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Irreducible Wakimoto-like Modules for the Lie Superalgebra $D(2,1;\alpha)$

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Abstract By using the idea of Wakimoto's free field, we construct a class of representations for the Lie superalgebra $D(2,1;\alpha)$ on the tensor product of a polynomial algebra and an exterior algebra involving one parameter λ. Then we obtain the necessary and sufficient condition for the representations to be irreducible. In fact, the representation is irreducible if and only if the parameter λ satisfies $(\lambda+m)(\lambda-\frac{1+\alpha}{\alpha}m)\neq 0$ for any $m\in\mathbb{Z}_+$.

Keywords Lie superalgebra, representation, Wakimoto's free field

MR(2010) Subject Classification 17B10, 17B25

1 Introduction

Lie algebras, Lie groups and their representation theories are important parts of modern mathematics. They played a central role in the description of symmetries. Lie superalgebras and their representations came from the understanding and exploitation of supersymmetry in physical systems. Since the classification of the simple complex finite-dimensional Lie superalgebras was completed by Kac [8] in 1977, these superalgebras have found applications in various areas including quantum mechanics, nuclear physics, particle physics, and string theory.

The representation theory of Lie superalgebra is so different from the one of complex semisimple Lie algebras. More and more works on the representation and character theory about Lie superalgebra are finished, such as [10, 11], even on the general Kac–Moody case [2, 7], and so on. While for the exceptional Lie superalgebra of type $D(2,1;\alpha)$, it is too discrete to describe the module theory. Understanding their module theory has been a very difficult problem, even at the level of the finite-dimensional simple modules.

In this paper, we aim to extend the understanding of the module theory of Lie superalgebras by drawing motivation from Wakimoto's free fields construction, which provides a remarkable way to realize affine Kac–Moody Lie algebras (see [4, 9, 12]). This approach has also been successfully used to construct the representation of $\widetilde{gl_N(\mathbb{C}_q)}$, the extended affine Lie algebra

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of type A coordinatized by a quantum torus, over a polynomial ring with infinitely many variables (see [5, 6, 14]). This idea has also been used to construct the representation of the Lie superalgebra $\widehat{\mathfrak{gl}_{m|n}}(\mathbb{C}_q)$ (see [1, 13]). Since the construction given in [6] is different from Wakimoto's original constructions, the researchers called this as Wakimoto-like construction. We construct a class of representations for the Lie superalgebra $D(2,1;\alpha)$ on the tensor product of a polynomial algebra and an exterior algebra involving one parameter λ, and we also obtain the necessary and sufficient condition for the representations to be irreducible.

The organization of this paper is as follows. In Section 2, we introduce the exceptional Lie superalgebra $D(2,1;\alpha)$. In the first part of Section 3, we construct a family of representations for the Lie superalgebra $D(2,1;\alpha)$ by using the idea of Wakimoto's free field. We end the paper by discussing the irreducibility of the representations.

Throughout this paper, we denote by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{C}$ the sets of integers, nonnegative integers and complex numbers respectively. All vector spaces and Lie (super)algebras are over $\mathbb{C}$.

2 The Exceptional Lie Superalgebra $D(2,1;\alpha)$

The exceptional Lie superalgebra $D(2,1;\alpha)$ is a deformation of the Lie superalgebra $osp\,(4|2)$ with a continuous parameter $\alpha \neq 0, -1, \infty$. It forms a one-parameter family of superalgebras of rank 3 and dimension 17 (see [3, 8]). The bosonic (or even) part is $sl_2 \oplus sl_2 \oplus sl_2$ and the fermionic (or odd) part is a spinorial representation $(2,2,2)$ of the even part.

In terms of the orthogonal basis vector $\epsilon_i\,(i = 1,2,3)$ with the bilinear form

$$(\epsilon_1,\epsilon_1) = -\frac{1+\alpha}{2}, \quad (\epsilon_2,\epsilon_2) = \frac{1}{2}, \quad (\epsilon_3,\epsilon_3) = \frac{\alpha}{2}, \quad (\epsilon_i,\epsilon_j) = 0 \quad \text{for } i \neq j,$$

the root system $\Delta = \Delta_0 \cup \Delta_1$ of $D(2,1;\alpha)$ is given by

$$\Delta_0 = \{\pm 2\epsilon_1, \pm 2\epsilon_2, \pm 2\epsilon_3\}, \quad \Delta_1 = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}.$$

Let $\prod = \{\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = 2\epsilon_2, \alpha_3 = 2\epsilon_3\}$ be the simple roots with α_1 being fermionic and α_2, α_3 being bosonic, the associated Cartan matrix is $(a_{ij}) = \begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.

The positive even roots Δ_0^+ and the positive odd roots Δ_1^+ are then given respectively by

$$\Delta_0^+ = \{\alpha_2, \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3\},$$
$$\Delta_1^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\},$$

the set of all positive roots Δ^+ is a union of the positive even and odd roots, namely, $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$.

Associated with each positive root δ, there is a raising operator e_δ, a lowering operator $f_\delta = e_{-\delta}$ and a Cartan generator h_δ. These operators have definite $\mathbb{Z}_2$-gradings:

$$[h_\delta] = 0, \quad [e_\delta] = [f_\delta] = \begin{cases} 0, & \delta \in \Delta_0^+, \\ 1, & \delta \in \Delta_1^+. \end{cases}$$

For any two homogenous elements (i.e., elements with definite $\mathbb{Z}_2$-gradings) $a, b \in D(2,1;\alpha)$, the (anti)commutator is defined by $[a,b] = ab - (-1)^{[a][b]}ba$, this commutator extends to inhomogenous elements through linearity.

Specifically, we describe all the relations of $D(2,1;\alpha)$ as the following.

Proposition 2.1 ([3])

$$
\begin{aligned}
&[e_{\alpha_i}, f_{\alpha_j}] = \delta_{ij} h_{\alpha_i},\\
&[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad i,j = 1,2,3,\\
&[h_{\alpha_i}, e_{\alpha_j}] = a_{ij} e_{\alpha_j},\\
&[h_{\alpha_i}, f_{\alpha_j}] = -a_{ij} f_{\alpha_j}, \quad i,j = 1,2,3,\\
&[e_{\alpha_1}, e_{\alpha_2}] = -e_{\alpha_1+\alpha_2}, \quad [e_{\alpha_1}, e_{\alpha_3}] = -e_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1}, e_{\alpha_1+\alpha_2+\alpha_3}] = -(1+\alpha) e_{2\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_1}, h_{\alpha_2}] = [e_{\alpha_1}, h_{\alpha_3}] = e_{\alpha_1},\\
&[e_{\alpha_1}, f_{\alpha_1+\alpha_2}] = f_{\alpha_2}, \quad [e_{\alpha_1}, f_{\alpha_1+\alpha_3}] = \alpha f_{\alpha_3},\\
&[e_{\alpha_1}, f_{2\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_2}, e_{\alpha_1+\alpha_3}] = e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_2}, h_{\alpha_1}] = -e_{\alpha_2}, \quad [e_{\alpha_2}, h_{\alpha_2}] = -2e_{\alpha_2},\\
&[e_{\alpha_2}, f_{\alpha_1+\alpha_2}] = f_{\alpha_1}, \quad [e_{\alpha_2}, f_{\alpha_1+\alpha_2+\alpha_3}] = f_{\alpha_1+\alpha_3},\\
&[e_{\alpha_3}, e_{\alpha_1+\alpha_2}] = e_{\alpha_1+\alpha_2+\alpha_3}, \quad [e_{\alpha_3}, h_{\alpha_1}] = -\alpha e_{\alpha_3},\\
&[e_{\alpha_3}, h_{\alpha_3}] = -2e_{\alpha_3}, \quad [e_{\alpha_3}, f_{\alpha_1+\alpha_3}] = f_{\alpha_1},\\
&[e_{\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}] = f_{\alpha_1+\alpha_2},\\
&[e_{\alpha_1+\alpha_2}, h_{\alpha_1}] = [e_{\alpha_1+\alpha_2}, h_{\alpha_2}] = -e_{\alpha_1+\alpha_2},\\
&[e_{\alpha_1+\alpha_2}, h_{\alpha_3}] = e_{\alpha_1+\alpha_2},\\
&[e_{\alpha_1+\alpha_2}, e_{\alpha_1+\alpha_3}] = (1+\alpha) e_{2\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_1+\alpha_2}, f_{\alpha_1}] = -e_{\alpha_2},\\
&[e_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_2}] = -h_{\alpha_1} + h_{\alpha_2},\\
&[e_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_2+\alpha_3}] = -\alpha f_{\alpha_3},\\
&[e_{\alpha_1+\alpha_2}, f_{2\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1+\alpha_3}, h_{\alpha_1}] = -\alpha e_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1+\alpha_3}, h_{\alpha_2}] = e_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1+\alpha_3}, h_{\alpha_3}] = -e_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1+\alpha_3}, f_{\alpha_1}] = -\alpha e_{\alpha_3}, \quad [e_{\alpha_1+\alpha_3}, f_{\alpha_3}] = -e_{\alpha_1},\\
&[e_{\alpha_1+\alpha_3}, f_{\alpha_1+\alpha_2}] = -h_{\alpha_1} + \alpha h_{\alpha_3},\\
&[e_{\alpha_1+\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_2},\\
&[e_{\alpha_1+\alpha_3}, f_{2\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1+\alpha_2},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, h_{\alpha_1}] = -(1+\alpha) e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, h_{\alpha_2}] = -e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, h_{\alpha_3}] = -e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_2}] = -e_{\alpha_1+\alpha_3},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_3}] = -e_{\alpha_1+\alpha_2},
\end{aligned}
$$

$$
\begin{aligned}
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_2}] = \alpha e_{\alpha_3},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_3}] = e_{\alpha_2},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}] = h_{\alpha_1} - h_{\alpha_2} - \alpha h_{\alpha_2},\\
&[e_{\alpha_1+\alpha_2+\alpha_3}, f_{2\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, h_{\alpha_1}] = -(1+\alpha) e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1}] = e_{\alpha_1+\alpha_2+\alpha_3},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_2}] = e_{\alpha_1+\alpha_3},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_3}] = e_{\alpha_1+\alpha_2},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}] = e_{\alpha_1},\\
&[e_{2\alpha_1+\alpha_2+\alpha_3}, f_{2\alpha_1+\alpha_2+\alpha_3}] = \frac{2}{1+\alpha} h_{\alpha_1} - \frac{1}{1+\alpha} h_{\alpha_2} - \frac{\alpha}{1+\alpha} h_{\alpha_3},\\
&[h_{\alpha_1}, f_{\alpha_2}] = -f_{\alpha_2}, \quad [h_{\alpha_1}, f_{\alpha_3}] = -\alpha f_{\alpha_3},\\
&[h_{\alpha_1}, f_{\alpha_1+\alpha_2}] = -f_{\alpha_1+\alpha_2},\\
&[h_{\alpha_1}, f_{\alpha_1+\alpha_3}] = -\alpha f_{\alpha_1+\alpha_3},\\
&[h_{\alpha_1}, f_{\alpha_1+\alpha_2+\alpha_3}] = -(1+\alpha) f_{\alpha_1+\alpha_2+\alpha_3},\\
&[h_{\alpha_1}, f_{2\alpha_1+\alpha_2+\alpha_3}] = -(1+\alpha) f_{2\alpha_1+\alpha_2+\alpha_3},\\
&[h_{\alpha_2}, f_{\alpha_1}] = f_{\alpha_1}, \quad [h_{\alpha_2}, f_{\alpha_2}] = -2 f_{\alpha_2},\\
&[h_{\alpha_2}, f_{\alpha_1+\alpha_2}] = -f_{\alpha_1+\alpha_2}, \quad [h_{\alpha_2}, f_{\alpha_1+\alpha_3}] = f_{\alpha_1+\alpha_3},\\
&[h_{\alpha_2}, f_{\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1+\alpha_2+\alpha_3},\\
&[h_{\alpha_3}, f_{\alpha_1}] = f_{\alpha_1}, \quad [h_{\alpha_3}, f_{\alpha_3}] = -2 f_{\alpha_3},\\
&[h_{\alpha_3}, f_{\alpha_1+\alpha_2}] = f_{\alpha_1+\alpha_2},\\
&[h_{\alpha_3}, f_{\alpha_1+\alpha_3}] = -f_{\alpha_1+\alpha_3},\\
&[h_{\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}] = -f_{\alpha_1+\alpha_2+\alpha_3},\\
&[f_{\alpha_1}, f_{\alpha_2}] = -f_{\alpha_1+\alpha_2}, \quad [f_{\alpha_1}, f_{\alpha_3}] = -f_{\alpha_1+\alpha_3},\\
&[f_{\alpha_1}, f_{\alpha_1+\alpha_2+\alpha_3}] = (1+\alpha) f_{2\alpha_1+\alpha_2+\alpha_3},\\
&[f_{\alpha_2}, f_{\alpha_1+\alpha_3}] = f_{\alpha_1+\alpha_2+\alpha_3},\\
&[f_{\alpha_3}, f_{\alpha_1+\alpha_2}] = f_{\alpha_1+\alpha_2+\alpha_3},\\
&[f_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_3}] = -(1+\alpha) f_{2\alpha_1+\alpha_2+\alpha_3}.
\end{aligned}
$$

3 A Family of Modules and Their Structure

3.1 Construction of the Wakimoto-like Modules

Let

$$W = \mathbb{C}[x_1, x_2] \otimes \Lambda(y_3, y_4)$$

be the tensor product of a polynomial algebra with two variables x_1, x_2 and an exterior algebra with two variables y_3, y_4 over $\mathbb{C}$.

Then we can get

$$[x_i, x_j] = [x_i, y_k] = \{y_k, y_l\} = \left\{\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right\} = 0,$$
$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_k}\right] = \left[\frac{\partial}{\partial x_i}, y_k\right] = \left[\frac{\partial}{\partial y_k}, x_i\right] = 0,$$
$$\left[\frac{\partial}{\partial x_i}, x_j\right] = \delta_{i,j}, \quad \left\{y_k, \frac{\partial}{\partial y_l}\right\} = \delta_{k,l},$$

where i, $j = 1, 2$; k, $l = 3, 4$; and $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_k}$ are the partial differential operators on W, $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ denote the Lie-bracket, Jordan-bracket respectively.

With $\tau_i \in \{\pm 1\}, i = 1, 2, 3, 4$, we define the following operators on W:

$$\begin{cases} E_{\alpha_1} = -\tau_1\tau_3 y_3 \dfrac{\partial}{\partial x_1} - \tau_2\tau_4(1+\alpha)x_2 \dfrac{\partial}{\partial y_4}, \\ E_{\alpha_2} = \tau_3\tau_4 y_4 \dfrac{\partial}{\partial y_3}, \\ E_{\alpha_3} = \tau_1 x_1, \\ E_{\alpha_1+\alpha_2} = -\tau_1\tau_4 y_4 \dfrac{\partial}{\partial x_1} + \tau_2\tau_3(1+\alpha)x_2 \dfrac{\partial}{\partial y_3}, \\ E_{\alpha_1+\alpha_3} = \tau_3 y_3, \\ E_{\alpha_1+\alpha_2+\alpha_3} = \tau_4 y_4, \\ E_{2\alpha_1+\alpha_2+\alpha_3} = \tau_2 x_2, \end{cases} \tag{3.1}$$

$$\begin{cases} F_{\alpha_1} = -\alpha\tau_1\tau_3 x_1 \dfrac{\partial}{\partial y_3} - \tau_2\tau_4 y_4 \dfrac{\partial}{\partial x_2}, \\ F_{\alpha_2} = \tau_3\tau_4 y_3 \dfrac{\partial}{\partial y_4}, \\ F_{\alpha_3} = -\tau_1\left(\lambda + x_1\dfrac{\partial}{\partial x_1} + y_3\dfrac{\partial}{\partial y_3} + y_4\dfrac{\partial}{\partial y_4}\right)\dfrac{\partial}{\partial x_1} + \tau_2\tau_3\tau_4(1+\alpha)x_2\dfrac{\partial}{\partial y_3}\dfrac{\partial}{\partial y_4}, \\ F_{\alpha_1+\alpha_2} = \alpha\tau_1\tau_4 x_1 \dfrac{\partial}{\partial y_4} - \tau_2\tau_3 y_3 \dfrac{\partial}{\partial x_2}, \\ F_{\alpha_1+\alpha_3} = \tau_3\left(\alpha\lambda + \alpha x_1\dfrac{\partial}{\partial x_1} - (1+\alpha)x_2\dfrac{\partial}{\partial x_2} - y_4\dfrac{\partial}{\partial y_4}\right)\dfrac{\partial}{\partial y_3} + \tau_1\tau_2\tau_4 y_4\dfrac{\partial}{\partial x_1}\dfrac{\partial}{\partial x_2}, \\ F_{\alpha_1+\alpha_2+\alpha_3} = -\tau_4\left(\alpha\lambda + \alpha x_1\dfrac{\partial}{\partial x_1} - (1+\alpha)x_2\dfrac{\partial}{\partial x_2} - y_3\dfrac{\partial}{\partial y_3}\right)\dfrac{\partial}{\partial y_4} \\ \qquad +\tau_1\tau_2\tau_3 y_3\dfrac{\partial}{\partial x_1}\dfrac{\partial}{\partial x_2}, \\ F_{2\alpha_1+\alpha_2+\alpha_3} = -\tau_2\left(-\dfrac{\alpha}{1+\alpha}\lambda + x_2\dfrac{\partial}{\partial x_2} + y_3\dfrac{\partial}{\partial y_3} + y_4\dfrac{\partial}{\partial y_4}\right)\dfrac{\partial}{\partial x_2} \\ \qquad -\alpha\tau_1\tau_3\tau_4 x_1\dfrac{\partial}{\partial y_3}\dfrac{\partial}{\partial y_4}, \end{cases} \tag{3.2}$$

$$\begin{cases} H_{\alpha_1} = \{E_{\alpha_1}, F_{\alpha_1}\} = \alpha\left(x_1\dfrac{\partial}{\partial x_1} + y_3\dfrac{\partial}{\partial y_3}\right) + (1+\alpha)\left(x_2\dfrac{\partial}{\partial x_2} + y_4\dfrac{\partial}{\partial y_4}\right), \\ H_{\alpha_2} = [E_{\alpha_2}, F_{\alpha_2}] = -y_3\dfrac{\partial}{\partial y_3} + y_4\dfrac{\partial}{\partial y_4}, \\ H_{\alpha_3} = [E_{\alpha_3}, F_{\alpha_3}] = \lambda + 2x_1\dfrac{\partial}{\partial x_1} + y_3\dfrac{\partial}{\partial y_3} + y_4\dfrac{\partial}{\partial y_4}. \end{cases} \tag{3.3}$$

We may view W as a superalgebra with $|x_i| = \bar{0}$ for $i = 1, 2$ and $|y_k| = \bar{1}$ for $k = 3, 4$. It follows that $\mathfrak{gl}(W)$ is a Lie superalgebra. Then by the above actions, we have the following theorem:

Theorem 3.1 *The linear map*

$$\varphi : D(2,1;\alpha) \longrightarrow \mathfrak{gl}(W)$$

given by

$$\begin{aligned}
&\varphi(e_{\alpha_i}) = E_{\alpha_i}, \quad i = 1,2,3,\\
&\varphi(f_{\alpha_i}) = F_{\alpha_i}, \quad i = 1,2,3,\\
&\varphi(h_{\alpha_i}) = H_{\alpha_i}, \quad i = 1,2,3,\\
&\varphi(e_{\alpha_1+\alpha_2}) = E_{\alpha_1+\alpha_2},\\
&\varphi(f_{\alpha_1+\alpha_2}) = F_{\alpha_1+\alpha_2},\\
&\varphi(e_{\alpha_1+\alpha_3}) = E_{\alpha_1+\alpha_3},\\
&\varphi(f_{\alpha_1+\alpha_3}) = F_{\alpha_1+\alpha_3},\\
&\varphi(e_{\alpha_1+\alpha_2+\alpha_3}) = E_{\alpha_1+\alpha_2+\alpha_3},\\
&\varphi(f_{\alpha_1+\alpha_2+\alpha_3}) = F_{\alpha_1+\alpha_2+\alpha_3},\\
&\varphi(e_{2\alpha_1+\alpha_2+\alpha_3}) = E_{2\alpha_1+\alpha_2+\alpha_3},\\
&\varphi(f_{2\alpha_1+\alpha_2+\alpha_3}) = F_{2\alpha_1+\alpha_2+\alpha_3},
\end{aligned}$$

is a Lie superalgebra homomorphism. That is, W is a $D(2,1;\alpha)$-module.

Proof It suffices to show that these corresponding operators satisfy all the commutator relations given in Proposition 2.1. We complete the proof by checking these formulas case by case. We give the proofs for some cases, the others are either similar or easier.

We begin with the cases of two odd roots:

$$\begin{aligned}
\{E_{\alpha_1}, F_{\alpha_1}\} &= \alpha\Big\{y_3\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial y_3}\Big\} + (1+\alpha)\Big\{x_2\frac{\partial}{\partial y_4}, y_4\frac{\partial}{\partial x_2}\Big\}\\
&= \alpha\Big(x_1\frac{\partial}{\partial x_1} + y_3\frac{\partial}{\partial y_3}\Big) + (1+\alpha)\Big(x_2\frac{\partial}{\partial x_2} + y_4\frac{\partial}{\partial y_4}\Big)\\
&= H_{\alpha_1},
\end{aligned}$$

we have $\{\varphi(e_{\alpha_1}), \varphi(f_{\alpha_1})\} = \varphi(\{e_{\alpha_1}, f_{\alpha_1}\})$.

$$\begin{aligned}
\{E_{\alpha_1}, E_{\alpha_1+\alpha_2+\alpha_3}\} &= \Big\{-\tau_1\tau_3 y_3\frac{\partial}{\partial x_1} - \tau_2\tau_4(1+\alpha)x_2\frac{\partial}{\partial y_4}, \tau_4 y_4\Big\}\\
&= -\tau_2(1+\alpha)\Big\{x_2\frac{\partial}{\partial y_4}, y_4\Big\}\\
&= -\tau_2(1+\alpha)x_2\\
&= -(1+\alpha)E_{2\alpha_1+\alpha_2+\alpha_3},
\end{aligned}$$

that is $\{\varphi(e_{\alpha_1}), \varphi(e_{\alpha_1+\alpha_2+\alpha_3})\} = \varphi(\{e_{\alpha_1}, e_{\alpha_1+\alpha_2+\alpha_3}\})$.

Other cases are calculated:

$$[E_{\alpha_1}, F_{2\alpha_1+\alpha_2+\alpha_3}] = \Big[-\tau_1\tau_3 y_3\frac{\partial}{\partial x_1} - \tau_2\tau_4(1+\alpha)x_2\frac{\partial}{\partial y_4},$$

$$
\begin{aligned}
&\quad -\tau_2\Big(-\frac{\alpha}{1+\alpha}\lambda+x_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4}\Big)\frac{\partial}{\partial x_2}-\alpha\tau_1\tau_3\tau_4x_1\frac{\partial}{\partial y_3}\frac{\partial}{\partial y_4}\Big]\\
&=\tau_1\tau_2\tau_3\Big[y_3\frac{\partial}{\partial x_1},y_3\frac{\partial}{\partial y_3}\frac{\partial}{\partial x_2}\Big]+\tau_4\alpha\Big[y_3\frac{\partial}{\partial x_1},x_1\frac{\partial}{\partial y_3}\frac{\partial}{\partial y_4}\Big]\\
&\quad+\tau_4(1+\alpha)\Big[x_2\frac{\partial}{\partial y_4},\Big(-\frac{\alpha}{1+\alpha}\lambda+x_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4}\Big)\frac{\partial}{\partial x_2}\Big]\\
&=-\tau_1\tau_2\tau_3y_3\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}+\tau_4\alpha\Big(y_3\frac{\partial}{\partial y_3}\frac{\partial}{\partial y_4}+x_1\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_4}\Big)\\
&\quad+\tau_4(1+\alpha)\Big(\frac{\alpha}{1+\alpha}\lambda-x_2\frac{\partial}{\partial x_2}-y_3\frac{\partial}{\partial y_3}\Big)\frac{\partial}{\partial y_4}\\
&=\tau_4\Big(\alpha\lambda+\alpha x_1\frac{\partial}{\partial x_1}-(1+\alpha)x_2\frac{\partial}{\partial x_2}-y_3\frac{\partial}{\partial y_3}\Big)\frac{\partial}{\partial y_4}-\tau_1\tau_2\tau_3y_3\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\\
&=-F_{\alpha_1+\alpha_2+\alpha_3},
\end{aligned}
$$

hence $[\varphi(e_{\alpha_1}),\varphi(f_{2\alpha_1+\alpha_2+\alpha_3})]=\varphi([e_{\alpha_1},f_{2\alpha_1+\alpha_2+\alpha_3}])$.

$$
\begin{aligned}
[H_{\alpha_1},E_{\alpha_1+\alpha_2}]&=\Big[\alpha\Big(x_1\frac{\partial}{\partial x_1}+y_3\frac{\partial}{\partial y_3}\Big),-\tau_1\tau_4y_4\frac{\partial}{\partial x_1}\Big]\\
&\quad+\Big[\alpha\Big(x_1\frac{\partial}{\partial x_1}+y_3\frac{\partial}{\partial y_3}\Big),\tau_2\tau_3(1+\alpha)x_2\frac{\partial}{\partial y_3}\Big]\\
&\quad+\Big[(1+\alpha)\Big(y_4\frac{\partial}{\partial y_4}+x_2\frac{\partial}{\partial x_2}\Big),-\tau_1\tau_4y_4\frac{\partial}{\partial x_1}\Big]\\
&\quad+\Big[(1+\alpha)\Big(y_4\frac{\partial}{\partial y_4}+x_2\frac{\partial}{\partial x_2}\Big),\tau_2\tau_3(1+\alpha)x_2\frac{\partial}{\partial y_3}\Big]\\
&=-\tau_1\tau_4y_4\frac{\partial}{\partial x_1}+\tau_2\tau_3(1+\alpha)x_2\frac{\partial}{\partial y_3}\\
&=E_{\alpha_1+\alpha_2}
\end{aligned}
$$

so $[\varphi(h_{\alpha_1}),\varphi(e_{\alpha_1+\alpha_2})]=\varphi([h_{\alpha_1},e_{\alpha_1+\alpha_2}])$.

And

$$
\begin{aligned}
[H_{\alpha_3},F_{\alpha_1+\alpha_2+\alpha_3}]&=\tau_2\Big(-\frac{\alpha}{1+\alpha}\lambda+x_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4}\Big)\frac{\partial}{\partial x_2}+\alpha\tau_1\tau_3\tau_4x_1\frac{\partial}{\partial y_3}\frac{\partial}{\partial y_4}\\
&=-F_{\alpha_1+\alpha_2+\alpha_3},
\end{aligned}
$$

we get $[\varphi(h_{\alpha_3}),\varphi(f_{\alpha_1+\alpha_2+\alpha_3})]=\varphi([h_{\alpha_3},f_{\alpha_1+\alpha_2+\alpha_3}])$.

$$
\begin{aligned}
&[E_{2\alpha_1+\alpha_2+\alpha_3},F_{2\alpha_1+\alpha_2+\alpha_3}]\\
&\quad=\Big[\tau_2x_2,-\tau_2\Big(-\frac{\alpha}{1+\alpha}\lambda+x_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4}\Big)\frac{\partial}{\partial x_2}-\alpha\tau_1\tau_3\tau_4x_1\frac{\partial}{\partial y_3}\frac{\partial}{\partial y_4}\Big]\\
&\quad=-\frac{\alpha}{1+\alpha}\lambda+2x_2\frac{\partial}{\partial x_2}+y_3\frac{\partial}{\partial y_3}+y_4\frac{\partial}{\partial y_4}\\
&\quad=\frac{2}{1+\alpha}H_{\alpha_1}-\frac{1}{1+\alpha}H_{\alpha_2}-\frac{a}{1+\alpha}H_{\alpha_3},
\end{aligned}
$$

which means

$$[\varphi(e_{2\alpha_1+\alpha_2+\alpha_3}),\varphi(f_{2\alpha_1+\alpha_2+\alpha_3})]=\varphi([e_{2\alpha_1+\alpha_2+\alpha_3},f_{2\alpha_1+\alpha_2+\alpha_3}]). \qquad \square$$

3.2 Irreducibility

Let $\mathfrak{h} = \text{span}_{\mathbb{C}}\{h_{\alpha_1}, h_{\alpha_2}, h_{\alpha_3}\}$, so $\mathfrak{h}$ is the Cartan subalgebra of $D(2,1;\alpha)$.

Let $\mathfrak{n}_- = \text{span}_{\mathbb{C}}\{f_{\alpha_1}, f_{\alpha_2}, f_{\alpha_3}, f_{\alpha_1+\alpha_2}, f_{\alpha_1+\alpha_3}, f_{\alpha_1+\alpha_2+\alpha_3}, f_{2\alpha_1+\alpha_2+\alpha_3}\}$.

Notice that the operators actions are given in (3.1)–(3.3), we have the following lemma immediately.

Lemma 3.2 *$\mathfrak{h}$ acts diagonally on W, which means that W is a weight module of $D(2,1;\alpha)$. As a $D(2,1;\alpha)$-module, W is a lowest weight module with lowest weight vector 1 of weight $(0,0,\lambda)$.*

Theorem 3.3 *W is an irreducible $D(2,1;\alpha)$-module if and only if λ satisfies $(\lambda+m)(\lambda-\frac{1+\alpha}{\alpha}m) \neq 0$ for any $m \in \mathbb{Z}_+$.*

Proof If $\lambda = -m$ for some $m \in \mathbb{Z}_+$, notice that $\mathfrak{n}_-.x_1^{m+1} = 0$, then we get the submodule generated by x_1^{m+1} is a lowest weight module with lowest weight vector x_1^{m+1}, which is a proper submodule of W.

If $\lambda = \frac{1+\alpha}{\alpha}m$ for some $m \in \mathbb{Z}_+$, notice that $\mathfrak{n}_-.x_2^{m+1} = 0$, then we get the submodule generated by x_2^{m+1} is a lowest weight module with lowest weight vector x_2^{m+1}, which is a proper submodule of W.

Suppose $(\lambda+m)(\lambda-\frac{1+\alpha}{\alpha}m) \neq 0$ for any $m \in \mathbb{Z}_+$. Let U be a nonzero submodule of W. Then we know that U is a weight module of $D(2,1;\alpha)$ by the above lemma.

Notice that in fact there are only four possible monomials classes in W as follows

$$x_1^{i_1}x_2^{i_2}, \quad x_1^{j_1}x_2^{j_2}y_3, \quad x_1^{k_1}x_2^{k_2}y_4, \quad x_1^{l_1}x_2^{l_2}y_3y_4.$$

Moreover, the weights of $x_1^{i_1}x_2^{i_2}, x_1^{j_1}x_2^{j_2}y_3, x_1^{k_1}x_2^{k_2}y_4, x_1^{l_1}x_2^{l_2}y_3y_4$ are $(\alpha(i_1+i_2)+i_2, 0, \lambda+2i_1)$, $(\alpha(j_1+j_2+1)+j_2, -1, \lambda+2j_1+1)$, $(\alpha(k_1+k_2+1)+k_2+1, 1, \lambda+2k_1+1)$, $(\alpha(l_1+l_2+2)$ $+l_2+1, 0, \lambda+2l_1+2)$ respectively.

Given a weight vector $v \in U$, there are only three cases as follows:

(1) $h_{\alpha_2}.v = v$, then we can assume $v = x_1^{k_1}x_2^{k_2}y_4, k_1, k_2 \in \mathbb{Z}_+$ up to a scalar multiplier. Notice that

$$\begin{aligned}&f_{\alpha_1+\alpha_2+\alpha_3}.(f_{2\alpha_1+\alpha_2+\alpha_3}^{k_2}.(f_{\alpha_3}^{k_1}.v))\\&\quad = (-\tau_1)^{k_1}k_1!(\lambda+k_1)\cdots(\lambda+1)(-\tau_2)^{k_2}k_2!\Big(-\frac{\alpha}{1+\alpha}\lambda+k_2\Big)\cdots\Big(-\frac{\alpha}{1+\alpha}\lambda+1\Big)(-\tau_4\alpha\lambda),\end{aligned}$$

now we get $1 \in U$ up to a scalar multiplier since $\tau_1, \tau_2, \tau_4, \alpha$ are all nonzero and $(\lambda+m)$ $(\lambda-\frac{1+\alpha}{\alpha}m) \neq 0$ for any $m \in \mathbb{Z}_+$.

(2) $h_{\alpha_2}.v = -v$, then we can assume $v = x_1^{j_1}x_2^{j_2}y_3, j_1, j_2 \in \mathbb{Z}_+$ up to a scalar multiplier. Notice that $e_{\alpha_2}.v = \tau_3\tau_4x_1^{j_1}x_2^{j_2}y_4 \in U$, then from the first case, we get $1 \in U$.

(3) $h_{\alpha_2}.v = 0$, then we can assume $v = \mu x_1^{i_1+1}x_2^{i_2+1} + \nu x_1^{i_1}x_2^{i_2}y_3y_4, i_1, i_2 \in \mathbb{Z}_+, \mu \neq 0$ or $\nu \neq 0$ up to a scalar multiplier. If $\mu \neq 0$, notice that $e_{\alpha_1+\alpha_2+\alpha_3}.v = \tau_4\mu x_1^{i_1+1}x_2^{i_2+1}y_4$, from the first case, we get $1 \in U$. Otherwise, $\mu = 0$, then $\nu \neq 0$. Notice that $e_{\alpha_1+\alpha_2}.v = \tau_2\tau_3$ $(1+\alpha)\nu x_1^{i_1}x_2^{i_2+1}y_4$, from the first case again, we get $1 \in U$ too.

To summarize, 1 must belong to the nonzero submodule U, then $U = W$, so W is an irreducible module if $(\lambda+m)(\lambda-\frac{1+\alpha}{\alpha}m) \neq 0$ for any $m \in \mathbb{Z}_+$. □

For convenience, we set $x_1^k = x_2^k = 0$ if $k \notin \mathbb{Z}_+$.

Theorem 3.4 (1) *If $\lambda = -m$ for some $m \in \mathbb{Z}_+$, let $n = -\frac{\alpha m}{1+\alpha}$, then the submodule $U(\lambda)$ of W generated by x_1^{m+1}, x_2^{n+1} is the unique maximal proper submodule of W, so $W/U(\lambda)$ is an irreducible $D(2,1;\alpha)$-module.*

(2) *If $\lambda = \frac{1+\alpha}{\alpha}m$ for some $m \in \mathbb{Z}_+$, let $n = -\frac{1+\alpha}{\alpha}m$, then the submodule $U(\lambda)$ of W generated by x_1^{n+1}, x_2^{m+1} is the unique maximal proper submodule of W, so $W/U(\lambda)$ is an irreducible $D(2,1;\alpha)$-module.*

Proof If $n \in \mathbb{Z}_+$, set $w := f_{\alpha_1}.(f_{\alpha_1+\alpha_2}.(x_2^{n+1})) = \alpha(n+1)\tau_1\tau_2x_1x_2^n - (n+1)n\tau_3\tau_4x_2^{n-1}y_3y_4 \in U(\lambda)$. We have

$$e_{\alpha_1}.w = -\alpha(n+1)(m+1)\tau_2\tau_3x_2^ny_3 \neq 0,$$
$$e_{\alpha_1+\alpha_2}.x_2^ny_3 = \tau_2\tau_3(1+\alpha)x_2^{n+1} \neq 0.$$

Let V_0 be the submodule of W generated by x_1^{m+1} and w. Then we have $U(\lambda) = V_0$.

Notice that

$$f_{\alpha_1}.w = f_{\alpha_2}.w = f_{\alpha_3}.w = f_{\alpha_1+\alpha_2}.w = 0,$$
$$f_{\alpha_1+\alpha_3}.w = f_{\alpha_1+\alpha_2+\alpha_3}.w = f_{2\alpha_1+\alpha_2+\alpha_3}.w = 0,$$

together with $\mathfrak{n}_-.x_1^{m+1} = 0$, we have $U(\lambda) = V_0$ is a proper submodule of W.

If $\lambda = 0$, then the submodule $U(0)$ of W generated by x_1, x_2 equals the set of polynomials without constant term forms. Obviously, it is the unique maximal proper submodule of W, hence $W/U(0)$ is an irreducible $D(2,1;\alpha)$-module.

Suppose $\lambda = -m$ for some $m \in \mathbb{Z}_+\backslash\{0\}$, let $n = -\frac{\alpha m}{1+\alpha}$.

Notice that

$$e_{\alpha_1}.x_1^{m+1} = -\tau_1\tau_3(m+1)x_1^my_3,$$

then $x_1^my_3 \in U(\lambda)$. Since

$$e_{\alpha_2}.x_1^my_3 = \tau_3\tau_4x_1^my_4,$$

then $x_1^my_4 \in U(\lambda)$. So by the actions of $e_{\alpha_3}, e_{2\alpha_1+\alpha_2+\alpha_3}$, we get $x_1^{m+l_1}x_2^{l_2}y_4 \in U(\lambda)$ for any $l_1, l_2 \in \mathbb{Z}_+$. Similarly, we have $x_1^{l_2}x_2^{n+l_1}y_4 \in U(\lambda)$ for any $l_1, l_2 \in \mathbb{Z}_+$.

Let $v \in W\backslash U(\lambda)$ be a non-zero weight vector, V is the submodule of W generated by v, we claim $1 \in V$ (which means $V = W$).

We separate it to three cases for discussing:

(I) $h_{\alpha_2}.v = v$, then we may assume $v = x_1^{k_1}x_2^{k_2}y_4, k_1, k_2 \in \mathbb{Z}_+$ up to a scalar multiplier.

We get $k_1 < m$ since $x_1^{m+l_1}x_2^{l_2}y_4 \in U(\lambda)$ for any $l_1, l_2 \in \mathbb{Z}_+$. If $n \in \mathbb{Z}_+$, we also have $k_2 < n$ since $x_1^{l_2}x_2^{n+l_1}y_4 \in U(\lambda)$ for any $l_1, l_2 \in \mathbb{Z}_+$, or $n \notin \mathbb{Z}_+$.

Notice that

$$\begin{aligned}&f_{\alpha_1+\alpha_2+\alpha_3}.(f_{2\alpha_1+\alpha_2+\alpha_3}^{k_2}.(f_{\alpha_3}^{k_1}.v))\\&\quad = (-\tau_1)^{k_1}k_1!(-m+k_1)\cdots(-m+1)(-\tau_2)^{k_2}k_2!(-n+k_2)\cdots(-n+1)(-\tau_4\alpha(-m)),\end{aligned}$$

we get $1 \in V$ since $\tau_1, \tau_2, \tau_4, \alpha$ are all nonzero, $m \in \mathbb{Z}_+\backslash\{0\}, k_1 < m, k_1, k_2 \in \mathbb{Z}_+$, and $k_2 < n$ or $n \notin \mathbb{Z}_+$.

(II) $h_{\alpha_2}.v = -v$, then we assume $v = x_1^{j_1}x_2^{j_2}y_3, j_1, j_2 \in \mathbb{Z}_+$ up to a scalar multiplier.

Notice that $e_{\alpha_2}.v = \tau_3\tau_4x_1^{j_1}x_2^{j_2}y_4 \in V$, $f_{\alpha_2}.(\tau_3\tau_4x_1^{j_1}x_2^{j_2}y_4) = v \notin U(\lambda)$, now we have $x_1^{j_1}x_2^{j_2}y_4 \in V\backslash U(\lambda)$, then from the first case, we get $1 \in V$.

(III) $h_{\alpha_2}.v = 0$, then we can assume $v = \mu x_1^{i_1+1} x_2^{i_2+1} + \nu x_1^{i_1} x_2^{i_2} y_3 y_4, i_1, i_2 \in \mathbb{Z}_+$, and at least one of μ, ν is non-zero.

We get $i_1 < m$ since both $x_1^{m+1} x_2^{i_2+1}$ and $x_1^m x_2^{i_2} y_3 y_4$ are in $U(\lambda)$ for any $i_2 \in \mathbb{Z}_+$. If $n \in \mathbb{Z}_+$, we also have $i_2 < n$ since $x_1^{i_1+1} x_2^{n+1}$, $x_1^{i_1} x_2^n y_3 y_4 \in U(\lambda)$ for any $i_1 \in \mathbb{Z}_+$.

(i) If $\nu = 0, \mu \neq 0$, notice that $f_{2\alpha_1+\alpha_2+\alpha_3}^{i_2+1}.(f_{\alpha_3}^{i_1+1}.v) = \mu(-\tau_1)^{i_1+1}(i_1+1)!(-m+i_1)\cdots(-m)$ $(-\tau_2)^{i_2+1}(i_2+1)!(-n+i_2)\cdots(-n)$, we get $1 \in V$ because τ_1, τ_2, μ are all nonzero, $i_1 < m, i_1, i_2 \in \mathbb{Z}_+$, and $i_2 < n$ or $n \notin \mathbb{Z}_+$.

(ii) If $\nu \neq 0, \mu = 0$, $i_1 < m-1$. Notice that $f_{\alpha_1}.v = -\nu\alpha\tau_1\tau_3 x_1^{i_1+1} x_2^{i_2} y_4$, then from (I), we have $1 \in V$.

(iii) If $\nu \neq 0, \mu = 0$, $i_1 = m-1$. Since $e_{\alpha_3}.(f_{\alpha_3}.v) = \nu\tau_1\tau_2\tau_3\tau_4(1+\alpha)x_1^{i_1+1}x_2^{i_2+1}$, together with the discussion of (i), we have $1 \in V$.

(iv) If $\nu \neq 0, \mu \neq 0$, $i_1 = m-1$. Notice that $x_1^m y_4 \in U(\lambda)$, $e_{\alpha_1}.(x_1^m y_4) = -\tau_1\tau_3 m x_1^{m-1} y_3 y_4 - \tau_2\tau_4(1+\alpha)x_1^m x_2 \in U(\lambda)$, we have $\tau_1\tau_3 m x_1^{m-1} x_2^{i_2} y_3 y_4 + \tau_2\tau_4(1+\alpha)x_1^m x_2^{i_2+1} \in U(\lambda)$ for any $i_2 \in \mathbb{Z}_+$.

Since $v = \mu x_1^m x_2^{i_2+1} + \nu x_1^{m-1} x_2^{i_2} y_3 y_4 \notin U(\lambda)$ and $\tau_1, \tau_2, \tau_3, \tau_4 \in \{\pm 1\}$, we have $\tau_1\tau_4\mu m \neq \tau_2\tau_3(1+\alpha)\nu$. Notice that $e_{\alpha_1+\alpha_2}.v = (-\tau_1\tau_4\mu m + \tau_2\tau_3(1+\alpha)\nu)x_1^{m-1}x_2^{i_2+1}y_4$, now we have $x_1^{m-1}x_2^{i_2+1}y_4 \in V$, if $n \in \mathbb{Z}_+, i_2 < n-1$ or $n \notin \mathbb{Z}_+$, then from (I), we have $1 \in V$.

If $n \in \mathbb{Z}_+, i_2 = n-1$, $f_{\alpha_1+\alpha_3}.v = (-\tau_3(1+\alpha)\nu + \tau_1\tau_2\tau_4\mu m)n x_1^{m-1}x_2^{n-1}y_4$, then we have $x_1^{m-1}x_2^{n-1}y_4 \in V$ since $n \neq 0$ and $\tau_1\tau_4\mu m \neq \tau_2\tau_3(1+\alpha)\nu$. Now from (I), we have $1 \in V$.

(v) If $\nu \neq 0, \mu \neq 0$, $i_1 < m-1$. Notice that $e_{\alpha_3}^{m-1-i_1}.v = \tau_1^{m-1-i_1}(\mu x_1^m x_2^{i_2+1} + \nu x_1^{m-1} x_2^{i_2} y_3 y_4)$ $\in V$, then from (iv), we have $1 \in V$.

Summarize all the discussion above, we have that 1 belongs to any nonzero submodule of W, except for $U(\lambda)$.

So $U(\lambda)$ is the unique maximal proper submodule of W, and $W/U(\lambda)$ is an irreducible $D(2,1;\alpha)$-module.

The proof for (2) is completely analogous to (1), we omit it. ☐

Hence we get all irreducible quotients of this class of modules.

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