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Generalized Gerstewitz's Functions and Vector Variational Principle for ϵ-Efficient Solutions in the Sense of Németh

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Abstract In this paper, we first generalize Gerstewitz's functions from a single positive vector to a subset of the positive cone. Then, we establish a partial order principle, which is indeed a variant of the pre-order principle [Qiu, J. H.: A pre-order principle and set-valued Ekeland variational principle. *J. Math. Anal. Appl.*, **419**, 904–937 (2014)]. By using the generalized Gerstewitz's functions and the partial order principle, we obtain a vector EVP for ϵ-efficient solutions in the sense of Németh, which essentially improves the earlier results by completely removing a usual assumption for boundedness of the objective function. From this, we also deduce several special vector EVPs, which improve and generalize the related known results.

Keywords Ekeland variational principle, partial order principle, ϵ-efficient solutions in the sense of Németh, Gerstewitz's function, convex cone

MR(2010) Subject Classification 46A03, 49J53, 58E30, 65K10

1 Introduction

In 1972, Ekeland (see [14, 15]) presented a variational principle, now known as Ekeland variational principle (briefly, denoted by EVP), which says that for any lower semi-continuous function f bounded from below on a complete metric space, there exists a slightly perturbed version of this function that has a strict minimum. In the last four decades, the famous EVP emerged as one of the most important results of nonlinear analysis and its application covers numerous areas such as optimization, optimal control theory, fixed point theory, nonsmooth analysis, Banach space geometry, game theory, nonlinear equations, dynamical systems, etc.; for example, see [4, 11, 15, 16, 20, 35, 50]. Motivated by its wide usefulness, many authors have been interested in extending EVP to the case with vector-valued maps or set-valued maps; see, for example [3, 5–8, 10–13, 17, 18, 20, 21, 24–27, 29, 30, 33, 34, 39–41, 43–45, 47, 48, 51] and the references therein.

In this paper, we consider extensions of EVP when the objective function is a vector-valued map $f : (X, d) \to Y$, where (X, d) is a complete metric space and Y is a real quasi-ordered (topological) vector space. A systematization of such results can be found in, for example, [11, 20, 21]. The common feature of these results is the presence of a certain term $d(x, x')\, k_0$ in the perturbation, where $k_0 \in D\backslash\{0\}$ and D is an ordering cone. Bednarczuk and Zagrodny (see

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[8, Theorem 4.1]) proved a vector EVP, where the perturbation is given by a bounded convex subset H of the ordering cone D multiplied by the distance function $d(x, x')$, i.e., its form is as $d(x, x')\, H$. This generalizes the case where directions of the perturbations are singleton $\{k_0\}$. Tammer and Zălinescu also considered this type of EVPs and gave an improvement of the above result; see [48, Theorem 6.2]. More generally, Gutiérrez, Jiménez and Novo [24] introduced a set-valued metric, which takes values in the set family of all subsets of the ordering cone and satisfies the triangle inequality. By using it they gave an original approach to extending the scalar-valued EVP to a vector-valued map, where the perturbation contains a set-valued metric. From this, they deduced several special versions of EVP involving approximate solutions for vector optimization problems. However, in their work the assumption that the ordering cone D is w-normal is required (see [24]). This requirement restricts the applicable extent of the new version of EVP. Qiu [41] introduced a slightly more general notion: set-valued quasi-metrics, and proposed the notion of compatibility between a set-valued quasi-metric and the original metric d. By means of these notions, Qiu proved a general vector EVP, where the perturbation contains a set-valued quasi-metric compatible with the original metric. Here, one needs not assume that the ordering cone is w-normal. From the general EVP, Qiu deduced a number of special vector EVPs, which improve the related known results. Particularly, Qiu obtained several EVPs for ϵ-efficient solutions in the sense of Németh, which improve the related results in [24].

In order to express our purpose clearly, we recall some details on this topic. Let Y be a locally convex Hausdorff topological vector space (briefly, denoted by locally convex space) and Y^* be its topological dual. For any $\xi \in Y^*$, we define a continuous semi-norm p_ξ on Y as follows: $p_\xi(y) := |\xi(y)|,\ \forall y \in Y$. The semi-norm family $\{p_\xi : \xi \in Y^*\}$ generates a locally convex Hausdorff topology on Y (see, e.g., [28, 31, 32, 49]), which is called the weak topology on Y and denoted by $\sigma(Y, Y^*)$. For any nonempty subset F of Y^*, the semi-norm family $\{p_\xi : \xi \in F\}$ can also generate a locally convex topology (which need not be Hausdorff) on Y, which is denoted by $\sigma(Y, F)$. In [31], the topology $\sigma(Y, F)$ is called the F-projective topology. If A, $B \subset Y$ and $\alpha \in R$, the sets $A + B$ and αA are defined as follows:

$$A + B := \{z \in Y : \exists x \in A,\ \exists y \in B \text{ such that } z = x + y\},$$
$$\alpha A := \{z \in Y : \exists x \in A \text{ such that } z = \alpha x\}.$$

A nonempty subset D of Y is called a cone if $\alpha D \subset D$ for any $\alpha \geq 0$. And D is called a convex cone if $D + D \subset D$ and $\alpha D \subset D$ for any $\alpha \geq 0$. Moreover, a convex cone is called a pointed convex cone if $D \cap (-D) = \{0\}$. A pointed convex cone D can specify a partial order in Y as follows.

$$y_1, y_2 \in Y, \quad y_1 \leq_D y_2 \Leftrightarrow y_1 - y_2 \in -D.$$

In this case, D is called the ordering cone or positive cone. The positive polar cone of D is denoted by D^+, that is, $D^+ = \{\xi \in Y^* : \xi(d) \geq 0,\ \forall d \in D\}$. For $H \subset D\backslash\{0\}$, the set $\{\xi \in Y^* : \inf\{\xi(h) : h \in H\} > 0\}$ is denoted by H^{+s}. A nonempty set $M \subset Y$ is said to be D-bounded by scalarization (briefly, denoted by D-bounded) if (see [24, Definition 3.3])

$$\inf\{\xi(y) : y \in M\} > -\infty, \quad \forall \xi \in D^+.$$

Let us consider the following vector optimization problem:

$$\mathrm{Min}\{f(x) : x \in S\}, \tag{1.1}$$

where $f : X \to Y$ is a vector-valued map and S is a nonempty closed subset of X. A point $x_0 \in S$ is called an efficient solution of (1.1) if

$$(f(S) - f(x_0)) \cap (-D\backslash\{0\}) = \emptyset,$$

where $f(S)$ denotes the set $\bigcup_{x\in S}\{f(x)\}$.

Gutiérrez, Jiménez and Novo introduced the (C,ϵ)-efficiency concept, which extends and unifies several ϵ-efficiency notions (see [22, 23]).

Definition 1.1 ([23]) *A nonempty set $C \subset Y$ is coradiant if $\bigcup_{\beta\geq 1}\beta C = C$.*

Definition 1.2 ([23]) *Let D be an ordering cone, $C \subset D\backslash\{0\}$ be a coradiant set and $\epsilon > 0$. A point $x_0 \in S$ is a (C,ϵ)-efficient solution of Problem* (1.1) *if $(f(S) - f(x_0)) \cap (-\epsilon C) = \emptyset$. In this case, we also denote $x_0 \in AE(C,\epsilon)$.*

In particular, if $C := H + D$, where $H \subset D\backslash\{0\}$, then we can easily verify that C is a coradiant set and $C \subset D\backslash\{0\}$. Thus, we obtain the concept of approximate efficiency due to Németh.

Definition 1.3 ([23, 24, 34, 41]) *Let $H \subset D\backslash\{0\}$ and $\epsilon > 0$. A point $x_0 \in S$ is said to be an ϵ-efficient solution of* (1.1) *in the sense of Németh* (*with respect to H*) *if $(f(S) - f(x_0)) \cap (-\epsilon H - D) = \emptyset$. In this case, we also denote $x_0 \in AE(C_H,\epsilon)$, where $C_H = H + D$.*

Usually, we assume that $H \subset D\backslash\{0\}$ is a D-convex set, i.e., $H + D$ is a convex set. Let H be a D-convex set and $\gamma > 0$. For any $x \in S$, put

$$S(x) := \{z \in S : f(x) \in f(z) + \gamma d(x,z)H + D\}. \tag{1.2}$$

It is easy to verify that $x \in S(x)$ and $S(x) \neq \emptyset$ for all $x \in S$. Moreover, $S(z) \subset S(x)$ for all $z \in S(x)$.

Definition 1.4 ([41]) *Let X be a metric space and let $S(\cdot) : X \to 2^X\backslash\{\emptyset\}$ be a set-valued map. The set-valued map $S(\cdot)$ is said to be dynamically closed at $x \in X$ if $(x_n) \subset S(x), S(x_{n+1}) \subset S(x_n) \subset S(x)$ for all n and $x_n \to \bar{x}$ then $\bar{x} \in S(x)$. In this case, we also say that $S(x)$ is dynamically closed.*

We remark that a property similar to the above dynamical closedness, i.e., the so-called limiting monotonicity property, was also introduced in [5, 6]. Let's recall the following assumption (see [41]):

(Q3) For any $x \in S(x_0)$, $S(x)$ is dynamically closed.

Now we can recall a vector EVP in [41] for ϵ-efficient solutions in the sense of Németh.

Theorem 1.5 ([41, Theorem 6.3]) *Let $H \subset D\backslash\{0\}$ be a D-convex set such that $0 \notin \mathrm{cl}(H+D)$, let $\gamma > 0$ and let Assumption* (Q3) *be satisfied by considering $S(\cdot)$ determined by* (1.2). *Let $x_0 \in S$ be an ϵ-efficient solution of* (1.1) *in the sense of Németh with respect to H, and assume that the set $(f(S) - f(x_0)) \cap (-\epsilon(\mathrm{cone}(C_H)\backslash C_H))$ is D-bounded, where $C_H = H + D$ and* $\mathrm{cone}(C_H)$ *denotes the cone generated by C_H. Then, there exists $\hat{x} \in S$ such that*

(a) $f(x_0) \in f(\hat{x}) + \gamma d(x_0,\hat{x})H + D$;

(b) $d(x_0,\hat{x})H \cap (\epsilon/\gamma)\,(\mathrm{cone}(C_H)\backslash C_H) \neq \emptyset$;

(c) $\forall x \in S\backslash\{\hat{x}\}, f(\hat{x}) \notin f(x) + \gamma d(\hat{x}, x)H + D$.

As we have seen, [41, Theorem 6.3] improves [24, Theorem 5.11] by removing the condition that D is w-normal and the condition that the cone(C_H) is based. However, whether in [41, Theorem 6.3] or in [24, Theorem 5.11], the assumption that $(f(S)-f(x_0))\cap(-\epsilon(\text{cone}(C_H)\backslash C_H))$ is D-bounded is necessary. In fact, in the proofs of the above two theorems, we need to verify that assumption (A6), i.e., $x_0 \in AE(C_H, \epsilon)$ and $(f(S) - f(x_0)) \cap (-\epsilon(\text{cone}(C_H)\backslash C_H))$ being D-bounded, is satisfied (for more details, see [24, 41]). Thus, the above assumption on D-boundedness is indispensable. In this paper, we shall follow another way of deriving this sort of results. First, we generalize Gerstewitz's functions from a single positive vector k_0 to a cone-convex subset H of the positive cone, i.e., generalizing ξ_{k_0} to ξ_H (see the following Section 2). As we shall see, this generalization destroys the subadditivity and the lower semi-continuity (when Y is a topological vector space) of ξ_{k_0}. But, not being too bad, it remains satisfying the subadditivity under some condition. There might still be a ray of hope of deriving new vector EVP by using the generalized Gerstewitz's functions. Then, we present a partial order principle, which consists of a partial order set $(X, \preceq)$ and an extended real-valued function η which is monotone with respect to $\preceq$. The partial order principle states that there exists a strong minimal point dominated by any given point provided that the monotone function η satisfies three general conditions. The partial order principle is indeed a variant of [43, Theorem 2.1]. By using the generalized Gerstewitz's functions and the partial order principle, we obtain a vector EVP for ϵ-efficient solutions in the sense of Németh, which essentially improves Theorem 1.5. To our surprise, we find out that even though the assumption $(f(S)-f(x_0))\cap(-\epsilon(\text{cone}(C_H)\backslash C_H))$ being D-bounded is completely removed, the result of Theorem 1.5 remains true. From this, we also deduce several results, which improve [41, Theorem 6.5] and [24, Theorem 5.12]. Moreover, by developing the above method, we obtain a vector EVP, where the perturbation contains a σ-convex set, which improves [8, Theorems 4.1 and 5.1], [48, Theorem 6.2] and [41, Theorem 6.8] by relaxing the lower boundedness on ranges of objective functions.

This paper is structured as follows. In Section 2, we generalize Gerstewitz's functions from a singleton $\{k_0\}$ to a set H and discuss the basic properties of generalized Gerstewitz's functions. In Section 3, we present a partial order principle, which is useful for deriving vector EVPs. In Section 4, by using the generalized Gerstewitz's functions and the partial order principle, we obtain a vector EVP for ϵ-efficient solutions in the sense of Németh, which improves the earlier results by removing a usual assumption on D-boundedness of the range of the objective function. From this, we also deduce several interesting EVPs, which improve related known results. Finally, in Section 5, we obtain a vector EVP, where the perturbation contains a σ-convex set (i.e., cs-complete bounded set; see [48], also see Section 5). The EVP improves several known EVPs by relaxing the lower boundedness for the range of the objective function.

2 Generalized Gerstewitz's Functions and Their Properties

A useful approach for solving a vector problem is to reduce it to a scalar problem. Gerstewitz's functions introduced in [19] are often used as the basis of the scalarization. In the framework of topological vector spaces, Gerstewitz's functions generated by closed convex (solid) cones and their properties have been investigated thoroughly, for example, see [11, 19, 20] and the references therein. In this section, we consider Gerstewitz's functions and their generalizations in a more general framework.

In the following, we always assume that Y is a real vector space. For a nonempty subset $A \subset Y$, the vector closure of A is defined as follows (refer to [1, 2]):

$$\mathrm{vcl}(A) = \{y \in Y : \exists v \in Y, \exists \lambda_n \geq 0, \lambda_n \to 0 \text{ such that } y + \lambda_n v \in A, \forall n \in N\}.$$

For any given $v_0 \in Y$, we define the v_0-vector closure (briefly, v_0-closure) of A as follows (refer to [42, 44]):

$$\mathrm{vcl}_{v_0}(A) = \{y \in Y : \exists \lambda_n \geq 0, \lambda_n \to 0 \text{ such that } y + \lambda_n v_0 \in A, \forall n \in N\}.$$

Obviously,

$$A \subset \mathrm{vcl}_{v_0}(A) \subset \bigcup_{v \in Y} \mathrm{vcl}_v(A) = \mathrm{vcl}(A).$$

All the above inclusions are proper. Moreover, if Y is a Hausdorff topological vector space (briefly, denoted by t.v.s.) and $\mathrm{cl}(A)$ denotes the closure of A, then $\mathrm{vcl}(A) \subset \mathrm{cl}(A)$ and the inclusion is also proper; for details, see [2, 44]. A subset A of Y is said to be v_0-closed if $A = \mathrm{vcl}_{v_0}(A)$; to be vectorially closed if $A = \mathrm{vcl}(A)$; to be (topologically) closed if $A = \mathrm{cl}(A)$. Next, we discuss the problem in a setting which is slightly more general than one in Section 1. Let Y be a real vector space and $D \subset Y$ be a convex cone. D can specify a quasi-order $\leq_D$ as follows:

$$y_1, y_2 \in Y, \quad y_1 \leq_D y_2 \Leftrightarrow y_1 - y_2 \in -D.$$

In this case, D is also called the ordering cone or positive cone (as in Section 1). We always assume that D is nontrivial, i.e., $D \neq \{0\}$ and $D \neq Y$. Let $k_0 \in D \setminus -D$ be given. For any $y \in Y$, if there exists $t \in \mathbf{R}$ such that $y \in tk_0 - D$, then for any $t' > t$, $y \in t'k_0 - D$. Thus, we can define a function $\xi_{k_0} : Y \to \mathbf{R} \cup \{\pm\infty\}$ as follows: if there exists $t \in \mathbf{R}$ such that $y \in tk_0 - D$, then define $\xi_{k_0}(y) = \inf\{t \in \mathbf{R} : y \in tk_0 - D\}$; or else, define $\xi_{k_0}(y) = +\infty$. Such a function is called a Gerstewitz's function generated by D and k_0.

The following results concerning Gerstewitz's functions originate from [19, 20].

Proposition 2.1 ([44, Lemma 2.6]) *There exists $z \in Y$ such that $\xi_{k_0}(z) = -\infty$ iff $k_0 \in -\mathrm{vcl}(D)$.*

Proposition 2.2 ([11, 20, 44]) *Let $D \subset Y$ be a convex cone and $k_0 \in D \setminus -\mathrm{vcl}(D)$. Then, the Gerstewitz's function ξ_{k_0} has the following properties*:

(i) $y_1 \leq_D y_2 \ \Rightarrow\ \xi_{k_0}(y_1) \leq \xi_{k_0}(y_2),\ \forall y_1, y_2 \in Y$;

(ii) $\xi_{k_0}(\alpha y) = \alpha\, \xi_{k_0}(y),\ \forall y \in Y,\ \forall \alpha \geq 0$, *where we assume that* $0 \cdot \infty = 0$ *if necessary*;

(iii) $\xi_{k_0}(y_1 + y_2) \leq \xi_{k_0}(y_1) + \xi_{k_0}(y_2),\ \forall y_1, y_2 \in Y$;

Let $y \in Y$ and $r \in \mathbf{R}$. Then, we have:

(iv) $\xi_{k_0}(y) < r \ \Leftrightarrow\ y \in rk_0 - \mathrm{vint}_{k_0}(D)$, *where* $\mathrm{vint}_{k_0}(D) = (0, +\infty)k_0 + D$;

(v) $\xi_{k_0}(y) \leq r \ \Leftrightarrow\ y \in rk_0 - \mathrm{vcl}_{k_0}(D)$;

(vi) $\xi_{k_0}(y) = r \ \Leftrightarrow\ y \in rk_0 - (\mathrm{vcl}_{k_0}(D) \setminus \mathrm{vint}_{k_0}(D))$;

Particularly, $\xi_{k_0}(0) = 0, \xi_{k_0}(k_0) = 1$;

(vii) $\xi_{k_0}(y) \geq r \ \Leftrightarrow\ y \notin rk_0 - \mathrm{vint}_{k_0}(D)$;

(viii) $\xi_{k_0}(y) > r \ \Leftrightarrow\ y \notin rk_0 - \mathrm{vcl}_{k_0}(D)$;

(ix) $\xi_{k_0}(y + rk_0) = \xi_{k_0}(y) + r$.

As we have seen, the Gersterwitz's function ξ_{k_0} plays an important role in deriving EVPs where perturbations contain a singleton $\{k_0\}$. Now, we consider EVPs where the $\{k_0\}$ in perturbations is replaced by a subset H of the ordering cone D; for example, see Theorem 1.5. Thus, we need to extend the notion of Gerstewitz's functions.

Let $H \subset D \backslash -D$ be a D-convex set. For any $y \in Y$, if there exists $t \in \mathbf{R}$ such that $y \in t\,H - D$, then for any $t' > t$, $y \in t'\,H - D$. Thus, we can define a function $\xi_H : Y \to \mathbf{R} \cup \{\pm\infty\}$ as follows: if there exists $t \in \mathbf{R}$ such that $y \in t\,H - D$, then define $\xi_H(y) = \inf\{t \in \mathbf{R} : y \in t\,H - D\}$; or else, define $\xi_H(y) = +\infty$. We call such a function a generalized Gerstewitz's function generated by D and H. Next, we give some properties of generalized Gerstewitz's functions.

Proposition 2.3 *There exists* $z \in Y$ *such that* $\xi_H(z) = -\infty$ *iff* $0 \in \mathrm{vcl}(H + D)$.

Proof Assume that there exists $z \in Y$ such that $\xi_H(z) = -\infty$. Then, for any $n \in \mathbf{N}$, $z \in -nH - D$. Thus, $z/n \in -H - D$. Letting $n \to \infty$, we have

$$0 \in \mathrm{vcl}(-H - D) = -\mathrm{vcl}(H + D).$$

Hence, $0 \in \mathrm{vcl}(H + D)$.

Conversely, assume that $0 \in \mathrm{vcl}(H + D)$. Then, there exists $v \in Y$ and a sequence (λ_n) with $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that $\lambda_n v \in H + D$. Since $0 \notin H + D$, we have $\lambda_n > 0$, $\forall n$. Thus,

$$-v \in -\frac{1}{\lambda_n} H - D, \quad \forall n.$$

Put $z = -v$. Then $\xi_H(z) = -\infty$. □

Proposition 2.4 *Let* $D \subset Y$ *be a convex cone and* $H \subset D$ *be a* D*-convex set such that* $0 \notin \mathrm{vcl}(H + D)$. *Then, the generalized Gerstewitz's function* ξ_H *has the following properties*:

(i) $y_1 \leq_D y_2 \implies \xi_H(y_1) \leq \xi_H(y_2)$, $\forall y_1, y_2 \in Y$;

(ii) $\xi_H(0) = 0$;

(iii) $\xi_H(\alpha y) = \alpha\, \xi_H(y)$, $\forall y \in Y$, $\forall \alpha \geq 0$, *where we assume that* $0 \cdot \infty = 0$ *if necessary*;

(iv) $\xi_H(y_1 + y_2) \leq \xi_H(y_1) + \xi_H(y_2)$ *if* $\xi_H(y_1) < 0$ *and* $\xi_H(y_2) < 0$.

Proof (i) Without loss of generality, we may assume that $\xi_H(y_2) < +\infty$. For any $\epsilon > 0$,

$$y_2 \in (\xi_H(y_2) + \epsilon)H - D.$$

Since $y_1 \leq_D y_2$, we have $y_1 \in y_2 - D$. Thus,

$$y_1 \in (\xi_H(y_2) + \epsilon)H - D - D = (\xi_H(y_2) + \epsilon)H - D.$$

Hence $\xi_H(y_1) \leq \xi_H(y_2) + \epsilon$, which leads to $\xi_H(y_1) \leq \xi_H(y_2)$.

(ii) Obviously, $0 \in 0 \cdot H - D$, so $\xi_H(0) \leq 0$. Assume that $\xi_H(0) < 0$. Then, there exists $\epsilon > 0$ such that $\xi_H(0) + \epsilon < 0$. Thus,

$$0 \in (\xi_H(0) + \epsilon)H - D = -(\xi_H(0) + \epsilon)(-H - D).$$

Since $-(\xi_H(y) + \epsilon) > 0$, we have $0 \in -H - D$ and $0 \in H + D$, contradicting $0 \notin \mathrm{vcl}(H + D)$.

(iii) If $\alpha = 0$, then from (ii), $\xi_H(\alpha y) = \xi_H(0) = 0$. Also, $\alpha \cdot \xi_H(y) = 0 \cdot \xi_H(y) = 0$. Hence, $\xi_H(\alpha y) = \alpha\, \xi_H(y)$ holds for $\alpha = 0$.

If $\alpha > 0$ and $\xi_H(y) < +\infty$, then for any $\epsilon > 0$, $y \in (\xi_H(y) + \epsilon)H - D$. Thus, $\alpha y \in \alpha(\xi_H(y) + \epsilon)H - D$. Hence, $\xi_H(\alpha y) \leq \alpha\xi_H(y) + \alpha\epsilon$. Since $\epsilon > 0$ may be arbitrary small, we have

$$\xi_H(\alpha y) \leq \alpha\xi_H(y). \tag{2.1}$$

Also,

$$\xi_H(y) = \xi_H\left(\frac{1}{\alpha}\,\alpha y\right) \leq \frac{1}{\alpha}\,\xi_H(\alpha y).$$

From this,

$$\xi_H(\alpha y) \;\geq\; \alpha\xi_H(y). \tag{2.2}$$

Combining (2.1) and (2.2), we have $\xi_H(\alpha y) \;=\; \alpha\xi_H(y)$.

If $\alpha > 0$ and $\xi_H(y) = +\infty$, then for any $t \in \mathbf{R}$, $y \notin tH - D$. Thus, for any $t \in \mathbf{R}$, $\alpha y \notin t\,H - D$. Hence $\xi_H(\alpha y) = +\infty$ and $\xi_H(\alpha y) = \alpha\xi_H(y)$ holds.

(iv) Assume that $\xi_H(y_1) < 0$ and $\xi_H(y_2) < 0$. Then, there exists $\epsilon > 0$ such that $\xi_H(y_1) + \epsilon < 0$ and $\xi_H(y_2) + \epsilon < 0$. Thus,

$$y_1 \in (\xi_H(y_1) + \epsilon)H - D \;=\; -(\xi_H(y_1) + \epsilon)\,(-H - D)$$

and

$$y_2 \in (\xi_H(y_2) + \epsilon)H - D \;=\; -(\xi_H(y_2) + \epsilon)\,(-H - D).$$

Since $-H - D$ is convex, we have

$$\begin{aligned} y_1 + y_2 &\in -(\xi_H(y_1) + \epsilon)(-H - D) - (\xi_H(y_2) + \epsilon)(-H - D) \\ &= -(\xi_H(y_1) + \xi_H(y_2) + 2\epsilon)(-H - D) \\ &= (\xi_H(y_1) + \xi_H(y_2) + 2\epsilon)H - D. \end{aligned}$$

From this,

$$\xi_H(y_1 + y_2) \;\leq\; \xi_H(y_1) + \xi_H(y_2) + 2\epsilon.$$

Since $2\epsilon > 0$ may be arbitrary small, we have

$$\xi_H(y_1 + y_2) \;\leq\; \xi_H(y_1) + \xi_H(y_2).$$

□

Remark 2.5 Proposition 2.4 (iv) points out that ξ_H satisfies the subadditivity only if the two items $\xi_H(y_1)$ and $\xi_H(y_2)$ are both negative. Being different from ξ_{k_0} (see Proposition 2.2 (iii)), in general, ξ_H does not satisfy the subadditivity. For example, let $Y := \mathbf{R}^2$ with the ordering cone $D := \{(\eta_1, \eta_2) \in Y \,:\, \eta_1 \geq 0,\, \eta_2 \geq 0\}$. Let H be the set $\{(\eta_1, \eta_2) \in Y \,:\, 1 \leq \eta_1 + \eta_2 \leq 2, \eta_1 \geq 0,\, \eta_2 \geq 0\}$. Then $H \subset D$ is a convex set and $0 \notin \text{vcl}(H + D)$. Put $y_1 = (1, 1)$ and put $y_2 = (-1, -1)$. It is easy to verify that

$$\xi_H(y_1) = \inf\{t \in \mathbf{R} :\, y_1 = (1, 1) \in tH - D\} = 1$$

and

$$\xi_H(y_2) = \inf\{t \in \mathbf{R} :\, y_2 = (-1, -1) \in tH - D\} = -2.$$

We see that

$$\xi_H(y_1 + y_2) = \xi_H((0, 0)) = 0 > 1 + (-2) = \xi_H(y_1) + \xi_H(y_2),$$

that is, the subadditivity does not hold.

3 A Partial Order Principle

In this section, we present a partial order principle, which is a useful tool of deriving EVPs. In fact, it is a variant of the pre-order principle in [43].

Let X be a nonempty set. As in [18], a binary relation $\preceq$ on X is called a pre-order if it satisfies the transitive property; a quasi order if it satisfies the reflexive and transitive properties; a partial order if it satisfies the antisymmetric, reflexive and transitive properties. Let $(X, \preceq)$ be a partial order set. An extended real-valued function $\eta : (X, \preceq) \to \mathbf{R} \cup \{\pm\infty\}$ is called monotone with respect to $\preceq$ if for any $x_1, x_2 \in X$,

$$x_1 \preceq x_2 \Longrightarrow \eta(x_1) \le \eta(x_2).$$

For any given $x_0 \in X$, denote $S(x_0)$ the set $\{x \in X : x \preceq x_0\}$. First we give a partial order principle, which is indeed a variant of [43, Theorem 2.1].

Theorem 3.1 *Let $(X, \preceq)$ be a partial order set, $x_0 \in X$ be given and $\eta : (X, \preceq) \to \mathbf{R} \cup \{\pm\infty\}$ be an extended real-valued function which is monotone with respect to $\preceq$.*

Suppose that the following conditions are satisfied:

(A) $-\infty < \inf\{\eta(x) : x \in S(x_0)\} < +\infty$;

(B) *For any $x \in S(x_0)\backslash\{x_0\}$ with $-\infty < \eta(x) < +\infty$ and any $x' \in S(x)\backslash\{x\}$, one has $\eta(x) > \eta(x')$*;

(C) *For any sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})\backslash\{x_{n-1}\}$, $\forall n$, such that $\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \to 0$ $(n \to \infty)$, there exists $u \in X$ such that $u \in S(x_n)$, $\forall n$.*

Then there exists $\hat{x} \in X$ such that

(a) $\hat{x} \in S(x_0)$;

(b) $S(\hat{x}) = \{\hat{x}\}$.

Proof For brevity, denote $\inf\{\eta(x) : x \in S(x_0)\}$ by $\inf \eta \circ S(x_0)$. By (A), we know that

$$-\infty < \inf \eta \circ S(x_0) < +\infty. \tag{3.1}$$

If $S(x_0) = \{x_0\}$, then we may take $\hat{x} := x_0$. Clearly, it satisfies (a) and (b). If $S(x_0) \neq \{x_0\}$, then by (3.1) we may take $x_1 \in S(x_0)\backslash\{x_0\}$ such that

$$\eta(x_1) < \inf \eta \circ S(x_0) + \frac{1}{2}. \tag{3.2}$$

By the transitive property of $\preceq$, we have

$$S(x_1) \subset S(x_0). \tag{3.3}$$

If $S(x_1) = \{x_1\}$, then we may take $\hat{x} := x_1$. Clearly, it satisfies (a) and (b). If $S(x_1) \neq \{x_1\}$, then by (3.1), (3.2) and (3.3) we conclude that

$$-\infty < \inf \eta \circ S(x_1) < +\infty.$$

We may take $x_2 \in S(x_1)\backslash\{x_1\}$ such that

$$\eta(x_2) < \inf \eta \circ S(x_1) + \frac{1}{2^2}.$$

In general, let $x_{n-1} \in X$ $(n \ge 1)$ be given. If $S(x_{n-1}) = \{x_{n-1}\}$, then we may take $\hat{x} := x_{n-1}$. Clearly, it satisfies (a) and (b). If $S(x_{n-1}) \neq \{x_{n-1}\}$, then we conclude that

$$-\infty < \inf \eta \circ S(x_{n-1}) < +\infty.$$

We may take $x_n \in S(x_{n-1})\backslash\{x_{n-1}\}$ such that

$$\eta(x_n) < \inf \eta \circ S(x_{n-1}) + \frac{1}{2^n}. \tag{3.4}$$

Without loss of generality, we assume that $S(x_n) \neq \{x_n\}$ for every n. Thus, we obtain a sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})\backslash\{x_{n-1}\}$, $\forall n$, such that

$$\eta(x_n) - \inf \eta \circ S(x_{n-1}) < \frac{1}{2^n} \to 0 \quad (n \to \infty).$$

By (C), there exists $\hat{x} \in X$ such that

$$\hat{x} \in S(x_n), \quad \forall n. \tag{3.5}$$

Clearly, $\hat{x} \in S(x_0)$, that is, $\hat{x}$ satisfies (a). Next, we show that $\hat{x}$ satisfies (b), that is, $S(\hat{x}) = \{\hat{x}\}$. If not, there exists $\bar{x} \in S(\hat{x})$ and $\bar{x} \neq \hat{x}$. By (B),

$$\eta(\hat{x}) > \eta(\bar{x}). \tag{3.6}$$

On the other hand, by $\bar{x} \in S(\hat{x})$ and (3.5) we have

$$\bar{x} \in S(x_n), \quad \forall n. \tag{3.7}$$

Since η is monotone with respect to $\preceq$, by (3.4), (3.5) and (3.7) we have

$$\begin{aligned} \eta(\hat{x}) \le \eta(x_n) &< \inf \eta \circ S(x_{n-1}) + \frac{1}{2^n} \\ &\le \eta(\bar{x}) + \frac{1}{2^n}, \quad \forall n. \end{aligned}$$

Letting $n \to \infty$, we have $\eta(\hat{x}) \le \eta(\bar{x})$, which contradicts (3.6). □

4 Vector EVPs for ϵ-Efficient Solutions in the Sense of Németh

In this section, we assume that (X, d) is a complete metric space, Y is a real vector space, $D \subset Y$ is a convex pointed cone, $f : X \to Y$ is a vector-valued map and S is a nonempty closed subset of X. By using the partial order principle (i.e., Theorem 3.1) and generalized Gerstewitz's functions, we obtain the following vector EVP, which improves Theorem 1.5 by removing the assumption that the set $(f(S) - f(x_0)) \cap (-\epsilon(\text{cone}(C_H)\backslash C_H))$ is D-bounded. Besides, the assumption in Theorem 1.5 that $0 \notin \text{cl}(H + D)$ is replaced by a weaker one that $0 \notin \text{vcl}(H + D)$.

Theorem 4.1 *Let $H \subset D$ be a D-convex set such that $0 \notin \text{vcl}(H + D)$, let $\gamma > 0$ and let Assumption* (Q3) *be satisfied by considering $S(\cdot)$ determined by* (1.2). *Let $x_0 \in S$ be an ϵ-efficient solution of* (1.1) *in the sense of Németh with respect to H, i.e., $(f(S) - f(x_0)) \cap (-\epsilon H - D) = \emptyset$* (see Definition 1.3). *Then, there exists $\hat{x} \in S$ such that*

(a) $f(x_0) \in f(\hat{x}) + \gamma d(x_0, \hat{x})H + D$;

(b) $d(x_0, \hat{x})H \cap (\epsilon/\gamma)\,(\text{cone}(C_H)\backslash C_H) \neq \emptyset$;

(c) $\forall x \in S\backslash\{\hat{x}\}, f(\hat{x}) \notin f(x) + \gamma d(\hat{x}, x)H + D$.

Proof For $x, x' \in S$, define $x' \preceq x$ iff $f(x) \in f(x') + \gamma d(x, x')H + D$. It is easy to verify that $\preceq$ is a partial order on S. Obviously, $\preceq$ satisfies the reflexive property and the transitive property. Next, we show that $\preceq$ satisfies the antisymmetric property. In fact, if $x' \preceq x$ and $x \preceq x'$, then

$$f(x) \in f(x') + \gamma d(x, x')H + D$$

and

$$f(x') \in f(x) + \gamma d(x', x)H + D.$$

Combining the above two belonging relations, we have

$$0 \in 2\gamma d(x, x')H + D.$$

Since $0 \notin H + D$, this leads to that $d(x, x') = 0$ and $x = x'$. If $x' \preceq x$ and $x' \neq x$, we denote $x' \prec x$. Define an extended real-valued function $\eta : (S, \preceq) \to \mathbf{R} \cup \{\pm\infty\}$ as follows:

$$\eta(x) := \xi_H(f(x) - y_0), \quad x \in S,$$

where $y_0 = f(x_0)$. Since $0 \notin \mathrm{vcl}(H + D)$, by Proposition 2.3, $\xi_H(f(x) - y_0) \neq -\infty$, that is, $\eta(x) \neq -\infty$, $\forall x \in S$. Let $x' \preceq x$. Then

$$f(x) \in f(x') + \gamma d(x, x')H + D.$$

Thus,

$$f(x') - f(x) \in -\gamma d(x, x')H - D \subset -D$$

and

$$f(x') - y_0 \leq_D f(x) - y_0.$$

By Proposition 2.4 (i), we have

$$\xi_H(f(x') - y_0) \leq \xi_H(f(x) - y_0), \quad \text{that is, } \eta(x') \leq \eta(x).$$

Hence, η is monotone with respect to $\preceq$. We denote the set $\{x' \in X : x' \preceq x\}$ by $S(x)$. Next, we prove that Assumptions (A), (B) and (C) in Theorem 3.1 are satisfied.

Proof of (A) Since $y_0 = f(x_0) \notin f(S) + \epsilon H + D$, for any $x \in S(x_0) \subset S$,

$$f(x) - y_0 \notin -\epsilon H - D, \quad \text{so } \eta(x) = \xi_H(f(x) - y_0) \geq -\epsilon.$$

Also, by Proposition 2.4 (ii),

$$\eta(x_0) = \xi_H(f(x_0) - y_0) = \xi_H(0) = 0.$$

Hence,

$$-\infty < -\epsilon \leq \inf\{\eta(x) : x \in S(x_0)\} \leq \eta(x_0) = 0 < +\infty.$$

That is, (A) is satisfied.

Proof of (B) Let $x \in S(x_0)\backslash\{x_0\}$ with $-\infty < \eta(x) < +\infty$ and let $x' \in S(x)\backslash\{x\}$. By $x \in S(x_0)\backslash\{x_0\}$, we have

$$f(x_0) \in f(x) + \gamma d(x_0, x)H + D \quad \text{and} \quad x \neq x_0. \tag{4.1}$$

By $x' \in S(x)\backslash\{x\}$, we have

$$f(x) \in f(x') + \gamma d(x, x')H + D \quad \text{and} \quad x' \neq x. \tag{4.2}$$

By (4.1), we know that

$$f(x) - y_0 \in -\gamma d(x_0, x)H - D,$$

and so

$$\xi_H(f(x) - y_0) \leq -\gamma d(x_0, x) < 0. \tag{4.3}$$

By (4.2), we know that

$$f(x') - f(x) \in -\gamma d(x, x')H - D,$$

and so

$$\xi_H(f(x') - f(x)) \leq -\gamma d(x, x') < 0. \tag{4.4}$$

Remarking (4.3) and (4.4), and using Proposition 2.4 (iv), we have

$$\xi_H(f(x') - y_0) \leq \xi_H(f(x') - f(x)) + \xi_H(f(x) - y_0).$$

From this and using (4.4), we have

$$\xi_H(f(x') - y_0) - \xi_H(f(x) - y_0) \leq \xi_H(f(x') - f(x)) \leq -\gamma d(x, x').$$

That is,

$$\eta(x') - \eta(x) \leq -\gamma d(x, x'),$$

and so

$$\eta(x') \leq \eta(x) - \gamma d(x, x') < \eta(x).$$

Thus, (B) is satisfied.

Proof of (C) Let a sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1})\backslash\{x_{n-1}\}$, $\forall n$, such that $\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \to 0$ $(n \to \infty)$. Since $x_0 \succ x_1 \succ x_2 \succ \cdots \succ x_n \succ \cdots$, by the transitive property and the antisymmetric property, we have $x_m \prec x_n$, $\forall m > n$. That is,

$$f(x_n) \in f(x_m) + \gamma d(x_n, x_m)H + D.$$

From this,

$$\xi_H(f(x_m) - f(x_n)) \leq -\gamma d(x_n, x_m),$$

and so

$$\gamma d(x_n, x_m) \leq -\xi_H(f(x_m) - f(x_n)). \tag{4.5}$$

Since $\xi_H(f(x_n) - y_0) \leq -\gamma d(x_0, x_n) < 0$ and $\xi_H(f(x_m) - f(x_n)) \leq -\gamma d(x_n, x_m) < 0$, by Proposition 2.4 (iv), we have

$$\xi_H(f(x_m) - y_0) \leq \xi_H(f(x_m) - f(x_n)) + \xi_H(f(x_n) - y_0),$$

and so

$$-\xi_H(f(x_m) - f(x_n)) \leq \xi_H(f(x_n) - y_0) - \xi_H(f(x_m) - y_0). \tag{4.6}$$

Combining (4.5) and (4.6), and remarking that $x_m \in S(x_{n-1})$, we have

$$\begin{aligned}\gamma d(x_n, x_m) &\leq \xi_H(f(x_n) - y_0) - \xi_H(f(x_m) - y_0)\\ &= \eta(x_n) - \eta(x_m)\\ &\leq \eta(x_n) - \inf \eta \circ S(x_{n-1}) \;\to\; 0 \quad (n \to \infty).\end{aligned}$$

Hence (x_n) is a Cauchy sequence. Since (X, d) is complete and $S \subset X$ is closed, there exists $\hat{x} \in S$ such that $x_n \to \hat{x}$ $(n \to \infty)$. For any given n, $S(x_n) \subset S(x_0)$. We observe that $(x_{n+p})_{p\in\mathbf{N}} \subset S(x_n)$ and $x_{n+p+1} \in S(x_{n+p}), \forall p$. Since $x_{n+p} \to \hat{x}$ $(p \to \infty)$ and $S(x_n)$ is dynamically closed by (Q3), we have $\hat{x} \in S(x_n)$, $\forall n$. That is, (C) is satisfied.

Now, applying Theorem 3.1 we conclude that there exists $\hat{x} \in S$ such that $\hat{x} \in S(x_0)$ and $S(\hat{x}) = \{\hat{x}\}$. That is, $\hat{x}$ satisfies (a) and (c). Finally we show that $\hat{x}$ satisfies (b). By (a),

$$f(x_0) \in f(\hat{x}) + \gamma d(x_0, \hat{x})H + D.$$

Hence, there exists $h_0 \in H$ and $d_0 \in D$ such that

$$f(x_0) = f(\hat{x}) + \gamma d(x_0, \hat{x})h_0 + d_0. \tag{4.7}$$

Clearly,

$$d(x_0, \hat{x})h_0 \in d(x_0, \hat{x})H \tag{4.8}$$

and

$$d(x_0, \hat{x})h_0 \in \text{cone}(H + D). \tag{4.9}$$

Next we show that

$$d(x_0, \hat{x})h_0 \notin \frac{\epsilon}{\gamma}(H + D). \tag{4.10}$$

Assume that

$$d(x_0, \hat{x})h_0 \in \frac{\epsilon}{\gamma}(H + D).$$

Then

$$\gamma d(x_0, \hat{x})h_0 \in \epsilon(H + D) = \epsilon H + D.$$

Thus,

$$\gamma d(x_0, \hat{x})h_0 + d_0 \in \epsilon H + D + D = \epsilon H + D.$$

Combining this with (4.7), we have

$$f(x_0) - f(\hat{x}) \in \epsilon H + D,$$

which contradicts the assumption that

$$f(x_0) \notin f(S) + \epsilon H + D.$$

Now, combining (4.8), (4.9) and (4.10), we have

$$\begin{aligned} d(x_0, \hat{x})h_0 &\in d(x_0, \hat{x})H \cap \left(\text{cone}(H + D) \backslash \frac{\epsilon}{\gamma}(H + D)\right) \\ &= d(x_0, \hat{x})H \cap \left(\frac{\epsilon}{\gamma}\right)(\text{cone}(H + D) \backslash (H + D)). \end{aligned}$$

This means that (b) is satisfied. □

Remark 4.2 As we have seen in the proof of Theorem 4.1, we applied Theorem 3.1 instead of [43, Theorem 2.1] to prove Theorem 4.1. Here, Theorem 3.1 is specially made for deriving Theorem 4.1. As shown in Section 2, being different from the Gerstewitz's function ξ_{k_0} (where k_0 is a single point), the generalized Gerstewitz's function ξ_H (where H is a cone-convex set) satisfies the subadditivity $\xi_H(y_1 + y_2) \leq \xi_H(y_1) + \xi_H(y_2)$ only when the two items $\xi_H(y_1)$ and $\xi_H(y_2)$ are both negative (compare Proposition 2.2 (iii) and Proposition 2.4 (iv)). In Proof of (B) in the proof of Theorem 4.1, we need to use the inequality

$$\xi_H(f(x') - y_0) \leq \xi_H(f(x') - f(x)) + \xi_H(f(x) - y_0). \tag{4.11}$$

In order to make the above inequality hold, we need to have

$$\xi_H(f(x') - f(x)) < 0 \quad \text{and} \quad \xi_H(f(x) - y_0) < 0.$$

By (4.3) and (4.4), we have

$$\xi_H(f(x) - y_0) \le -\gamma\, d(x_0, x) \quad \text{and} \quad \xi_H(f(x') - f(x)) \le -\gamma\, d(x, x').$$

Thus, in order to make (4.11) hold, it is sufficient to assume that $d(x_0, x) > 0$ and $d(x, x') > 0$, that is, $x_0 \ne x$ and $x' \ne x$. Hence, in Condition (B) of Theorem 3.1 we need to assume that $x \in S(x_0)\backslash\{x_0\}$ and $x' \in S(x)\backslash\{x\}$. Recalling Condition (B) of [43, Theorem 2.1], there we only assumed that $x \in S(x_0)$ and $x' \in S(x)\backslash\{x\}$. This is the reason why we need to change "$x \in S(x_0)$" in Condition (B) of [43, Theorem 2.1] into "$x \in S(x_0)\backslash\{x_0\}$" in Condition (B) of Theorem 3.1.

Similarly, in Proof of (C) in the proof of Theorem 4.1, we need to use the inequality

$$\xi_H(f(x_m) - y_0) \ \le\ \xi_H(f(x_m) - f(x_n)) + \xi_H(f(x_n) - y_0). \tag{4.12}$$

In order to make the above inequality hold, we need to have

$$\xi_H(f(x_m) - f(x_n)) < 0 \quad \text{and} \quad \xi_H(f(x_n) - y_0) < 0.$$

Since

$$\xi_H(f(x_m) - f(x_n)) \le -\gamma d(x_n, x_m) \quad \text{and} \quad \xi_H(f(x_n) - y_0) \le -\gamma d(x_n, x_0),$$

in order to make (4.12) hold, we only need to assume that $d(x_n, x_m) > 0$ $(\forall m > n)$ and $d(x_0, x_n) > 0$ $(\forall n > 0)$, that is, $x_m \ne x_n$ $(\forall m > n)$ and $x_n \ne x_0$ $(\forall n > 0)$. Hence, we need to assume that "$\preceq$" is a partial order (not only a pre-order) and $x_n \in S(x_{n-1})\backslash\{x_{n-1}\}, \forall n$ (not only $x_n \in S(x_{n-1}), \forall n$). This is just the difference between Condition (C) of Theorem 3.1 and Condition (C) of [43, Theorem 2.1].

In a word, from [43, Theorem 2.1] we can't prove Theorem 4.1 directly. But, from its variant, i.e., Theorem 3.1, we can deduce Theorem 4.1.

As in [24, 41], a vector-valued map $f : X \to Y$ is said to be sequentially submonotone with respect to D (briefly, denoted by submonotone) if for every $x \in X$ and for each sequence (x_n) such that $x_n \to x$ and $f(x_m) \le_D f(x_n)$, $\forall m > n$, it follows that $f(x) \le_D f(x_n)$, $\forall n$. Sometimes, a submonotone vector-valued map is said to be D-sequentially lower monotone (briefly, denoted by D-slm or slm); see, for example [26]. In [8], a submonotone vector-valued map is called a monotonically semi-continuous (denoted by msc) with respect to D map; in [21] it is called a map with Property (H4); and in [30] it is called a lower semi-continuous from above (briefly, denoted by lsca). Let us observe that a D-lower semi-continuous vector-valued map $f : X \to Y$, i.e., f such that the sets $\{x \in X : f(x) \le_D y\}$ are closed for all $y \in Y$, is submonotone. But the converse is not true even Y is the real number space with the usual order, for example, see [9].

Next, we present a particular version of vector EVP for ϵ-efficient solutions by giving a certain condition for (Q3) fulfilled.

Theorem 4.3 *Let $H \subset D$ be a D-convex set such that $0 \notin \mathrm{vcl}(H + D)$, and let $x_0 \in S$ be an ϵ-efficient solution of* (1.1) *in the sense of Németh with respect to H. Moreover, assume that*

$H+D$ is h_0-closed for a certain $h_0 \in H$ and f is submonotone. Then, for any $\gamma > 0$, there exists $\hat{x} \in S$ such that

(a) $f(x_0) \in f(\hat{x}) + \gamma d(x_0, \hat{x})H + D$;

(b) $d(x_0, \hat{x}) < \epsilon/\gamma$;

(c) $\forall x \in S \backslash \{\hat{x}\}, f(\hat{x}) \notin f(x) + \gamma d(\hat{x}, x)H + D$.

Proof For any $x \in S$, put

$$S(x) := \{z \in S : f(x) \in f(z) + \gamma\, d(x,z)\, H + D\}.$$

From Theorem 4.1, we only need to prove that (Q3) is satisfied. Let $x \in S(x_0)$, $(x_n) \subset S(x)$ with $x_{n+1} \in S(x_n)$ and $x_n \to u$. For any given n and for every $m > n$, we have $x_m \in S(x_n)$ and hence $f(x_m) \leq_D f(x_n)$. Since f is submonotone and $x_m \to u$ $(m \to \infty)$, we have $f(u) \leq_D f(x_n)$. For $m > n$, $x_m \in S(x_n)$. Thus,

$$\begin{aligned} f(x_n) &\in f(x_m) + \gamma d(x_n, x_m)H + D \\ &\subset f(u) + \gamma d(x_n, x_m)H + D. \end{aligned}$$

Next, we show the result according to the following two cases.

Case 1 There exists $m > n$ such that $d(x_n, x_m) \geq d(x_n, u)$. Then

$$\begin{aligned} f(x_n) &\in f(u) + \gamma d(x_n, x_m)H + D \\ &\subset f(u) + \gamma d(x_n, u)H + D. \end{aligned}$$

That is, $u \in S(x_n) \subset S(x)$.

Case 2 For every $m > n$, $d(x_n, x_m) < d(x_n, u)$. Then, from

$$f(x_n) \in f(u) + \gamma d(x_n, x_m)H + D,$$

we have

$$\begin{aligned} & f(x_n) + \gamma(d(x_n, u) - d(x_n, x_m))h_0 \\ &\quad \in f(u) + \gamma d(x_n, x_m)H + D + \gamma(d(x_n, u) - d(x_n, x_m))h_0 \\ &\quad \subset f(u) + \gamma d(x_n, x_m)(H + D) + \gamma(d(x_n, u) - d(x_n, x_m))(H + D) \\ &\quad = f(u) + \gamma d(x_n, u)(H + D) \\ &\quad = f(u) + \gamma d(x_n, u)H + D. \end{aligned}$$

From this,

$$f(x_n) - f(u) + \gamma(d(x_n, u) - d(x_n, x_m))h_0 \in \gamma d(x_n, u)H + D. \tag{4.13}$$

Since $d(x_n, u) - d(x_n, x_m) \to 0$ $(m \to \infty)$ and $\gamma d(x_n, u)H + D$ is h_0-closed, by (4.13) we have

$$f(x_n) - f(u) \in \gamma d(x_n, u)H + D$$

and

$$f(x_n) \in f(u) + \gamma d(x_n, u)H + D.$$

That is, $u \in S(x_n) \subset S(x)$. Thus, we have shown that (Q3) is satisfied. Applying Theorem 4.1, there exists $\hat{x} \in S$ such that (a) and (c) are satisfied. Next we show that (b) is satisfied. If not, assume that $d(x_0, \hat{x}) \geq \epsilon/\gamma$. Then from (a), we have

$$\begin{aligned} f(x_0) &\in f(\hat{x}) + \gamma d(x_0, \hat{x})H + D \\ &\subset f(\hat{x}) + \gamma \frac{\epsilon}{\gamma} H + D \\ &= f(\hat{x}) + \epsilon H + D, \end{aligned}$$

which contradicts the assumption that $x_0 \in S$ is an ϵ-efficient solution of (1.1) in the sense of Németh with respect to H, i.e., $f(x_0) \notin f(S) + \epsilon H + D$. □

Theorem 4.4 *Let Y be a locally convex space, $D \subset Y$ be a closed convex cone and $H \subset D\backslash -D$ be a $\sigma(Y, D^+)$-countably compact, D-convex set. Suppose that $f : X \to Y$ is submonotone and x_0 is an ϵ-efficient solution of* (1.1) *in the sense of Németh with respect to H. Then, for any $\gamma > 0$, there exists $\hat{x} \in S$ such that*

(a) $f(x_0) \in f(\hat{x}) + \gamma d(x_0, \hat{x})H + D$;

(b) $d(x_0, \hat{x}) < \epsilon/\gamma$;

(c) $\forall x \in S\backslash\{\hat{x}\}, f(\hat{x}) \notin f(x) + \gamma d(\hat{x}, x)H + D$.

Proof By Theorem 4.3, we only need to prove that $H + D$ is vectorially closed. Let $z \in \mathrm{vcl}(H + D)$. Then, there exists $v_0 \in Y$ and a sequence (ϵ_n) with $\epsilon_n > 0$ and $\epsilon_n \to 0$ such that $z + \epsilon_n v_0 \in H + D$. For each n, there exists $h_n \in H$ such that

$$z + \epsilon_n v_0 \in h_n + D,$$

that is,

$$z - h_n + \epsilon_n v_0 \in D. \tag{4.14}$$

Since H is $\sigma(Y, D^+)$-countably compact, the sequence $(h_n) \subset H$ has a $\sigma(Y, D^+)$-cluster point $h' \in H$. Take any $\xi \in D^+$. From (4.14), we have

$$\xi(z) - \xi(h_n) + \epsilon_n \xi(v_0) \geq 0. \tag{4.15}$$

Since a continuous map preserves cluster points, $\xi(h')$ is a cluster point of $(\xi(h_n))_n$ in $\mathbf{R}$. Hence, there exists a subsequence $n_1 < n_2 < n_3 < \cdots$ such that $\xi(h_{n_i}) \to \xi(h')$ $(i \to \infty)$. By (4.15), we have

$$\xi(z) - \xi(h_{n_i}) + \epsilon_{n_i} \xi(v_0) \geq 0.$$

Letting $i \to \infty$, we have

$$\xi(z) - \xi(h') \geq 0, \quad \text{i.e., } \xi(z - h') \geq 0.$$

Since $\xi \in D^+$ is arbitrary and D is a closed convex cone, we have

$$z - h' \in D^{++} = D.$$

That is, $z \in h' + D \subset H + D$. Thus, we have shown that $H + D = \mathrm{vcl}(H + D)$. Now, from Theorem 4.3 we obtain the result. □

Remark 4.5 [41, Theorem 6.5] also gives the same result as in Theorem 4.4, but there one needs to assume that H is a base of D. Here, we have removed the assumption. Clearly, Theorem 4.4 improves [41, Theorem 6.5] and also improves [24, Theorem 5.12].

5 Vector EVP with Perturbation Containing a σ-Convex Set

Vector EVPs, where perturbations are of type $d(x, y)H$, were also considered by Bednarczuk and Zagrodny [8], Tammer and Zălinescu [48] and Qiu [41]. For details, see [8, theorem 4.1], [48, Theorem 6.2] and [41, Theorem 6.8]. We shall see that our partial order principle, i.e., Theorem 3.1, also implies this type of EVPs. We shall obtain a vector EVP, where the perturbation contains a σ-convex set, which improves the above three results. First, we recall some terms and notions. Let Y be a t.v.s. and $B \subset Y$ be nonempty. A convex series of points of B is a series of the form $\sum_{n=1}^{\infty} \lambda_n b_n$, where every $b_n \in B$, every $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$. B is said to be a σ-convex set if every convex series of its points converges to a point of B (see [36, 42] and the references therein). Let B be a σ-convex set. Then, for a sequence (b_n) in B and a real sequence (λ_n) with $\lambda_n \geq 0$ and $0 < \sum_{n=1}^{\infty} \lambda_n < +\infty$, $\sum_{n=1}^{\infty} \lambda_n b_n / \sum_{n=1}^{\infty} \lambda_n$ is a convex series in B and it converges to some point $\bar{b} \in B$. Thus, $\sum_{n=1}^{\infty} \lambda_n b_n$ converges to $(\sum_{n=1}^{\infty} \lambda_n)\bar{b} \in (\sum_{n=1}^{\infty} \lambda_n)B$. We call a set B sequentially complete iff every Cauchy sequence (b_n) in B, converges to a point of B. In [8], " sequentially complete" is called "semi-complete". It is easy to show that every sequentially complete, bounded convex set is a σ-convex set (see [48, Remark 6.1]). However, a σ-convex set need not be sequentially complete. For example, an open ball B in a Banach space is σ-convex, but it is not closed and hence is not sequentially complete (for details, see [36, 42]).

Some authors used the notion of cs-complete sets. In [48, 50], a set $B \subset Y$ is said to be cs-complete if for all sequence $(\lambda_n)_{n\in\mathbf{N}} \subset [0, \infty)$ and $(b_n)_{n\in\mathbf{N}} \subset B$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and the sequence $(\sum_{i=1}^{n} \lambda_i b_i)_{n\in\mathbf{N}}$ is Cauchy, the series $\sum_{n=1}^{\infty} \lambda_n b_n$ is convergent to a point of B. A set $B \subset Y$ is said to be cs-closed if the sum of the series $\sum_{n=1}^{\infty} \lambda_n b_n$ belongs to B whenever $\sum_{n=1}^{\infty} \lambda_n b_n$ is convergent and $(b_n)_{n\in\mathbf{N}} \subset B$, $(\lambda_n)_{n\in\mathbf{N}} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. As pointed out in [48], any cs-complete set is cs-closed; and any cs-closed set is convex. Hence, any cs-complete set is convex.

Next, we show that a set $B \subset Y$ is σ-convex iff B is cs-complete and bounded. First, assume that B is σ-convex. Then, clearly, it is cs-complete. Moreover, B is also bounded. If not, there exists a circled 0-neighborhood V such that for all $n \in \mathbf{N}$, $B \not\subset n^3 V$. Thus, there exists a sequence $(b_n)_{n\in\mathbf{N}} \subset B$ such that $b_n \notin n^3 V$. So, $b_n/n^2 \notin nV$. This leads to that the series $\sum_{n=1}^{\infty} b_n/n^2$ does not converge, which contradicts that B is σ-convex. Conversely, assume that $B \subset Y$ is cs-complete and bounded. Since a cs-complete set must be convex, B is a bounded convex set. Take any convex series $\sum_{n=1}^{\infty} \lambda_n b_n$, where $(b_n)_{n\in\mathbf{N}} \subset B$ and $(\lambda_n)_{n\in\mathbf{N}} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. For any $m, k \in \mathbf{N}$,

$$\begin{aligned}\sum_{n=1}^{m+k} \lambda_n b_n - \sum_{n=1}^{m} \lambda_n b_n &= \sum_{n=m+1}^{m+k} \lambda_n b_n \\ &= \left(\sum_{n=m+1}^{m+k} \lambda_n\right) \left(\sum_{n=m+1}^{m+k} \lambda_n b_n / \sum_{n=m+1}^{m+k} \lambda_n\right) \\ &= \left(\sum_{n=m+1}^{m+k} \lambda_n\right) b'_{m,k},\end{aligned}$$

where $b'_{m,k} := \sum_{n=m+1}^{m+k} \lambda_n b_n / \sum_{n=m+1}^{m+k} \lambda_n \in B$ since B is a convex set. Thus,

$$\sum_{n=1}^{m+k} \lambda_n b_n - \sum_{n=1}^{m} \lambda_n b_n \in \Big(\sum_{n=m+1}^{m+k} \lambda_n \Big) B.$$

Since B is bounded, for any circled 0-neighborhood V, there exists $\alpha > 0$ such that $B \subset \alpha V$. Since $\sum_{n=1}^{\infty} \lambda_n = 1$, for $\alpha > 0$, there exists $m_0 \in \mathbf{N}$ such that $\sum_{n=m+1}^{m+k} \lambda_n \le 1/\alpha$ for all $m \ge m_0$ and all $k \in \mathbf{N}$. Thus,

$$\begin{aligned}\sum_{n=1}^{m+k} \lambda_n b_n - \sum_{n=1}^{m} \lambda_n b_n &\in \Big(\sum_{n=m+1}^{m+k} \lambda_n \Big) B \\ &\subset \Big(\sum_{n=m+1}^{m+k} \lambda_n \Big) \alpha V \\ &\subset \frac{1}{\alpha} \alpha V \\ &= V.\end{aligned}$$

This means that $(\sum_{n=1}^{m} \lambda_n b_n)_{m \in \mathbf{N}}$ is a Cauchy sequence. Since B is cs-complete, we conclude that $\sum_{n=1}^{\infty} \lambda_n b_n$ is convergent to a point of B. That is, B is σ-convex.

Theorem 5.1 *Let (X, d) be a complete metric space, Y be a t.v.s., $D \subset Y$ be a closed convex cone, $H \subset D$ be a σ-convex set (i.e., a cs-complete bounded set) such that $0 \notin \mathrm{vcl}(H + D)$ and let $f : X \to Y$ be a submonotone vector-valued map. Suppose that $x_0 \in X$ and $\epsilon > 0$ such that*

$$(f(x_0) - \epsilon H - D) \cap f(X) = \emptyset.$$

Then, for any $\gamma > 0$, there exists $\hat{x} \in S$ such that

(a) $f(x_0) \in f(\hat{x}) + \gamma d(x_0, \hat{x}) H + D$;

(b) $d(x_0, \hat{x}) < \epsilon/\gamma$;

(c) $\forall x \in S \backslash \{\hat{x}\}, \ f(\hat{x}) \notin f(x) + \gamma d(\hat{x}, x) H + D$.

Proof For $x, x' \in X$, define $x' \preceq x$ iff $f(x) \in f(x') + \gamma d(x, x') H + D$. It is easy to show that $\preceq$ is a partial order on X. Here, in order to show that $\preceq$ satisfies antisymmetric property we only need to assume that $0 \notin H + D$. Hence we need not assume that D is pointed. As in the proof of Theorem 4.1, we define an extended real-valued function $\eta : (X, \preceq) \to \mathbf{R} \cup \{\pm\infty\}$ as follows:

$$\eta(x) := \xi_H(f(x) - y_0), \quad x \in X,$$

where $y_0 = f(x_0)$. For any $x \in X$, put $S(x) := \{x' \in X : x' \preceq x\}$. It's easy to prove that η is monotone with respect to $\preceq$, and Assumptions (A) and (B) in Theorem 3.1 are satisfied. It suffices to prove that Assumption (C) in Theorem 3.1 is satisfied. Let a sequence $(x_n) \subset S(x_0)$ with $x_n \in S(x_{n-1}) \backslash \{x_{n-1}\}$, $\forall n$, such that

$$\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \to 0 \quad (n \to \infty).$$

By $x_i \in S(x_{i-1})$ for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}&f(x_0) \in f(x_1) + \gamma d(x_0, x_1) H + D, \\ &f(x_1) \in f(x_2) + \gamma d(x_1, x_2) H + D,\end{aligned}$$

$$\cdots \cdots$$

$$f(x_{n-1}) \in f(x_n) + \gamma d(x_{n-1}, x_n)H + D.$$

By adding the two sides of the above n belonging relations, we have

$$f(x_0) \in f(x_n) + \gamma\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big)H + D$$

and

$$f(x_n) - y_0 = f(x_n) - f(x_0) \in -\gamma\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big)H - D.$$

From this,

$$\xi_H(f(x_n) - y_0) \leq -\gamma\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big).$$

Hence,

$$\gamma\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big) \leq -\xi_H(f(x_n) - y_0). \tag{5.1}$$

By the assumption,

$$y_0 = f(x_0) \notin f(x_n) + \epsilon H + D,$$

so

$$f(x_n) - y_0 \notin -\epsilon H - D.$$

Thus,

$$\xi_H(f(x_n) - y_0) \geq -\epsilon$$

and

$$-\xi_H(f(x_n) - y_0) \leq \epsilon. \tag{5.2}$$

Combining (5.1) and (5.2), we have

$$\gamma\Big(\sum_{i=1}^{n} d(x_{i-1}, x_i)\Big) \leq \epsilon$$

and

$$\sum_{i=1}^{n} d(x_{i-1}, x_i) \leq \frac{\epsilon}{\gamma}, \quad \forall n.$$

Thus,

$$\sum_{i=1}^{\infty} d(x_{i-1}, x_i) \leq \frac{\epsilon}{\gamma}.$$

Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$ in (X, d). Next, we show that $u \in S(x_n)$, $\forall n$. By $x_{i+1} \in S(x_i)$ for $i = n, n+1, \ldots, n+k-1$, we have

$$f(x_n) \in f(x_{n+1}) + \gamma d(x_n, x_{n+1})H + D,$$

$$f(x_{n+1}) \in f(x_{n+2}) + \gamma d(x_{n+1}, x_{n+2})H + D,$$

$$\cdots \cdots$$

$$f(x_{n+k-1}) \in f(x_{n+k}) + \gamma d(x_{n+k-1}, x_{n+k})H + D.$$

Thus, there exist h_{n+1}, h_{n+2}, $\dots$, $h_{n+k} \in H$ such that

$$\begin{aligned} &f(x_n) \in f(x_{n+1}) + \gamma d(x_n, x_{n+1})h_{n+1} + D,\\ &f(x_{n+1}) \in f(x_{n+2}) + \gamma d(x_{n+1}, x_{n+2})h_{n+2} + D,\\ &\cdots\ \cdots\\ &f(x_{n+k-1}) \in f(x_{n+k}) + \gamma d(x_{n+k-1}, x_{n+k})h_{n+k} + D.\end{aligned}$$

By adding the two sides of the above k belonging relations, we have

$$\begin{aligned} f(x_n) &\in f(x_{n+k}) + \gamma \sum_{i=n}^{n+k-1} d(x_i, x_{i+1})h_{i+1} + D\\ &= f(x_{n+k}) + \gamma \sum_{i=n+1}^{n+k} d(x_{i-1}, x_i)h_i + D.\end{aligned}$$

From this,

$$f(x_{n+k}) \in f(x_n) - \gamma \sum_{i=n+1}^{n+k} d(x_{i-1}, x_i)h_i - D. \tag{5.3}$$

Since f is submonotone, we have $f(u) \leq_D f(x_{n+k})$. Combining this with (5.3), we have

$$f(u) \in f(x_{n+k}) - D \subset f(x_n) - \gamma \sum_{i=n+1}^{n+k} d(x_{i-1}, x_i)h_i - D. \tag{5.4}$$

Remarking that H is σ-convex, we conclude that there exists $h'_n \in H$ such that

$$\sum_{i=n+1}^{n+k} d(x_{i-1}, x_i)h_i \to \left(\sum_{i=n+1}^{\infty} d(x_{i-1}, x_i)\right)h'_n \quad (k \to \infty). \tag{5.5}$$

Since D is closed, by (5.4) and (5.5) we have

$$f(u) \in f(x_n) - \gamma\left(\sum_{i=n+1}^{\infty} d(x_{i-1}, x_i)\right)h'_n - D,$$

and so

$$f(x_n) \in f(u) + \gamma\left(\sum_{i=n+1}^{\infty} d(x_{i-1}, x_i)\right)h'_n + D. \tag{5.6}$$

On the other hand,

$$\begin{aligned} d(x_n, u) &= \lim_{k\to\infty} d(x_n, x_{n+k})\\ &\leq \lim_{k\to\infty} (d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}))\\ &= \sum_{i=n+1}^{\infty} d(x_{i-1}, x_i).\end{aligned} \tag{5.7}$$

By (5.6) and (5.7) we have

$$f(x_n) \in f(u) + \gamma\left(\sum_{i=n+1}^{\infty} d(x_{i-1}, x_i)\right)h'_n + D$$

$$\subset f(u) + \gamma d(x_n, u)h'_n + D$$
$$\subset f(u) + \gamma d(x_n, u)H + D.$$

Thus, $u \in S(x_n)$ and Assumption (C) is satisfied. Now, applying Theorem 3.1, there exists $\hat{x} \in X$ such that $\hat{x} \in S(x_0)$ and $S(\hat{x}) = \{\hat{x}\}$. From this, we can easily show that $\hat{x}$ satisfies (a), (b) and (c). □

As is well-known, for locally convex spaces, there are various notions of completeness. The weakest one seems to be local completeness (see [36, 37, 46]). A locally convex space Y is locally complete iff it is l^1-complete, i.e., for each bounded sequence $(b_n) \subset Y$ and each $(\lambda_n) \subset l^1$, the series $\sum_{n=1}^{\infty} \lambda_n b_n$ converges in Y. Thus, if Y is a locally complete locally convex space and H is a locally closed, bounded convex set, then we can show that H is a σ-convex set. Concerning local completeness and local closedness, please refer to [36, Chapter 5] and [37, 38, 46].

As we have seen, the assumption in [41, Theorem 6.8] that there exists $\xi \in D^+ \cap H^{+s}$ such that $(f(X) - f(x_0)) \cap (-(\bigcup_{\lambda>0} \lambda H + D))$ is ξ-lower bounded has been replaced by here one that there exists $\epsilon > 0$ such that $(f(x_0) - \epsilon H - D) \cap f(X) = \emptyset$. We shall see that the latter is strictly weaker than the former from the following Proposition 5.2 and Example 5.3. Hence, Theorem 5.1 improves [41, Theorem 6.8], and also improves [8, Theorem 4.1] and [48, Theorem 6.2].

Proposition 5.2 *Assume that there exists $\xi \in D^+ \cap H^{+s}$ such that $(f(X) - f(x_0)) \cap (-(\bigcup_{\lambda>0} \lambda H + D))$ is ξ-lower bounded. Then, there exists $\epsilon > 0$ such that $(f(x_0) - \epsilon H - D) \cap f(X) = \emptyset$.*

Proof If not, for every $n \in \mathbf{N}$,

$$(f(x_0) - nH - D) \cap f(X) \neq \emptyset.$$

From this,

$$(f(X) - f(x_0)) \cap (-nH - D) \neq \emptyset, \quad \forall n.$$

For each n, there exists $z_n \in f(X) - f(x_0)$ such that

$$z_n \in -nH - D. \tag{5.8}$$

Clearly,

$$z_n \in (f(X) - f(x_0)) \cap \Big(- \Big(\bigcup_{\lambda>0} \lambda H + D \Big) \Big). \tag{5.9}$$

Since $\xi \in D^+ \cap H^{+s}$, we have $\xi(d) \geq 0,\ \forall d \in D$ and $\alpha := \inf\{\xi(h) : h \in H\} > 0$. Combining this with (5.8), we have

$$\xi(z_n) \leq -n\alpha, \quad \forall n.$$

This with (5.9) contradicts the assumption that $(f(X) - f(x_0)) \cap (-(\bigcup_{\lambda>0} \lambda H + D))$ is ξ-lower bounded. □

The following example shows that there is such a vector-valued map $f : X \to Y$ and $x_0 \in X$ such that there exists $\epsilon > 0$ such that $(f(X) - f(x_0)) \cap (-\epsilon H - D) = \emptyset$, but for every $\xi \in D^+ \cap H^{+s}$, $(f(X) - f(x_0)) \cap (-(\bigcup_{\lambda>0} \lambda H + D))$ is not ξ-lower bounded.

Example 5.3 Let X be $\mathbf{R}$ with the usual metric, i.e., $d(x, x') = |x - x'|,\ x, x' \in \mathbf{R}$, let Y be $\mathbf{R}^2$ with the usual topology and with the partial order generated by the closed convex pointed cone $D = \{(y_1, y_2) \in \mathbf{R}^2 : y_1 \geq 0,\ y_2 \geq 0\}$, and let $H \subset D\backslash\{0\}$ be a singleton $H = \{k_0\}$, where $k_0 = (1,1) \in D \subset \mathbf{R}^2$. For any $\xi \in D^+ \cap H^{+s} = D^+ \cap \{k_0\}^{+s}$, there exists a unique $(\alpha, \beta) \in \mathbf{R}^2$ such that

$$\xi(y) \;=\; \alpha y_1 + \beta y_2, \quad \forall y = (y_1, y_2) \in Y = \mathbf{R}^2.$$

From $\xi \in D^+ \cap \{k_0\}^{+s}$, we conclude that $\alpha \geq 0$, $\beta \geq 0$ and at least one of α and β is strictly greater than 0, i.e., $\alpha > 0$, $\beta \geq 0$ or $\alpha \geq 0$, $\beta > 0$. Let $f : X = \mathbf{R} \to Y = \mathbf{R}^2$ be defined as follows:

$$f(x) = \begin{cases} (-x, -1), & \text{if } x > 0; \\ (0,0), & \text{if } x = 0; \\ (-1, x), & \text{if } x < 0. \end{cases}$$

Put $x_0 := 0$ and $\epsilon = 2$. Then

$$\begin{aligned} f(X) - f(x_0) &= \{(-x,-1) : x > 0\} \cup \{(-1,x) : x < 0\} \cup \{(0,0)\} \\ &= \{(x,-1) : x < 0\} \cup \{(-1,x) : x < 0\} \cup \{(0,0)\}. \end{aligned}$$

Also,

$$\begin{aligned} -\epsilon H - D &= -2k_0 - D \\ &= -2(1,1) + \{(y_1,y_2) \in \mathbf{R}^2 :\, y_1 \leq 0,\ y_2 \leq 0\} \\ &= \{(y_1 - 2,\ y_2 - 2) :\, y_1 \leq 0, y_2 \leq 0\} \\ &= \{(y_1, y_2) :\, y_1 \leq -2,\ y_2 \leq -2\}. \end{aligned}$$

Obviously,

$$(f(X) - f(x_0)) \,\cap\, (-\epsilon H - D) \;=\; \emptyset.$$

On the other hand,

$$\begin{aligned} &(f(X) - f(x_0)) \cap \Big(- \Big(\bigcup_{\lambda > 0} \lambda H + D \Big) \Big) \\ &= (f(X) - f(x_0)) \cap \Big(- \Big(\bigcup_{\lambda > 0} \lambda k_0 + D \Big) \Big) \\ &= (\{(x,-1) : x < 0\} \cup \{(-1,x) : x < 0\} \cup \{(0,0)\}) \\ &\quad \cap\ \{(y_1,y_2) \in \mathbf{R}^2 :\, y_1 < 0, y_2 < 0\} \\ &= \{(x,-1) :\, x < 0\} \cup \{(-1,x) :\, x < 0\}. \end{aligned}$$

For any $\xi \in D^+ \cap \{k_0\}^{+s}$, there exists a unique $(\alpha, \beta) \in \mathbf{R}^2$ such that $\xi(y) = \alpha y_1 + \beta y_2$, $\forall y = (y_1, y_2) \in Y = \mathbf{R}^2$, where $\alpha \geq 0$, $\beta \geq 0$ and at least one of α, β is strictly greater than 0. Thus,

$$\begin{aligned} &\xi \circ \Big((f(X) - f(x_0)) \cap \Big(- \Big(\bigcup_{\lambda > 0} \lambda H + D \Big) \Big) \Big) \\ &= \{\alpha x - \beta :\, x < 0\} \cup \{-\alpha + \beta x :\, x < 0\}, \end{aligned}$$

where $\alpha \geq 0$ and $\beta \geq 0$. If $\alpha > 0$, then $\{\alpha x - \beta :\, x < 0\}$ is not lower bounded. If $\beta > 0$, then $\{-\alpha + \beta x :\, x < 0\}$ is not lower bounded. Hence, for any $\xi \in D^+ \cap H^{+s}$,

$$(f(X) - f(x_0)) \cap \Big(- \Big(\bigcup_{\lambda > 0} \lambda H + D \Big) \Big)$$

is not ξ-lower bounded.

Remark 5.4 In [41, Theorem 6.8], we imposed that $\exists\,\xi \in D^+ \cap H^{+s}$. Now, we imposed that $0 \notin \mathrm{vcl}(H+D)$. In fact, $\exists\,\xi \in D^+ \cap H^{+s} \;\Rightarrow\; 0 \notin \mathrm{vcl}(H+D)$. First, since $\xi \in D^+ \cap H^{+s}$, there exists $\alpha > 0$ such that $\xi(d) \geq 0,\ \forall d \in D$ and $\xi(h) \geq \alpha,\ \forall h \in H$. Assume that $0 \in \mathrm{vcl}(H+D)$. Then, there exists a $v \in Y$ and a real sequence (λ_n) with every $\lambda_n \geq 0$ and $\lambda_n \to 0$ such that $0 + \lambda_n v \in H + D$. Thus, we have

$$\xi(\lambda_n v) = \lambda_n \xi(v) \to 0 \quad (n \to \infty). \tag{5.10}$$

On the other hand, since $\lambda_n v \in H+D$, we have $\xi(\lambda_n v) \geq \alpha > 0$, $\forall n$, which contradicts (5.10). In general, $0 \notin \mathrm{vcl}(H+D) \;\not\Rightarrow\; \exists\xi \in D^+ \cap H^{+s}$. In fact, a non-locally convex Hausdorff topological vector space may have no nonzero continuous linear functional. Combining this with Proposition 5.2 and Example 5.3, we conclude that Theorem 5.1 indeed improves [41, Theorem 6.8].

Finally, we give a corollary of Theorem 5.1, which improves [8, Theorem 5.1].

Corollary 5.5 *Let (X, d) be a complete metric space, Y be a locally convex space, $D \subset Y$ be a closed convex cone, $H \subset D$ be a σ-convex set such that $0 \notin \mathrm{vcl}(H+D)$ and let $f : X \to Y$ be a submonotone vector-valued map. Suppose that $x_0 \in X$, $\epsilon > 0$ and $\lambda > 0$ such that x_0 is an $\epsilon\lambda$-approximate solution with respect to H, i.e.,*

$$(f(X) - f(x_0)) \cap (-\epsilon\lambda H - D) = \emptyset.$$

Then, there exists $\hat{x} \in X$ such that

(a) $f(x_0) \in f(\hat{x}) + \epsilon d(x_0, \hat{x})\, H + D$;

(b) $d(x_0, \hat{x}) < \lambda$;

(c) $\forall x \in X \backslash \{\hat{x}\}, f(\hat{x}) \notin f(x) + \epsilon d(\hat{x}, x) H + D$.

Comparing [8, Theorem 5.1] with Corollary 5.5, we have seen that the assumption that $H \subset D$ is a closed semi-complete (i.e., sequentially complete) convex and bounded set in [8, Theorem 5.1] is replaced by a weaker one: H is a σ-convex set (see the beginning of this section). And the assumption that $0 \notin \mathrm{cl}(H+D)$ is replaced by a weaker one: $0 \notin \mathrm{vcl}(H+D)$. Furthermore, when $x_0 \in X$ is an $\epsilon\lambda$-approximate solution with respect to H, the assumption that $f : X \to Y$ is D-bounded has been completely removed.

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