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Qualitative Analysis of a Belousov–Zhabotinskii Reaction Model

Aung Zaw MYINT **Li LI**[1)] **Ming Xin WANG**
Department of Mathematics, Harbin Institute of Technology, Harbin 150001*, P. R. China*
E-mail: *aungzawmyint@hit.edu.cn* *leeeee@hit.edu.cn* *mxwang@hit.edu.cn*

Abstract This paper deals with one kind of Belousov–Zhabotinskii reaction model. Linear stability is discussed for the spatially homogeneous problem firstly. Then we focus on the stationary problem with diffusion. Non-existence and existence of non-constant positive solutions are obtained by using implicit function theorem and Leray–Schauder degree theory, respectively.

Keywords Belousov–Zhabotinskii reaction, stability, positive stationary solutions

MR(2010) Subject Classification 35J57, 35B09, 35B35, 92E20

1 Introduction

The Belousov–Zhabotinskii reaction (BZ reaction) is one of a class of reactions that serve as a classical example of non-equilibrium thermodynamics. The system is important to theoretical chemistry in that it shows that chemical reactions do not have to be dominated by equilibrium thermodynamic behavior. One kind of the spatially homogeneous BZ reaction model reads as [27]

$$\begin{cases} \dfrac{du}{dt} = u(1-u-cv), & t>0, \\ \dfrac{dv}{dt} = bw - v - kuv, & t>0, \\ \dfrac{dw}{dt} = h(u-w), & t>0, \end{cases} \tag{1.1}$$

where b, c, k, h are positive constants which are related to the speed of reactions, and (u, v, w) represent the concentration of reactants, so $u, v, w > 0$. Although the distribution differences in space are neglected in (1.1), the ODE system (1.1) still implies the existence of chemical oscillators, which is of interest for the chemists. See [13, 25] for more descriptions on the model.

The BZ reaction was first discovered by Belousov in 1951 in an unpublished paper. His discovery was briefly reported in a Russian medical meeting in 1959, see [1]. A translation of the original work can be found in [6]. The study was continued by Zhabotinskii in [29], and the model is now known as the Belousov–Zhabotinskii reaction or simply the BZ reaction. Later, some variants of the original model are derived, for example, Noyes–Field model of BZ reaction is derived in [7].

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1) Corresponding author

In the spatial inhomogeneous case, the diffusion terms play important roles. In this situation, the model (1.1) with diffusions is

$$\begin{cases} u_t - d_1\Delta u = u(1-u-cv), & x\in\Omega,\ t>0,\\ v_t - d_2\Delta v = bw - v - kuv, & x\in\Omega,\ t>0,\\ w_t - d_3\Delta w = h(u-w), & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t\geq 0,\\ (u,v,w) = (u_0,v_0,w_0), & x\in\bar{\Omega},\ t=0, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, positive constants d_1, d_2, d_3 are diffusion coefficients and the Neumann boundary conditions mean that there is no flux through the boundary.

The corresponding stationary problem of (1.2) is

$$\begin{cases} -d_1\Delta u = u(1-u-cv), & x\in\Omega,\\ -d_2\Delta v = bw - v - kuv, & x\in\Omega,\\ -d_3\Delta w = h(u-w), & x\in\Omega,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

Denote the solution of (1.3) by $U = (u, v, w)$. It is obvious that system (1.3) has only one positive constant solution $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w})$:

$$\tilde{u} = \tilde{w} = 1 - c\tilde{v}, \quad \tilde{v} = \frac{bc+k+1-\sqrt{(bc+k+1)^2-4bck}}{2ck}. \tag{1.4}$$

Although the biologists or chemists are more interested in the oscillation of the states of reaction, the model itself produces rich research materials, especially the diffusion effect, for mathematicians. Historically, Turing in [24] first realized that the diffusion plays an important role in the formation of patterns. The diffusion effect has been extensively studied for many models ever since, which include the Sel'kov model [5, 9, 18, 26], the Brusselator model [2, 20, 31], the chemotactic diffusion model [28], the competition model [10–12], the predator-prey model [4, 8, 15, 16, 22, 23, 30], as well as models of semiconductors, plasmas, chemical waves, combustion systems, embryogenesis, etc., see e.g., [3] and references therein.

In 2004 [19], Peng and Wang studied one version of the Noyes–Field model arising from BZ reaction. They proved the existence and non-existence of the nontrivial stationary solution.

In this paper, we first analyze the stability of $\tilde{U}$ to the spatially homogeneous problem (1.1). Then we prove the non-existence and existence of nontrivial positive solutions of (1.3) under some conditions. The starting point to study the solutions of (1.3) is to derive the *a priori* estimates, where the maximum principle and Hopf's Lemma are the fundamental tools. Then we use the implicit function theorem and Leray–Schauder theory to obtain the non-existence and existence of nontrivial positive solutions, respectively.

The organization of the paper is as follows. In Section 2 we consider the stability of constant solutions $\tilde{U}$ for the spatially homogeneous BZ reaction model (1.1). Then we concentrate on the stationary problem (1.3) in the following sections. The *a priori* estimates are derived in Section 3. The non-existence and existence of non-constant positive solutions of (1.3) are obtained in Sections 4 and 5, respectively.

2 Stability of $\tilde{U}$ for the Problem (1.1)

When $U = (u, v, w)$ is independent of space variable x, we write model (1.1) as $U_t = F(U)$, where

$$F(U) = F(u, v, w) = \begin{pmatrix} u(1-u-cv) \\ bw - v - kuv \\ h(u-w) \end{pmatrix}.$$

Clearly,

$$F_U(\tilde{U}) = \begin{pmatrix} 1-2\tilde{u}-c\tilde{v} & -c\tilde{u} & 0 \\ -k\tilde{v} & -(1+k\tilde{u}) & b \\ h & 0 & -h \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{(bc-k+1)-Q(b,c,k)}{2k} & \dfrac{c}{2k}[(bc-k+1)-Q(b,c,k)] & 0 \\ \dfrac{-(bc+k+1)+Q(b,c,k)}{2c} & \dfrac{1}{2}[(bc-k-1)-Q(b,c,k)] & b \\ h & 0 & -h \end{pmatrix},$$

where

$$Q(b,c,k) = \sqrt{(bc+k+1)^2 - 4bck}. \tag{2.1}$$

It can be checked that the corresponding eigenpolynomial of $F_U(\tilde{U})$ is

$$P(\lambda) = \lambda^3 + A_2\lambda^2 + A_1\lambda + A_0,$$

with

$$\begin{aligned} A_0 &= \frac{h}{2k}[b^2c^2 - 2bc(k-1) + (k+1)^2 + (k-bc-1)Q] > 0, \\ A_1 &= \frac{1}{2k}[(k+1)^2 + (3+2bc-3k)bc + [k^2+2k-1-bc(k+1)]h \\ &\quad + [k-2bc-1+(k+1)h]Q], \\ A_2 &= \frac{1}{2k}[(k^2+2k-1) - (k+1)bc + 2kh + (k+1)Q] > 0, \end{aligned}$$

On the other hand, the roots $\lambda_1, \lambda_2, \lambda_3$ of $P(\lambda) = 0$ satisfy

$$\begin{aligned} A_0 &= -\lambda_1\lambda_2\lambda_3 > 0, \\ A_1 &= \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \\ A_2 &= -(\lambda_1 + \lambda_2 + \lambda_3) > 0. \end{aligned}$$

According to Routh–Hurwitz conditions, the necessary and sufficient conditions for the constant solution to be stable are:

$$A_1A_2 - A_0 > 0.$$

Therefore, we can conclude the following stability results:

Theorem 2.1 *Let $b, c, k, h > 0$, and consider the ODE system* (1.1).

(i) *Fix b, c, k, then for h sufficiently large, we have $A_1A_2 - A_0 > 0$, and the constant solution $\tilde{U}$ is locally stable.*

(ii) *Fix b, c, h, then for k sufficiently large, we have $A_1A_2 - A_0 > 0$, and the constant solution $\tilde{U}$ is locally stable.*

(iii) *Fix k, h, then for bc sufficiently large, we have $A_1A_2 - A_0 > 0$, and the constant solution $\tilde{U}$ is locally stable.*

Proof (i) When b, c, k are fixed and h is large, the leading term of $A_1A_2 - A_0$ is

$$A_1A_2 - A_0 = \frac{h^2}{2k}\left[k^2 + 2k - 1 - bc(k+1) + (k+1)Q\right] + O(1)h, \tag{2.2}$$

where $Q = Q(b, c, k)$ is defined in (2.1) and $O(1)$ represents a quantity which is bounded by a constant independent of h. It can be checked by direct computation that

$$(k+1)^2Q^2 > (k^2 + 2k - 1 - bc(k+1))^2.$$

The coefficient of h^2 in (2.2) is always positive. Thus $\tilde{U}$ is stable.

(ii) Assume that b, c, h are fixed and k is large. Notice that

$$\lim_{k\to\infty}(Q - k) = \lim_{k\to\infty}(\sqrt{(bc + k + 1)^2 - 4bck} - k) = 1 - bc.$$

Then the leading terms of A_0, A_1, A_2 are

$$\begin{aligned} A_0 &= \frac{h}{2k}(k^2 + kQ + O(1)k) = hk + O(1), \\ A_1 &= \frac{1}{2k}(h+1)(k^2 + kQ + O(1)k) = (h+1)k + O(1), \\ A_2 &= \frac{1}{2k}(k^2 + kQ + O(1)k) = k + O(1), \end{aligned}$$

where $O(1)$ represents a quantity which is bounded by a constant independent of k. So the leading term of $(A_1A_2 - A_0)$ is $(h+1)k^2 > 0$ for k sufficiently large, $\tilde{U}$ is stable.

(iii) It is easy to see that the sign of $(A_1A_2 - A_0)$ depends only on bc, h, k. Assume that h, k are fixed and bc is large. Notice that

$$\lim_{bc\to\infty}(Q - bc) = \lim_{bc\to\infty}(\sqrt{(bc + k + 1)^2 - 4bck} - bc) = 1 - k,$$

and

$$\lim_{bc\to\infty}[bc(Q - bc - 1 + k)] = 2k.$$

Thus we have

$$Q = bc + 1 - k + \frac{2k}{bc} + o(1)\frac{1}{bc},$$

where $o(1)$ means a quantity which goes to zero as $bc \to \infty$. Therefore, the leading terms of A_0, A_1, A_2 are

$$\begin{aligned} A_0 &= \frac{h}{2k}\left[b^2c^2 - 2bc(k-1) + (k+1)^2 + (k - bc - 1)\left(bc + 1 - k + \frac{2k}{bc} + o(1)\frac{1}{bc}\right)\right] \\ &= h + o(1). \end{aligned}$$

Similarly,

$$A_1 = h + o(1), \quad A_2 = (1 + h) + o(1).$$

So $A_1A_2 - A_0 = h^2 + o(1) > 0$ when bc is sufficiently large. Thus $\tilde{U}$ is stable when bc is large. The theorem is proved. □

3 The *a Priori* Estimates for the Positive Solution of (1.3)

The positive solution (u, v, w) of (1.3) to be mentioned throughout this paper always refers to classical solutions with u, v and $w > 0$ on $\bar{\Omega}$. It should also be noted that the well-known maximum principle ensures that a nonnegative classical solution of (1.3) with $u, v, w \not\equiv 0$ must be positive one.

Lemma 3.1 (Maximum Principle [11]) *Suppose that* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $g \in C(\bar{\Omega} \times \mathbb{R})$.

(i) *If u satisfies*

$$\Delta u + g(x, u(x)) \geq 0 \ \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} \leq 0 \ \text{on } \partial\Omega,$$

and $u(x_0) = \max_{\bar{\Omega}} u$, *then* $g(x_0, u(x_0)) \geq 0$.

(ii) *If u satisfies*

$$\Delta u + g(x, u(x)) \leq 0 \ \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} \geq 0 \ \text{on } \partial\Omega,$$

and $u(x_0) = \min_{\bar{\Omega}} u$, *then* $g(x_0, u(x_0)) \leq 0$.

We first give the upper bound of positive solutions of (1.3). Let $(u, v, w) \in [C^2(\Omega) \cap C(\bar{\Omega})]^3$ be a positive solution of (1.3) and suppose

$$u(x_1) = \max_{\bar{\Omega}} u, \quad u(x_2) = \min_{\bar{\Omega}} u,$$
$$v(y_1) = \max_{\bar{\Omega}} v, \quad v(y_2) = \min_{\bar{\Omega}} v,$$
$$w(z_1) = \max_{\bar{\Omega}} w, \quad w(z_2) = \min_{\bar{\Omega}} w.$$

Applying the maximum principle (Lemma 3.1) to the equation of u, v and w, respectively, we have

$$\begin{aligned}
&1 - u(x_1) - cv(x_1) \geq 0, \\
&1 - u(x_2) - cv(x_2) \leq 0, \\
&bw(y_1) - v(y_1) - ku(y_1)v(y_1) \geq 0, \\
&bw(y_2) - v(y_2) - ku(y_2)v(y_2) \leq 0, \\
&u(z_1) - w(z_1) \geq 0, \\
&u(z_2) - w(z_2) \leq 0,
\end{aligned}$$

which imply that

$$1 - cv(x_2) \leq u(x_2) \leq u(x_1) \leq 1 - cv(x_1) \leq 1, \tag{3.1}$$

$$\frac{bw(y_2)}{1 + ku(y_2)} \leq v(y_2) \leq v(y_1) \leq \frac{bw(y_1)}{1 + ku(y_1)} \leq bw(y_1), \tag{3.2}$$

$$0 \leq u(x_2) \leq w(z_2) \leq w(z_1) \leq u(z_1) \leq u(x_1) \leq 1. \tag{3.3}$$

Therefore,

$$(u, v, w) \leq (1, b, 1) \quad \text{on } \bar{\Omega}. \tag{3.4}$$

In order to derive the positive lower bound of positive solutions of (1.3), we first give a lemma, whose proof is the same as that of Lemma 2 in [17] and the details are omitted here.

Lemma 3.2 *Let $d_{ij} \in (0, +\infty)$, $i = 1, 2, 3$, and (u_j, v_j, w_j) be the positive solutions of (1.3) with $d_i = d_{ij}$. Assume that $\lim_{j\to\infty} d_{ij} = d_i \in [0, +\infty]$, and $(u_j, v_j, w_j) \to (u^*, v^*, w^*)$ uniformly on $\bar{\Omega}$. If u^*, v^* and w^* are constants, then (u^*, v^*, w^*) must satisfy*

$$1 - u^* - cv^* = 0, \quad bw^* - v^* - ku^*v^* = 0, \quad u^* = w^*.$$

In particular, if u^*, v^* and w^* are nonnegative constants, then $(u^*, v^*, w^*) = (\tilde{u}, \tilde{v}, \tilde{w})$, the unique positive constant solution of (1.3).

Lemma 3.3 (Lower bound) *Let $d > 0$ be a constant. Then there is a positive constant $\varepsilon(d)$ such that for any $d_1, d_2, d_3 \geq d$, every positive solution (u, v, w) of (1.3) satisfies*

$$\min\left\{ \min_{\bar{\Omega}} u,\ \min_{\bar{\Omega}} v,\ \min_{\bar{\Omega}} w \right\} \geq \varepsilon(d).$$

Proof We will use contradiction argument. Suppose the theorem is not true. Then there exist a sequence $\{(d_{1j}, d_{2j}, d_{3j})\}_{j=1}^{\infty}$ and the corresponding positive solutions (u_j, v_j, w_j) of (1.3) with $(d_1, d_2, d_3) = (d_{1j}, d_{2j}, d_{3j})$, such that

$$d_{1j}, d_{2j}, d_{3j} \geq d, \quad \lim_{j\to\infty} \min\{\min_{\bar{\Omega}} u_j,\ \min_{\bar{\Omega}} v_j,\ \min_{\bar{\Omega}} w_j\} = 0.$$

By (3.4), (u_j, v_j, w_j) is bounded. We may assume, by passing to a subsequence if necessary, that as $j \to \infty$,

$$\begin{aligned} &\lim_{j\to\infty} d_{ij} = d_i \in [d, +\infty],\ i = 1, 2, 3, \quad (u_j, v_j, w_j) \to (u, v, w) \in [C^{2+\alpha}(\Omega)]^3, \\ &\min\left\{ \min_{\bar{\Omega}} u, \min_{\bar{\Omega}} v, \min_{\bar{\Omega}} w \right\} = 0, \end{aligned} \tag{3.5}$$

where u, v, w are non-negative functions.

We will separate the proof into the following cases:

Case 1 $d_1, d_2, d_3 < \infty$. In such a case, (u, v, w) satisfies

$$-d_1\Delta u = u(1 - u - cv) \quad \text{in } \Omega, \tag{3.6}$$

$$-d_2\Delta v = bw - v - kuv \quad \text{in } \Omega, \tag{3.7}$$

$$-d_3\Delta w = h(u - w) \quad \text{in } \Omega, \tag{3.8}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{3.9}$$

If $w(x_0) = \min_{\bar{\Omega}} w = 0$ for some $x_0 \in \bar{\Omega}$, using Hopf's lemma and the boundary condition we conclude $x_0 \in \Omega$. It then follows from (3.8) that $u(x_0) = 0$, and so $\min_{\bar{\Omega}} u = 0$.

If $\min_{\bar{\Omega}} v = 0$, similar to the above we have $\min_{\bar{\Omega}} w = 0$, and then $\min_{\bar{\Omega}} u = 0$.

Therefore, we always have $\min_{\bar{\Omega}} u = 0$. Applying the strong maximum principle and Hopf boundary lemma to (3.6), we have $u \equiv 0$. Then $w \equiv v \equiv 0$ by use of (3.8) and (3.7), respectively. This is a contradiction to Lemma 3.2.

Case 2 $d_1 = \infty$.

(2a) The subcase $d_2, d_3 < \infty$. Then u satisfies

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

So $u = u^* \geq 0$, where u^* is a constant. Then v, w satisfies

$$\begin{cases} -d_2\Delta v = bw - (1+ku^*)v & \text{in } \Omega, \\ -d_3\Delta w = h(u^* - w) & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

If $u^* = 0$, then (3.10) and the maximum principle implies that $v = w = 0$. This is a contradiction to Lemma 3.2.

If $u^* > 0$, then either $\min_{\bar{\Omega}} v = 0$ or $\min_{\bar{\Omega}} w = 0$. Similarly to the above we can deduce $w(x_0) = \min_{\bar{\Omega}} w = 0$ for some $x_0 \in \Omega$. It contradicts with the second equation of (3.10).

(2b) The subcase $d_2 = \infty$. Now with $d_1 = d_2 = \infty$, u, v satisfy

$$\Delta u = \Delta v = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

This implies $u = u^*$, $v = v^*$ for some nonnegative constants u^*, v^*.

If $d_3 = \infty$. Then $w = w^*$ similarly to the above. So $(u, v, w) = (u^*, v^*, w^*)$ for some nonnegative constants u^*, v^* and w^*. By Lemma 3.2, $(u^*, v^*, w^*) = (\tilde{u}, \tilde{v}, \tilde{w})$. This contradicts (3.5).

If $d_3 < \infty$. Then w satisfies

$$-d_3\Delta w = h(u^* - w) \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.11}$$

If $u^* = 0$, then $w \equiv 0$. So $(u, v, w) = (0, v^*, 0)$. This is a contradiction to Lemma 3.2. If $v^* = 0$, then by the second equation of (1.3), $w \equiv 0$. So $(u, v, w) = (u^*, 0, 0)$. This is a contradiction to Lemma 3.2. If $u^* > 0$, $v^* > 0$, by (3.5), $\min_{\bar{\Omega}} w = 0$. Similarly to the above, $w(x_0) = 0$ for some $x_0 \in \Omega$. Then the differential equation of (3.11) does not hold at x_0 since $u^* > 0$. This is a contradiction.

(2c) The subcase $d_2 < \infty$, $d_3 = \infty$. In the present case, $u = u^*$ and $w = w^*$ are nonnegative constants. Noticing

$$-d_{3j}\Delta w_j = h(u_j - w_j) \text{ in } \Omega, \quad \frac{\partial w_j}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

we have $\int_\Omega (u_j - w_j)dx = 0$, and so $u^* = w^*$. Thus v satisfies

$$-d_2\Delta v = bu^* - (1+ku^*)v \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.12}$$

If $u^* = 0$, then (3.12) implies $v \equiv 0$, so $u = v = w = 0$, which contradicts Lemma 3.2. If $u^* > 0$, then $\min_{\bar{\Omega}} u = \min_{\bar{\Omega}} w > 0$, and (3.5) implies $\min_{\bar{\Omega}} v = 0$. Similarly to the above, there exists $x_0 \in \Omega$ such that $v(x_0) = \min_{\bar{\Omega}} v = 0$. However, the differential equation of (3.12) does not hold at x_0. Now we have a contradiction.

Case 3 $d_1 < \infty$, d_2 or $d_3 = \infty$.

(3a) The subcase $d_2 = \infty$. In such a case, v satisfies $\Delta v = 0$ in Ω with homogeneous Neumann boundary condition. Therefore $v = v^*$ for some constant $v^* \geq 0$.

If $d_3 = \infty$, then $w = w^*$ for some constant $w^* \geq 0$. Because of $\int_\Omega (u_j - w_j)dx = 0$ and $u_j \to u$, $w_j \to w^*$, it follows that $u = w^*$. Hence $(u, v, w) = (w^*, v^*, w^*)$. Lemma 3.2 gives $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$, which contradicts (3.5).

When $d_3 < \infty$, we see that w satisfies

$$-d_3\Delta w = h(u - w) \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.13}$$

If $v^*=0$, then $w\equiv 0$ by the second equation of (1.3). Thus $u=0$. This is a contradiction. If $v^*>0$, then $\min_{\bar\Omega} u=0$ or $\min_{\bar\Omega} w=0$ by (3.5). Noticing that $\min_{\bar\Omega} w=0$ implies $\min_{\bar\Omega} u=0$, we always have $\min_{\bar\Omega} u=0$. Now since $d_1<\infty$, u satisfies

$$-d_1\Delta u+(cv^*+u)u=u \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu}=0 \text{ on } \partial\Omega.$$

The Hopf boundary lemma and strong maximum principle assert $u\equiv 0$, and then $w\equiv 0$ by (3.13). This is a contradiction.

(3b) The subcase $d_2<\infty$, $d_3=\infty$. Arguing as above we have $w=w^*$ for some constant $w^*\geq 0$. By the third equation of (1.3), $u=w=w^*$.

If $w^*=0$, then v satisfies $-d_2\Delta v=-v$ in Ω with homogeneous Neumann boundary condition. Thus, $v=v^*$. Now $(u,v,w)=(w^*,v^*,w^*)$ leads to a contradiction of Lemma 3.2 and (3.5).

If $w^*>0$, then $\min_{\bar\Omega} v=0$ by (3.5), and v satisfies

$$-d_2\Delta v=bw^*-(1+kw^*)v \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu}=0 \text{ on } \partial\Omega.$$

Similarly to the above, we can get a contradiction.

Now we have proved that no matter in which case, there is always a contradiction. The theorem is proved. □

From the above discussion we have

Theorem 3.4 (*A priori* estimate) *Every positive solution (u,v,w) of (1.3) satisfies $(u,v,w)\leq(1,b,1)$. Moreover, for the fixed $d>0$, there exists a constant $\varepsilon(d)>0$ such that any positive solution (u,v,w) of (1.3), with $d_1,d_2,d_3\geq d$, satisfies $(u,v,w)\geq\varepsilon(d)$.*

4 Non-existence of Non-constant Positive Solutions

In this section, we use the implicit function theorem [14] to investigate the non-existence of non-constance positive solution of (1.3) when c,k is simultaneously small, or one of d_1,d_2 is large. The technique comes from [16, 19, 21]. We first state some lemmas.

Lemma 4.1 *Fix b,h and $d_i, i=1,2,3$. Assume that (u_j,v_j,w_j) are solutions of (1.3) with $c=c_j, k=k_j$ and $c_j,k_j\to 0^+$ as $j\to\infty$. Then $(u_j,v_j,w_j)\to(1,b,1)$ in $[C^1(\bar\Omega)]^3$ as $j\to\infty$.*

Proof This lemma is followed from (3.1)–(3.3), Theorem 3.4 and the standard regularity theory of elliptic equations. □

Lemma 4.2 *Choose $b,c>0$. Then*

(i) *Fix k,h and $d_i, i=2,3$. Let (u_j,v_j,w_j) be positive solutions of (1.3) with $d_1=d_{1j}$ and $d_{1j}\to\infty$ as $j\to\infty$. Then $(u_j,v_j,w_j)\to(\tilde u,\tilde v,\tilde w)$ in $[C^1(\bar\Omega)]^3$ as $j\to\infty$.*

(ii) *Fix k,h and $d_i, i=1,3$. Let (u_j,v_j,w_j) be positive solutions of (1.3) with $d_2=d_{2j}$ and $d_{2j}\to\infty$ as $j\to\infty$. Then $(u_j,v_j,w_j)\to(\tilde u,\tilde v,\tilde w)$ in $[C^1(\bar\Omega)]^3$ as $j\to\infty$.*

Proof (i) By Theorem 3.4 and the standard regularity theory of elliptic equations, we can show that there exists a subsequence of (u_j,v_j,w_j), also labelled by itself, such that $(u_j,v_j,w_j)\to(u,v,w)$ in $[C^1(\bar\Omega)]^3$ as $j\to\infty$. Moreover, $u\equiv c_1$ which is a nonnegative constant and (c_1,v,w) satisfy

$$\int_\Omega(1-c_1-cv)dx=0, \tag{4.1}$$

$$-d_2\Delta v = bw - (1+kc_1)v \quad \text{in } \Omega, \tag{4.2}$$

$$-d_3\Delta w + hw = hc_1 \quad \text{in } \Omega, \tag{4.3}$$

$$\frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

In view of Lemma 3.2, it is easy to show that $c_1 > 0$. By (4.3), (4.4) and the uniqueness of solution we have $w \equiv c_1$.

Similarly, by (4.2) and (4.4), we have $v \equiv \frac{bc_1}{1+kc_1}$. Substituting the value of v into (4.1) one has $1 - c_1 - \frac{cbc_1}{1+kc_1} = 0$. Then

$$c_1 = \frac{k - bc - 1 + \sqrt{(k-bc-1)^2 + 4k}}{2k} = \tilde{u}, \quad v \equiv \tilde{v}.$$

Therefore, our result holds and the proof of (i) is complete.

Now we prove (ii). The same as the discussion of (i), there exists a subsequence of (u_j, v_j, w_j), still denoted by itself, such that $(u_j, v_j, w_j) \to (u, v, w)$ in $[C^1(\bar{\Omega})]^3$ as $j \to \infty$. Moreover $v \equiv c_2$ is a positive constant, $u, w \geq 0$ and satisfy

$$-d_1\Delta u = u(1 - u - cc_2) \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{4.5}$$

$$\int_\Omega (bw - c_2 - kuc_2)\, dx = 0, \tag{4.6}$$

$$-d_3\Delta w = h(u - w) \text{ in } \Omega, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{4.7}$$

We claim that $1 - cc_2 > 0$. If this is not true, then $u \equiv 0$ by (4.5), and in turn $w \equiv 0$ by (4.7). This contradicts with (4.6). Thus $1 - cc_2 > 0$, and then $u \equiv 1 - cc_2 := \hat{u}$. Using (4.7) and (4.6) successively, we have $w = \hat{u}$ and $b\hat{u} - c_2 - k\hat{u}c_2 = 0$. Noticing $1 - cc_2 > 0$, it solves

$$c_2 = \frac{(bc + k + 1) - \sqrt{(bc+k+1)^2 - 4bck}}{2ck} := \tilde{v}.$$

Therefore, $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$. The proof of part (ii) is finished. □

Now, we are ready to prove the following non-existence results of the non-constant solution of (1.3).

Theorem 4.3 (i) *Fix b, h and $d_i, i = 1, 2, 3$. Then there exists a positive constant ϵ_0 depending only on b, h and $d_i, i = 1, 2, 3$, such that* (1.3) *has no nontrivial positive solution provided that $0 < c, k < \epsilon_0$.*

(ii) *Let b, c, k, h and d_2, d_3 be fixed. Then there exists a constant $D_1 > 0$ such that, for any $d_1 > D_1$, the problem* (1.3) *has no nontrivial positive solution.*

(iii) *Let b, c, k, h and d_1, d_3 be fixed. If $bc \neq k + 1$, then there exists a positive constant D_2 such that, for any $d_2 > D_2$, the problem* (1.3) *has no nontrivial positive solution.*

Proof Define the Banach spaces:

$$H^2_\nu(\Omega) = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \nu}\bigg|_{\partial\Omega} = 0 \right\}, \quad L^2_0(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega f(x)dx = 0 \right\}$$

and $X = H^2_\nu(\Omega) \cap L^2_0(\Omega)$.

Proof of (i) Define $F : \mathbb{R}^+ \times \mathbb{R}^+ \times [H^2_\nu(\Omega)]^3 \to [L^2(\Omega)]^3$ with

$$F(c, k, u, v, w) = (f_1, f_2, f_3)^T (c, k, u, v, w),$$

$$\begin{aligned}
f_1(c,k,u,v,w) &= d_1\Delta u + u(1-u-cv),\\
f_2(c,k,u,v,w) &= d_2\Delta v + bw - v - kuv,\\
f_3(c,k,u,v,w) &= d_3\Delta w + h(u-w).
\end{aligned}$$

Notice that finding positive solutions of (1.3) is equivalent to finding positive solution of $F(c,k,u,v,w)=0$. It is easy to see that when $c=k=0$, $(\tilde{u},\tilde{v},\tilde{w})|_{c=k=0}=(1,b,1)$ is the unique positive solution of $F(0,0,u,v,w)=0$. In order to prove that $(\tilde{u},\tilde{v},\tilde{w})$ is the unique solution of $F(c,k,u,v,w)=0$ for c,k small, it is sufficient to prove that $\Psi = D_{(u,v,w)}F(0,0,1,b,1)$ is bijective. Actually,

$$\Psi(\eta,\phi,\psi)=\begin{pmatrix} d_1\Delta\eta-\eta\\ d_2\Delta\phi+b\psi-\phi\\ d_3\Delta\psi+h(\eta-\psi)\end{pmatrix}.$$

It is not hard to see that Ψ is bijective. By the implicit function theorem, for c,k sufficiently small, $(\tilde{u},\tilde{v},\tilde{w})$ is the unique solution of $F(c,k,u,v,w)=0$. Therefore, there exists no nontrivial solution of (1.3) when c,k are small.

Proof of (ii) We make a decomposition: $u=\hat{u}+\xi$ with $\int_\Omega \hat{u}dx=0$ and $\xi\in\mathbb{R}^+=[0,+\infty)$. We observe that finding positive solutions of (1.3) is equivalent to finding positive solution of

$$\begin{cases}
\Delta\hat{u}+\rho\mathbf{P}[(\hat{u}+\xi)(1-(\hat{u}+\xi)-cv)]=0 & \text{in } \Omega,\\
\displaystyle\int_\Omega[(\hat{u}+\xi)(1-(\hat{u}+\xi)-cv)]dx=0,\\
d_2\Delta v+bw-v-kv(\hat{u}+\xi)=0 & \text{in } \Omega,\\
d_3\Delta w+h(\hat{u}+\xi)-hw=0 & \text{in } \Omega,\\
\dfrac{\partial u}{\partial\nu}=\dfrac{\partial v}{\partial\nu}=\dfrac{\partial w}{\partial\nu}=0 & \text{on } \partial\Omega,
\end{cases}\tag{4.8}$$

where $\rho=d_1^{-1}$ and $\mathbf{P}z=z-\frac{1}{|\Omega|}\int_\Omega zdx$, i.e., $\mathbf{P}$ is the projective operator from $L^2(\Omega)$ to $L_0^2(\Omega)$. Define

$$\begin{aligned}
F(\rho,\hat{u},\xi,v,w) &= (f_1,f_2,f_3,f_4)^T(\rho,\hat{u},\xi,v,w),\\
f_1(\rho,\hat{u},\xi,v,w) &= \Delta\hat{u}+\rho\mathbf{P}[(\hat{u}+\xi)(1-(\hat{u}+\xi)-cv)],\\
f_2(\rho,\hat{u},\xi,v,w) &= \int_\Omega[(\hat{u}+\xi)(1-(\hat{u}+\xi)-cv)]dx,\\
f_3(\rho,\hat{u},\xi,v,w) &= d_2\Delta v+bw-v-kv(\hat{u}+\xi),\\
f_4(\rho,\hat{u},\xi,v,w) &= d_3\Delta w+h(\hat{u}+\xi)-hw.
\end{aligned}$$

Then

$$F:\mathbb{R}^+\times X\times\mathbb{R}^+\times[H_\nu^2(\Omega)]^2\to L_0^2(\Omega)\times\mathbb{R}\times[L^2(\Omega)]^2$$

is a well-defined mapping, and clearly, for any $\rho>0$, (u,v,w) solves (1.3) if and only if $F(\rho,\hat{u},\xi,v,w)=0$.

It is easy to prove that the equation (4.8) has a unique positive solution $(0,\tilde{u},\tilde{v},\tilde{w})$ when $\rho=0$. Let Ψ be the Fréchet derivative of F at $(0,0,\tilde{u},\tilde{v},\tilde{w})$ with respect to $(\hat{u},\xi,v,w)$. A direct computation shows that

$$\Psi:=D_{(\hat{u},\xi,v,w)}F(0,0,\tilde{u},\tilde{v},\tilde{w}):X\times\mathbb{R}^+\times[H_\nu^2(\Omega)]^2\to L_0^2(\Omega)\times\mathbb{R}\times[L^2(\Omega)]^2,$$

where

$$\Psi(\eta, z, \phi, \psi) = \begin{pmatrix} \Delta\eta \\ \displaystyle\int_\Omega [(1-2\tilde{u}-c\tilde{v})\eta + (1-2\tilde{u}-c\tilde{v})z - c\tilde{u}\phi]dx \\ d_2\Delta\phi - k\tilde{v}\eta - k\tilde{v}z - (1+k\tilde{u})\phi + b\psi \\ d_3\Delta\psi + h\eta + hz - h\psi \end{pmatrix}.$$

In order to use the implicit function theorem, we have to verify that Ψ is both injective and surjective. In the following, we will prove that the equation $\Psi(\eta, z, \phi, \psi) = (0,0,0,0)$ has only solution $(0,0,0,0)$, so Ψ is injective.

Suppose $\Psi(\eta, z, \phi, \psi) = (0,0,0,0)$. By $\Delta\eta = 0$, $\frac{\partial\eta}{\partial\nu} = 0$ and $\int_\Omega \eta dx = 0$, we have $\eta \equiv 0$. Thus, $\Psi(\eta, z, \phi, \psi) = (0,0,0,0)$ becomes

$$\int_\Omega [(1-2\tilde{u}-c\tilde{v})z - c\tilde{u}\phi]dx = 0, \tag{4.9}$$

$$d_2\Delta\phi - k\tilde{v}z - (1+k\tilde{u})\phi + b\psi = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0, \tag{4.10}$$

$$d_3\Delta\psi + hz - h\psi = 0 \quad \text{in } \Omega, \quad \left.\frac{\partial\psi}{\partial\nu}\right|_{\partial\Omega} = 0. \tag{4.11}$$

It follows from $z \in \mathbb{R}$ that the problem (4.11) has a unique solution $\psi = z$. Hence, the problem (4.10) becomes

$$-d_2\Delta\phi + (1+k\tilde{u})\phi = (b-k\tilde{v})z \quad \text{in } \Omega, \quad \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0,$$

which also has a unique solution $\phi = \frac{(b-k\tilde{v})z}{1+k\tilde{u}}$.

By substituting the value of ϕ into (4.9) we have

$$\int_\Omega \left(1 - 2\tilde{u} - c\tilde{v} - c\tilde{u}\frac{b-k\tilde{v}}{1+k\tilde{u}}\right) z dx = 0.$$

Noting that $1 - \tilde{u} - c\tilde{v} = 0$, we have

$$\begin{aligned} 1 - 2\tilde{u} - c\tilde{v} - c\tilde{u}\frac{b-k\tilde{v}}{1+k\tilde{u}} &= -\tilde{u}\left(1 + c\frac{b-k\tilde{v}}{1+k\tilde{u}}\right) \\ &= -\frac{\tilde{u}}{1+k\tilde{u}}(1 + k\tilde{u} + bc - ck\tilde{v}) \\ &= -\frac{\tilde{u}}{1+k\tilde{u}}(1 + k(1-c\tilde{v}) + bc - ck\tilde{v}) \\ &= -\frac{\tilde{u}}{1+k\tilde{u}}(1 + k + bc - 2ck\tilde{v}) \\ &= -\frac{\tilde{u}}{1+k\tilde{u}}\sqrt{(bc+k+1)^2 - 4bck} \neq 0. \end{aligned}$$

Since $z \in \mathbb{R}$, we conclude $z = 0$ from the above integral. Therefore, $\eta = z = \phi = \psi = 0$ and Ψ is invertible. On the other hand, we can easily verify that Ψ is also a surjection.

By the implicit function theorem, there exist positive constants ρ_0 and ε_0 such that, for each $\rho \in [0, \rho_0]$, $(0, \tilde{u}, \tilde{v}, \tilde{w})$ is the unique solution of $F(\rho, \hat{u}, \xi, v, w) = 0$ in $B_{\varepsilon_0}(0, \tilde{u}, \tilde{v}, \tilde{w})$, where $B_{\varepsilon_0}(0, \tilde{u}, \tilde{v}, \tilde{w})$ is the ball in $X \times \mathbb{R} \times H^2_\nu \times H^2_\nu$ centered at $(0, \tilde{u}, \tilde{v}, \tilde{w})$ with radius ε_0. Take small ρ_0 and ε_0, part (ii) can be proved by use of Lemma 4.2.

Proof of (iii) Similarly, we decompose $v = \hat{v} + \gamma$, where $\int_\Omega \hat{v} dx = 0$ and $\gamma \in \mathbb{R}^+$. Then finding positive solutions of (1.3) is equivalent to finding positive solution of

$$\begin{cases} d_1 \Delta u + u(1 - u - c(\hat{v} + \gamma)) = 0 & \text{in } \Omega, \\ \Delta \hat{v} + \rho \mathbf{P}(bw - (1 + ku)(\hat{v} + \gamma)) = 0 & \text{in } \Omega, \\ \displaystyle\int_\Omega [bw - (1 + ku)(\hat{v} + \gamma)] dx = 0, & \\ d_3 \Delta w + h(u - w) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\rho = d_2^{-1}$ and $\mathbf{P} z = z - \frac{1}{|\Omega|} \int_\Omega z dx$. Define

$$\begin{aligned} F(\rho, u, \hat{v}, \gamma, w) &= (f_1, f_2, f_3, f_4)^T (\rho, u, \hat{v}, \gamma, w), \\ f_1(\rho, u, \hat{v}, \gamma, w) &= d_1 \Delta u + u(1 - u - c(\hat{v} + \gamma)), \\ f_2(\rho, u, \hat{v}, \gamma, w) &= \Delta \hat{v} + \rho \mathbf{P}(bw - (1 + ku)(\hat{v} + \gamma)), \\ f_3(\rho, u, \hat{v}, \gamma, w) &= \int_\Omega [bw - (1 + ku)(\hat{v} + \gamma)] dx, \\ f_4(\rho, u, \hat{v}, \gamma, w) &= d_3 \Delta w + h(u - w). \end{aligned}$$

Then

$$F : \mathbb{R}^+ \times H_\nu^2(\Omega) \times X \times \mathbb{R}^+ \times H_\nu^2(\Omega) \to L^2(\Omega) \times L_0^2(\Omega) \times \mathbb{R} \times L^2(\Omega)$$

is a well-defined mapping, and clearly, for any $\rho > 0$, (u, v, w) solves (1.3) if and only if $F(\rho, u, \hat{v}, \gamma, w) = 0$.

Similarly as part (ii), it can be proved that $(\tilde{u}, 0, \tilde{v}, \tilde{w})$ is a solution of $F(\rho, u, \hat{v}, \gamma, w) = 0$ for any ρ and the solution is unique when $\rho = 0$. In order to prove that $(\tilde{u}, 0, \tilde{v}, \tilde{w})$ is the only solution for ρ small by implicit function theorem, it is sufficient to prove that $D_{(u, \hat{v}, \gamma, w)} F(0, \tilde{u}, 0, \tilde{v}, \tilde{w})$ is a bijection. Actually, let $\Psi = D_{(u, \hat{v}, \gamma, w)} F(0, \tilde{u}, 0, \tilde{v}, \tilde{w})$, then

$$\Psi(\eta, \phi, z, \psi) = \begin{pmatrix} d_1 \Delta \eta + \eta(1 - 2\tilde{u} - c\tilde{v}) - c\tilde{u}\phi - c\tilde{u}z \\ \Delta \phi \\ \displaystyle\int_\Omega [b\psi - k\tilde{v}\eta + (1 + k\tilde{u})(\phi + z)] dx \\ d_3 \Delta \psi + h\eta - h\psi \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix},$$

where $(\eta, \phi, z, \psi) \in H_\nu^2(\Omega) \times X \times \mathbb{R}^+ \times H_\nu^2(\Omega)$. We first solve $\Psi(\eta, \phi, z, \psi) = 0$. By $\Psi_2 = 0$ in Ω, $\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ and $\int_\Omega \phi dx = 0$, we have $\phi = 0$. Then $\Psi_1 = \Psi_4 = 0$ becomes

$$d_1 \Delta \eta - \tilde{u}\eta - c\tilde{u}z = 0 \text{ in } \Omega, \quad \frac{\partial \eta}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{4.12}$$

$$\frac{d_3}{h} \Delta \psi + \eta - \psi = 0 \text{ in } \Omega, \quad \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{4.13}$$

where the equality $1 - \tilde{u} - c\tilde{v} = 0$ is used. Since $c, \tilde{u}, z$ are constants and $\tilde{u} > 0$, by the uniqueness of solution we can derive from (4.12) that $\eta = -cz$. Similarly, $\psi = -cz$ by (4.13). Using $1 - \tilde{u} - c\tilde{v} = 0$ we have

$$\Psi_3 = \int_\Omega (1 + k - bc) z dx.$$

Therefore, $\Psi_3 = 0$ and $bc \neq k+1$ implies $z = 0$, and then $\eta = \phi = z = \psi = 0$. This shows that Ψ is injection. On the other hand, it can be prove that Ψ is surjective under the condition $bc \neq k+1$. Then using similar arguments as in part (ii), we can prove the conclusion (iii). □

5 Existence of Non-constant Positive Solutions

Recall that $U = (u, v, w)^T$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w})^T$. Let $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$ be the complete set of eigenvalues of the operator $-\Delta$ in Ω with homogeneous Neumann boundary condition, l_i be the multiplicity of λ_i.

We shall derive the existence of non-constant positive solutions to (1.3) in terms of diffusion coefficient by topology degree theory. It is easy to see that (1.3) are equivalent to

$$\begin{cases} u = (I - d_1\Delta)^{-1}(2u - u^2 - cuv), \\ v = (I - d_2\Delta)^{-1}(bw - kuv), \\ w = (I - \dfrac{d_3}{h}\Delta)^{-1}u, \end{cases} \tag{5.1}$$

with Neumann boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0$. Denote the right hand side of (5.1) by

$$G(U) := \begin{pmatrix} (I - d_1\Delta)^{-1}(2u - u^2 - cuv) \\ (I - d_2\Delta)^{-1}(bw - kuv) \\ (I - \frac{d_3}{h}\Delta)^{-1}u \end{pmatrix},$$

then (5.1) can be written as $(I - G)U = 0$. It is obvious that $G : [C(\bar{\Omega})]^3 \to [C(\bar{\Omega})]^3$ is compact.

In order to apply the degree theory to obtain the existence of non-constant positive solutions, our first aim is to compute the index of $I - G$ at $(\tilde{u}, \tilde{v}, \tilde{w})$. Consider the eigenvalue problem for the linearized system,

$$(I - D_{(u,v,w)}G(\tilde{U}))V = -\mu V \tag{5.2}$$

where $V = (\xi, \eta, \zeta)$, μ is some constant. By Leray–Schauder theorem, we know that if zero is not an eigenvalue of (5.2), then

$$\text{index}(I - G, \tilde{U}) = (-1)^\gamma, \quad \text{with } \gamma = \sum_{\mu > 0} n_\mu, \tag{5.3}$$

where n_μ is the multiplicity of the eigenvalue μ of (5.2).

Direct computation shows that (5.2) is equivalent to

$$\begin{cases} (\mu+1)(I - d_1\Delta)\xi - (2\xi - 2\tilde{u}\xi - c\tilde{u}\eta - c\tilde{v}\xi) = 0 & \text{in } \Omega, \\ (\mu+1)(I - d_2\Delta)\eta - (b\zeta - k\tilde{u}\eta - k\tilde{v}\xi) = 0 & \text{in } \Omega, \\ (\mu+1)\left(I - \dfrac{d_3}{h}\Delta\right)\zeta - \xi = 0 & \text{in } \Omega, \\ \dfrac{\partial \xi}{\partial \nu} = \dfrac{\partial \eta}{\partial \nu} = \dfrac{\partial \zeta}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Suppose that (ξ, η, ζ) are multiples of the eigenfunction of $-\Delta$ corresponding to eigenvalue λ_i under the Neumann boundary condition, i.e.

$$-\Delta\xi = \lambda_i\xi, \quad -\Delta\eta = \lambda_i\eta, \quad -\Delta\zeta = \lambda_i\zeta \text{ in } \Omega, \quad \frac{\partial \xi}{\partial \nu} = \frac{\partial \eta}{\partial \nu} = \frac{\partial \zeta}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Put the above relations into (5.4), we get a system

$$\begin{cases} (\mu+1)(1+d_1\lambda_i)\xi + (-2+2\tilde{u}+c\tilde{v})\xi + c\tilde{u}\eta = 0, \\ (\mu+1)(1+d_2\lambda_i)\eta + k\tilde{v}\xi + k\tilde{u}\eta - b\zeta = 0, \\ (\mu+1)(1+\frac{d_3}{h}\lambda_i)\zeta - \xi = 0. \end{cases} \tag{5.5}$$

System (5.5) has a solution if and only if $B_i(\mu) = 0$ for some $i \geq 1$, where

$$B_i(\mu) = \mathrm{Det}\begin{pmatrix} \mu + \dfrac{d_1\lambda_i + \tilde{u}}{d_1\lambda_i + 1} & \dfrac{c\tilde{u}}{d_1\lambda_i+1} & 0 \\ \dfrac{k\tilde{v}}{d_2\lambda_i+1} & \mu + \dfrac{d_2\lambda_i + 1 + k\tilde{u}}{d_2\lambda_i+1} & \dfrac{-b}{d_2\lambda_i+1} \\ \dfrac{-h}{d_3\lambda_i + h} & 0 & \mu+1 \end{pmatrix}$$

is a polynomial of μ of degree 3, i.e.,

$$B_i(\mu) = \mu^3 + c_{i2}\mu^2 + c_{i1}\mu + c_{i0}.$$

Denote the multiplicity of μ, which is a root of $B_i(\mu) = 0$, by $m_{i\mu}$. By Lemma 5.1 in [16] and (5.3), we have that if $B_i(0) \neq 0$ for all $i \geq 1$,

$$\mathrm{index}(I - G, \tilde{U}) = (-1)^\gamma, \quad \gamma = \sum_{i=1}^{\infty} \sum_{\mu>0,\, B_i(\mu)=0} m_{i\mu} l_i. \tag{5.6}$$

Lemma 5.1 *There exists a positive constant $\bar{D}_1$ depending only on b, c, k, h, d_2 and d_3 such that* $\mathrm{index}(I - G, \tilde{U}) = 1$, *for all $d_1 > \bar{D}_1$.*

Proof For $i = 1$, we have $\lambda_1 = 0$ and $B_1(\mu)$ is given by

$$B_1(\mu) = \mu^3 + (2+\tilde{u}+k\tilde{u})\mu^2 + (1+2\tilde{u}+k\tilde{u}+k\tilde{u}^2 - ck\tilde{u}\tilde{v})\mu + (\tilde{u} + bc\tilde{u} + k\tilde{u}^2 - ck\tilde{u}\tilde{v}).$$

Using (1.4) we can deduce

$$B_1(\mu) = \mu^3 + (2+\tilde{u}+k\tilde{u})\mu^2 + (1+2\tilde{u}+2k\tilde{u}^2)\mu + (\tilde{u}+bc\tilde{u} - k\tilde{u} + 2k\tilde{u}^2) > 0$$

for all $\mu \geq 0$.

For $i \geq 2$, we have $\lambda_i > \lambda_1 = 0$ and

$$\lim_{d_1\to\infty} B_i(\mu) = (\mu+1)^2\left(\mu + \frac{d_2\lambda_i + 1 + k\tilde{u}}{d_2\lambda_i+1}\right).$$

Therefore, there exists a large positive constant $\bar{D}_1$ depending only on b, c, k, h, d_2 and d_3 such that $B_i(\mu) > 0$ for all $\mu \geq 0$, $d_1 \geq \bar{D}_1$ and $i \geq 2$.

The above arguments show that $B_i(\mu) > 0$ for all $\mu \geq 0$, $d_1 \geq \bar{D}_1$ and $i \geq 1$, and the number γ in (5.6) is zero. The desired conclusion follows from (5.6). □

A direct computation gives

$$c_{i0} = \frac{bch\tilde{u}}{(1+d_1\lambda_i)(1+d_2\lambda_i)(h+d_3\lambda_i)} + \frac{(1+d_2\lambda_i+k\tilde{u})(d_1\lambda_i+\tilde{u}) - ck\tilde{u}\tilde{v}}{(1+d_1\lambda_i)(1+d_2\lambda_i)},$$

$$c_{i1} = \frac{1+d_2\lambda_i+k\tilde{u}}{1+d_2\lambda_i} + \frac{d_1\lambda_i+\tilde{u}}{1+d_1\lambda_i} + \frac{(1+d_2\lambda_i+k\tilde{u})(d_1\lambda_i+\tilde{u}) - ck\tilde{u}\tilde{v}}{(1+d_1\lambda_i)(1+d_2\lambda_i)}$$

$$
\begin{aligned}
&> \frac{1+d_2\lambda_i+k\tilde{u}}{1+d_2\lambda_i} - \frac{ck\tilde{u}\tilde{v}}{(1+d_1\lambda_i)(1+d_2\lambda_i)}\\
&= \frac{(1+d_2\lambda_i+k\tilde{u})(1+d_1\lambda_i) - k\tilde{u} + k\tilde{u}^2}{(1+d_1\lambda_i)(1+d_2\lambda_i)} > 0,\\
c_{i2} &= 1 + \frac{1+d_2\lambda_i+k\tilde{u}}{1+d_2\lambda_i} + \frac{d_1\lambda_i+\tilde{u}}{1+d_1\lambda_i} > 0,
\end{aligned}
$$

here $c\tilde{v} = 1 - \tilde{u}$ is used. Therefore, the equation

$$B_i(\mu) = \mu^3 + c_{i2}\mu^2 + c_{i1}\mu + c_{i0} = 0$$

has positive root if and only if $c_{i0} < 0$. Moreover, when $c_{i0} < 0$, the positive root of $B_i(\mu) = 0$ exists uniquely and is simple. We write

$$c_{i0} = \frac{H(\lambda_i)}{(1+d_1\lambda_i)(1+d_2\lambda_i)(h+d_3\lambda_i)},$$

where

$$
\begin{aligned}
H(\lambda_i) = {} & d_1d_2d_3\lambda_i^3 + [d_1d_2h + d_1d_3(1+k\tilde{u}) + d_2d_3\tilde{u}]\lambda_i^2\\
& + [(1+k\tilde{u})hd_1 + hd_2\tilde{u} + \tilde{u}(1-k+2k\tilde{u})d_3]\,\lambda_i\\
& + h\tilde{u}(1+bc-k+2k\tilde{u}).
\end{aligned}
$$

The direct calculation gives $1 + bc - k + 2k\tilde{u} > 0$. Denote

$$\tilde{C} := 1 - k + 2k\tilde{u}.$$

Then

$$\lim_{d_3\to\infty} \frac{H(\lambda)}{d_3} = \lambda\{d_1d_2\lambda^2 + [d_1(1+k\tilde{u}) + d_2\tilde{u}]\lambda + \tilde{C}\tilde{u}\}.$$

Thus we have

Proposition 5.2 *Assume $\tilde{C} < 0$. For any fixed $d_1, d_2 > 0$, there exists a positive constant D_3 such that when $d_3 \geq D_3$, the equation $H(\lambda) = 0$ has three real solutions λ_j^*, $j = 1, 2, 3$, with the following properties*:

(i) $-\infty < \lambda_1^* < 0 < \lambda_2^* < \lambda_3^* < \infty$,

(ii) $H(\lambda) < 0$ *in* $(-\infty, \lambda_1^*) \cup (\lambda_2^*, \lambda_3^*)$; $H(\lambda) > 0$ *in* $(\lambda_1^*, \lambda_2^*) \cup (\lambda_3^*, \infty)$,

(iii) $\lim_{d_3\to\infty} \lambda_2^* = 0$, $\lim_{d_3\to\infty} \lambda_1^* = \lambda_-^*$ *and* $\lim_{d_3\to\infty} \lambda_3^* = \lambda_+^*$ *where*

$$\lambda_\pm^* = \frac{1}{2d_1d_2}\Big(-\Theta \pm \sqrt{\Theta^2 - 4d_1d_2\tilde{C}}\Big), \quad \Theta = d_1(1+k\tilde{u}) + d_2\tilde{u}.$$

Lemma 5.3 *Assume that $\tilde{C} < 0$ and $\lambda_+^* \in (\lambda_m, \lambda_{m+1})$ for some $m \geq 2$. Then there exists a positive constant D_3 depending only on b, c, k, h, d_1 and d_2 such that when $d_3 \geq D_3$,*

$$\mathrm{index}(I - G, \tilde{U}) = (-1)^\sigma, \quad \sigma = \sum_{i=2}^{m} l_i.$$

Proof In view of Proposition 5.2, when $d_3 \geq D_3$, $H(\lambda_i) < 0$ if and only if $2 \leq i \leq m$. Moreover, for each $2 \leq i \leq m$, the positive root μ of $B_i(\mu) = 0$ exists uniquely and is simple, i.e., $m_{i\mu} = 1$. Using (5.6) we get the desired conclusion. □

Theorem 5.4 *Assume that $\tilde{C} < 0$ and $\lambda_+^* \in (\lambda_m, \lambda_{m+1})$ for some $m \geq 2$. If $\sum_{i=2}^m l_i$ is odd, then there exists a positive constant D_3 depending only on b, c, k, h, d_1 and d_2 such that* (1.3) *has at least one nontrivial solution provided that $d_3 \geq D_3$.*

Proof Let $\tilde{D}_1 := \max\{D_1, \bar{D}_1\} + 1$, where $D_1, \bar{D}_1$ are given in (ii) of Theorem 4.3 and Lemma 5.1 respectively for fixed b, c, k, h, d_2 and $d_3 = 1$. Then (1.3) with $d_1 = \tilde{D}_1$, $d_3 = 1$ has only trivial solution $(\tilde{u}, \tilde{v}, \tilde{w})$ by (ii) of Theorem 4.3 and Lemma 4.2.

For all $0 \leq t \leq 1$, we define

$$G(U; t) = \begin{pmatrix} (I - (td_1 + (1-t)\tilde{D}_1)\Delta)^{-1}(2u - u^2 - cuv) \\ (I - d_2\Delta)^{-1}(bw - kuv) \\ (I - (t\frac{d_3}{h} + (1-t))\Delta)^{-1}(u) \end{pmatrix}.$$

In view of the *a priori* estimates Theorem 3.4, there exists a positive constant M depending only on b, c, k, h such that (1.3) has no solution on $\partial\Lambda$, where

$$\Lambda = \left\{ u, v, w \in C(\bar{\Omega}) : \frac{1}{M} < u, v, w < M \right\}.$$

Since $G(U; t) : \Lambda \times [0, 1] \to [C(\bar{\Omega})]^3$ is compact, $\deg(I - G(U; t), \Lambda, 0)$ is well-defined. By the homotopy invariance of the topological degree,

$$\deg(I - G(U; 0), \Lambda, 0) = \deg(I - G(U; 1), \Lambda, 0). \tag{5.7}$$

Since (1.3) with $d_1 = \tilde{D}_1$, $d_3 = 1$ has only trivial solution $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{w})$, we have that

$$\deg(I - G(U; 0), \Lambda, 0) = \text{index}(G(U; 0), \tilde{U}) = 1.$$

On the contrary, we assume that for some $d_3 \geq D_3$, (1.3) has no nontrivial solution. By Lemma 5.3, we have

$$\deg(I - G(U; 1), \Lambda, 0) = \text{index}(G(U; 1), \tilde{U}) = (-1)^\sigma = -1,$$

where $\sigma = \sum_{i=2}^m l_i$ is odd. This contradicts (5.7), and the proof is complete. □

Remark 5.5 It can be proved that when $k > 1$ and $bc > \frac{(k+1)^2}{2(k-1)}$, we have $\tilde{C} < 0$. Theorem 5.4 gives the existence of nontrivial solutions under the condition $\tilde{C} < 0$. In other cases when $\tilde{C} \geq 0$, the existence of nontrivial solutions is not clear.

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