchon connection. This is related to a question asked by Yau. The cases with $s = 0$ (a flat Chern connection) or $s = 2$ (a flat Bismut connection) are classified respectively by Boothby in the 1950s or by the authors in a recent joint work with Q. Wang. In this article, we observe that if either $s \geq 4+2\sqrt{3} \approx 7.46$ or $s \leq 4-2\sqrt{3} \approx 0.54$ and $s \neq 0$, then g is Kähler. We also show that, when $n = 2$, g is always Kähler unless $s = 2$. Therefore non-Kähler compact Gauduchon flat surfaces are exactly isosceles Hopf surfaces.

Keywords Hermitian manifolds, Hermitian connections, Kähler manifolds

MR(2010) Subject Classification 53C55

1 Introduction

Yau [13] asked an interesting question on Hermitian geometry.

Question 1.1 (Problem 87 in [13]) If the holonomy group of a compact Hermitian manifold can be reduced to a proper subgroup of $U(n)$, can we say something nontrivial about the manifold? The problem is that the connection need not to be Riemannian.

Recall that on a Hermitian manifold (M^n, J, g) where J is the complex structure on M^n, there are a lot of Hermitian connections, namely, linear connections ∇ satisfying $\nabla g = 0$ and $\nabla J = 0$.

An important special case of Question 1.1 is when the Hermitian connection under consideration is flat, hence its holonomy group is discrete. The question states:

Received September 8, 2017, accepted November 27, 2017

Question 1.2 Classify compact Hermitian manifolds which admit flat Hermitian connections.

One of the main difficulties in answering Question 1.2 is that the linear space of Hermitian connections on a general Hermitian manifold is of infinite dimension. Note that there are three Hermitian connections relatively well studied in the literature: the Chern connection ∇^c, the Bismut connection ∇^b (introduced in [1] and [12]), and ∇^{lv} which is the projection onto the holomorphic tangent bundle of the Riemannian (Levi–Civita) connection of g. The latter was also called the associated connection (see [6]), Levi–Civita connection (see [7, 8]), or the first canonical connection, etc. It is well known that these three connections lie on a straight line in the space of all Hermitian connections, namely, ∇^{lv} is the arithmetic average of the other two. In the rest of this paper, we will fix the following notation which is partly motivated by the work of Gauduchon [5]:

Definition 1.3 (Gauduchon connections) *The* s-Gauduchon connection *of* (M^n, g) *is defined to be the Hermitian connection* $\nabla^s = (1 - \frac{s}{2})\nabla^c + \frac{s}{2}\nabla^b$*, where* $s \in \mathbb{R}$.

So $\nabla^0 = \nabla^c$, $\nabla^2 = \nabla^b$, and $\nabla^1 = \nabla^{lv}$. When g is Kähler, every s-Gauduchon connection $\nabla^s \equiv \nabla^c$ and is equal to the Riemannian connection. When g is not Kähler, $\nabla^s \neq \nabla^{s'}$ whenever $s \neq s'$. From the differential geometric point of view, we want to study the curvature tensor R^s of ∇^s. As a first step to understand Question 1.2, we would like to know what kind of compact Hermitian manifolds can have flat s-Gauduchon connection ∇^s.

When $s = 0$, the question is well understood and well known. In 1958, Boothby [2] proved that compact Hermitian manifolds with flat Chern connection are exactly the quotients of complex Lie groups, equipped with left invariant Hermitian metrics. An important subset of this is the complex parallelizable manifolds, which are classified by Wang [9].

For $s = 2$, the recent joint work [10] of Wang and the authors classified all (including noncompact ones) Bismut flat manifolds. Those compact Bismut flat manifolds are exactly (finite undercover of) compact local Samelson spaces. In more details, given any compact Bismut flat manifold M^n, its universal cover is a Samelson space, namely, $G \times \mathbb{R}^k$ equipped with a bi-invariant metric and a left invariant complex structure. Here G is a simply-connected compact semisimple Lie group, and $0 \leq k \leq 2n$. In particular, compact non-Kähler Bismut flat surfaces are exactly those isosceles Hopf surfaces, and in dimension three, their universal cover is either a central Calabi–Eckmann threefold $S^3 \times S^3$, or $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$. The readers are referred to [10] for more details.

For $s = 1$, Ganchev and Kassabov [6] proved an interesting local characterization theorem for Hermitian manifolds with flat ∇^1, which they called the associated connection. More precisely, they showed that for any $n \geq 2$, if a Hermitian manifold (M^n, g) has flat ∇^1 and is conformal to a Kähler metric $\tilde{g} = e^{2u}g$, then $\tilde{g}$ has constant holomorphic sectional curvature. Conversely, given any Kähler metric $\tilde{g}$ with constant holomorphic sectional curvature, there always exists a Hermitian metric g which is conformal to $\tilde{g}$ such that g has flat ∇^1.

To sum up, we propose the following version of Question 1.2:

Conjecture 1.4 *If* $s \neq 0, 2$*, then any compact Hermitian manifold* (M^n, g) *which admits a flat s-Gauduchon connection must be Kähler, thus being a finite undercover of a flat complex torus.*

The main purpose of this article is to confirm the above conjecture in the $n = 2$ case, namely, we have the following

Theorem 1.5 *Let (M^2, g) be a compact Hermitian surface with a flat s-Gauduchon connection ∇^s. If $s \neq 2$, then g is Kähler.*

Therefore M^2 in the above theorem is either a flat complex torus or a flat hyperelliptic surface. Note that any Chern flat complex surface must be Kähler. Combining the result in [10] we conclude that the only non-Kähler surfaces which are s-Gauduchon flat are isosceles Hopf surfaces. For $n \geq 3$, we are able to prove the following:

Theorem 1.6 *Let (M^n, g) be a compact Hermitian manifold with a flat s-Gauduchon connection ∇^s. If either $s \geq 4 + 2\sqrt{3}$, or $s \leq 4 - 2\sqrt{3}$ and $s \neq 0$, then g is Kähler.*

In other words, if (M^n, g) is compact, s-Gauduchon flat with $s \neq 0$, and is non-Kähler, then s must lie in the interval $(4 - 2\sqrt{3}, 4 + 2\sqrt{3})$. Note however that this interval contains the interesting cases $s = 2$ (Bismut), $s = 1$, and $s = \frac{2}{3}$ (see below).

If we take the dimension into account, then the two constants $4 \pm 2\sqrt{3}$ can be slightly improved. For $n \geq 3$, let us denote by

$$a_n^{\pm} = \frac{1}{n}[4(n-1) \pm 2\sqrt{3n^2 - 7n + 4}], \quad b_n^{\pm} = \frac{1}{(n+1)}[4(n-1) \pm 2\sqrt{3n^2 - 8n + 5}].$$

Then we have $a_n^- < b_n^- < 0.6$ and both sequences (a_n^-) and (b_n^-) are monotonically decreasing and approaching $4 - 2\sqrt{3}$ when $n \to \infty$. Similarly, $3.4 < b_n^+ < a_n^+$, and both sequences (a_n^+) and (b_n^+) are monotonically increasing and approaching $4 + 2\sqrt{3}$ when $n \to \infty$. The statement of Theorem 1.6 can be slightly improved by

Theorem 1.7 *Let (M^n, g) be a compact Hermitian manifold with a flat s-Gauduchon connection ∇^s. If either $s > a_n^+$, or $s < a_n^-$ and $s \neq 0$, then g is Kähler.*

When the metric g is locally conformally Kähler, its torsion tensor takes a simple form, so the above type consideration leads to the conclusion that g will be Kähler for all values of s except possibly when $s = b_n^+$ or $s = b_n^-$. That is, we have

Theorem 1.8 *Let (M^n, g) be a compact Hermitian manifold with a flat s-Gauduchon connection ∇^s. Assume $n \geq 3$ and g is locally conformal Kähler. If $s \neq b_n^{\pm}$, then g is Kähler.*

The main idea of proving Theorem 1.5 is: when $n = 2$, the torsion 1-form η contains all the information about the torsion tensor. So the above type of consideration in Theorem 1.8 can be pushed further to lead to the Kählerness of g for all s values (other than 0 and 2) except one value: $s = \frac{2}{3}$. By a Bochner type argument, one can conclude the Kählerness in the case $s = \frac{2}{3}$ as well, thus proving Theorem 1.5.

Note that in all dimensions, the connection $\nabla^{\frac{2}{3}}$ distinguishes itself with the property that it has the smallest total torsion amongst all Gauduchon connections. For that reason, we will also call the $\frac{2}{3}$-Gauduchon connection $\nabla^{\frac{2}{3}}$ the *minimal Gauduchon connection.*

Recall that a Hermitian metric g is called *balanced,* if $d\omega^{n-1} = 0$, where ω is the Kähler form of g. g is said to be *Gauduchon* if $\partial\overline{\partial}\omega^{n-1} = 0$. A local property about Gauduchon flat manifolds worth mentioning is the following:

Proposition 1.9 *Let (M^n, g) be a Hermitian manifold with a flat s-Gauduchon connection. If $s = \frac{1}{2}$, then g is Kähler. Also, if $s \neq 0$ and g is balanced, then g is Kähler.*

Let us remark that a noncompact version of Conjecture 1.4 is much more subtle. If we focus on the Chern flat case ($s = 0$), there are example of noncompact Hermitian surfaces (incomplete ones in [2] and complete ones in [10]) without parallel Chern torsion, hence they do not come from quotients of complex Lie groups with left invariant metrics. On the other hand, noncompact (not necessarily complete) Bismut flat ($s = 2$) Hermitian manifolds have been classified in [10]. It is an interesting question if such a difference also exists for noncompact s-Gauduchon flat Hermitian manifolds when s is other than 0 and 2.

The paper is organized as follows. In Section 2, we collect some known results and fix the notations. In Section 3, we give proofs to Theorem 1.6 through Proposition 1.9. In Section 4, we prove Theorem 1.5.

2 Preliminaries

We begin with a Hermitian manifold (M^n, g). We will follow the notations of [11] for the most part. Denote by ∇, ∇^c the Riemannian (aka Levi–Civita) and the Chern connection, respectively. Denote by R, R^c the curvature tensors of these two connections, and by T^c the torsion tensor of ∇^c. Under a local unitary frame $\{e_1, \ldots, e_n\}$ of type $(1,0)$ tangent vectors, T^c has components

$$T^c(e_i, e_j) = \sum_{k=1}^{n} 2\, T_{ij}^k e_k, \quad T^c(e_i, \overline{e_j}) = 0.$$

Note the coefficient 2 above, which is unconventional but makes some of the subsequent formula simpler. We will write $e = {}^t(e_1, \ldots, e_n)$ as a column vector. Write $\varphi = {}^t(\varphi_1, \ldots, \varphi_n)$ the column vector of local $(1,0)$-forms that are dual to e. As in [11], let us write

$$\nabla^c e = \theta e, \quad \nabla e = \theta_1 e + \overline{\theta_2} \overline{e},$$

so θ and $\Theta = d\theta - \theta \wedge \theta$ are the matrices of connection and curvature of ∇^c under the unitary frame e, while $\hat{\theta}$ and $\hat{\Theta} = d\hat{\theta} - \hat{\theta} \wedge \hat{\theta}$ are the matrices of connection and curvature of ∇ under the frame $\{e, \overline{e}\}$, with

$$\hat{\theta} = \begin{bmatrix} \theta_1 & \overline{\theta_2} \\ \theta_2 & \overline{\theta_1} \end{bmatrix}, \qquad \hat{\Theta} = \begin{bmatrix} \Theta_1 & \overline{\Theta}_2 \\ \Theta_2 & \overline{\Theta}_1 \end{bmatrix}.$$

The structure equations are:

$$d\varphi = -\,{}^t\theta \wedge \varphi + \tau, \tag{1}$$

$$d\theta = \theta \wedge \theta + \Theta, \tag{2}$$

where τ is the column vector of the torsion 2-forms under the local frame e, and

$$d\varphi = -\,{}^t\theta_1 \wedge \varphi - {}^t\theta_2 \wedge \overline{\varphi}, \tag{3}$$

$$\Theta_1 = d\theta_1 - \theta_1 \wedge \theta_1 - \overline{\theta_2} \wedge \theta_2, \tag{4}$$

$$\Theta_2 = d\theta_2 - \theta_2 \wedge \theta_1 - \overline{\theta_1} \wedge \theta_2. \tag{5}$$

The entries of τ are $(2,0)$ forms, and the entries of Θ are all $(1,1)$ forms. Taking exterior differentiation of the above equations, we get the two Bianchi identities:

$$d\tau = -\,{}^t\theta \wedge \tau + {}^t\Theta \wedge \varphi, \tag{6}$$

$$d\Theta = \theta \wedge \Theta - \Theta \wedge \theta. \tag{7}$$

From [11], we know that when e is unitary, we have the following simple formula

$$\tau_k = \sum_{i,j=1}^{n} T_{ij}^k \varphi_i \wedge \varphi_j, \quad (\theta_2)_{ij} = \sum_{k=1}^{n} \overline{T_{ij}^k} \varphi_k, \quad \gamma'_{ij} = \sum_{k=1}^{n} T_{ik}^j \varphi_k, \tag{8}$$

where γ' is the $(1,0)$ part of $\gamma = \theta_1 - \theta = \gamma' - \gamma'^*$. We will also denote by $\eta = \mathrm{tr}(\gamma')$ the torsion 1-form, aka Gauduchon 1-form ([4]), so we have

$$\eta = \sum_{i=1}^{n} \eta_i \varphi_i = \sum_{i,k=1}^{n} T_{ki}^k \varphi_i, \tag{9}$$

$$\partial \omega^{n-1} = -2\ \eta \wedge \omega^{n-1}, \tag{10}$$

where $\omega = \sqrt{-1}\ {}^t\varphi \wedge \overline{\varphi}$ is the Kähler $(1,1)$-form. From the last equation above, we get

$$\partial\overline{\partial}\omega^{n-1} = 2\ (\overline{\partial}\eta + 2\eta \wedge \overline{\eta}) \wedge \omega^{n-1}. \tag{11}$$

By Lemma 2 of [10], we know that the matrix of connection for ∇^b under e is given by $\theta + 2\gamma$, therefore we obtain the following:

Lemma 2.1 *Given a Hermitian manifold (M^n, g), the matrix of connection forms for the s-Gauduchon connection under the frame e is given by*

$$\theta^s = \theta + s\gamma. \tag{12}$$

Next let us recall the conformal change formula. Let $\tilde{g} = e^{2u} g$ be a metric conformal to g, where u is a smooth real valued function. Locally we can take $\tilde{\varphi} = e^u \varphi$ and $\tilde{e} = e^{-u} e$, so $\tilde{e}$ is a local unitary frame for $(M^n, \tilde{g})$, with $\tilde{\varphi}$ its dual coframe.

Let $\tilde{\theta}$, $\widetilde{\Theta}$, and $\tilde{\tau}$ be respectively the matrices or column vector of the Chern connection, curvature, and torsion for the metric $\tilde{g}$ under the unitary frame $\tilde{e}$. From §5 of [11], we have the following:

$$\tilde{\theta} = \theta + (\partial u - \overline{\partial} u) I, \quad \widetilde{\Theta} = \Theta - 2\partial\overline{\partial} u I,$$

and

$$\tilde{\tau} = e^u (\tau + 2\partial u \wedge \varphi).$$

This leads to the following

$$\widetilde{T_{ij}^k} = e^{-u} [T_{ij}^k + u_i \delta_{jk} - u_j \delta_{ik}], \tag{13}$$

where $u_i = e_i(u)$.

Next, recall that $\eta_i = \sum_k T_{ki}^k$, and let us denote

$$|\eta|^2 = \sum_{i=1}^{n} |\eta_i|^2, \quad |T|^2 = \sum_{i,j,k=1}^{n} |T_{jk}^i|^2.$$

Both are independent of the choice of unitary frames thus are well defined global functions on the manifold M^n.

If $\tilde{g}$ is Kähler, namely, if $\tilde{\tau} = 0$, then we have

$$T_{ij}^k = u_j \delta_{ik} - u_i \delta_{jk}$$

for any indices i, j, k. This leads to the following:

Lemma 2.2 *If (M^n, g) is a Hermitian manifold that is locally conformally Kähler, then it holds*

$$|T|^2 = \frac{2}{n-1}|\eta|^2.$$

Proof From the identity right above the statement of the lemma, we know that $T^k_{ij} = 0$ when $k \notin \{i, j\}$, and $T^i_{ij} = u_j$ if $i \neq j$. So $\eta_j = (n-1)u_j$. From this, we get $|\eta|^2 = (n-1)^2|du|^2$, and $|T|^2 = 2(n-1)|du|^2$, where $|du|^2 = |u_1|^2 + \cdots + |u_n|^2$. Thus $|T|^2 = \frac{2}{n-1}|\eta|^2$, and the lemma is proved. □

3 The s-Gauduchon Flat Manifolds

Throughout this section, we will assume that (M^n, g) is a Hermitian manifold with flat s-Gauduchon connection ∇^s, where $s \neq 0$. For any $p \in M$, there always exists a unitary frame e in a neighborhood of p such that is e is ∇^s-parallel. That is, $\theta^s = 0$. Note that such a frame is unique up to changes by constant valued unitary matrices. Let us fix such a local frame e. Specializing the structure equations and Bianchi identities with our condition $\theta = -s\gamma$, and using the fact that ${}^t\gamma'\varphi = -\tau$, we get

$$\partial\varphi = (s-1)\ {}^t\gamma'\varphi, \tag{14}$$

$$\overline{\partial}\varphi = -s\ \overline{\gamma'}\varphi, \tag{15}$$

$$\partial\gamma' = -s\ \gamma'\gamma', \tag{16}$$

$${}^t\Theta = -s\ (\overline{\partial}\ {}^t\gamma' - \partial\overline{\gamma'}) - s^2(\overline{\gamma'}\ {}^t\gamma' +\ {}^t\gamma'\overline{\gamma'}), \tag{17}$$

where the third equation is because of the vanishing of the $(2,0)$-component of Θ and the fact that $s \neq 0$. The $(2,1)$-part of the first Bianchi identity $d\tau =\ {}^t\Theta\varphi -\ {}^t\theta\tau$ leads to

$$[\ (s-1)\ \overline{\partial}\ {}^t\gamma' - s\ \partial\overline{\gamma'} + s(s-1)\ (\ {}^t\gamma'\overline{\gamma'} + \overline{\gamma'}{}^t\gamma')]\varphi = 0. \tag{18}$$

Lemma 3.1 *Suppose a Hermitian manifold (M^n, g) is s-Gauduchon flat, where $s \neq 0$. Let e be a local ∇^s-parallel unitary frame. Then the torsion components satisfy*

$$T^\ell_{ij,k} - T^\ell_{ik,j} = \sum_{r=1}^n \{2(1-s)T^\ell_{ir}T^r_{jk} + sT^\ell_{jr}T^r_{ik} - sT^\ell_{kr}T^r_{ij}\}, \tag{19}$$

$$(n-2)(s-1)\sum_{r=1}^n \{T^\ell_{ir}T^r_{jk} + T^\ell_{jr}T^r_{ki} + T^\ell_{kr}T^r_{ij}\} = 0, \tag{20}$$

$$\begin{aligned} 2(s-1)T^k_{ij,\bar\ell} + s(\overline{T^i_{k\ell,\bar j}} - \overline{T^j_{k\ell,\bar i}}) = \sum_{r=1}^n \{ & 2(s-s^2)T^r_{ij}\overline{T^r_{k\ell}} + 2(s-s^2)(T^k_{ri}\overline{T^j_{r\ell}} - T^k_{rj}\overline{T^i_{r\ell}}) \\ & + s^2(T^\ell_{ri}\overline{T^j_{rk}} - T^\ell_{rj}\overline{T^i_{rk}})\} \end{aligned} \tag{21}$$

for any $1 \leq i, j, k, \ell \leq n$, where the index after comma denotes the covariant derivative with respect to ∇^s.

Proof The first identity comes from the fact that $\partial\gamma' = -s\gamma'\gamma'$ since $s \neq 0$. When $n \geq 3$, if we combine the identity $\partial\gamma' = -s\gamma'\gamma'$ with the $(3,0)$-part of the first Bianchi identity, $\partial\tau = s{}^t\gamma'\tau$, we get $(s-1)\ {}^t\gamma'\ {}^t\gamma'\varphi = 0$. This leads to the second equation. The third equation comes from (18). □

By the first two equations in the above lemma, we get

Lemma 3.2 *Let (M^n, g) be a Hermitian manifold that is s-Gauduchon flat, where $s \neq 1$ and $n \geq 3$. Then for any indices i, j, k, ℓ, it holds*

$$T^{\ell}_{ij,k} - T^{\ell}_{ik,j} = (2-s)\sum_{r=1}^{n} T^{\ell}_{ir} T^{r}_{jk}. \tag{22}$$

This identity shows that the case of the Bismut connection ($s = 2$) is special as the right-hand side would vanish when $s = 2$. Next, recall that $\eta_i = \sum_k T^k_{ki}$, and let us denote by

$$|\eta|^2 = \sum_{i=1}^{n} |\eta_i|^2, \quad |T|^2 = \sum_{i,j,k=1}^{n} |T^i_{jk}|^2, \quad \text{and} \quad \chi = \sum_{i=1}^{n} \eta_{i,\bar{i}}.$$

In the last equation of Lemma 3.1, choosing $i = k$ and $j = \ell$ and summing them up from 1 to n, we get

$$2(s-1)\chi + 2s\overline{\chi} = s^2|T|^2 + 2\left(s - \frac{3}{2}s^2\right)|\eta|^2.$$

Since the right-hand side is a real number, we see that χ must be real, so we get

Lemma 3.3 *Under a local unitary ∇^s-parallel frame e, the quantity $\chi = \sum_{i=1}^{n} \eta_{i,\bar{i}}$ satisfies the identity*

$$(2s-1)\chi = \frac{s^2}{2}|T|^2 + s\left(1 - \frac{3}{2}s\right)|\eta|^2. \tag{23}$$

Now we are ready to prove Proposition 1.9. For $s = \frac{1}{2}$, the above identity gives $\frac{1}{8}|\eta|^2 + \frac{1}{8}|T|^2 = 0$, so $T = 0$ everywhere, and g is Kähler. If $s \neq 0$ and $\eta = 0$, then $\chi = 0$, so the above equation leads to $T = 0$ again. This proves Proposition 1.9.

Next, by the equation $\overline{\partial}\varphi = -s\overline{\gamma'}\varphi$, we get

$$\overline{\partial}\eta = -\sum_{i,j=1}^{n} \left(\eta_{i,\bar{j}} + s\sum_{k=1}^{n} \eta_k \overline{T^i_{jk}} \right) \varphi_i \wedge \overline{\varphi_j}.$$

Since $n\sqrt{-1}\,(\sum_{i,j=1}^{n} a_{ij}\varphi_i \wedge \overline{\varphi_j}) \wedge \omega^{n-1} = (\sum_{i=1}^{n} a_{ii})\,\omega^n$, we obtain the following

$$n\sqrt{-1}\,\overline{\partial}\eta \wedge \omega^{n-1} = -(\chi + s|\eta|^2)\,\omega^n. \tag{24}$$

Combining this with Lemma 3.3, we get

Lemma 3.4 *Let (M^n, g) be a ∇^s-flat Hermitian manifold. Then it holds*

$$(2s-1)n\sqrt{-1}\partial\overline{\partial}\omega^{n-1} = [(8s - s^2 - 4)\ |\eta|^2 - s^2|T|^2]\ \omega^n. \tag{25}$$

When the s-Gauduchon flat manifold M^n is compact, the integral of the left-hand side is zero. Thus we immediately get the following

Lemma 3.5 *Let (M^n, g) be a compact ∇^s-flat Hermitian manifold. Then it holds*

$$(8s - s^2 - 4)\int_M |\eta|^2\ \omega^n = s^2 \int_M |T|^2\ \omega^n. \tag{26}$$

Since the two roots of $8s - s^2 - 4$ are $4 \pm 2\sqrt{3}$, when either $s \geq 4 + 2\sqrt{3}$, or $s \leq 4 - 2\sqrt{3}$ and $s \neq 0$, the left-hand side is nonpositive, which will force T to be identically zero, meaning that g is Kähler. So we have proved Theorem 1.6.

Theorem 1.6 means that when M^n is compact, we may assume that $4-2\sqrt{3} < s < 4+2\sqrt{3}$, or approximately,

$$0.54 < s < 7.46.$$

Next, note that $\eta_i = \sum_k T^k_{ki}$ is the sum of at most $(n-1)$ terms, since $T^i_{ii} = 0$. By the inequality $|a_1 + \cdots + a_{n-1}|^2 \leq (n-1)(|a_1|^2 + \cdots + |a_{n-1}|^2)$, we know that

$$|\eta|^2 \leq (n-1)|T|^2.$$

Plugging this into the identity in Lemma 3.5, we get

$$F(s) \int_M |T|^2 \omega^n \leq 0, \quad \text{where } F(s) = ns^2 - 8(n-1)s + 4(n-1). \tag{27}$$

Clearly, when $F(s) > 0$, the above inequality implies $T = 0$ everywhere. The two roots of F are exactly the two dimension-dependent constants

$$a_n^{\pm} := \frac{1}{n}[4(n-1) \pm 2\sqrt{3n^2 - 7n + 4}]$$

given in the introduction. Thus we have completed the proof of Theorem 1.7.

Note that the improvement of Theorem 1.7 to Theorem 1.6 works better when n is smaller, as $a_n^{\pm} \to 4 \pm 2\sqrt{3}$ when $n \to \infty$. When $n = 3$, for instance, $a_3^{\pm} = \frac{2}{3}(4 \pm \sqrt{10})$, so we just need to consider the range

$$0.56 \leq s \leq 4.77$$

for any potential non-Kähler s-Gauduchon flat compact threefold.

Next let us consider the locally conformally Kähler case. In this case we have the relationship $|\eta|^2 = \frac{(n-1)}{2}|T|^2$ by Lemma 2.2. Plug this again into the identity in Lemma 3.5, we get

$$[8(n-1)s - 4(n-1) - (n+1)s^2] \int_M |T|^2 \omega^n = 0 \tag{28}$$

for compact Hermitian manifold (M^n, g) that is ∇^s-flat and locally conformally Kähler. When the coefficient is not zero, it will force $T = 0$, so g would be Kähler. The zeroes of the polynomial are precisely the two constant $b_n^{\pm}$ given in the introduction. So we have completed the proof of Theorem 1.8 here.

Now let us restrict ourselves to the two dimensional case. In this case, $\eta_1 = -T^2_{12}$, $\eta_2 = T^1_{12}$, so $|T|^2 = 2|\eta|^2$. Plug this into the identity in Lemma 3.5, we get

$$(3s-2)(s-2) \int_{M^2} |T|^2 \omega^2 = 0. \tag{29}$$

Therefore we obtained the following

Lemma 3.6 *Let (M^2, g) be a compact Hermitian surface with flat s-Gauduchon connection. If $s \neq 2$ and $s \neq \frac{2}{3}$, then g is Kähler.*

Therefore in order to prove Theorem 1.5, we just need to deal with the $s = \frac{2}{3}$ case, namely, the minimal Gauduchon connection $\nabla^{\frac{2}{3}}$.

4 Surfaces with Flat Minimal Connection

In this section, we will assume that (M^2, g) is a compact Hermitian surface with flat $\nabla^{\frac{2}{3}}$, and our goal is to use the Bochner formula to conclude that g must be Kähler, thus proving Theorem 1.5 stated in the introduction.

Let $s = \frac{2}{3}$, and e be a ∇^s-parallel local unitary frame. Let us denote $a = T^1_{12} = \eta_2$, $b = T^2_{12} = -\eta_1$. Then $\lambda = |a|^2 + |b|^2 = |\eta|^2$ is a nonnegative smooth function on M^2. Since $2(1-s) = s$, in the first identity of Lemma 3.1, if we let $i = k = 1$, $j = 2$, we get

$$T^\ell_{12,1} = -2sT^\ell_{12}T^2_{12}.$$

Similarly, if we let $i = k = 2$ and $j = 1$, we get

$$T^\ell_{12,2} = 2sT^\ell_{12}T^1_{12}.$$

That is, we have the following

$$a_1 = -2s\ ab, \quad b_1 = -2s\ b^2, \quad a_2 = 2s\ a^2, \quad b_2 = 2s\ ab. \tag{30}$$

Next, we look at the last identity in Lemma 3.1. Letting $i = 1$, $j = 2$, and $k = \ell$, we get

$$T^k_{12,\bar{k}} = 2sT^k_{12}(\overline{T^2_{2k}} + \overline{T^1_{1k}}),$$

that is,

$$a_{\bar{1}} = -2s\ a\bar{b}, \quad b_{\bar{2}} = 2s\ \bar{a}b. \tag{31}$$

Similarly, by letting respectively $i = k = 1$, $j = \ell = 2$ or $i = \ell = 1$, $j = k = 2$ in the last identity of Lemma 3.1, and using the results for $a_{\bar{1}}$ and $b_{\bar{2}}$ above, we get

$$a_{\bar{2}} = 2s\ |a|^2, \quad b_{\bar{1}} = -2s\ |b|^2. \tag{32}$$

From this, we get

$$\begin{aligned} \lambda_1 = -4s\lambda b, \quad \lambda_2 &= 4s\lambda a, \\ \lambda_{1\bar{1}} = 24s^2\lambda|b|^2, \quad \lambda_{2\bar{2}} &= 24s^2\lambda|a|^2. \end{aligned}$$

Thus

$$\sum_i \lambda_{i\bar{i}} = 24s^2\lambda^2, \quad \sum_i \lambda_i\overline{\eta_i} = 4s\lambda^2, \quad \sum_i |\lambda_i|^2 = 16s^2\lambda^3.$$

Now let us consider the smooth function $f = \log(\lambda + \epsilon)$ on M^2, where $\epsilon > 0$ is a constant. Since $\overline{\partial}\varphi = -s\overline{\gamma'}\varphi$, we have

$$\begin{aligned} \partial\overline{\partial} f &= -\overline{\partial}\Big(\sum_i f_i\varphi_i\Big) \\ &= \sum_{i,j}\Big(f_{i\bar{j}} + s\sum_k f_k\overline{T^i_{jk}}\Big)\ \varphi_i \wedge \overline{\varphi_j}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\sqrt{-1}\partial\overline{\partial} f \wedge \omega &= \Big(\sum_i f_{i\bar{i}} + s\sum_i f_i\overline{\eta_i}\Big)\ \omega^2 \\ &= \Big(\frac{1}{\lambda+\epsilon}\sum_i \lambda_{i\bar{i}} - \frac{1}{(\lambda+\epsilon)^2}\sum_i |\lambda_i|^2 + \frac{s}{\lambda+\epsilon}\sum_i \lambda_i\overline{\eta_i}\Big)\ \omega^2 \end{aligned}$$

$$= 4s^2 \frac{(3\lambda + 7\epsilon)}{(\lambda + \epsilon)^2} \lambda^2 \omega^2.$$

On the other hand, by Lemma 3.4, specialized to our case of $n = 2$ and $s = \frac{2}{3}$, we get $\partial\overline{\partial}\omega = 0$. So if we integrate the above equality, we get $\lambda \equiv 0$ on M^2, that is, (M^2, g) is Kähler. This completes the proof of Theorem 1.5.

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