Acta Mathematica Sinica, English Series
Oct., 2018, Vol. 34, No. 10, pp. 1611–1625
Published online: February 25, 2018
https://doi.org/10.1007/s10114-018-7273-9
http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Cohomology of the Universal Enveloping Algebras of Certain Bigraded Lie Algebras

Li Nan ZHONG
School of Mathematical Sciences, South China Normal University, Guangzhou 510631*, P. R. China*
Department of Mathematics, Yanbian University, Yanji 133000*, P. R. China*
E-mail: *zhonglinan*2000*@*126*.com*

Hao ZHAO[1)] **Wen Huai SHEN**
School of Mathematical Sciences, South China Normal University, Guangzhou 510631*, P. R. China*
E-mail: *zhaohao@scnu.edu.cn* *shenn@scnu.edu.cn*

Abstract Let p be an odd prime and $q = 2(p-1)$. Up to total degree $t - s < \max\{(5p^3 + 6p^2 + 6p + 4)q - 10, p^4 q\}$, the generators of $H^{s,t}(U(L))$, the cohomology of the universal enveloping algebra of a bigraded Lie algebra L, are determined and their convergence is also verified. Furthermore our results reveal that this cohomology satisfies an analogous Poinćare duality property. This largely generalizes an earlier classical results due to J. P. May.

Keywords Steenrod algebra, Hopf algebra, Lie algebra, spectral sequence, stable homotopy groups of sphere

MR(2010) Subject Classification 55Q45, 55S10, 17B35

1 Introduction

Let p be an odd prime and let $\mathcal{A}$ be the mod p Steenrod algebra. To determine the stable homotopy groups of sphere is one of the central problems in homotopy theory. One of the main tools to approach the computation of the stable homotopy groups of sphere is the classical Adams spectral sequence

$$\{E_r^{s,t}; d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}\} \Longrightarrow \pi_{t-s}S.$$

The most important term of the Adams spectral sequence is its E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) = H^{s,t}(\mathcal{A}),$$

which is the cohomology of $\mathcal{A}$. In order to compute $\pi_{t-s}S$, we first need to know the explicit structure of $\mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ and then verify the convergence of the corresponding generators. Up to now, only partial results about $\mathrm{Ext}_{\mathcal{A}}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ have been known except the case $s \le 3$ (refer to [1, 3]).

Received May 31, 2017, accepted December 8, 2017
This paper was supported by NSFC (Grant Nos. 11671154 and 11761072) and General Financial Grant from the China Postdoctoral Science Foundation (Grant No. 2017M622721)
1) Corresponding author

Since it is difficult to consider $H^{*,*}(\mathcal{A})$, we can study it by considering the cohomology of some Hopf-subalgebra of $\mathcal{A}$. From this view of point, May [4, 5] studied $H^{s,t}(\mathbb{P}) = \mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$, the cohomology of a Hopf-subalgebra of $\mathcal{A}$ which is generated by the reduced power operations $\mathcal{P}^i$ $(i \geq 0)$. Knowing well $\mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ is related to the existence of important Smith–Toda spectra [9] which realizing the exterior part of the dual mod-p Steenrod algebra. It was shown in [9] that the Smith–Toda spectra $V(n)$ exists for $p > 2n$ when $n = 0, 1, 2, 3$. Later, Nave [7] showed that not all Smith–Toda spectra exist.

In the following we recall some results on $\mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$ due to May [4, 5]. Let $\varepsilon\colon \mathbb{P} \to \mathbb{Z}/p$ be the argumentation. Let I be the kernel of ε and define $F_0\mathbb{P} = \mathbb{P}$ and $F_{-i}\mathbb{P} = I \cdot F_{-i+1}\mathbb{P}$ inductively for $i > 0$. Associated with the filtration $\mathbb{P} = F_0\mathbb{P} \supset F_{-1}\mathbb{P} \supset \cdots$, there is a graded Hopf algebra $E^0\mathbb{P} = \sum_i F_i\mathbb{P}/F_{i-1}\mathbb{P}$. Then by the corresponding exact couple there is a spectral sequence

$$E_2^{s,t} = H^{s,t}(E^0\mathbb{P}) \Longrightarrow H^{s,t}(\mathbb{P}) = \mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p). \tag{1.1}$$

According to Milnor–Moore's theorem [6], $E^0\mathbb{P}$ is a primitively generated Hopf algebra of characteristic p and it is isomorphic to the restricted enveloping Hopf algebra $V(L) = U(L)/J$, where $L = \overline{P}(E^0\mathbb{P})$ is the restricted Lie algebra which consists of the primitive elements of $E^0\mathbb{P}$, and

$$U(L) = T(L)/\{xy - yx - [x, y]\}$$

is the universal enveloping algebra of L as a Lie algebra and J is the ideal generated by $\xi(x) - x^p$ for $x \in L$ and a certain self-map $\xi\colon L_n \to L_{pn}$ (refer to [6]). Lemma 9 in [5] implies that there exists another multipliticable spectral sequence

$$E_2^{*,*} = P[b_i^j] \otimes H^{*,*}(U(L)) \Longrightarrow H^{*,*}(V(L)) = H^{*,*}(E^0\mathbb{P}) \tag{1.2}$$

where $P[\,]$ denotes the polynomial algebra and b_i^j is a generator of bidegree $(2, 2(p^{i+j+1} - p^{j+1}))$. From these two spectral sequences we obtain an estimate on $\mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$.

One critical thing is to determine the structure of $H^{*,*}(U(L))$. Define a differential bigraded exterior algebra $(E(R_i^j)\delta)$ generated by R_i^j of bidegree $(1, 2(p^{i+j} - p^j))$. The differential δ is given by

$$\delta(R_i^j) = \sum_{k=1}^{i-1} R_{i-k}^{j+k} R_k^j.$$

May [4] showed that there is an isomorphism $H^{*,*}(U(L)) \cong H^{*,*}(E(R_i^j), \delta)$ such that the determination of $H^{*,*}(U(L))$ is transformed into determining $H^{*,*}(E(R_i^j), \delta)$. Along this idea, May [4] computed the generators of $H^{s,t}(U(L))$ for the range $t - s < (p^3 + 2p^2 + 2p + 1)q - 4$. In [10] the above results were generalized to the case $t - s < (2p^3 + 2p^2)q - 3$. But unfortunately both of these two papers did not give the details of proof. Considering this, we go further to determine the generators of $H^{*,*}(U(L))$ in a greater range. Furthermore we show that the obtained generators converge nontrivially into $\mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. We state our main results as follows.

Theorem 1.1 *Up to total degree $t - s < \max\{(5p^3 + 6p^2 + 6p + 4)q - 10, p^4q\}$, $H^{s,t}(U(L))$ is multiplicatively generated by the following classes*:

$$h_i = \{R_1^i\}\ (0 \leq i \leq 3), \quad g_i = \{R_2^i R_1^i\}\ (0 \leq i \leq 2), \quad k_i = \{R_2^i R_1^{i+1}\}\ (0 \leq i \leq 2);$$

$l_1 = \{R_3^0 R_2^0 R_1^0\}, \quad l_2 = \{R_2^1 R_2^0 R_1^1\}, \quad l_3 = \{R_3^0 R_1^2 R_1^0\}, \quad l_4 = \{R_3^0 R_2^1 R_1^2\},$
$l_5 = \{R_3^1 R_2^1 R_1^1\}, \quad l_6 = \{R_2^2 R_2^1 R_1^2\}, \quad l_7 = \{R_3^1 R_1^3 R_1^1\}, \quad l_8 = \{R_3^1 R_2^2 R_1^3\};$

$m_1 = \{R_3^0 R_2^1 R_2^0 R_1^1\}, \quad m_2 = \{R_4^0 R_3^0 R_2^0 R_1^0\}, \quad m_3 = \{R_3^1 R_2^1 R_2^0 R_1^1\},$
$m_4 = \{R_2^2 R_3^0 R_1^2 R_1^0\}, \quad m_5 = \{R_2^2 R_3^0 R_2^1 R_1^2\}, \quad m_6 = \{R_3^1 R_1^3 R_2^0 R_1^1\},$
$m_7 = \{R_4^0 R_1^3 R_2^0 R_1^0\}, \quad m_8 = \{R_3^1 R_2^2 R_2^1 R_1^2\}, \quad m_9 = \{R_4^0 R_2^2 R_1^3 R_1^0\},$
$m_{10} = \{R_4^0 R_3^1 R_2^2 R_1^3\};$

$n_1 = \{R_3^1 R_3^0 R_2^1 R_2^0 R_1^1\}, \quad n_2 = \{R_3^1 R_3^0 R_2^1 R_1^2 R_1^1\}, \quad n_3 = \{R_3^1 R_2^1 R_1^3 R_2^0 R_1^1\},$
$n_4 = \{R_3^1 R_2^2 R_3^0 R_2^1 R_1^2\}, \quad n_5 = \{R_4^0 R_2^2 R_1^3 R_2^0 R_1^0\}, \quad n_6 = \{R_4^0 R_3^1 R_1^3 R_2^0 R_1^1\},$
$n_7 = \{R_4^0 R_2^2 R_3^0 R_1^2 R_1^0\};$

$u_1 = \{R_3^1 R_3^0 R_2^1 R_1^2 R_2^0 R_1^1\}, \quad u_2 = \{R_4^0 R_3^1 R_1^3 R_2^1 R_2^0 R_1^1\}, \quad u_3 = \{R_4^0 R_3^1 R_2^2 R_1^3 R_2^0 R_1^0\},$
$u_4 = \{R_4^0 R_2^2 R_3^0 R_2^1 R_1^2 R_1^0\}, \quad u_5 = \{R_4^0 R_2^2 R_1^3 R_3^0 R_1^2 R_2^0\}, \quad u_6 = \{R_4^0 R_3^1 R_2^2 R_3^0 R_2^1 R_1^2\};$

$v_1 = \{R_3^1 R_2^2 R_1^3 R_3^0 R_2^0 R_1^1 R_1^0\}, \quad v_2 = \{R_4^0 R_3^1 R_3^0 R_2^1 R_1^2 R_2^0 R_1^1\}, \quad v_3 = \{R_4^0 R_3^1 R_1^3 R_3^0 R_2^1 R_2^0 R_1^1\}.$

For the above generators, the nontrivial multiplications among them are given as follows:

Dimension	Multiplication
2	$h_0h_2 \quad h_0h_3 \quad h_1h_3$
3	$h_0g_2 \quad h_0k_2 \quad h_1g_0 = h_0k_0 \quad h_1k_1 \quad h_3g_0 \quad h_3g_2 \quad h_3k_0$
4	$h_0h_2k_2 = h_0h_3g_2 \quad h_0l_4 \quad h_1l_1 \quad h_1l_4 \quad h_1l_8 \quad h_2l_1 \quad h_2l_5 \quad h_2l_8$ $h_1h_3g_0 = h_0h_3k_0$
5	$h_0m_1 = g_1l_1 \quad h_0m_5 \quad h_0m_{10} \quad h_1m_2 \quad h_1m_7 \quad h_1m_8 \quad h_1m_{10}$ $h_2m_1 = k_0l_4 \quad h_2m_2 \quad h_2m_9 \quad h_2m_{10} \quad h_3m_2 \quad g_0l_4 = k_1l_1$ $g_1l_8 = k_2l_5 \quad k_1l_8 = h_3m_8 \quad k_2l_1$
6	$h_0h_2m_{10} \quad h_0n_6 \quad h_1k_1l_8 = h_1h_3m_8 = h_1k_2l_5 \quad h_1h_3m_2 \quad h_1k_1l_1 \quad h_3n_7$ $g_1m_2 \quad g_1m_{10} \quad g_2m_2 \quad k_0m_{10} \quad k_1m_2 \quad k_1m_{10} \quad k_2m_2$
7	$h_0u_6 \quad h_1g_0m_2 \quad h_1k_1m_{10} = h_2g_1m_{10} \quad h_1u_6 \quad h_3g_2m_2 \quad h_3u_6 \quad l_2m_{10}$ $l_3m_{10} \quad l_5m_2 \quad l_6m_2 \quad l_7m_2 \quad l_8m_2$
8	$h_0l_4m_{10} \quad h_1h_3u_6 = h_1l_4m_{10} \quad h_1l_8m_2 \quad h_2l_5m_2 \quad h_2l_8m_2 \quad h_3l_5m_2$ $k_0u_6 \quad m_1m_{10} \quad m_2m_8$
9	$h_1m_2m_8 \quad h_2m_1m_{10} \quad h_3m_2m_8 = k_1l_8m_2 \quad g_1l_8m_2 = k_2l_5m_2$
10	$h_1k_1l_8m_2 = h_1h_3m_2m_8 = h_1k_2l_5m_2$

Corollary 1.2 ([4]) *Up to total degree* $t - s < (p^3 + 2p^2 + 2p + 1)q - 4$, $H^{s,t}(U(L))$ *is multiplicatively generated by the following classes*:

$h_i = \{R_1^i\}\,(0 \le i \le 3), \quad g_i = \{R_2^i R_1^i\}\,(0 \le i \le 2), \quad k_i = \{R_2^i R_1^{i+1}\}\,(0 \le i \le 1);$

$l_1 = \{R_3^0 R_2^0 R_1^0\}, \quad l_2 = \{R_2^1 R_2^0 R_1^1\}, \quad l_3 = \{R_3^0 R_1^2 R_1^0\},$
$l_4 = \{R_3^0 R_2^1 R_1^2\}, \quad l_5 = \{R_3^1 R_2^1 R_1^1\}, \quad l_6 = \{R_2^2 R_2^1 R_1^2\};$
$m_1 = \{R_3^0 R_2^1 R_2^0 R_1^1\}, \quad m_2 = \{R_4^0 R_3^0 R_2^0 R_1^0\}, \quad m_3 = \{R_3^1 R_2^1 R_2^0 R_1^1\}, \quad m_4 = \{R_2^2 R_3^0 R_1^2 R_1^0\};$

and we have additively

$$H^{*,*}(U(L)) \cong \{1, l_4, h_3\} \otimes \{1, h_0, h_1, g_0, k_0, k_0h_0\} + \{h_2, h_2h_0, g_1, l_1, l_2, l_1h_1, k_1, l_3, k_1h_1, l_1h_2, m_1, m_1h_0, g_2, g_2h_0, l_5, m_2, m_3, l_6, m_4\}.$$

Corollary 1.3 $\mathrm{Rank}(\mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \leq \mathrm{Rank}([P[b_{ij}] \otimes H^{*,*}(U(L))]^{s,t}))$.

Remark 1.4 The Poincaré duality property has already been shown on the above table. It seems that the Poincaré duality property holds for this kind of cohomology in a greater range. We believe that our method is valid for an even larger range, but it seems that the number of the obtained generators will become huge and we still do not know what is it used for in that case.

This paper is organized as follows. In Section 2, we will inductively compute out the desired generators shown as Theorem 1.1. In Section 3, we will verify that all the obtained generators can converge nontrivially to $\mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$.

2 Computation of Generators

We take the notations defined in Section 1. Recall that there is an isomorphism $H^{*,*}(U(L)) \cong H^{*,*}(E(R_i^j), \delta)$ and the differential δ is given by

$$\delta(R_i^j) = \sum_{k=1}^{i-1} R_{i-k}^{j+k} R_k^j.$$

Our method of computing $H^{*,*}(U(L))$ is to break the exterior algebra $E(R_i^j)$ into four summands and compute the generators of the cohomology of each summand. The follows are our proof.

Proof of Theorem 1.1 (i) Let $K^0 = E[R_i^j | i + j \leq 4, j > 0]$. Define a chain of increasing complexes $K^1 \subset K^2 \subset K^3 \subset K^4$ by

$$K^1 = \{R_1^0\} \otimes K^0; \quad K^2 = \{R_2^0, R_1^0\} \otimes K^0;$$
$$K^3 = \{R_3^0, R_2^0, R_1^0\} \otimes K^0; \quad K^4 = \{R_4^0, R_3^0, R_2^0, R_1^0\} \otimes K^0.$$

There is a short exact sequence $0 \to K^{l-1} \to K^l \to K^l/K^{l-1} \to 0$ for $2 \leq l \leq 4$ with $K^l/K^{l-1} = \{R_l^0\} \otimes K^0$. We will compute each $H^{*,*}(K^l)$ by the induced cohomological long exact sequence

$$\longrightarrow H^{*-1,*}(K^l/K^{l-1}) \xrightarrow{\delta} H^{*,*}(K^{l-1}) \xrightarrow{i} H^{*,*}(K^l) \xrightarrow{j} H^{*,*}(K^l/K^{l-1}) \xrightarrow{\delta} H^{*+1,*}(K^l) \longrightarrow \tag{2.1}$$

Let us first compute $H^{*,*}(K^0)$. There are six generators for K^0: R_3^1, R_2^1, R_1^1, R_2^2, R_1^2, R_1^3. In what follows we list the first May differential on elements of K^0.

$R_1^1 \longrightarrow 0 \quad R_1^2 \longrightarrow 0$
$R_1^3 \longrightarrow 0 \quad R_2^1 \longrightarrow R_1^2 R_1^1$

$$R_2^2 \longrightarrow R_1^3R_1^2 \quad R_3^1 \longrightarrow R_2^2R_1^1 - R_2^1R_1^3$$

$$\begin{aligned}
&R_1^3R_1^1 \longrightarrow 0 \quad R_2^1R_1^1 \longrightarrow 0\\
&R_2^1R_1^2 \longrightarrow 0 \quad R_2^1R_1^3 \longrightarrow R_1^3R_1^2R_1^1\\
&R_2^2R_1^2 \longrightarrow 0 \quad R_2^2R_1^3 \longrightarrow 0\\
&R_2^2R_2^1 \longrightarrow -R_2^2R_1^2R_1^1 + R_2^1R_1^3R_1^2 \quad R_3^1R_1^1 \longrightarrow -R_2^1R_1^3R_1^1\\
&R_3^1R_1^2 \longrightarrow -R_2^2R_1^2R_1^1 - R_2^1R_1^3R_1^2 \quad R_3^1R_1^3 \longrightarrow -R_2^2R_1^3R_1^1\\
&R_3^1R_2^1 \longrightarrow -R_3^1R_1^2R_1^1 - R_2^2R_2^1R_1^1 \quad R_3^1R_2^2 \longrightarrow -R_3^1R_1^3R_1^2 - R_2^2R_2^1R_1^3
\end{aligned}$$

$$R_3^1R_1^2 - R_2^2R_2^1 \longrightarrow -2R_2^1R_1^3R_1^2 \quad R_3^1R_1^2 + R_2^2R_2^1 \longrightarrow 2R_2^2R_1^2R_1^1$$

$$\begin{aligned}
&R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_2^2R_1^3R_1^2 \longrightarrow 0\\
&R_2^2R_2^1R_1^1 \longrightarrow R_2^1R_1^3R_1^2R_1^1 \quad R_2^2R_2^1R_1^2 \longrightarrow 0\\
&R_2^2R_2^1R_1^3 \longrightarrow -R_2^2R_1^3R_1^2R_1^1 \quad R_3^1R_1^3R_1^1 \longrightarrow 0\\
&R_3^1R_2^1R_1^1 \longrightarrow 0 \quad R_3^1R_2^1R_1^2 \longrightarrow R_2^2R_2^1R_1^2R_1^1\\
&R_3^1R_2^1R_1^3 \longrightarrow -R_3^1R_1^3R_1^2R_1^1 + R_2^2R_2^1R_1^3R_1^1 \quad R_3^1R_2^2R_1^1 \longrightarrow -R_3^1R_1^3R_1^2R_1^1 - R_2^2R_2^1R_1^3R_1^1\\
&R_3^1R_2^2R_1^2 \longrightarrow -R_2^2R_2^1R_1^3R_1^2 \quad R_3^1R_2^2R_1^3 \longrightarrow 0\\
&R_3^1R_2^2R_2^1 \longrightarrow R_3^1R_2^2R_1^2R_1^1 - R_3^1R_2^1R_1^3R_1^2\\
&R_3^1R_2^2R_1^1 - R_3^1R_2^1R_1^3 \longrightarrow -2R_2^2R_2^1R_1^3R_1^1 \quad R_3^1R_2^2R_1^1 + R_3^1R_2^1R_1^3 \longrightarrow -2R_3^1R_1^3R_1^2R_1^1
\end{aligned}$$

$$\begin{aligned}
&R_3^1R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_3^1R_2^1R_1^3R_1^1 \longrightarrow 0\\
&R_3^1R_2^1R_1^3R_1^2 \longrightarrow R_2^2R_2^1R_1^3R_1^2R_1^1 \quad R_3^1R_2^2R_1^3R_1^1 \longrightarrow 0\\
&R_3^1R_2^2R_1^3R_1^2 \longrightarrow 0 \quad R_3^1R_2^2R_2^1R_1^1 \longrightarrow -R_3^1R_2^1R_1^3R_1^2R_1^1\\
&R_3^1R_2^2R_2^1R_1^2 \longrightarrow 0 \quad R_3^1R_2^2R_2^1R_1^3 \longrightarrow R_3^1R_2^2R_1^3R_1^2R_1^1\\
&R_3^1R_2^2R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_3^1R_2^2R_2^1R_1^3R_1^1 \longrightarrow 0\\
&R_3^1R_2^2R_2^1R_1^3R_1^2 \longrightarrow 0
\end{aligned}$$

$$R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1 \longrightarrow 0$$

The above elements with trivial differentials are the all possible generators of $H^{*,*}(K^0)$. It is known that the Poinćare series for $H^{*,*}(K^0)$ is $1+3t+5t^2+6t^3+5t^4+3t^5+t^6$. Thus it follows the generators of $H^{*,*}(K^0)$:

$$\begin{aligned}
&\dim 1 \quad R_1^1 \quad R_1^2 \quad R_1^3\\
&\dim 2 \quad R_1^1\cdot R_1^3 \quad R_2^1R_1^1 \quad R_2^1R_1^2 \quad R_2^2R_1^2 \quad R_2^2R_1^3\\
&\dim 3 \quad R_1^1\cdot R_2^1R_1^2 \quad R_1^3\cdot R_2^2R_1^2 \quad R_2^2R_2^1R_1^2 \quad R_3^1R_1^3R_1^1 \quad R_3^1R_2^1R_1^1 \quad R_3^1R_2^2R_1^3\\
&\dim 4 \quad R_1^2\cdot R_3^1R_2^1R_1^1 \quad R_1^3\cdot R_3^1R_2^1R_1^1 \quad R_1^1\cdot R_3^1R_2^2R_1^3 \quad R_1^2\cdot R_3^1R_2^2R_1^3 \quad R_3^1R_2^2R_2^1R_1^2\\
&\dim 5 \quad R_1^1\cdot R_3^1R_2^2R_2^1R_1^2 \quad R_2^1R_1^1\cdot R_3^1R_2^2R_1^3 \quad R_2^1R_1^2\cdot R_3^1R_2^2R_1^3\\
&\dim 6 \quad R_1^1\cdot R_2^1R_1^2\cdot R_3^1R_2^2R_1^3
\end{aligned}$$

Since $d_1(R_1^0)=0$, it follows that $H^{*,*}(K^1)=\{R_1^0\}\otimes H^{*,*}(K^0)$. For $2\le l\le 4$, since there is

$d_1(R_1^0) \notin E[R_i^j | i+j \le 4, j>0]$, we have $H^{*,*}(K^l/K^{l-1}) = \{R_1^0\} \otimes H^{*,*}(K^0)$. Considering the long exact sequence (2.1), we see that if one element of $H^{*,*}(K^l/K^{l-1})$ hit another one element of $H^{*,*}(K^{l-1})$ under the connecting homomorphism δ, then both of them vanish in $H^{*,*}(K^l)$ (exclude $H^{*,*}(K^0)$). The remaining elements are the generators of $H^{*,*}(K^l)$. This idea will also be used to compute $H^{*,*}(F^l)$ and $H^{*,*}(G^l)$ later. Following this idea we first compute out $H^{*,*}(K^2)$. We list out the actions of δ on $H^{*,*}(K^2/K^1)$ in the following table:

$$R_2^0 \longrightarrow R_1^1R_1^0 \quad R_2^0R_1^1 \longrightarrow 0$$
$$R_2^0R_1^2 \longrightarrow R_1^2R_1^1R_1^0 \quad R_2^0R_1^3 \longrightarrow R_1^3R_1^1R_1^0$$

$$R_2^0R_1^3R_1^1 \longrightarrow 0 \quad R_2^0R_2^1R_1^1 \longrightarrow 0$$
$$R_2^0R_2^1R_1^2 \longrightarrow R_2^1R_1^2R_1^1R_1^0 \quad R_2^0R_2^2R_1^2 \longrightarrow R_2^2R_1^2R_1^1R_1^0$$
$$R_2^0R_2^2R_1^3 \longrightarrow R_2^2R_1^3R_1^1R_1^0$$

$$R_2^0R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_2^0R_2^2R_1^3R_1^2 \longrightarrow R_2^2R_1^3R_1^2R_1^1R_1^0$$
$$R_2^0R_2^2R_2^1R_1^2 \longrightarrow R_2^2R_2^1R_1^2R_1^1R_1^0 \quad R_2^0R_3^1R_1^3R_1^1 \longrightarrow 0$$
$$R_2^0R_3^1R_2^1R_1^1 \longrightarrow 0 \quad R_2^0R_3^1R_2^2R_1^3 \longrightarrow R_3^1R_2^2R_1^3R_1^1R_1^0$$

$$R_2^0R_3^1R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_2^0R_3^1R_2^1R_1^3R_1^1 \longrightarrow 0$$
$$R_2^0R_3^1R_2^2R_1^3R_1^1 \longrightarrow 0 \quad R_2^0R_3^1R_2^2R_1^3R_1^2 \longrightarrow R_3^1R_2^2R_1^3R_1^2R_1^1R_1^0$$
$$R_2^0R_3^1R_2^2R_2^1R_1^2 \longrightarrow R_3^1R_2^2R_2^1R_1^2R_1^1R_1^0$$

$$R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1 \longrightarrow 0$$
$$R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2 \longrightarrow R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1R_1^0$$

$$R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1 \longrightarrow 0$$

By checking the generators of $H^{*,*}(K^1)$ and $H^{*,*}(K^2/K^1)$ according to the above table, we obtain the generators of $H^{*,*}(K^2)$ as follows:

dim 1 $\quad R_1^0$

dim 2 $\quad R_2^0R_1^1 \quad R_2^0R_1^2 \quad R_1^2R_1^0 \quad R_1^3R_1^0$

dim 3 $\quad R_2^0R_1^3R_1^1 \quad R_2^0R_2^1R_1^1 \quad R_2^0R_2^2R_1^2 \quad R_2^0R_2^2R_1^3 \quad R_2^1R_1^1R_1^0 \quad R_2^1R_1^2R_1^0 \quad R_2^2R_1^2R_1^0 \quad R_2^2R_1^3R_1^0$

dim 4 $\quad R_2^0R_2^1R_1^2R_1^1 \quad R_2^0R_3^1R_1^3R_1^1 \quad R_2^0R_3^1R_2^1R_1^1 \quad R_2^0R_2^2R_1^3R_1^2 \quad R_2^0R_2^2R_2^1R_1^2$
$\quad R_2^2R_1^3R_1^2R_1^0 \quad R_2^2R_2^1R_1^2R_1^0 \quad R_3^1R_1^3R_1^1R_1^0 \quad R_3^1R_2^1R_1^1R_1^0 \quad R_3^1R_2^2R_1^3R_1^0$

dim 5 $\quad R_2^0R_3^1R_2^1R_1^2R_1^1 \quad R_2^0R_3^1R_2^1R_1^3R_1^1 \quad R_2^0R_3^1R_2^2R_1^3R_1^1 \quad R_2^0R_3^1R_2^2R_1^3R_1^2$
$\quad R_3^1R_2^1R_1^2R_1^1R_1^0 \quad R_3^1R_2^1R_1^3R_1^1R_1^0 \quad R_3^1R_2^2R_1^3R_1^2R_1^0 \quad R_3^1R_2^2R_2^1R_1^2R_1^0$

dim 6 $\quad R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1 \quad R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1 \quad R_3^1R_2^2R_2^1R_1^3R_1^1R_1^0 \quad R_3^1R_2^2R_2^1R_1^3R_1^2R_1^0$

dim 7 $\quad R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1$

Following the above method we can similarly use $H^{*,*}(K^2)$ and $H^{*,*}(K^3/K^2)$ to compute out $H^*(K^3)$ as follows:

dim 1 $\quad R_1^0$

dim 2 $R_3^0R_1^3$ $R_2^0R_1^1$ $R_1^2R_1^0$ $R_1^3R_1^0$

dim 3 $R_3^0R_2^1R_1^2$ $R_3^0R_1^3R_1^1$ $R_2^0R_1^3R_1^1$ $R_2^0R_2^1R_1^1$ $R_2^0R_2^2R_1^2$ $R_2^0R_2^2R_1^3$ $R_2^2R_1^2R_1^0$ $R_2^2R_1^3R_1^0$

dim 4 $R_3^0R_2^1R_1^2R_1^1$ $R_3^0R_2^2R_2^1R_1^2$ $R_3^0R_2^2R_1^3R_1^2$ $R_2^0R_3^1R_1^3R_1^1$ $R_2^0R_3^1R_2^1R_1^1$
$R_2^0R_2^2R_2^1R_1^2$ $R_2^2R_1^3R_1^2R_1^0$ $R_3^1R_1^3R_1^1R_1^0$ $R_3^1R_2^1R_1^1R_1^0$ $R_3^1R_2^2R_1^3R_1^0$

dim 5 $R_3^0R_3^1R_2^1R_1^2R_1^1$ $R_3^0R_3^1R_2^2R_2^1R_1^2$ $R_3^0R_3^1R_2^1R_1^3R_1^1$ $R_2^0R_3^1R_2^1R_1^3R_1^1$
$R_2^0R_3^1R_2^2R_1^3R_1^1$ $R_3^1R_2^1R_1^2R_1^1R_1^0$ $R_3^1R_2^2R_1^3R_1^2R_1^0$ $R_3^1R_2^2R_2^1R_1^2R_1^0$

dim 6 $R_3^0R_3^1R_2^2R_2^1R_1^2R_1^1$ $R_3^0R_3^1R_2^2R_2^1R_1^3R_1^2$ $R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1$ $R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1$

dim 7 $R_3^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1$

We use $H^{*,*}(K^3)$ and $H^{*,*}(K^4/K^3)$ to compute out $H^{*,*}(K^4)$ as follows:

dim 1 R_1^0

dim 2 $R_2^0R_1^1$ $R_1^0 \cdot R_1^2$ $R_1^0 \cdot R_1^3$

dim 3 $R_3^0R_2^1R_1^2$ $R_1^3 \cdot R_2^0R_1^1$ $R_2^1R_2^0R_1^1$ $R_1^0 \cdot R_2^2R_1^2$ $R_1^0 \cdot R_2^2R_1^3$

dim 4 $R_4^0R_3^1R_2^2R_1^3$ $R_1^1 \cdot R_3^0R_2^1R_1^2$ $R_2^2R_3^0R_2^1R_1^2$ $R_3^1R_1^3R_2^0R_1^1$ $R_3^1R_2^1R_2^0R_1^1$ $R_1^0 \cdot R_1^2 \cdot R_2^2R_1^3$

dim 5 $R_1^1 \cdot R_4^0R_3^1R_2^2R_1^3$ $R_1^2 \cdot R_4^0R_3^1R_2^2R_1^3$ $R_3^1R_3^0R_2^1R_1^2R_1^1$ $R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_3^1R_1^3R_2^1R_2^0R_1^1$

dim 6 $R_2^1R_1^1 \cdot R_4^0R_3^1R_2^2R_1^3$ $R_2^1R_1^2 \cdot R_4^0R_3^1R_2^2R_1^3$ $R_3^1R_2^2R_3^0R_2^1R_1^2R_1^1$

dim 7 $R_1^1 \cdot R_2^1R_1^2 \cdot R_4^0R_3^1R_2^2R_1^3$

(ii) Define a chain of increasing complexes $F^1 \subset F^2 \subset F^3$ by

$$F^1 = \{R_2^0R_1^0\} \otimes K^0;$$
$$F^2 = \{R_3^0R_2^0, R_3^0R_1^0, R_2^0R_1^0\} \otimes K^0;$$
$$F^3 = \{R_4^0R_3^0, R_4^0R_2^0, R_4^0R_1^0, R_3^0R_2^0, R_3^0R_1^0, R_2^0R_1^0\} \otimes K^0.$$

There is a short exact sequence $0 \to F^{l-1} \to F^l \to F^l/F^{l-1} \to 0$ for $2 \le l \le 3$. We will compute each $H^{*,*}(F^l)$ by the induced cohomological long exact sequence

$$\longrightarrow H^{*-1}(F^l/F^{l-1}) \xrightarrow{\delta} H^*(F^{l-1}) \xrightarrow{i} H^*(F^l) \xrightarrow{j} H^*(F^l/F^{l-1}) \xrightarrow{\delta} H^{*+1}(F^l) \longrightarrow \quad (2.2)$$

Since $d_1(R_2^0R_1^0) = 0$, it follows $H^{*,*}(F^1) = \{R_2^0R_1^0\} \otimes H^{*,*}(K^0)$. For $F^2/F^1 = \{R_3^0\} \otimes K^2$, there is $H^{*,*}(F^2/F^1) = \{R_3^0\} \otimes H^{*,*}(K^2)$. Considering the long exact sequence (2.2), the actions of δ on $H^{*,*}(F^2/F^1) =$ are listed as follows.

$R_3^0R_1^0 \longrightarrow R_2^0R_1^0R_1^2$

$R_3^0R_2^0R_1^1 \longrightarrow -R_2^0R_1^0R_2^1R_1^1$ $R_3^0R_2^0R_1^2 \longrightarrow -R_2^0R_1^0R_2^1R_1^2$
$R_3^0R_1^2R_1^0 \longrightarrow 0$ $R_3^0R_1^3R_1^0 \longrightarrow R_2^0R_1^0R_1^3R_1^2$

$R_3^0R_2^0R_1^3R_1^1 \longrightarrow -R_2^0R_1^0R_2^1R_1^3R_1^1$ $R_3^0R_2^0R_2^1R_1^1 \longrightarrow 0$
$R_3^0R_2^0R_2^2R_1^2 \longrightarrow R_2^0R_1^0R_2^2R_2^1R_1^2$ $R_3^0R_2^0R_2^2R_1^3 \longrightarrow R_2^0R_1^0R_2^2R_2^1R_1^3$
$R_3^0R_2^1R_1^1R_1^0 \longrightarrow -R_2^0R_1^0R_2^1R_1^2R_1^1$ $R_3^0R_2^1R_1^2R_1^0 \longrightarrow 0$
$R_3^0R_2^2R_1^2R_1^0 \longrightarrow 0$ $R_3^0R_2^2R_1^3R_1^0 \longrightarrow R_2^0R_1^0R_2^2R_1^3R_1^2$

$$R_3^0R_2^0R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_3^0R_2^0R_3^1R_1^3R_1^1 \longrightarrow R_2^0R_1^0R_3^1R_2^1R_1^3R_1^1$$
$$R_3^0R_2^0R_3^1R_2^1R_1^1 \longrightarrow 0 \quad R_3^0R_2^0R_2^2R_1^3R_1^2 \longrightarrow R_2^0R_1^0R_2^2R_2^1R_1^3R_1^2$$
$$R_3^0R_2^2R_1^3R_1^2R_1^0 \longrightarrow 0$$
$$R_3^0R_2^2R_2^1R_1^2R_1^0 \longrightarrow 0 \quad R_3^0R_3^1R_1^3R_1^1R_1^0 \longrightarrow -R_2^0R_1^0R_3^1R_1^3R_1^2R_1^1$$
$$R_3^0R_3^1R_2^1R_1^1R_1^0 \longrightarrow -R_2^0R_1^0R_3^1R_2^1R_1^2R_1^1 \quad R_3^0R_3^1R_2^2R_1^3R_1^0 \longrightarrow R_2^0R_1^0R_3^1R_2^2R_1^3R_1^2$$

$$R_3^0R_2^0R_3^1R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_3^0R_2^0R_3^1R_2^1R_1^3R_1^1 \longrightarrow 0$$
$$R_3^0R_2^0R_3^1R_2^2R_1^3R_1^1 \longrightarrow -R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^1 \quad R_3^0R_2^0R_3^1R_2^2R_1^3R_1^2 \longrightarrow -R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2$$
$$R_3^0R_3^1R_2^1R_1^2R_1^1R_1^0 \longrightarrow 0 \quad R_3^0R_3^1R_2^1R_1^3R_1^1R_1^0 \longrightarrow -R_2^0R_1^0R_3^1R_2^1R_1^3R_1^2R_1^1$$
$$R_3^0R_3^1R_2^2R_1^3R_1^2R_1^0 \longrightarrow 0 \quad R_3^0R_3^1R_2^2R_2^1R_1^2R_1^0 \longrightarrow 0$$

$$R_3^0R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1 \longrightarrow 0 \quad R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1 \longrightarrow 0$$
$$R_3^0R_3^1R_2^2R_2^1R_1^3R_1^1R_1^0 \longrightarrow -R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1 \quad R_3^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^0 \longrightarrow 0$$

$$R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1 \longrightarrow 0$$

By checking the generators of $H^{*,*}(F^1)$ and $H^{*,*}(F^2/F^1)$ according to the above table, we obtain the generators of $H^{*,*}(F^2)$ as follows:

dim 2 $R_2^0R_1^0$

dim 3 $R_3^0R_1^0R_1^2$ $R_3^0R_1^0R_1^3$ $R_2^0R_1^0R_1^1$ $R_2^0R_1^0R_1^3$

dim 4 $R_3^0R_2^0R_1^3R_1^1$ $R_3^0R_2^0R_2^1R_1^1$ $R_3^0R_2^0R_2^2R_1^3$ $R_3^0R_1^0R_2^1R_1^2$
$R_3^0R_1^0R_2^2R_1^2$ $R_2^0R_1^0R_1^3R_1^1$ $R_2^0R_1^0R_2^2R_1^2$ $R_2^0R_1^0R_2^2R_1^3$

dim 5 $R_3^0R_2^0R_2^1R_1^2R_1^1$ $R_3^0R_2^0R_3^1R_2^1R_1^1$ $R_3^0R_2^0R_2^2R_1^3R_1^2$ $R_3^0R_2^0R_2^2R_2^1R_1^2$ $R_3^0R_1^0R_2^2R_1^3R_1^2$
$R_3^0R_1^0R_2^2R_2^1R_1^2$ $R_3^0R_1^0R_3^1R_1^3R_1^1$ $R_2^0R_1^0R_3^1R_1^3R_1^1$ $R_2^0R_1^0R_3^1R_2^1R_1^1$ $R_2^0R_1^0R_3^1R_2^2R_1^3$

dim 6 $R_3^0R_2^0R_3^1R_2^1R_1^2R_1^1$ $R_3^0R_2^0R_3^1R_2^1R_1^3R_1^1$ $R_3^0R_3^1R_2^1R_1^2R_1^1R_1^0$ $R_3^0R_3^1R_2^1R_1^3R_1^1R_1^0$
$R_3^0R_1^0R_3^1R_2^2R_1^3R_1^2$ $R_3^0R_1^0R_3^1R_2^2R_2^1R_1^2$ $R_2^0R_1^0R_3^1R_2^2R_1^3R_1^1$ $R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2$

dim 7 $R_3^0R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1$ $R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1$ $R_3^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2$ $R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2R_1^1$

dim 8 $R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1$

In a similar way, we use $H^{*,*}(F^2)$ and $H^{*,*}(F^3/F^2) = \{R_4^0\} \otimes H^{*,*}(K^3)$ to compute out the generators of $H^{*,*}(F^3)$ as follows:

dim 2 $R_2^0R_1^0$

dim 3 $R_3^0R_1^2R_1^0$ $R_1^1R_2^0R_1^0$ $R_1^3R_2^0R_1^0$

dim 4 $R_4^0R_2^2R_1^3R_1^0$ $R_3^0R_2^1R_2^0R_1^1$ $R_1^0R_3^0R_2^1R_1^2$ $R_2^2R_3^0R_1^2R_1^0$ $R_1^1R_1^3R_2^0R_1^0$

dim 5 $R_4^0R_3^1R_1^3R_2^0R_1^1$ $R_1^2R_4^0R_2^2R_1^3R_1^0$ $R_1^0R_4^0R_3^1R_2^2R_1^3$ $R_1^2R_3^0R_2^1R_2^0R_1^1$
$R_3^1R_3^0R_2^1R_2^0R_1^1$ $R_1^0R_2^2R_3^0R_2^1R_1^2$

dim 6 $R_4^0R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_4^0R_3^1R_1^3R_2^1R_2^0R_1^1$ $R_2^0R_1^1R_4^0R_3^1R_2^2R_1^3$ $R_1^0R_1^2R_4^0R_3^1R_2^2R_1^3$
$R_3^1R_3^0R_2^1R_1^2R_2^0R_1^1$

dim 7 $R_1^1R_4^0R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_1^3R_4^0R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_2^1R_2^0R_1^1R_4^0R_3^1R_2^2R_1^3$

dim 8 $R_1^1R_3^0R_2^1R_1^2R_4^0R_3^1R_2^2R_1^3$

(iii) Define two complexes $G^1 \subset G^2$ by

$$G^1 = \{R_3^0R_2^0R_1^0\} \otimes K^0;$$
$$G^2 = \{R_4^0R_3^0R_2^0, R_4^0R_3^0R_1^0, R_4^0R_2^0R_1^0, R_3^0R_2^0R_1^0\} \otimes K^0.$$

Since $d_1(R_3^0R_2^0R_1^0) = 0$, it follows that $H^{*,*}(G^1) = \{R_3^0R_2^0R_1^0\} \otimes H^{*,*}(K^0)$. It is easy to see that $G^2/G^1 = \{R_4^0\} \otimes F^2$, thus $H^{*,*}(G^2/G^1) = \{R_4^0\} \otimes H^{*,*}(F^2)$. For the induced cohomological long exact sequence

$$\longrightarrow H^{*-1,*}(G^2/G^1) \xrightarrow{\delta} H^{*,*}(G^1) \xrightarrow{i} H^{*,*}(G^2) \xrightarrow{j} H^{*,*}(G^2/G^1) \xrightarrow{\delta} H^{*+1,*}(G^1) \longrightarrow \quad (2.3)$$

let us consider the action of connecting homomorphism δ on $H^{*,*}(G^2/G^1)$. They are listed as follows:

$R_4^0R_2^0R_1^0 \longrightarrow -R_3^0R_2^0R_1^0R_1^3$

$R_4^0R_3^0R_1^0R_1^2 \longrightarrow R_3^0R_2^0R_1^0R_2^2R_1^2$ $\quad R_4^0R_3^0R_1^0R_1^3 \longrightarrow R_3^0R_2^0R_1^0R_2^2R_1^3 + R_4^0R_2^0R_1^0R_1^3R_1^2$

$R_4^0R_2^0R_1^0R_1^1 \longrightarrow -R_3^0R_2^0R_1^0R_1^3R_1^1$ $\quad R_4^0R_2^0R_1^0R_1^3 \longrightarrow 0$

$R_4^0R_3^0R_2^0R_1^3R_1^1 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_1^3R_1^1 + R_4^0R_2^0R_1^0R_2^1R_1^3R_1^1$

$R_4^0R_3^0R_2^0R_2^1R_1^1 \longrightarrow R_3^0R_2^0R_1^0R_3^1R_2^1R_1^1$

$R_4^0R_3^0R_2^0R_2^2R_1^3 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^2R_1^3 - R_4^0R_2^0R_1^0R_2^2R_2^1R_1^3 - R_4^0R_3^0R_1^0R_2^2R_1^3R_1^1$

$R_4^0R_3^0R_1^0R_2^1R_1^2 \longrightarrow R_3^0R_2^0R_1^0R_2^2R_2^1R_1^2$

$R_4^0R_3^0R_1^0R_2^2R_1^2 \longrightarrow 0$ $\quad R_4^0R_2^0R_1^0R_1^3R_1^1 \longrightarrow 0$

$R_4^0R_2^0R_1^0R_2^2R_1^2 \longrightarrow R_3^0R_2^0R_1^0R_2^2R_1^3R_1^2$ $\quad R_4^0R_2^0R_1^0R_2^2R_1^3 \longrightarrow 0$

$R_4^0R_3^0R_2^0R_2^1R_1^2R_1^1 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^1R_1^2R_1^1$ $\quad R_4^0R_3^0R_2^0R_3^1R_2^1R_1^1 \longrightarrow 0$

$R_4^0R_3^0R_2^0R_2^2R_1^3R_1^2 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^2R_1^3R_1^2 - R_4^0R_2^0R_1^0R_2^2R_2^1R_1^3R_1^2 + R_4^0R_3^0R_1^0R_2^2R_1^3R_1^2R_1^0$

$R_4^0R_3^0R_2^0R_2^2R_2^1R_1^2 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2$ $\quad R_4^0R_3^0R_1^0R_2^2R_1^3R_1^2 \longrightarrow 0$

$R_4^0R_3^0R_1^0R_2^2R_2^1R_1^2 \longrightarrow 0$ $\quad R_4^0R_3^0R_1^0R_3^1R_1^3R_1^1 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^2R_1^3R_1^1 + R_4^0R_2^0R_1^0R_3^1R_1^3R_1^2R_1^1$

$R_4^0R_2^0R_1^0R_3^1R_1^3R_1^1 \longrightarrow 0$ $\quad R_4^0R_2^0R_1^0R_3^1R_2^1R_1^1 \longrightarrow -R_3^0R_2^0R_1^0R_3^1R_2^1R_1^3R_1^1$

$R_4^0R_2^0R_1^0R_3^1R_2^2R_1^3 \longrightarrow 0$

$R_4^0R_3^0R_2^0R_3^1R_2^1R_1^2R_1^1 \longrightarrow 0$ $\quad R_4^0R_3^0R_2^0R_3^1R_2^1R_1^3R_1^1 \longrightarrow 0$

$R_4^0R_3^0R_3^1R_2^1R_1^2R_1^1R_1^0 \longrightarrow R_3^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2R_1^1$

$R_4^0R_3^0R_3^1R_2^1R_1^3R_1^1R_1^0 \longrightarrow R_3^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^1 + R_4^0R_2^0R_1^0R_3^1R_2^1R_1^3R_1^2R_1^1$

$R_4^0R_3^0R_1^0R_3^1R_2^2R_1^3R_1^2 \longrightarrow 0$ $\quad R_4^0R_3^0R_1^0R_3^1R_2^2R_2^1R_1^2 \longrightarrow 0$

$R_4^0R_2^0R_1^0R_3^1R_2^2R_1^3R_1^1 \longrightarrow 0$ $\quad R_4^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2 \longrightarrow R_3^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2$

$R_4^0R_3^0R_2^0R_3^1R_2^2R_2^1R_1^2R_1^1 \longrightarrow 0$ $\quad R_4^0R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^1 \longrightarrow 0$

$R_4^0R_3^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2 \longrightarrow 0$ $\quad R_4^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^2R_1^1 \longrightarrow R_3^0R_2^0R_1^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1$

$R_4^0R_3^0R_2^0R_3^1R_2^2R_2^1R_1^3R_1^2R_1^1 \longrightarrow 0$

According to the above table, the generators which are connected by δ will vanish in $H^{*,*}(G^2)$ and the remaining generators of $H^{*,*}(G^1)$ and $H^{*,*}(G^2/G^1)$ are the generators of $H^*(G^2)$. They are listed as follows:

dim 3 $R_3^0R_2^0R_1^0$

dim 4 $R_1^1\cdot R_3^0R_2^0R_1^0$ $R_1^2\cdot R_3^0R_2^0R_1^0$ $R_4^0R_1^3R_2^0R_1^0$

dim 5 $R_2^1R_1^1\cdot R_3^0R_2^0R_1^0$ $R_2^1R_1^2\cdot R_3^0R_2^0R_1^0$ $R_2^2R_1^3\cdot R_3^0R_2^0R_1^0$ $R_4^0R_2^2R_3^0R_1^2R_1^0$ $R_1^1\cdot R_4^0R_1^3R_2^0R_1^0$ $R_4^0R_2^2R_1^3R_2^0R_1^0$

dim 6 $R_1^1\cdot R_2^1R_1^2\cdot R_3^0R_2^0R_1^0$ $R_4^0R_2^2R_1^3R_3^0R_1^2R_2^0$ $R_1^3\cdot R_4^0R_2^2R_3^0R_1^2R_1^0$ $R_4^0R_2^2R_3^0R_2^1R_1^2R_1^0$ $R_4^0R_3^1R_2^2R_1^3R_2^0R_1^0$ $R_1^0\cdot R_4^0R_3^1R_1^3R_2^0R_1^1$

dim 7 $R_3^1R_2^2R_1^3R_3^0R_2^0R_1^1R_1^0$ $R_4^0R_3^1R_3^0R_2^1R_1^2R_2^0R_1^1$ $R_4^0R_3^1R_1^3R_3^0R_2^1R_2^0R_1^1$ $R_3^0R_1^2R_1^0\cdot R_4^0R_3^1R_2^2R_1^3$ $R_1^0\cdot R_4^0R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_1^1\cdot R_2^0R_1^0\cdot R_4^0R_3^1R_2^2R_1^3$

dim 8 $R_2^0R_1^1\cdot R_4^0R_3^1R_2^2R_3^0R_2^1R_1^2$ $R_4^0R_3^1R_2^2R_1^3\cdot R_3^0R_2^1R_2^0R_1^1$ $R_1^0\cdot R_3^0R_2^1R_1^2\cdot R_4^0R_3^1R_2^2R_1^3$

dim 9 $R_1^2\cdot R_4^0R_3^1R_2^2R_1^3\cdot R_3^0R_2^1R_2^0R_1^1$

(iv) Let us define $N=\{R_4^0R_3^0R_2^0R_1^0\}\otimes K^0$. Since $d_1(R_4^0R_3^0R_2^0R_1^0)=0$, it follows that

$$H^{*,*}(N)=\{R_4^0R_3^0R_2^0R_1^0\}\otimes H^{*,*}(K^0).$$

Since $E[R_i^j|0<i+j\le 4, j\ge 0]=K^0\oplus K^4\oplus F^4\oplus G^2\oplus N$, thus we have

$$H^{*,*}(U(L))=H^{*,*}(K^0)\oplus H^{*,*}(K^4)\oplus H^{*,*}(F^4)\oplus H^{*,*}(G^2)\oplus H^{*,*}(N).$$

The above computation in each part together gives the desired results. The multiplication among the generators of $H^{*,*}(U(L))$ has already been marked in the list of its each summand. □

3 Convergence of Generators

In this section, we will show that the obtained generators of $H^{*,*}(U(L))$ can converge nontrivially to $\mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$. By reference [8], there is a spectral sequence $\{E_r^{s,t,*},d_r\}$ which converges to $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$. Its E_1-term is

$$E_1^{*,*,*}=E(h_{m,i}|m>0,i\ge 0)\otimes P(b_{m,i}|m>0,i\ge 0)\otimes P(a_n|n\ge 0)$$

where

$$h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1},\quad b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},(2m-1)p},\quad a_n\in E_1^{1,2p^n-1,2n+1}.$$

One has the r-th differential $d_r\colon E_r^{s,t,M}\to E_r^{s+1,t,M-r}$ for $r\ge 1$. For $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$, there is $d_r(x\cdot y)=d_r(x)\cdot y+(-1)^s x\cdot d_r(y)$ and for $x,y\in\{h_{m,i},\ b_{m,i},\ a_n\}$ there is $x\cdot y=(-1)^{ss'+tt'}y\cdot x$. The first differential $d_1\colon E_1^{s,t,M}\to E_1^{s+1,t,M-1}$ is given by

$$d_1(h_{i,j})=\sum_{0<k<i}h_{i-k,k+j}h_{k,j},\quad d_1(a_i)=\sum_{0\le k<i}h_{i-k,k}a_k,\quad d_1(b_{i,j})=0.$$

For the May spectral sequence (1.2) in Section 1,

$$E_2^{*,*}=P(b_i^j)\otimes H^{*,*}(U(L))\cong P(b_i^j)\otimes H^{*,*}(E(R_i^j),\delta)\Longrightarrow H^{*,*}(V(L)),$$

by the reasons of degree and dimension, there is an isomorphism

$$P(b_{ij})\otimes H^{*,*}(E(h_{ij}),d_1)\cong P(b_i^j)\otimes H^{*,*}(E(R_i^j),\delta)$$

by identifying b_{ij} and sending h_{ij} to R_i^j. Hence every $\{R_i^j\}$ in $H^{*,*}(U(L))$ has a unique preimage h_{ij} in $E_1^{*,*,*}$. It follows that every generator in $H^{*,*}(U(L))$ has a unique preimage in $E_1^{*,*,*}$. Thus in order to prove that the generators of $H^{*,*}(U(L))$ converge into $\mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$, it is sufficient to prove that their corresponding preimages in $E_1^{*,*,*}$ converge into $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p)$. This relation can be shown in the following diagram:

$$\begin{tikzcd}
E_1^{*,*,*} \arrow[r, Rightarrow] \arrow[d] & \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p) \arrow[d] \\
P(b_i^j)\otimes H^{*,*}(U(L)) \arrow[r, Rightarrow] & \mathrm{Ext}_{\mathbb{P}}^{*,*}(\mathbb{Z}/p,\mathbb{Z}/p).
\end{tikzcd}$$

Theorem 3.1 *Each generator of $H^{s,t}(U(L))$ with $t-s<\max\{(5p^3+6p^2+6p+4)q-10, p^4q\}$ is a permanent cycle and converges nontrivially into $\mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$.*

Proof According to the above statement we use the same notations in $E_1^{*,*,*}$ to denote the preimages of the corresponding generators in Theorem 1.1. It is well known that h_i, g_i and k_i in $H^{*,*}(U(L))$ converge nontrivially to h_i, g_i and k_i in $\mathrm{Ext}_{\mathbb{P}}^{s,t}(\mathbb{Z}/p,\mathbb{Z}/p)$, respectively. Thus we need only to verify the convergence of the other generators of $H^{*,*}(U(L))$.

Suppose we are given a generator $x\in E_1^{s,t,M}$, in order to show the convergence of x we need to first show that any May differential $d_r\colon E_r^{s,t,M}\to E_r^{s+1,t,M-r}$ on x is trivial and x also can not be hit by any other May differential $d_r\colon E_r^{s-1,t,M+r}\to E_r^{s,t,M}$. For each preimage of $H^{*,*}(U(L))$ in $E_1^{*,*,*}$, we list a table as follows.

generators $\in E_1^{s,t,M}$	s	t	M	$E_1^{s-1,t,*}$	$E_1^{s+1,t,*}$
$l_1=h_{30}h_{20}h_{10}$	3	$(p^2+2p+3)q$	9	none	none
$l_2=h_{21}h_{20}h_{11}$	3	$(p^2+3p+1)q$	7	none	$h_{30}h_{11}b_{10}$ $h_{20}h_{21}b_{10}$ $h_{20}h_{11}b_{20}$
$l_3=h_{30}h_{12}h_{10}$	3	$(2p^2+p+2)q$	7	none	$h_{30}h_{10}b_{11}$
$l_4=h_{30}h_{21}h_{12}$	3	$(3p^2+2p+1)q$	9	none	$h_{30}h_{21}b_{11}$ $h_{30}h_{12}b_{20}$
$l_5=h_{31}h_{21}h_{11}$	3	$(p^3+2p^2+3p)q$	9	none	$h_{31}h_{21}b_{10}$ $h_{31}h_{11}b_{20}$ $h_{21}h_{11}b_{30}$
$l_6=h_{22}h_{21}h_{12}$	3	$(p^3+3p^2+p)q$	7	none	$h_{31}h_{12}b_{11}$ $h_{21}h_{22}b_{11}$ $h_{21}h_{12}b_{21}$
$l_7=h_{31}h_{13}h_{11}$	3	$(2p^3+p^2+2p)q$	7	none	$h_{31}h_{13}b_{10}$ $h_{11}h_{13}b_{30}$ $h_{31}h_{11}b_{12}$
$l_8=h_{31}h_{22}h_{13}$	3	$(3p^3+2p^2+p)q$	9	none	$h_{31}h_{13}b_{10}$ $h_{22}h_{13}b_{30}$ $h_{31}h_{22}b_{12}$
$m_1=h_{30}h_{21}h_{20}h_{11}$	4	$(2p^2+4p+2)q$	12	none	$h_{30}h_{20}h_{21}b_{10}$ $h_{30}h_{20}h_{11}b_{20}$
$m_2=h_{40}h_{30}h_{20}h_{10}$	4	$(p^3+2p^2+3p+4)q$	16	none	none
$m_3=h_{31}h_{21}h_{20}h_{11}$	4	$(p^3+2p^2+4p+1)q$	12	none	$h_{40}h_{21}h_{11}b_{10}$ $h_{20}h_{31}h_{11}b_{20}$ $h_{20}h_{21}h_{11}b_{30}$ $h_{31}h_{21}h_{20}b_{10}$

generators $\in E_1^{s,t,M}$	s	t	M	$E_1^{s-1,t,*}$	$E_1^{s+1,t,*}$
$m_4 = h_{22}h_{30}h_{12}h_{10}$	4	$(p^3+3p^2+p+2)q$	10	none	$h_{40}h_{10}h_{12}b_{10}$ $h_{30}h_{10}h_{22}b_{11}$ $h_{30}h_{10}h_{12}b_{21}$
$m_5 = h_{22}h_{30}h_{21}h_{12}$	4	$(p^3+4p^2+2p+1)q$	12	none	$h_{40}h_{21}h_{12}b_{11}$ $h_{30}h_{31}h_{12}b_{11}$ $h_{30}h_{21}h_{22}b_{11}$ $h_{30}h_{21}h_{12}b_{21}$
$m_6 = h_{40}h_{13}h_{20}h_{10}$	4	$(2p^3+p^2+2p+3)q$	12	none	$h_{40}h_{20}h_{10}b_{12}$
$m_7 = h_{31}h_{13}h_{20}h_{11}$	4	$(2p^3+p^2+3p+1)q$	12	none	$h_{20}h_{11}h_{13}b_{30}$ $h_{20}h_{31}h_{11}b_{12}$ $h_{31}h_{13}h_{20}b_{10}$ $h_{40}h_{13}h_{11}b_{10}$
$m_8 = h_{31}h_{22}h_{21}h_{12}$	4	$(2p^3+4p^2+2p)q$	12	none	$h_{31}h_{22}h_{21}b_{11}$ $h_{31}h_{22}h_{12}b_{20}$ $h_{31}h_{21}h_{12}b_{21}$ $h_{22}h_{21}h_{12}b_{30}$
$m_9 = h_{40}h_{22}h_{13}h_{10}$	4	$(3p^3+2p^2+p+2)q$	12	none	$h_{40}h_{22}h_{10}b_{12}$ $h_{40}h_{13}h_{10}b_{21}$
$m_{10} = h_{40}h_{31}h_{22}h_{13}$	4	$(4p^3+3p^2+2p+1)q$	16	none	$h_{40}h_{31}h_{22}b_{12}$ $h_{40}h_{31}h_{13}b_{21}$ $h_{40}h_{22}h_{13}b_{30}$
$n_1 = h_{31}h_{30}h_{21}h_{20}h_{11}$	5	$(p^3+4p^2+4p+1)q$	17	none	$h_{40}h_{30}h_{21}h_{11}b_{10}$ $h_{40}h_{21}h_{20}h_{11}b_{20}$ $h_{30}h_{20}h_{31}h_{21}b_{10}$ $h_{31}h_{30}h_{20}h_{11}h_{20}$ $h_{30}h_{21}h_{20}h_{11}h_{30}$
$n_2 = h_{31}h_{30}h_{21}h_{12}h_{11}$	5	$(p^3+4p^2+4p+1)q$	15	none	$h_{40}h_{21}h_{12}h_{11}b_{20}$ $h_{31}h_{30}h_{21}h_{12}b_{10}$ $h_{31}h_{30}h_{21}h_{11}b_{11}$ $h_{31}h_{30}h_{12}h_{11}b_{20}$ $h_{30}h_{21}h_{12}h_{11}b_{30}$ $h_{40}h_{21}b_{20}^2$ $h_{30}h_{31}b_{20}^2$ $h_{30}h_{21}b_{30}b_{20}$
$n_3 = h_{31}h_{13}h_{21}h_{20}h_{11}$	5	$(2p^3+2p^2+4p+1)q$	13	none	$h_{31}h_{13}h_{21}h_{20}b_{10}$ $h_{31}h_{13}h_{20}h_{11}b_{20}$ $h_{31}h_{21}h_{20}h_{11}b_{12}$ $h_{13}h_{21}h_{20}h_{11}b_{30}$ $h_{20}h_{11}b_{30}^2$
$n_4 = h_{31}h_{22}h_{30}h_{21}h_{12}$	5	$(2p^3+5p^2+3p+1)q$	17	none	$h_{31}h_{22}h_{30}h_{21}b_{11}$ $h_{31}h_{22}h_{30}h_{12}b_{20}$ $h_{31}h_{30}h_{21}h_{12}b_{21}$ $h_{22}h_{30}h_{21}h_{12}b_{30}$
$n_5 = h_{40}h_{22}h_{13}h_{20}h_{10}$	5	$(3p^3+2p^2+2p+3)q$	15	none	$h_{40}h_{22}h_{20}h_{10}b_{12}$ $h_{40}h_{13}h_{20}h_{10}b_{21}$

generators $\in E_1^{s,t,M}$	s	t	M	$E_1^{s-1,t,*}$	$E_1^{s+1,t,*}$
$n_6 = h_{40}h_{31}h_{13}h_{20}h_{11}$	5	$(3p^3+2p^2+4p+2)q$	17	none	$h_{40}h_{31}h_{13}h_{20}b_{10}$ $h_{40}h_{31}h_{20}h_{11}b_{12}$ $h_{40}h_{13}h_{20}h_{11}b_{30}$
$n_7 = h_{40}h_{22}h_{30}h_{12}h_{10}$	5	$(2p^3+4p^2+2p+3)q$	17	none	$h_{40}h_{22}h_{30}h_{10}b_{11}$ $h_{40}h_{30}h_{12}h_{10}b_{21}$
$u_1 = h_{31}h_{30}h_{21}h_{12}h_{20}h_{11}$	6	$(p^3+4p^2+5p+2)q$	18	none	$h_{31}h_{30}h_{21}h_{12}h_{20}b_{10}$ $h_{31}h_{30}h_{21}h_{20}h_{11}b_{11}$ $h_{31}h_{30}h_{12}h_{20}h_{11}b_{20}$ $h_{30}h_{21}h_{12}h_{20}h_{11}b_{30}$ $h_{40}h_{21}h_{12}h_{20}h_{11}b_{20}$ $h_{40}h_{30}h_{21}h_{12}h_{11}b_{10}$ $h_{31}h_{30}h_{20}b_{20}^2$
$u_2 = h_{40}h_{31}h_{13}h_{21}h_{20}h_{11}$	6	$(3p^3+3p^2+5p+2)q$	20	none	$h_{40}h_{31}h_{13}h_{21}h_{20}b_{10}$ $h_{40}h_{31}h_{13}h_{20}h_{11}b_{20}$ $h_{40}h_{31}h_{21}h_{20}h_{11}b_{12}$ $h_{40}h_{13}h_{21}h_{20}h_{11}b_{30}$ $h_{40}h_{31}h_{22}h_{20}h_{11}b_{10}$ $h_{40}h_{31}h_{13}h_{30}h_{11}b_{10}$ $h_{40}h_{20}h_{11}b_{30}^2$
$u_3 = h_{40}h_{31}h_{22}h_{13}h_{20}h_{10}$	6	$(4p^3+3p^2+3p+3)q$	20	none	$h_{40}h_{31}h_{22}h_{20}h_{10}b_{12}$ $h_{40}h_{31}h_{13}h_{20}h_{10}b_{21}$ $h_{40}h_{22}h_{13}h_{20}h_{10}b_{30}$
$u_4 = h_{40}h_{22}h_{30}h_{21}h_{12}h_{10}$	6	$(2p^3+5p^2+3p+3)q$	20	none	$h_{40}h_{22}h_{30}h_{21}h_{10}b_{11}$ $h_{40}h_{22}h_{30}h_{12}h_{10}b_{20}$ $h_{40}h_{30}h_{21}h_{12}h_{10}b_{21}$ $h_{40}h_{31}h_{30}h_{12}h_{10}b_{20}$
$u_5 = h_{40}h_{22}h_{13}h_{30}h_{12}h_{20}$	6	$(3p^3+4p^2+3p+3)q$	20	none	$h_{40}h_{22}h_{13}h_{30}h_{20}b_{11}$ $h_{40}h_{22}h_{30}h_{12}h_{20}b_{12}$ $h_{40}h_{13}h_{30}h_{12}h_{20}b_{21}$ $h_{40}h_{30}h_{20}b_{21}^2$
$u_6 = h_{40}h_{31}h_{22}h_{30}h_{21}h_{12}$	6	$(3p^3+6p^2+4p+2)q$	24	none	$h_{40}h_{31}h_{22}h_{30}h_{21}b_{11}$ $h_{40}h_{31}h_{22}h_{30}h_{12}b_{20}$ $h_{40}h_{31}h_{30}h_{21}h_{12}b_{21}$ $h_{40}h_{22}h_{30}h_{21}h_{12}b_{30}$

generators $\in E_1^{s,t,M}$	s	t	M	$E_1^{s-1,t,*}$	$E_1^{s+1,t,*}$
$v_1 = h_{31}h_{22}h_{13}h_{30}h_{20}h_{11}h_{10}$	7	$(3p^3+3p^2+4p+3)q$	19	none	$h_{31}h_{22}h_{13}h_{30}h_{20}h_{10}b_{10}$ $h_{31}h_{22}h_{30}h_{20}h_{11}h_{10}b_{12}$ $h_{31}h_{13}h_{30}h_{20}h_{11}h_{10}b_{21}$ $h_{22}h_{13}h_{30}h_{20}h_{11}h_{10}b_{30}$ $h_{13}h_{30}h_{20}h_{10}b_{30}^2$
$v_2 = h_{40}h_{31}h_{30}h_{21}h_{12}h_{20}h_{11}$	7	$(2p^3+5p^2+6p+3)q$	25	none	$h_{40}h_{31}h_{30}h_{21}h_{12}h_{20}b_{10}$ $h_{40}h_{31}h_{30}h_{21}h_{20}h_{11}b_{11}$ $h_{40}h_{31}h_{30}h_{12}h_{20}h_{11}b_{20}$ $h_{40}h_{30}h_{21}h_{12}h_{20}h_{11}h_{30}$ $h_{40}h_{31}h_{30}h_{21}h_{11}h_{10}b_{20}$ $h_{40}h_{31}h_{30}h_{20}b_{20}^2$
$n_7 = h_{40}h_{22}h_{30}h_{12}h_{10}$	5	$(2p^3+4p^2+2p+3)q$	17	none	$h_{40}h_{22}h_{30}h_{10}b_{11}$ $h_{40}h_{30}h_{12}h_{10}b_{21}$
$v_3 = h_{40}h_{31}h_{13}h_{30}h_{21}h_{20}h_{11}$	7	$(3p^3+4p^2+6p+3)q$	25	none	$h_{40}h_{31}h_{13}h_{30}h_{21}h_{20}b_{10}$ $h_{40}h_{31}h_{13}h_{30}h_{20}h_{11}b_{20}$ $h_{40}h_{31}h_{30}h_{21}h_{20}h_{11}b_{12}$ $h_{40}h_{13}h_{30}h_{21}h_{20}h_{11}b_{30}$ $h_{40}h_{31}h_{22}h_{30}h_{20}h_{11}b_{10}$ $h_{40}h_{30}h_{20}h_{11}b_{30}^2$

According to the above table, we see that for every preimage $x \in E_1^{s,t,M}$ and the corresponding r-th differential $d_r\colon E_r^{s,t,M} \to E_r^{s+1,t,M-r}$, the E_1-term $E_1^{s+1,t,M-r}$ is either zero or has generators with May filtrations greater than $M-r$. It follows that x has trivial May differential and then it is a permanent cycle in the spectral sequence $\{E_r^{s,t,*}, d_r\}$. Also since $E_1^{s-1,t,M+r}$ is trivial, it follows that x can not be hit by the May differential starting from $E_r^{s-1,t,M+r}$. Thus x converges nontrivially into $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$. This finishes our proof. $\square$

Acknowledgements The authors would like to give their thanks to the referee for his/her many helpful suggestions on the original version of this paper.

References

[1] Aikawa, T.: 3-dimensional cohomology of the mod p Steenrod algebra. *Math. Scand.*, **47**, 91–115 (1980)
[2] Cohen, F., Moore, J., Neisendorfer, J.: The double suspension and exponents of the homotopy groups of spheres. *Ann. of Math.*, **110**, 549–565 (1979)
[3] Liulevicius, A.: The factorizations of cyclic reduced powers by secondary cohomology operations. *Mem. Amer. Math. Soc.*, **42**, (1962)
[4] May, J. P.: The Cohomology of Restricted Lie Algebras and of Hopf Algebras, PhD Thesis, Princeton, 1964
[5] May, J. P.: The cohomology of restricted Lie algebras and of Hopf algebras. *J. Algebra*, **3**, 123–146 (1966)
[6] Milnor, J., Moore, J. C.: On the structure of Hopf algebra. *Ann. of Math.*, **81**, 211–264 (1965)
[7] Nave, L. S.: The Smith–Toda complex $V((p+1)/2)$ does not exist. *Ann. of Math.*, **171**, 491–509 (2010)

[8] Ravenel, D. C.: Complex Cobordism and Stable Homotopy Groups of Spheres, Academic Press, Orlando, 1986

[9] Toda, H.: On spectra realizing exterior parts of the Steenrod algebra. *Topology*, **10**, 53–65 (1971)

[10] Yu, H. B., Shen, W. H., Zhao, H.: Convergence of the generators of $H^{*,*}(U(L))$. *Southeast Asian Bull. Math.*, **35**, 185–190 (2011)