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Global Well-posedness of the 3D Generalized Rotating Magnetohydrodynamics Equations

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Abstract In this paper, we establish the global well-posedness of the generalized rotating magnetohydrodynamics equations if the initial data are in $\mathcal{X}^{1-2\alpha}$ defined by $\mathcal{X}^{1-2\alpha} = \{u \in \mathscr{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{1-2\alpha}|\hat{u}(\xi)|d\xi < +\infty\}$. In addition, we also give Gevrey class regularity of the solution.

Keywords Magnetohydrodynamics, fractional MHD, incompressible, rotation framework, Coriolis force

MR(2010) Subject Classification 35Q35, 42B37, 35Q86, 26A33

1 Introduction

This paper focuses on the generalized rotating magnetohydrodynamics equations in three dimensions:

$$\begin{cases} u_t + (u\cdot\nabla)u + \mu(-\Delta)^\alpha u + \Omega e_3 \times u + \nabla P = (B\cdot\nabla)B & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ B_t + (u\cdot\nabla)B + \gamma(-\Delta)^\alpha B = (B\cdot\nabla)u & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ \operatorname{div} u = 0,\ \operatorname{div} B = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}_+, \\ u|_{t=0} = u_0,\ B|_{t=0} = B_0 & \text{in } \mathbb{R}^3, \end{cases} \tag{grMHD}$$

where u is the velocity field of the fluid, $P = p + \frac{1}{2}|B|^2$ which p is pressure and B is the magnetic field, constant μ is the viscosity coefficient, $\Omega \in \mathbb{R}$ denotes twice the speed of rotation around the vertical unit vector $e_3 = (0, 0, 1)$, γ is the diffusion of the magnetic field and α is a positive parameter.

When $\alpha = 1$, from mathematical point of view, the equation (grMHD) explain why the earth has a non-zero large-scale magnetic field whose polarity turns out to invert over several hundred centuries. We refer to [6] and references therein.

When $\Omega = 0$, the equation (grMHD) reduces to the generalized Magnetohydrodynamics (abbreviated as MHD) equations, which is the study of the magnetic properties of electrically conducting fluids. Since Duvaut and Lions [9] constructed a global weak solution to MHD for initial data with finite energy, 3D MHD equation remains an outstanding mathematical

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problem whether there always exists a global smooth solution for smooth initial data. We refer to [16, 23, 26–30] and references therein.

When $\alpha = 1$, $B = 0$ and $\Omega \neq 0$, Babin et al. [1–3] proved the global existence and regularity of solution to the equation (grMHD) with the periodic initial velocity in the case $|\Omega|$ is enough large. Chemin et al. [5, 6] showed that for a given divergence free initial velocity u_0 belonging to $L^2(\mathbb{R}^2) + H^{\frac{1}{2}}(\mathbb{R}^3)$, there exists a unique solution in the case $|\Omega| > \Omega_0 > 0$. Hieber and Shibata [12] obtained the uniform global well-posedness for small initial data in $H^{\frac{1}{2}}(\mathbb{R}^3)^3$. Iwabuchi and Takada [13] proved the existence of global unique solutions to the Navier–Stokes equations with Coriolis force in Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{4}$ if the speed of rotation Ω is sufficiently large.

When $\alpha = 1$, $\Omega = 0$ and $B = 0$, the equation (grMHD) reduces to the classical Navier–Stokes equations, which is well known that the global regularity of its solution is still a famous open problem and one can gain the global regularity only for the equation with some special geometry structure (see e.g. [7, 8, 11, 15, 20, 21, 32] and references therein). For the case without the special geometric structure, Lei and Lin [17] gave global mild solutions of Navier–Stokes equations in $C(\mathbb{R}_+, \mathcal{X}^{-1}) \cap L^1(\mathbb{R}_+, \mathcal{X}^1)$ with initial data $\|u_0\|_{\mathcal{X}^{-1}} < \mu$, where $\mathcal{X}^s = \{u \in \mathscr{D}'(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi)| d\xi < +\infty\}$ with the norm $\|u\|_{\mathcal{X}^s} = \int_{\mathbb{R}^3} |\xi|^s |\hat{u}(\xi)| d\xi$. Recently, Bae [4] proved the work of Lei–Lin [17] in a slightly different setting, which can be specifically described as following:

$$\mathcal{L}_t^\infty \mathcal{X}^s = \left\{ f \in \mathscr{D}'(\mathbb{R}^3 \times \mathbb{R}_+) : \int_{\mathbb{R}^3} \Big[\sup_{0 \le t < +\infty} |\xi|^s |\hat{f}(\xi, t)| \Big] d\xi < +\infty \right\}$$

with $\|f\|_{\mathcal{L}_t^\infty \mathcal{X}^s} = \int_{\mathbb{R}^3} [\sup_{0 \le t < +\infty} |\xi|^s |\hat{f}(\xi, t)|] d\xi$ and

$$L_t^1 \mathcal{X}^1 = \left\{ f \in \mathscr{D}'(\mathbb{R}^3 \times \mathbb{R}_+) : \int_{\mathbb{R}^3} \int_0^{+\infty} |\xi| |\hat{f}(\xi, \tau)| d\tau d\xi < +\infty \right\}$$

with $\|f\|_{L_t^1 \mathcal{X}^1} = \int_{\mathbb{R}^3} \int_0^t |\xi| |\hat{f}(\xi, \tau)| d\tau d\xi$.

When $s = 1$, spaces $\mathcal{X}^1$ and $\mathcal{L}_t^\infty \mathcal{X}^1$ are discussed in detail in [17] and [4], respectively. In fact, $\mathcal{X}^1$ and $\mathcal{L}_t^\infty \mathcal{X}^1$ are scale-invariant function spaces that are natural with respect to the scaling of the Navier–Stokes equations.

In this paper, we will apply Lei–Lin and Bae's ideas and methods to the equation (grMHD). Considering the natural scale-invariant function spaces to the fractional Navier–Stokes equations, we should take $s = 1 - 2\alpha$ in $\mathcal{L}_t^\infty \mathcal{X}^s$. In contrast to Bae's results [4], we give the corresponding results of the equation (grMHD) but we have to be careful to deal with the rotating item, fractional diffusion terms and more complex nonlinear terms in the equation (grMHD). Moreover, we avoid discussing the value of Ω because our workspace is in the frequency space rather than the physical space. From the physical point of view, the value of Ω can not be great in some physical models, which is not suitable for the case in [5]. In fact, this paper also generalizes the results of [22]. Specifically, our results are following:

Theorem 1.1 (Global well posedness) *Assume that* $\frac{1}{2} \le \alpha \le 1$, *then there exists a positive constant* $\varepsilon_0 = \varepsilon_0(\mu, \gamma, \alpha) > 0$, *such that for any initial data* u_0, B_0 *in* $\mathcal{X}^{1-2\alpha}$ *with*

$$\|u_0\|_{\mathcal{X}^{1-2\alpha}} + \|B_0\|_{\mathcal{X}^{1-2\alpha}} < \varepsilon_0,$$

there is a unique global in time solution $u, B \in \mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha} \cap L_t^1 \mathcal{X}^1$ *such that*

$$\|u\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha}} + \|u\|_{L_t^1 \mathcal{X}^1} + \|B\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha}} + \|B\|_{L_t^1 \mathcal{X}^1} \lesssim \|u_0\|_{\mathcal{X}^{1-2\alpha}} + \|B_0\|_{\mathcal{X}^{1-2\alpha}}.$$

The research about Gevrey class regularity of the solution to Navier–Stokes equation is also an important topic, see [10, 18] and references therein. This approach enables one to avoid cumbersome recursive estimation of higher-order derivatives. By Bae's inspiration [4], we will prove the Gevrey class regularities for the equation (grMHD). And the specific results are as follows:

Theorem 1.2 (Gevrey class regularity) *Suppose that* $\frac{1}{2} \le \alpha \le 1$, *then there exists a constant* $\varepsilon_0' = \varepsilon_0'(\mu, \gamma, \alpha) > 0$ *such that for any initial data* u_0, B_0 *in* $\mathcal{X}^{1-2\alpha}$ *with*

$$\|u_0\|_{\mathcal{X}^{1-2\alpha}} + \|B_0\|_{\mathcal{X}^{1-2\alpha}} \le \varepsilon_0',$$

the solution obtained in Theorem 1.1 *is analytic in the sense that*

$$\begin{aligned} &\|e^{\sqrt{t}|D|^\alpha} u\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha}} + \|e^{\sqrt{t}|D|^\alpha} u\|_{L_t^1 \mathcal{X}^1} + \|e^{\sqrt{t}|D|^\alpha} B\|_{\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha}} + \|e^{\sqrt{t}|D|^\alpha} B\|_{L_t^1 \mathcal{X}^1} \\ &\quad \lesssim \|u_0\|_{\mathcal{X}^{1-2\alpha}} + \|B_0\|_{\mathcal{X}^{1-2\alpha}}, \end{aligned} \tag{1.1}$$

where $e^{\sqrt{t}|D|^\alpha}$ *is a Fourier multiplier whose symbol is given by* $e^{\sqrt{t}|\xi|^\alpha}$.

Remark 1.3 When we take $B = 0$, $\alpha = 1$ and $\Omega = 0$ in equations (grMHD), Theorems 1.1 and 1.2 become immediately Theorems 1.2 and 1.3 in [4], respectively.

Remark 1.4 Without loss of generality, we just need to take $\mu = \gamma = 1$ when we prove Theorems 1.1 and 1.2.

This paper is organized as follows. In Section 2, we introduce "semigroup" $T_{\Omega,\alpha}$ corresponding to the linearized equation of the equation (grMHD). In Section 3, we prove Theorem 1.1. And in Section 4, we give the proof of Theorem 1.2.

2 Preliminaries

When the magnetic field $B = 0$, the equation (grMHD) deduces to the fractional Navier–Stokes equations with Coriolis force. Hence we need to introduce the corresponding generalized Stokes–Coriolis semigroup T. In fact, we have to consider the following linear generalized problem:

$$\begin{cases} u_t + \mu(-\Delta)^\alpha u + \Omega e_3 \times u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0, +\infty), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^3. \end{cases} \tag{LNSC}$$

The solution of equation (LNSC) can be given by the generalized Stokes–Coriolis semigroup $T_{\Omega,\alpha}$, which has the explicit representation [12, 22]:

$$\begin{aligned} T_{\Omega,\alpha}(t)u &= \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^{2\alpha}t} I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)e^{-|\xi|^{2\alpha}t} R(\xi)\right] * u \\ &= \mathscr{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right) I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right) R(\xi)\right] * (e^{-(-\Delta)^\alpha t} u), \end{aligned}$$

where divergence free vector field $u \in \mathscr{S}(\mathbb{R}^3)$, I is the unit matrix in $M_{3\times 3}(\mathbb{R})$ and $R(\xi)$ is skew symmetric matrix defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Thus, we are easy to get a "semigroup":

$$\mathscr{A}_{\Omega,\alpha}(t) = \begin{pmatrix} T_{\Omega,\alpha}(t) & 0 \\ 0 & S_\alpha(t) \end{pmatrix}$$

where $S_\alpha(t) := \mathrm{e}^{-(-\Delta)^\alpha t} = \mathcal{F}^{-1}(\mathrm{e}^{-|\xi|^{2\alpha} t})$. With the help of $\mathscr{A}_{\Omega,\alpha}(t)$, we can rewrite the equation (grMHD) in the form of integral:

$$\begin{aligned}\begin{pmatrix} u \\ B \end{pmatrix} &= \mathscr{A}_{\Omega,\alpha}(t)\begin{pmatrix} u_0 \\ B_0 \end{pmatrix} \\ &\quad - \int_0^t \mathscr{A}_{\Omega,\alpha}(t-\tau)\mathbb{P}\begin{pmatrix} \operatorname{div}(u\otimes u - B\otimes B) \\ \operatorname{div}(u\otimes B - B\otimes u)\end{pmatrix}(\tau)d\tau \end{aligned} \tag{2.1}$$

where

$$\mathbb{P} = I - \nabla(-\Delta)^{-1}\mathrm{div}$$

is the Leray–Hopf projection.

3 Proof of Theorem 1.1

In order to estimate the nonlinear terms in the proof of Theorem 1.1, we need an inequality from [31]:

Lemma 3.1 ([31]) *Assume that* $\frac{1}{2} \le \alpha \le 1$, *then the following inequality holds*

$$|x|^{2(1-\alpha)} \le \frac{2^{2(1-\alpha)}}{2}(|y||x-y|^{1-2\alpha} + |y|^{1-2\alpha}|x-y|) \tag{3.1}$$

for any x, y *in* $\mathbb{R}^3$.

Remark 3.2 Ye [31] gave two counterexamples:

(1) If $\alpha > 1, x = (1-b, 0, -\sqrt{1-b^2}), y = (1,0,0)$ for $\frac{1}{2} < b \le 1$, then the inequality (3.1) fails;

(2) If $0 < \alpha < \frac{1}{2}, x = (1,0,0), y = (1-c, 0, -\sqrt{1-c^2})$ with c close enough to 1, then the inequality (3.1) fails.

These two counterexamples also explain the reason for the range of α in Theorems 1.1 and 1.2 in some sense.

Next, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Taking the Fourier transform to the equation (2.1), we have

$$\hat{u}(\xi,t) = \left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I + \sin\left(\Omega\frac{\xi_3}{|\xi|}t\right)R(\xi)\right]\mathrm{e}^{-t|\xi|^{2\alpha}}\hat{u}_0(\xi)$$

$$- \int_0^t \left\{ \cos\left[\Omega\frac{\xi_3}{|\xi|}(t-\tau)\right] I + \sin\left[\Omega\frac{\xi_3}{|\xi|}(t-\tau)\right] R(\xi) \right\} \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \mathbb{P}\mathrm{i}\xi$$
$$\cdot \int_{\mathbb{R}^3} [\hat{u}(\xi-\eta,\tau) \otimes \hat{u}(\eta,\tau) - \hat{B}(\xi-\eta,\tau) \otimes \hat{B}(\eta,\tau)] d\eta d\tau \tag{3.2}$$

and

$$\hat{B}(\xi,t) = \mathrm{e}^{-t|\xi|^{2\alpha}} \hat{B}_0(\xi) - \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \mathbb{P}\mathrm{i}\xi$$
$$\cdot \int_{\mathbb{R}^3} [\hat{u}(\xi-\eta,\tau) \otimes \hat{B}(\eta,\tau) - \hat{B}(\xi-\eta,\tau) \otimes \hat{u}(\eta,\tau)] d\eta d\tau. \tag{3.3}$$

Since $\mathbb{P}$ and $T_{\Omega,\alpha}$ are bounded Fourier multiplier, we ignore these terms in the following estimations. Hence, (3.2) and (3.3) yield

$$\begin{aligned}
&|\hat{u}(\xi,t)| + |\hat{B}(\xi,t)| \\
&\quad \le \mathrm{e}^{-t|\xi|^{2\alpha}} [|\hat{u}_0(\xi)| + |\hat{B}_0(\xi)|] \\
&\quad\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi| [|\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| + |\hat{B}(\xi-\eta,\tau)|\hat{B}(\eta,\tau)|] d\eta d\tau \\
&\quad\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi| [|\hat{B}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| + |\hat{u}(\xi-\eta,\tau)|\hat{B}(\eta,\tau)|] d\eta d\tau.
\end{aligned} \tag{3.4}$$

Step 1 Estimate of u, B in $\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha}$.

Multiplying the equations (3.4) by $|\xi|^{1-2\alpha}$, we have

$$\begin{aligned}
|\xi|^{1-2\alpha}[|\hat{u}(\xi,t)| + |\hat{B}(\xi,t)|] &\le \mathrm{e}^{-t|\xi|^{2\alpha}} |\xi|^{1-2\alpha} [|\hat{u}_0(\xi)| + |\hat{B}_0(\xi)|] \\
&\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| d\eta d\tau \\
&\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\hat{B}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)| d\eta d\tau \\
&\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\hat{B}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| d\eta d\tau \\
&\quad + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\hat{u}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)| d\eta d\tau \\
&= \mathrm{e}^{-t|\xi|^{2\alpha}} |\xi|^{1-2\alpha} [|\hat{u}_0(\xi)| + |\hat{B}_0(\xi)|] + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By Lemma 3.1, we may estimate the nonlinear term I_1, I_2, I_3, I_4, respectively:

$$\begin{aligned}
I_1 &= \int_0^t \left[\int_{\mathbb{R}^3} |\xi|^{2-2\alpha} |\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| d\eta \right] d\tau \\
&\le \frac{2^{2(1-\alpha)}}{2} \int_0^t \int_{\mathbb{R}^3} (|\eta||\xi-\eta|^{1-2\alpha} + |\eta|^{1-2\alpha}|\xi-\eta|) |\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| d\eta d\tau \\
&\le 2^{2(1-\alpha)} \left[\int_0^\infty |\cdot||\hat{u}(\cdot,\tau)| d\tau \right] * \left[\sup_{0\le t<+\infty} |\cdot|^{1-2\alpha} |\hat{u}(\cdot,t)| \right], \\
I_2 &\le 2^{2(1-\alpha)} \left[\int_0^\infty |\cdot||\hat{B}(\cdot,\tau)| d\tau \right] * \left[\sup_{0\le t<+\infty} |\cdot|^{1-2\alpha} |\hat{B}(\cdot,t)| \right]
\end{aligned}$$

and

$$I_3 + I_4 \le \frac{2^{2(1-\alpha)}}{2} \left\{ \left[\int_0^\infty |\cdot||\hat{B}(\cdot,\tau)| d\tau \right] * \left[\sup_{0\le t<+\infty} |\cdot|^{1-2\alpha} |\hat{u}(\cdot,t)| \right] \right.$$

$$+\left[\int_0^{\infty}|\cdot||\hat{u}(\cdot,\tau)|d\tau\right]*\left[\sup_{0\le t<+\infty}|\cdot|^{1-2\alpha}|\hat{B}(\cdot,t)|\right]\Big\}.$$

Hence, we can get

$$\begin{aligned}
&\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}\\
&=\int_{\mathbb{R}^3}\sup_{0\le t<+\infty}|\xi|^{1-2\alpha}|\hat{u}(\xi,t)|d\xi+\int_{\mathbb{R}^3}\sup_{0\le t<+\infty}|\xi|^{1-2\alpha}|\hat{u}(\xi,t)|d\xi\\
&\le\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|B_0\|_{\mathcal{X}^{1-2\alpha}}\\
&\quad+2^{2(1-2\alpha)}(\|u\|_{L_t^1\mathcal{X}^1}\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{L_t^1\mathcal{X}^1}\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}})\\
&\quad+2^{2(1-\alpha)}(\|B\|_{L_t^1\mathcal{X}^1}\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}})\\
&\le\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|B_0\|_{\mathcal{X}^{1-2\alpha}}\\
&\quad+2^{2(1-2\alpha)}(\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1})(\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}).
\end{aligned}\tag{3.5}$$

Step 2 Estimate of u,B in $L_t^1\mathcal{X}^1$.

Multiplying the equation (3.4) by $|\xi|$, we can gain

$$\begin{aligned}
&|\xi|[|\hat{u}(\xi,t)|+|\hat{B}(\xi,t)|]\\
&\quad\le|\xi|^{2\alpha}\mathrm{e}^{-t|\xi|^{2\alpha}}|\xi|^{1-2\alpha}[|\hat{u}_0(\xi)|+|\hat{B}_0(\xi)|]\\
&\qquad+\int_0^t|\xi|^{2\alpha}\mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}}\int_{\mathbb{R}^3}|\xi|^{2-2\alpha}[|\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)|+|\hat{B}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)|]d\eta d\tau\\
&\qquad+\int_0^t|\xi|^{2\alpha}\mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}}\int_{\mathbb{R}^3}|\xi|^{2-2\alpha}[|\hat{B}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)|+|\hat{u}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)|]d\eta d\tau.
\end{aligned}$$

Since

$$\int_0^{+\infty}|\xi|^{2\alpha}\mathrm{e}^{-t|\xi|^{2\alpha}}dt<\infty,$$

we obtain

$$\begin{aligned}
&\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1}\\
&\lesssim\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|u_0\|_{\mathcal{X}^{1-2\alpha}}\\
&\quad+2^{2(1-2\alpha)}[\|u\|_{L_t^1\mathcal{X}^1}\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{L_t^1\mathcal{X}^1}\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}]\\
&\quad+2^{2(1-2\alpha)}[\|B\|_{L_t^1\mathcal{X}^1}\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}]\\
&\le\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|B_0\|_{\mathcal{X}^{1-2\alpha}}\\
&\quad+2^{2(1-2\alpha)}(\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1})(\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}).
\end{aligned}\tag{3.6}$$

Step 3 Estimate of $\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1}$.

Let $Y=\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1}$. (3.5) and (3.6) yield

$$\begin{aligned}
&\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1}\\
&\quad\lesssim2(\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|B_0\|_{\mathcal{X}^{1-2\alpha}})\\
&\qquad+2\cdot2^{2(1-2\alpha)}(\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1})(\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}})\\
&\quad\lesssim\|u_0\|_{\mathcal{X}^{1-2\alpha}}+2^{2(1-2\alpha)}(\|u\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|B\|_{\mathcal{L}_t^\infty\mathcal{X}^{1-2\alpha}}+\|u\|_{L_t^1\mathcal{X}^1}+\|B\|_{L_t^1\mathcal{X}^1})^2,
\end{aligned}$$

that is

$$Y\lesssim2^{2(1-2\alpha)}Y^2+(\|u_0\|_{\mathcal{X}^{1-2\alpha}}+\|B_0\|_{\mathcal{X}^{1-2\alpha}}).$$

Denote

$$\delta = 1 - 4 \cdot 2^{2(1-2\alpha)}(\|u_0\|_{\mathcal{X}^{1-2\alpha}} + \|B_0\|_{\mathcal{X}^{1-2\alpha}}),$$

then it is not hard to get the existence of global solution in $\mathcal{L}_t^\infty \mathcal{X}^{1-2\alpha} \cap L_t^1 \mathcal{X}^1$ for small initial data in $\mathcal{X}^{1-2\alpha}$ by standard fixed point argument.

4 Proof of Theorem 1.2

Firstly, we need an inequality [22], which is used to estimate nonlinear term in the proof of Theorem 1.2.

Lemma 4.1 ([22]) *Assume* $0 < s \le t < +\infty$ *and* $0 \le \alpha \le 1$, *then the following inequality holds*

$$t|x|^\alpha - \frac{1}{2}(t^2 - s^2)|x|^{2\alpha} - s|x-y|^\alpha - s|y|^\alpha \le \frac{1}{2}$$

for any $x, y \in \mathbb{R}^3$.

With the help of Lemma 4.1, we give the proof of Theorem 1.2.

Proof of Theorem 1.2 We utilize the following Gevrey class regularity:

$$\sup_{0<t<\infty} \sup_{\xi\in\mathbb{R}^3} \mathrm{e}^{\sqrt{t}|\xi|^\alpha} |\xi|^{2\alpha} |\hat{u}(\xi,t)| + \sup_{0<t<\infty} \sup_{\xi\in\mathbb{R}^3} \mathrm{e}^{\sqrt{t}|\xi|^\alpha} |\xi|^{2\alpha} |\hat{B}(\xi,t)| < +\infty,$$

which is an extension of Gevrey class regularity of the solution that Le Jan and Sznitman constructed in [18]:

$$\sup_{0<t<\infty} \sup_{\xi\in\mathbb{R}^3} \mathrm{e}^{\sqrt{t}|\xi|} |\xi|^2 |\hat{u}(\xi,t)| < +\infty.$$

Let $\binom{v}{b} = \mathrm{e}^{\sqrt{t}|D|^\alpha} \binom{u}{B}$. On account of the equation (2.1), we have

$$\begin{aligned}\begin{pmatrix} v \\ b \end{pmatrix} &= \mathrm{e}^{\sqrt{t}|D|^\alpha} \mathscr{A}_{\Omega,\alpha}(t) \begin{pmatrix} u_0 \\ B_0 \end{pmatrix} \\ &\quad - \int_0^t \mathrm{e}^{\sqrt{t}|D|^\alpha} \mathscr{A}_{\Omega,\alpha}(t-\tau) \mathbb{P} \begin{pmatrix} \operatorname{div}(u\otimes u - B\otimes B) \\ \operatorname{div}(u \otimes B - B \otimes u) \end{pmatrix} (\tau) d\tau.\end{aligned}$$

Similar to the method of deriving the inequality (3.4), we can get

$$\begin{aligned}&|\hat{v}(\xi,t)| + |\hat{b}(\xi,t)| \\ &\quad\lesssim \mathrm{e}^{\sqrt{t}|\xi|^\alpha - t|\xi|^{2\alpha}} [|\hat{u}_0(\xi)| + |\hat{B}_0(\xi)|] \\ &\qquad + \int_0^t \mathrm{e}^{\sqrt{t}|\xi|^\alpha - (t-\tau)|\xi|^{2\alpha}} |\xi| \\ &\qquad \left\{ \int_{\mathbb{R}^3} [|\hat{u}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| + |\hat{B}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)|] d\eta \right\} d\tau \\ &\qquad + \int_0^t \mathrm{e}^{\sqrt{t}|\xi|^\alpha - (t-\tau)|\xi|^{2\alpha}} |\xi| \\ &\qquad \left\{ \int_{\mathbb{R}^3} [|\hat{B}(\xi-\eta,\tau)||\hat{u}(\eta,\tau)| + |\hat{u}(\xi-\eta,\tau)||\hat{B}(\eta,\tau)|] d\eta \right\} d\tau \\ &\quad= \mathrm{e}^{\sqrt{t}|\xi|^\alpha - t|\xi|^{2\alpha}} [|\hat{u}_0(\xi)| + |\hat{B}_0(\xi)|] \\ &\qquad + \int_0^t \mathrm{e}^{\sqrt{t}|\xi|^\alpha - (t-\tau)|\xi|^{2\alpha} - \sqrt{\tau}(|\xi-\eta|^\alpha + |\eta|^\alpha)} |\xi|\end{aligned}$$

$$\left\{ \int_{\mathbb{R}^3} [|\hat{v}(\xi-\eta,\tau)||\hat{v}(\eta,\tau)| + |\hat{b}(\xi-\eta,\tau)||\hat{b}(\eta,\tau)|]d\eta \right\} d\tau$$
$$+ \int_0^t \mathrm{e}^{\sqrt{t}|\xi|^\alpha - (t-\tau)|\xi|^{2\alpha} - \sqrt{\tau}(|\xi-\eta|^\alpha + |\eta|^\alpha)} |\xi|$$
$$\left\{ \int_{\mathbb{R}^3} [|\hat{b}(\xi-\eta,\tau)||\hat{v}(\eta,\tau)| + |\hat{v}(\xi-\eta,\tau)||\hat{b}(\eta,\tau)|]d\eta \right\} d\tau. \tag{4.1}$$

Note

$$\mathrm{e}^{\sqrt{t}|\xi|^\alpha - \frac{1}{2}t|\xi|^{2\alpha}} = \mathrm{e}^{-\frac{1}{2}(\sqrt{t}|\xi|^\alpha - 1)^2 + \frac{1}{2}} \leq \mathrm{e}^{\frac{1}{2}}$$

and

$$\mathrm{e}^{\sqrt{t}|\xi|^\alpha - \frac{1}{2}(t-\tau)|\xi|^{2\alpha} - \sqrt{\tau}(|\xi-\eta|^\alpha + |\eta|^\alpha)} \leq \mathrm{e}^{\frac{1}{2}}$$

by Lemma 4.1. Hence, we substitute the above two inequalities in the inequality (4.1) and obtain

$$\begin{aligned} &|\hat{v}(\xi,t)| + |\hat{b}(\xi,t)| \\ &\quad \lesssim \mathrm{e}^{-\frac{1}{2}t|\xi|^\alpha} [|\hat{v}_0(\xi)| + |\hat{v}_0(\xi)|] \\ &\qquad + \int_0^t \mathrm{e}^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}} |\xi| \left\{ \int_{\mathbb{R}^3} [|\hat{v}(\xi-\eta,\tau)||\hat{v}(\eta,\tau)| + |\hat{b}(\xi-\eta,\tau)||\hat{b}(\eta,\tau)|]d\eta \right\} d\tau \\ &\qquad + \int_0^t \mathrm{e}^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}} |\xi| \left\{ \int_{\mathbb{R}^3} [|\hat{b}(\xi-\eta,\tau)||\hat{v}(\eta,\tau)| + |\hat{u}(\xi-\eta,\tau)||\hat{b}(\eta,\tau)|]d\eta \right\} d\tau. \end{aligned}$$

The remaining part of the proof is similar to Theorem 1.1, and thus we omit the details.

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