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Homogenization of Elliptic Problems with Neumann Boundary Conditions in Non-smooth Domains

Jun GENG
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000*, P. R. China*
E-mail: *gengjun@lzu.edu.cn*

Abstract We consider a family of second-order elliptic operators $\{\mathcal{L}_\varepsilon\}$ in divergence form with rapidly oscillating and periodic coefficients in Lipschitz and convex domains in $\mathbb{R}^n$. We are able to show that the uniform $W^{1,p}$ estimate of second order elliptic systems holds for $\frac{2n}{n+1} - \delta < p < \frac{2n}{n-1} + \delta$ where $\delta > 0$ is independent of ε and the ranges are sharp for $n = 2, 3$. And for elliptic equations in Lipschitz domains, the $W^{1,p}$ estimate is true for $\frac{3}{2} - \delta < p < 3 + \delta$ if $n \geq 4$, similar estimate was extended to convex domains for $1 < p < \infty$.

Keywords Homogenization, elliptic, non-smooth

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1 Introduction

In this paper, we carry out a study of the periodic homogenization problem subjected to Neumann boundary condition. Precisely, let Ω be a bounded Lipschitz or convex domain in $\mathbb{R}^n$ and $N = (N_1, N_2, \dots, N_n)$ the outward unit normal to $\partial\Omega$, assume $f \in L^p(\Omega)$ and $g \in B^{-1/p,p}(\partial\Omega)$, consider

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = \mathrm{div} f & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -f \cdot N + g & \text{on } \partial\Omega, \end{cases} \tag{N$_\mathrm{p}$}$$

where

$$\left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha = N_i(x) a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon^\beta}{\partial x_j} \tag{1.1}$$

denotes the conormal derivative with respect to $\mathcal{L}_\varepsilon$ and $1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq m$. Here $\mathcal{L}_\varepsilon$ is a family of second order elliptic operator with rapidly oscillating, periodic coefficients, arising in the theory of homogenization,

$$\mathcal{L}_\varepsilon = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right] = -\mathrm{div}\left[A\left(\frac{x}{\varepsilon}\right)\nabla\right]. \tag{1.2}$$

Suppose that the coefficient matrix $A(y) = a_{ij}^{\alpha\beta}(y) (1 \leq i, j \leq n, 1 \leq \alpha, \beta \leq m)$ is real, bounded, measurable. Here and thereafter we will suppose that A is elliptic, i.e.,

$$\mu|\xi|^2 \leq a_{ij}^{\alpha\beta}(x)\xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu}|\xi|^2, \quad \text{for } \xi = (\xi_i^\alpha) \in \mathbb{R}^{nm},\ x \in \mathbb{R}^n,\ \mu > 0, \tag{1.3}$$

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and the periodicity condition

$$A(y+z)=A(y), \quad \text{for } z\in\mathbb{Z}^n,\ y\in\mathbb{R}^n, \tag{1.4}$$

as well as the smoothness condition

$$|A(x)-A(y)|\le\tau|x-y|^\eta, \quad \text{for some } \eta\in(0,1) \text{ and } \tau\ge 0. \tag{1.5}$$

The symmetry condition $A^*=A$, i.e.,

$$a_{ij}^{\alpha\beta}(x)=a_{ji}^{\beta\alpha}(x) \tag{1.6}$$

is also needed.

Letting $1<p<\infty$, for any $f\in L^p(\Omega)$ and $g\in B^{-1/p,p}(\partial\Omega)$, we say u_ε is a solution to $(\mathrm{N})_\mathrm{p}$ if $u_\varepsilon\in W^{1,p}(\Omega)$ and

$$\int_\Omega a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon^\beta}{\partial x_j}\frac{\partial\phi^\alpha}{\partial x_i}dx=\int_\Omega f_i^\alpha\frac{\partial\phi^\alpha}{\partial x_i}dx+\langle g^\alpha,\phi^\alpha\rangle_{B^{-1/p,p}(\partial\Omega)\times B^{1/p,p'}(\partial\Omega)} \tag{1.7}$$

for any $\phi\in C^\infty(\mathbb{R}^n)$, where $\langle\cdot,\cdot\rangle$ denotes the paring between $B^{1/p,p'}(\partial\Omega)$ and its dual $B^{-1/p,p}(\partial\Omega)$. Recall that for any $0<\alpha<1$, the Besov spaces $B^{\alpha,p}$ is defined to mean the collection of functions u on $\partial\Omega$ with the norm

$$\|u\|_{B^{\alpha,p}(\partial\Omega)}=\|u\|_{L^p(\partial\Omega)}+\left(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{n-1+\alpha p}}\right)^{1/p}<\infty. \tag{1.8}$$

The following are the main results of the paper.

Theorem 1.1 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n (n\ge 2)$. Let $m>1$ and $\mathcal{L}_\varepsilon$ be defined as in (1.2) with A satisfying (1.3)–(1.6). Suppose that $f\in L^p(\Omega)$ and $g\in B^{-1/p,p}(\partial\Omega)$ where $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$. Then if g satisfies the compatibility condition*

$$\langle g^\alpha,1\rangle_{B^{-1/p,p}(\partial\Omega)\times B^{1/p,p'}(\partial\Omega)}=0, \tag{1.9}$$

the weak solutions to the Neumann problem $(\mathrm{N})_\mathrm{p}$ satisfy the $W^{1,p}$ estimate

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)}\le C\{\|f\|_{L^p(\Omega)}+\|g\|_{B^{-1/p,p}(\partial\Omega)}\}, \tag{1.10}$$

where constants $\delta, C>0$ are independent of ε.

In the case of scalar equation ($m=1$), we have the following theorem.

Theorem 1.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n (n\ge 2)$. Let $m=1$ and $\mathcal{L}_\varepsilon$ be defined as in (1.2) with A satisfying (1.3)–(1.6). Let $f\in L^p(\Omega)$ and $g\in B^{-1/p,p}(\partial\Omega)$ where $\frac{3}{2}-\delta<p<3+\delta$ if $n\ge 3$ ($\frac{4}{3}-\delta<p<4+\delta$ if $n=2$). Then if g satisfies the compatibility condition*

$$\langle g,1\rangle_{B^{-1/p,p}(\partial\Omega)\times B^{1/p,p'}(\partial\Omega)}=0, \tag{1.11}$$

the weak solutions to the Neumann problem $(\mathrm{N})_\mathrm{p}$ satisfy the $W^{1,p}$ estimate

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)}\le C\{\|f\|_{L^p(\Omega)}+\|g\|_{B^{-1/p,p}(\partial\Omega)}\}, \tag{1.12}$$

where constant $\delta=\delta(\mu,n,\tau,\Omega,\eta)>0$ and $C>0$ depends only on n, p, A and the Lipschitz character of Ω.

The sharp ranges of $p's$ are obtained in convex domains, as follows.

Theorem 1.3 *Let $1 < p < \infty$. Assume that Ω is a bounded convex domain in $\mathbb{R}^n (n \geq 2)$. Let $m = 1$ and $\mathcal{L}_\varepsilon$ be defined as in (1.2) with A satisfying (1.3)–(1.6). Let $f \in L^p(\Omega)$ and $g \in B^{-1/p,p}(\partial\Omega)$. Then if g satisfies the compatibility condition*

$$\langle g, 1\rangle_{B^{-1/p,p}(\partial\Omega)\times B^{1/p,p'}(\partial\Omega)} = 0, \tag{1.13}$$

the weak solutions to the Neumann problem $(\mathrm{N})_\mathrm{p}$ *satisfy the* $W^{1,p}$ *estimate*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|g\|_{B^{-1/p,p}(\partial\Omega)}\}, \tag{1.14}$$

where constant $\delta = \delta(\mu, n, \tau, \Omega, \eta) > 0$ and $C > 0$ depending only on n, p, A and the Lipschitz character of Ω.

Remark 1.4 In Theorems 1.3 and 1.2, the range of $p's$ is sharp even for the Laplacian.

Uniform regularity estimates in non-smooth domains play an essential role in the study of the convergence problems in homogenization, we refer the reader to [7], also see related work in [2–6, 20] and [22]. In [23] the L^2 Dirichlet, Regularity and Neumann problems in Lipschitz domain for elliptic systems with rapidly oscillating coefficients satisfying (1.3) are solved by the method of layer potentials. The uniform $W^{1,p}$ estimate (1.10) of $(\mathrm{N})_\mathrm{p}$ was established by Kenig, et al. in [20] for $1 < p < \infty$ in $C^{1,\alpha}$ domain under the assumption that A satisfies (1.3), (1.4), (1.5) and (1.6), the non-tangential maximal function estimates and Lipschitz estimates were also obtained there via a three-step compactness argument. In the case of Dirichlet boundary condition on $C^{1,\alpha}$ domain, the uniform $W^{1,p}$ estimate (1.10) was obtained in [2] under the same assumption on A as [20] without $A^* = A$. We also refer the reader to [3, 5, 6] for various estimates in elliptic homogenization.

In the case of second order elliptic systems subject to Dirichlet boundary conditions in Lipschitz domains, in [16], the authors were able to show that the uniform $W^{1,p}$ estimate (1.10) holds on Lipschitz domains for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2n} + \delta$ under the assumption that A satisfies (1.5), (1.4) and (1.3) as well as $A^* = A$. Similar results for linear elasticity problem are also proved in [16] by different approach. In the case of scalar equation ($m = 1$) on Lipschitz domain, the $W^{1,p}$ estimate (1.10) for the elliptic homogenization problem $\mathcal{L}_\varepsilon u_\varepsilon = \mathrm{div} f$ in Ω was proved in [26] for $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$ if $n = 2$ and $\frac{4}{3} - \varepsilon < p < 4 + \varepsilon$ when $n \geq 3$, and the range of $p's$ is sharp. The L^p boundedness of Riesz transforms associated with $\{\mathcal{L}_\varepsilon\}$ is also established for the same optimal range of p in [26]. However, if the equation is subject to the Neumann boundary, it is more difficult to obtain such estimate than the Dirichlet problem since the Neumann condition in (1.1) involves ε.

Our approach is to reduce the $W^{1,p}$ estimate to the weak reverse Hölder inequality (locally) via a duality argument of the Calderón–Zygmund decomposition, see Theorem 2.2. Basically, to obtain the $W^{1,p}$ solvability of $(N)_p$ in Lipschitz domains, we rewrite $|\nabla u_\varepsilon|^p = |\nabla u_\varepsilon|^{p-2}|\nabla u_\varepsilon|^2$, and it is noted that the well-known interior estimate

$$\sup_B |\nabla u_\varepsilon| \leq \frac{C}{r}\bigg\{\fint_{2B} |u_\varepsilon|^2\bigg\}^{1/2}, \tag{1.15}$$

is proved by Avellaneda and Lin [2] if $\mathcal{L}_\varepsilon u_\varepsilon = 0$ in $B(x, 4r)$ with A satisfying (1.3)–(1.5) and the constant C is independent of ε. Consequently, $|\nabla u_\varepsilon|^{p-2}$ will be estimated by (1.15) and

$|\nabla u_\varepsilon|^2$ be estimated by the Rellich estimate (see [23])

$$\int_{\partial\Omega} |(\nabla u_\varepsilon)^*|^2 \le C \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2.$$

This allows us to solve the $W^{1,p}$ estimate for homogenization problems $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in Ω with Neumann boundary data in $L^p(\partial\Omega)$.

Our proof of Theorem 1.3 follows the same line of argument as in the proof of Theorem 1.2, but one may notice that a convex domain must be Lipschitz domain, but a Lipschitz domain may not be convex. And it worthwhile to point out that, in [15], by using some elegant analysis tools such as co-area formula, weighted inequality, together with the convexity of Ω, we adapt a complete different approach to establish the weak reverse Hölder estimates

$$\left(\fint_{Z(r)} |\nabla u|^p\right)^{\frac{1}{p}} \le C\left(\fint_{Z(2r)} |\nabla u|^2\right)^{\frac{1}{2}} \tag{1.16}$$

holds for any $1 < p < \infty$, where u is a weak solution of $\Delta u = 0$ in Ω and $\frac{\partial u}{\partial N} = 0$ on $\partial\Omega$, and this results has been extended to the constant coefficient case in this paper, see Theorem 6.1.

Next, it follows by a real-variable perturbation argument, we obtain the weak reverse Hölder estimates (1.16) for u satisfying $\mathcal{L}u = 0$ in $Z(3r)$ and $\frac{\partial u}{\partial \nu} = 0$ on $S(3r)$ with $A \in \mathrm{VMO}$, then the same argument as in Theorem 1.2 deduces that the $W^{1,p}$ estimates (1.14) holds for any $2 < p < \infty$, this gives the small scale $W^{1,p}$ estimates. To handle the large scale $W^{1,p}$ estimates, we fist utilize a three step compactness argument to obtain the boundary Hölder estimates

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\left(\frac{|x-y|}{r}\right)^{\eta}\left(\fint_{Z(r)} |u_\varepsilon|^2\right)^{1/2}, \tag{1.17}$$

where $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $Z(2r)$ and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $S(2r)$. This, together with the L^2 non-tangential maximal function estimates, leads to the large scale estimates. See Section 6.

Compared to $C^{1,\alpha}$ domain, as expected for non-smooth domains, the main difficulty lies in the boundary estimates and one may not expect that the boundary $W^{1,p}$ estimates hold for all $1 < p < \infty$ in Lipschitz domains. Let ψ be a Lipschitz mapping $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$, for $r > 0$, set

$$Z(r) = \{(x', x_n) \in \mathbb{R}^n : |x'| < r\}, \tag{1.18}$$

$$S(r) = \{(x', x_n) \in \mathbb{R}^n : |x'| < r \text{ and } \psi(x') < x_n < \psi(x') + (M + 10n)r\} \tag{1.19}$$

denote the Lipschitz cylinder and its surface, respectively. And

$$B(Q, r_0) \cap \Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \psi(x')\} \cap B(Q, r_0), \tag{1.20}$$

$$B(Q, r_0) \cap \partial\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n = \psi(x')\} \cap B(Q, r_0), \tag{1.21}$$

where $B(Q, r_0) = \{x \in \mathbb{R}^n : |x - Q| < r_0\}$ denotes a ball centered at Q with radius r_0.

2 A Real Variable Argument

The following theorem is a refined real variable argument which established in [25, Theorem 3.2], and can be seen as a duality argument of the Calderón–Zygmund decomposition. It plays an important role in the proof of Theorems 2.2 and 3.1, with the help of it, the $W^{1,p}$ estimates follow from the locally weak reverse Hölder inequality consequently, we generalize [25, Theorem 3.2] and provide the proof for the sake of completeness.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz domain and $F \in L^2(\Omega)$. Let $p > 2$ and $f \in L^q(\Omega)$ for some $2 < q < p$. Suppose that for each ball B with $|B| \le \beta|\Omega|$, there exist two measurable functions F_B, R_B on $2B$ such that $|F| \le |F_B| + |R_B|$ on $2B \cap \Omega$,*

$$\left\{ \fint_{2B\cap\Omega} |R_B|^p dx \right\}^{\frac{1}{p}} \le C_1 \left\{ \left(\fint_{\alpha B\cap\Omega} |F|^2 dx \right)^{\frac{1}{2}} + \sup_{B\subset B'} \left(\fint_{B'\cap\Omega} |f|^2 dx \right)^{\frac{1}{2}} \right\} \tag{2.1}$$

and

$$\fint_{2B\cap\Omega} |F_B|^2 dx \le C_2 \sup_{B\subset B'} \fint_{B'} |f|^2 dx + \sigma \fint_{\alpha B} |F|^2 dx, \tag{2.2}$$

where $C_1, C_2 > 0$ and $0 < \beta < 1 < \alpha$. Then, if $0 \le \sigma < \sigma_0 = \sigma_0(C_1, C_2, n, p, q, \alpha, \beta)$, we have

$$\left\{ \fint_{\Omega} |F|^q dx \right\}^{\frac{1}{q}} \le C \left\{ \left(\fint_{\Omega} |F|^2 dx \right)^{\frac{1}{2}} + \left(\fint_{\Omega} |f|^q dx \right)^{\frac{1}{q}} \right\}, \tag{2.3}$$

where $C > 0$ depends only on $C_1, C_2, n, p, q, \alpha, \beta$.

Proof We will argue by the Calderón–Zygmund decomposition and "Good-λ" inequality. For the convenience of dyadic decomposition, we may choose a cube Q_0 such that $\Omega \subset Q_0$ and $|Q_0| \sim |\Omega|$, and we will not make the effort to distinguish the "cube" and "ball". Then for $\lambda > 0$, consider

$$E(\lambda) = \{x \in Q_0 : M_{2Q_0}(|F|^2)(x) > \lambda\}, \tag{2.4}$$

where M_{2Q_0} denotes the Hardy–Littlewood maximal function defined on $2Q_0$.

Observe that the weak $(1,1)$ estimate of M_{2Q_0} implies that

$$|E(\lambda)| \le \frac{C_n}{\lambda} \int_{2Q_0} |F|^2 dx. \tag{2.5}$$

By letting $\lambda_0 = \frac{C_0}{|2Q_0|} \int_{2Q_0} |F|^2 dx$ and choosing $C_0 = 2^{n+1} C_n/\epsilon$, combining with (2.5) we obtain

$$|E(\lambda)| < \epsilon |Q_0| \tag{2.6}$$

for any $\lambda > \lambda_0$, where $\epsilon \in (0,1)$ is a constant to be determined later.

Next, we apply the Calderón–Zygmund decomposition to Q_0, let Q_k be the maximal dyadic subcubes of Q_0 and $\widetilde{Q}_k$ denote the "*parent*" of Q_k, by choosing ϵ sufficiently small, we may assume that $|Q_k| \le \beta |Q_0|$, we then have the following properties:

$$\Big| E(\lambda) \backslash \bigcup_k Q_k \Big| = 0, \tag{2.7}$$

and

$$\fint_{\widetilde{Q}_k} |F|^2 dx \le \lambda, \quad \text{since } \widetilde{Q}_k \subsetneq E(\lambda) \tag{2.8}$$

as well as

$$\fint_{Q'} |F|^2 dx \le C_n \lambda, \quad \text{if } Q' \cap Q_k \ne \emptyset \quad \text{and } |Q'| \ge c_n |Q_k|. \tag{2.9}$$

Hence it is not hard to see that if $x \in Q_k$,

$$M_{2Q_0}(|F|^2)(x) \le \max\{M_{2Q_k}(|F|^2), C_n \lambda\}. \tag{2.10}$$

Next, we claim that if $\{x \in Q_k : M_{2Q_0}(|f|^2) \le \lambda\gamma\} \ne \emptyset$, then

$$|E(A\lambda) \cap Q_k| \le \epsilon |Q_k|, \tag{2.11}$$

where $A = (2\epsilon)^{-\frac{2}{q}}$. To establish (2.11), we may assume that $A > C_n$. Then weak $(1,1)$, weak $(\frac{p}{2}, \frac{p}{2})$ type estimates of M_{2Q_k} and the fact $|F| \le |F_{Q_k}| + |R_{Q_k}|$ on $2Q_k$ imply that

$$\begin{aligned} |E(A\lambda) \cap Q_k| &\le |\{x \in Q_k : M_{2Q_k}(|F|^2)(x) > A\lambda\}| \\ &\le \left|\left\{x \in Q_k : M_{2Q_k}(|F_{Q_k}|^2)(x) > \frac{A\lambda}{4}\right\}\right| \\ &\quad + \left|\left\{x \in Q_k : M_{2Q_k}(|R_{Q_k}|^2)(x) > \frac{A\lambda}{4}\right\}\right| \\ &\le \frac{C_n}{A\lambda} \int_{2Q_k} |F_{Q_k}|^2 dx + \frac{C_{n,p}}{(A\lambda)^{\frac{p}{2}}} \int_{2Q_k} |R_{Q_k}|^p dx. \end{aligned} \tag{2.12}$$

It follows from (2.1) and the fact $\{x \in Q_k : M_{2Q_0}(|f|^2) \le \lambda\gamma\} \ne \emptyset$ that

$$\begin{aligned} \int_{2Q_k} |R_{Q_k}|^p dx &\le C_1^p |2Q_k| \left\{ \left(\fint_{\alpha Q_k} |F|^2 dx \right)^{\frac{1}{2}} + \sup_{Q_k \subset Q'} \left(\fint_{Q'} |f|^2 dx \right)^{\frac{1}{2}} \right\}^p \\ &\le C_{n,p} C_1^p |Q_k| \{\lambda^{\frac{1}{2}} + (\gamma\lambda)^{\frac{1}{2}}\}^p. \end{aligned} \tag{2.13}$$

On the other hand, (2.2) implies that

$$\begin{aligned} \int_{2Q_k} |F_{Q_k}|^2 dx &\le C_2 |Q_k| \sup_{Q_k \subset Q'} \fint_{Q'} |f|^2 dx + \sigma |Q_k| \fint_{\alpha Q_k} |F|^2 dx \\ &\le C_2 |Q_k| \gamma\lambda + \sigma |Q_k| C_n \lambda \\ &\le \lambda |Q_k| \big\{ C_2\gamma + \sigma C_n \big\}. \end{aligned} \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we then have

$$|E(A\lambda) \cap Q_k| \le \epsilon |Q_k| \{C_n C_2 \gamma \epsilon^{\frac{2}{q}-1} + C_n \sigma \epsilon^{\frac{2}{q}-1} + C_{n,p} C_1^p \epsilon^{\frac{p}{q}-1}\}. \tag{2.15}$$

One may notice that it is possible for us to choose δ and then $\gamma, \sigma_0 > 0$ such that $\{C_n C_2 \gamma \epsilon^{\frac{2}{q}-1} + C_n \sigma_0 \epsilon^{\frac{2}{q}-1} + C_{n,p} C_1^p \epsilon^{\frac{p}{q}-1}\} < \frac{1}{2}$, which gives the proof of (2.11). By taking the summation with index k of (2.11) and using the fact that Q_k is contained in $E(\lambda)$, one obtains

$$|E(A\lambda)| \le \epsilon |E(\lambda)| + |\{x \in Q_0 : M_{2Q_0}(|f|^2)(x) > \gamma\lambda\}|. \tag{2.16}$$

The rest of the proof is exactly the same as in the case $\sigma = 0$. We refer the reader to [25, Theorem 3.2] for details. □

Armed with Theorem 2.1, the $W^{1,p}$ estimates for second order elliptic equations or systems with divergence form subjected to Neumann boundary could be reduced to show that the weak reverse Hölder inequality holds for $p > 2$, as follows.

Theorem 2.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n, n \ge 2$ and $p > 2$. Let $\mathcal{L} = div(A(x)\nabla)$ with $A = (a_{ij}^{\alpha\beta}(x))$ satisfies* (1.3). *Assume that for any $B(x_0, r)$ with the property that $0 < r < r_0/8$ and either $x_0 \in \partial\Omega$ or $B(x_0, 2r) \subset \Omega$, the weak reverse Hölder inequality*

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla v|^p \right)^{\frac{1}{p}} \le C_0 \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla v|^2 \right)^{\frac{1}{2}} \tag{2.17}$$

holds, whenever $v \in W^{1,2}(B(x_0,2r)\cap\Omega)$ *satisfies* $\mathcal{L}(v)=0$ *in* $B(x_0,2r)\cap\Omega$ *and* $\frac{\partial v}{\partial\nu}=0$ *on* $B(x_0,2r)\cap\partial\Omega$ (*if* $x_0\in\partial\Omega$). *Let* u *be a* H^1 *solution of* $(N)_2$ *with* $f\in L^p(\Omega)$ *and* $g=0$. *Then* $u\in W^{1,p}(\Omega)$ *and*

$$\|\nabla u\|_{L^p(\Omega)}\le C\|f\|_{L^p(\Omega)}, \tag{2.18}$$

with constants $C>0$ *depending only on* n,p,μ,C_0 *and the Lipschitz character of* Ω.

Proof The proof is directly follows from Theorem 2.1, see [14]. □

Lemma 2.3 *Let* $u_\varepsilon\in H^1(Z(3r))$ *be a weak solution of* $\mathcal{L}_\varepsilon u_\varepsilon=0$ *in* $Z(3r)$ *and* $\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=0$ *in* $S(3r)$. *Then* ∇u_ε *is locally* L^p*-integrable, for some* $p>2$, *we have*

$$\Big\{\fint_{Z(r)}|\nabla u_\varepsilon|^p\Big\}^{\frac{1}{p}}\le C\Big\{\fint_{Z(2r)}|\nabla u_\varepsilon|^2\Big\}^{\frac{1}{2}}. \tag{2.19}$$

Proof The proof is is essentially due to Giaquinta (see [17]), which follows from the Cacciopoli's inequality and Sobolev–Poincaré inequality. □

3 The VMO Coefficient Case

This section is devoted to the $W^{1,p}$ estimates for second order elliptic systems with VMO coefficients in nonsmooth domains. Armed with the regularity theory of second order elliptic systems in divergence form with constant coefficient in Lipschitz domain [18] (see [14, 19, 25] and the references there), for elliptic systems with VMO coefficient it has become customary to use perturbation method, first introduced in [12], also see [1, 8–11, 13] and [24].

The following theorem was stated in [14] without a proof, we sketch out the proof here. Recall that we say $f\in \mathrm{VMO}(\mathbb{R}^n)$ if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\fint_{B(x,r)}\Big|f-\fint_{B(y,r)}f\,dy\Big|dx=0, \tag{3.1}$$

where $\fint_{B(x,r)}f\,dx$ denotes the L^1 average of f over $B(x,r)$.

Theorem 3.1 *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$, $n\ge 2$ *and* $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$, *where* $\delta>0$. *Let* $\mathcal{L}$ *be a second order elliptic operator of divergence form with real-valued, bounded, measurable coefficients. Also assume that the coefficient matrix* A *is symmetric and in* $\mathrm{VMO}(\mathbb{R}^n)$. *Then there exists a unique* (*up to constants*) $u\in W^{1,p}(\Omega)$ *such that* $\mathcal{L}u=\operatorname{div}f$ *in* Ω *and* $\frac{\partial u}{\partial\nu}=-f\cdot N$ *on* $\partial\Omega$. *Moreover, the* $W^{1,p}$ *estimate*

$$\|\nabla u\|_{L^p(\Omega)}\le C\{\|f\|_{L^p(\Omega)}+\|g\|_{B^{-1/p,p}(\partial\Omega)}\} \tag{3.2}$$

holds with constant C *depends only on* n, p, A *and* Ω.

The following perturbation argument originating in [12] plays an important role in the proof of Theorem 3.1.

Lemma 3.2 *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n(n\ge 2)$. *Assume that* $\mathcal{L}$ *satisfies the same conditions as in Theorem* 3.1. *Let* u *be a* $W^{1,2}$ *solution of* $\mathcal{L}(u)=0$ *in* $B(x_0,8r)\cap\Omega$ *and* $\frac{\partial u}{\partial\nu}=0$ *on* $B(x_0,8r)\cap\partial\Omega$. *Then there exist a function* $\theta(r)$ *such that* $\lim_{r\to 0}\theta(r)=0$ *and a function* $v\in W^{1,p}(B(x_0,r)\cap\Omega)$ *as well as* $p>\frac{2n}{n-1}$ *satisfy*:

$$\Big\{\fint_{B(x_0,r)\cap\Omega}|\nabla v|^p dx\Big\}^{\frac{1}{p}}\le C\Big\{\fint_{B(x_0,8r)\cap\Omega}|\nabla u|^2 dx\Big\}^{\frac{1}{2}}, \tag{3.3}$$

$$\left\{ \fint_{B(x_0,r)\cap\Omega} |\nabla u - \nabla v|^2 dx \right\}^{\frac{1}{2}} \le \theta(r) \left\{ \fint_{B(x_0,8r)\cap\Omega} |\nabla u|^2 dx \right\}^{\frac{1}{2}}. \tag{3.4}$$

Proof Consider

$$\begin{cases} L(v) = 0 & \text{in } \Omega \cap B(x_0, 4r), \\ c_{ij}^{\alpha\beta} \dfrac{\partial v}{\partial x_j} N_i = a_{ij}^{\alpha\beta} \dfrac{\partial u}{\partial x_j} N_i & \text{on } \partial(\Omega \cap \mathrm{B}(\mathrm{x}_0, 4\mathrm{r})), \end{cases} \tag{3.5}$$

where $L = \frac{\partial}{\partial x_i}(c_{ij}^{\alpha\beta} \frac{\partial}{\partial x_j})$ and $c_{ij}^{\alpha\beta} = \fint_{B(x_0,8r)} a_{ij}^{\alpha\beta}(x) dx$ is a constant.

By using $v - u$ as a test function, we derive

$$\int_{\Omega\cap B(x_0,4r)} c_{ij}^{\alpha\beta} \frac{\partial(u-v)^\beta}{\partial x_j} \frac{\partial(u-v)^\alpha}{\partial x_i} dx = \int_{\Omega\cap B(x_0,4r)} (c_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}) \frac{\partial u^\beta}{\partial x_j} \frac{\partial(u-v)^\alpha}{\partial x_i} dx. \tag{3.6}$$

We will show that v satisfies estimates (3.3) and (3.4).

Thus, in view of ellipticity and Cauchy inequality with ε, we have

$$\begin{aligned} &\mu \int_{\Omega\cap B(x_0,4r)} |\nabla(v-u)|^2 dx \\ &\le \int_{\Omega\cap B(x_0,4r)} \sum_{i,j} |a_{ij}^{\alpha\beta} - c_{ij}^{\alpha\beta}| |\nabla u| |\nabla(v-u)| dx \\ &\le C \int_{\Omega\cap B(x_0,4r)} |a_{ij}^{\alpha\beta} - c_{ij}^{\alpha\beta}|^2 |\nabla u|^2 dx + \varepsilon \int_{\Omega\cap B(x_0,4r)} |\nabla(u-v)|^2 dx. \end{aligned} \tag{3.7}$$

It follows from the Hölder inequality and Lemma 2.3, one obtains

$$\begin{aligned} &\left\{ \fint_{\Omega\cap B(x_0,4r)} |\nabla(v-u)|^2 dx \right\}^{\frac{1}{2}} \\ &\le C \left\{ \fint_{\Omega\cap B(x_0,4r)} |\nabla u|^{2q} dx \right\}^{\frac{1}{2q}} \times \left\{ \fint_{\Omega\cap B(x_0,4r)} |a_{ij}^{\alpha\beta} - c_{ij}^{\alpha\beta}|^{2q'} dx \right\}^{\frac{1}{2q'}} \\ &\le \theta(r) \times \left\{ \fint_{\Omega\cap B(x_0,8r)} |\nabla u|^2 dx \right\}^{\frac{1}{2}}, \end{aligned} \tag{3.8}$$

where

$$\theta(r) = C \sup_{x_0 \in \bar\Omega} \sum_{i,j} \left\{ \fint_{\Omega\cap B(x_0,4r)} |a_{ij}^{\alpha\beta} - c_{ij}^{\alpha\beta}|^{2q'} dx \right\}^{\frac{1}{2q'}}$$

and Lemma 2.3 was used in the last inequality. It is also known that the John–Nirenberg inequality implies that $\theta(r) \to 0$ as $r \to 0$ since $a_{ij}^{\alpha\beta} \in \mathrm{VMO}$. This gives (3.4).

Observing that v is a solution to $L(v) = 0$ in $\Omega \cap B(x_0, 4r)$ and $c_{ij}^{\alpha\beta} \frac{\partial v}{\partial x_j} N_i = 0$ on $B(x_0, 4r) \cap \partial\Omega$, by the regularity theory of second order elliptic systems in divergence form with constant coefficients [19], we obtain

$$\left\{ \fint_{B(x_0,r)\cap\Omega} |\nabla v|^p dx \right\}^{\frac{1}{p}} \le C \left\{ \fint_{B(x_0,4r)\cap\Omega} |\nabla v|^2 dx \right\}^{\frac{1}{2}} \le C \left\{ \fint_{B(x_0,8r)\cap\Omega} |\nabla u|^2 dx \right\}^{\frac{1}{2}},$$

where $p = \frac{2n}{n-1} + \delta$. This gives (3.3) and completes the proof. □

Now, we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1 In view of Theorem 2.2, it suffices to show $\mathcal{L}$ satisfies the weak reverse Hölder inequality (2.17) for $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$ if $n\geq 2$. Since that u,v satisfy (3.3)–(3.4), it follows from Theorem 2.1 with $R_B=\nabla v, F=\nabla u$ and $f=0$ as well as $F_B=F-R_B$, we then obtain desired estimate (2.17), thus Theorem 2.2 implies the gradient estimate

$$\|\nabla u\|_{L^p(\Omega)}\leq C\|f\|_{L^p(\Omega)} \tag{3.9}$$

in the case of $p>2$, $g=0$.

The uniform $W^{1,p}$ estimate in the case of $p<2$, and $g\neq 0$ will be obtained by a duality argument (see Section 4 for details), i.e., for any $\frac{2n}{n+1}-\delta<p<\frac{2n}{n-1}+\delta$, we have

$$\|\nabla u\|_{L^p(\Omega)}\leq C\{\|f\|_{L^p(\Omega)}+\|g\|_{B^{-1/p,p}(\partial\Omega)}\}. \tag{3.10}$$

Thus we complete the proof.

4 Proof of Theorem 1.1

In view of Theorem 2.2, it suffices to show that for any weak solution $u_\varepsilon\in W^{1,2}(Z(3r))$ to $\mathcal{L}_\varepsilon u_\varepsilon=0$ in $Z(3r)$ and $\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=0$ on $S(3r)$, then we have $\nabla u_\varepsilon\in L^{p_n}(Z(r))$ and estimate (2.17) holds for $p=p_n=\frac{2n}{n-1}$, i.e.,

$$\left\{⨍_{Z(r)}|\nabla u_\varepsilon|^p\right\}^{\frac{1}{p}}\leq C\left\{⨍_{Z(2r)}|\nabla u_\varepsilon|^2\right\}^{\frac{1}{2}}, \tag{4.1}$$

where the constant C independent of ε.

Lemma 4.1 *Let $u_\varepsilon\in H^1(Z(3r))$ be a weak solution of $\mathcal{L}_\varepsilon u_\varepsilon=0$ in $Z(3r)$ and $\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=0$ in $S(3r)$, where the coefficient $a_{ij}^{\alpha\beta}$ of $\mathcal{L}_\varepsilon$ satisfies* (1.5), (1.6), (1.3) *and* (1.4). *Let $2<p=p_n=\frac{2n}{n-1}$. Then we have*

$$\left(⨍_{Z(r)}|\nabla u_\varepsilon|^p\right)^{\frac{1}{p}}\leq C\left(⨍_{Z(2r)}|\nabla u_\varepsilon|^2\right)^{\frac{1}{2}}. \tag{4.2}$$

Proof We divide the proof into two steps.

Step 1 Let $p=p_1$ be the same as in Lemma 2.3. It follows from the interior estimate and Lemma 2.3 that

$$|\nabla u_\varepsilon|\leq C\left\{⨍_{B(x,cd(x))}|\nabla u_\varepsilon|^{p_1}\right\}^{\frac{1}{p_1}}\leq C\left\{\frac{r}{d(x)}\right\}^{\frac{n}{p_1}}\left\{⨍_{B(x,cr)}|\nabla u_\varepsilon|^2\right\}^{\frac{1}{2}}, \tag{4.3}$$

where $d(x)=|x_n-\psi(x')|$.

It follows from [23] that the L^2 Neumann problem $\mathcal{L}_\varepsilon u_\varepsilon=0$ on Ω and $\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}=g$ on $\partial\Omega$ is uniquely solvable under the assumption that the coefficient A satisfies (1.3)–(1.6); moreover, the solution u_ε satisfies the estimate

$$\left\{\int_{\partial\Omega}|(\nabla u_\varepsilon)^*|^{2+\delta}\right\}^{\frac{1}{2+\delta}}\leq C\left\{\int_{\partial\Omega}\left|\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}\right|^{2+\delta}\right\}^{\frac{1}{2+\delta}}, \tag{4.4}$$

where $\delta>0$ and $(\nabla u_\varepsilon)^*$ denotes the non-tangential maximal function of ∇u_ε. By a perturbation argument on $Z(tr)$ with $t\in(1,2)$, we have

$$\int_{S(r)}|(\nabla u_\varepsilon)^*|^{2+\delta}d\sigma\leq\frac{C}{r}\int_{Z(2r)}|\nabla u_\varepsilon|^{2+\delta}dx. \tag{4.5}$$

Letting $p_0 = 2+\delta$, it follows from Lemma 2.3 and (4.5) that

$$\Big\{ \fint_{S(r)} |(\nabla u_\varepsilon)^*|^{p_0} d\sigma \Big\}^{\frac{1}{p_0}} \le C\Big\{ \fint_{Z(2r)} |\nabla u_\varepsilon|^2 \Big\}^{\frac12}. \tag{4.6}$$

Next, by using (4.6) and (4.3), if $0 < n(p-p_0) < p_1$ we have that

$$\begin{aligned} &\Big\{ \fint_{Z(r)} |\nabla u_\varepsilon|^p dx \Big\}^{\frac1p} \\ &= \Big\{ \fint_{Z(r)} |\nabla u_\varepsilon|^{p_0} |\nabla u_\varepsilon|^{p-p_0} dx \Big\}^{\frac1p} \\ &\le C\Big\{ \frac{1}{r^{n-1}} \int_{S(r)} |(\nabla u_\varepsilon)^*|^{p_0} d\sigma \frac1r \int_0^{cr} (r/t)^{\frac{n(p-p_0)}{p_1}} dt \Big\}^{\frac1p} \cdot \Big\{ \fint_{Z(3r)} |\nabla u_\varepsilon|^2 dx \Big\}^{\frac{p-p_0}{2p}} \\ &\le C\Big\{ \fint_{Z(3r)} |\nabla u_\varepsilon|^2 dx \Big\}^{\frac12}. \end{aligned} \tag{4.7}$$

It is easy to see that $n(p-p_0) < p_1$ implies that $p < 2+\delta+\frac{p_1}{n}$.

Step 2 (Iteration) Assume $p_1 < \frac{2n}{n-1}$, in view of Step 1 and Lemma 2.3, we know that there exists some $p = p_2 > 2+\frac{\delta}{2}+\frac{p_1}{n} > p_1$ such that

$$\Big\{ \fint_{Z(r)} |\nabla u_\varepsilon|^{p_2} \Big\}^{\frac{1}{p_1}} \le C_0\Big\{ \fint_{Z(2r)} |\nabla u_\varepsilon|^2 \Big\}^{\frac12}. \tag{4.8}$$

If $p_2 < \frac{2n}{n-1}$, by Step 1 and Lemma 2.3, we know that there exists some $p = p_3 > 2+\frac{\delta}{2}+\frac{p_1}{n}$ $> p_1$ such that

$$\Big\{ \fint_{Z(r)} |\nabla u_\varepsilon|^{p_3} \Big\}^{\frac{1}{p_3}} \le C_0\Big\{ \fint_{Z(2r)} |\nabla u_\varepsilon|^2 \Big\}^{\frac12}. \tag{4.9}$$

Continuing this process, we claim that there exists some $p = p_j > \frac{2n}{n-1}$ such that

$$\Big\{ \fint_{Z(r)} |\nabla u_\varepsilon|^{p_3} \Big\}^{\frac{1}{p_3}} \le C_0\Big\{ \fint_{Z(2r)} |\nabla u_\varepsilon|^2 \Big\}^{\frac12}. \tag{4.10}$$

Otherwise, we may have a bounded increasing sequence $\{p_j\}$ such that $p_{j+1} > 2+\frac{\delta}{2}+\frac{p_j}{n}$, let p be the limit of p_j, which implies that $p > p_n = \frac{2n}{n-1}$. It follows that $p_j > p_n$ if j sufficiently large. Thus (4.7) holds for some $p = p_j > p_n$. Thus we complete the proof.

We need the following two duality lemmas to finish the proof of Theorem 1.1 in the case of $g \ne 0$ and $p < 2$, as follows.

Lemma 4.2 *Let Ω be a bounded Lipschitz domain and $\frac{2n}{n+1}-\delta < p < \frac{2n}{n-1}+\delta$ and $f \in L^p(\Omega)$. Let $\mathcal{L}_\varepsilon$ be defined as in* (1.2)*. Suppose that A satisfies* (1.3)–(1.6)*. Let $u_\varepsilon \in W^{1,p}(\Omega)$ be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = \mathrm{div} f & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = -f\cdot N & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

Then

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{4.12}$$

where C independent of ε.

Proof The case $p > 2$ was proved in Lemma 4.1. Let $\mathcal{L}_\varepsilon^*$ denote the adjoint operator of $\mathcal{L}_\varepsilon$ and v_ε be a weak solution of $\mathcal{L}_\varepsilon^*(v_\varepsilon) = \operatorname{div}(g)$ in Ω and $\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon^*} = 0$ on $\partial\Omega$ where $g \in L^q(\Omega)$. Suppose that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{2n}{n+1} - \delta < p < 2$. Since A^* is also elliptic, periodic and Hölder continuous, by using integration by parts we have that

$$\bigg| \int_\Omega g_i^\alpha \frac{\partial u_\varepsilon^\alpha}{\partial x_i} dx \bigg| = \bigg| \int_\Omega f_i^\alpha \frac{\partial v_\varepsilon^\alpha}{\partial x_i} dx \bigg|. \tag{4.13}$$

Then the desired estimate (4.12) will be deduced from Lemma 4.1 and the fact

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} = \sup_{\|g\|_{L^q(\Omega)} \le 1} \bigg| \int_\Omega \nabla u_\varepsilon \cdot g dx \bigg|. \tag{4.14}$$

Thus we finish the proof. □

Similar argument will lead to the following lemma and we leave details to the reader.

Lemma 4.3 *Let Ω be a bounded Lipschitz domain and $\frac{2n}{n+1} - \delta < p < \frac{2n}{n-1} + \delta$. Suppose that the coefficients of the operator $\mathcal{L}_\varepsilon$ in* (1.2) *satisfy* (1.3)–(1.6). *Then for any $g^\alpha \in B^{-\frac{1}{p},p}(\partial\Omega)$, there exists a unique* (*up to constants*) *$u_\varepsilon \in W^{1,p}(\Omega)$ such that*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where g satisfies the compatible condition $\langle g, 1\rangle = 0$. Moreover,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C\|g\|_{B^{-\frac{1}{p},p}(\partial\Omega)}, \tag{4.16}$$

where C independent on ε.

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 The desired result follows from Lemmas 4.2 and 4.3. Let v_ε be a weak solution of $\mathcal{L}_\varepsilon(v_\varepsilon) = \operatorname{div} f$ in Ω and $\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = f \cdot N$ on $\partial\Omega$. Let $h_\varepsilon = u_\varepsilon - v_\varepsilon$. Then $\mathcal{L}_\varepsilon(h_\varepsilon) = 0$ in Ω and $\frac{\partial h_\varepsilon}{\partial \nu_\varepsilon} = g$. Hence

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^p(\Omega)} &\le \|\nabla h_\varepsilon\|_{L^p(\Omega)} + \|\nabla v_\varepsilon\|_{L^p(\Omega)} \\ &\le C\{\|g\|_{B^{-\frac{1}{p},p}(\partial\Omega)} + \|f\|_{L^p(\Omega)}\}. \end{aligned} \tag{4.17}$$

Thus we finish the proof.

5 Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, in view of Theorem 2.2, it suffices to show that the weak reverse Hölder inequality

$$\bigg\{ \fint_{Z(r)} |\nabla u_\varepsilon|^p \bigg\}^{\frac{1}{p}} \le C \bigg\{ \fint_{Z(2r)} |\nabla u_\varepsilon|^2 \bigg\}^{\frac{1}{2}} \tag{5.1}$$

holds for $\frac{3}{2} - \delta < p < 3 + \delta$ if $n \ge 4$ with the constant C independent of ε whenever $u_\varepsilon \in W^{1,2}(B(x_0, 2r) \cap \Omega)$ satisfies $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, 2r) \cap \Omega$ and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $B(x_0, 2r) \cap \partial\Omega$ (if $x_0 \in \partial\Omega$).

Proof of Theorem 1.2 We divide the proof into two parts.

Step 1 ($p > 2$ and $g = 0$) For $x = (x', x_n) \in Z(2r)$, let $d(x) = |x_n - \psi(x')|$. It follows from the interior estimate Lemma 1.15, De Giorgi–Nash estimate and Poincaré inequality, we have

$$|\nabla u_\varepsilon| \leq C\Big\{\frac{d(x)}{r}\Big\}^{\eta-1}\Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dy\Big\}^{\frac{1}{2}}, \tag{5.2}$$

where $\eta > 0$.

Since A satisfies (1.3)–(1.6), it follows from the L^2 estimate (see [22]) that

$$\|(\nabla u_\varepsilon)^*\|_{L^2(\partial\Omega)} \leq C\Big\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\Big\|_{L^2(\partial\Omega)}. \tag{5.3}$$

Integrating with respect to $t \in (1, 2)$ on $Z(tr)$, we obtain

$$\int_{S(r)} |(\nabla u_\varepsilon)_r^*|^2 d\sigma \leq \frac{C}{r}\int_{Z(2r)} |\nabla u_\varepsilon|^2 dx, \tag{5.4}$$

here $(\nabla u_\varepsilon)_r^*(x', \psi(x')) = \sup\{|\nabla u_\varepsilon(x', x_n)| : (x', x_n) \in Z(tr)\}$.

Note that if $(p-2)(\eta-1) > -1$, we have

$$\begin{aligned}
\int_{Z(r)} |\nabla u_\varepsilon|^p dx &\leq \frac{C}{r^{(p-2)(\eta-1)}}\int_{Z(r)} |\nabla u_\varepsilon|^2 (d(x))^{(p-2)(\eta-1)} dx \Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dx\Big\}^{\frac{p}{2}-1}\\
&\leq \frac{C}{r^{(p-2)(\eta-1)}}\int_{S(r)} |(\nabla u_\varepsilon)_r^*|^2 d\sigma \int_0^{cr} t^{(p-2)(\eta-1)} dt \Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dy\Big\}^{\frac{p}{2}-1}\\
&\leq Cr\int_{S(r)} |(\nabla u_\varepsilon)_r^*|^2 d\sigma \Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dy\Big\}^{\frac{p}{2}-1},
\end{aligned} \tag{5.5}$$

where (5.2) was used in the first inequality.

Plugging (5.4) into (5.5), we obtain the desired estimate

$$\Big\{\fint_{Z(r)} |\nabla u_\varepsilon|^p dx\Big\}^{\frac{1}{p}} \leq C\Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dx\Big\}^{\frac{1}{2}} \tag{5.6}$$

for $2 < p < 2 + \frac{1}{1-\eta}$. Typically, we may take $p = 3$. This gives the desired result for $n \geq 4$.

Step 2 (Duality) Theorem 1.2 was partially proved if $p > 2$ and $g = 0$ in Step 1, the remaining proof of Theorem 1.2 in the case of $p < 2$ and $g \neq 0$ could be deduced from the duality argument as in Section 4. Thus we complete the proof.

6 Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3, in view of Theorem 2.2, it suffices to show that the weak reverse Hölder inequality

$$\Big\{\fint_{Z(r)} |\nabla u_\varepsilon|^p\Big\}^{\frac{1}{p}} \leq C\Big\{\fint_{Z(2r)} |\nabla u_\varepsilon|^2\Big\}^{\frac{1}{2}} \tag{6.1}$$

holds for $2 < p < \infty$ if $n \geq 2$ with the constant C independent of ε whenever $u_\varepsilon \in W^{1,2}(B(x_0, 2r) \cap \Omega)$ satisfies $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $B(x_0, 2r) \cap \Omega$ and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $B(x_0, 2r) \cap \partial\Omega$ (if $x_0 \in \partial\Omega$).

We first establish the $W^{1,p}$ regularity result of Neumann problem for second order elliptic equation with constant coefficients in convex domains, as follows.

Lemma 6.1 *Let $2 < p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain. Suppose that u is a weak solution of $\mathcal{L}u = 0$ in $B(x_0, 2r) \cap \Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $B(x_0, 2r) \cap \partial\Omega$, where $\mathcal{L} = -\mathrm{div}(A\nabla)$ with $A = (a_{ij})$ being constant and $A = A^*$. Then*

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u|^p\right)^{\frac{1}{p}} \leq C_0 \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla u|^2\right)^{\frac{1}{2}}, \tag{6.2}$$

where the constant $\delta > 0$ and C_0 depends only on n, the Lipschitz character of Ω and the ellipticity constant μ.

Proof Since A is constant, symmetric and positive definite, by a change of the coordinate system, we may assume that $\mathcal{L} = \Delta$. (6.2) was shown in [15] in the case of Laplace equation subject to Neumann boundary and we outline the proof here for the sake of completeness.

Let $g = |\mathrm{v}|^2$ where v is a C^2 vector which to be determined later, and assume that $\mathrm{v} \cdot n = 0$ on $\partial\Omega$. Set Φ defined on $[0, \infty)$ as a Lipschitz function. Use integration by parts twice we obtain that

$$\begin{aligned}
&\int_\Omega \Phi(|\mathrm{v}|^2)\left\{\{\mathrm{div}(\mathrm{v})\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\right\} \\
&\quad = \int_{\partial\Omega} \Phi(|\mathrm{div}|^2)\left\{n_i v_i \mathrm{div}(\mathrm{v}) - n_j v_i \frac{\partial v_j}{\partial x_i}\right\} \\
&\qquad + 2\int_\Omega \Phi'(|\mathrm{v}|^2)\left\{v_k \frac{\partial v_k}{\partial x_j} \cdot v_i \cdot \frac{\partial v_j}{\partial x_i} - v_k \frac{\partial v_k}{\partial x_i} \cdot v_i \cdot \mathrm{div}(\mathrm{v})\right\}.
\end{aligned} \tag{6.3}$$

Notice that $\mathrm{v} \cdot n = 0$, combine with the convexity $-\beta(v_T; v_T) = n_i v_i \mathrm{div}(\mathrm{v}) - n_j v_i \frac{\partial v_j}{\partial x_i} \geq 0$, i.e., $\beta(v_T; v_T) \leq 0$ on $\partial\Omega$, and choose

$$\Phi = \begin{cases} 1 & s \geq \tau, \\ \text{linear} & t \leq s \leq \tau, \\ 0 & s \leq t, \end{cases}$$

this, together with (6.3) as well as some direct computation leads that

$$\begin{aligned}
\frac{1}{2(\tau - t)} \int_{t<|g|^2<\tau} |\nabla g|^2 &\leq \frac{1}{(\tau - t)} \int_{t<g<\tau} |\nabla g||\mathrm{v}||\mathrm{div}(\mathrm{v})| \\
&\quad + \frac{1}{(\tau - t)} \int_{t<g<\tau} |\nabla g||\mathrm{v}|\left\{\left|\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right|^2\right\}^{1/2} \\
&\quad + \int_{g>t} \Phi(g)\left\{\{\mathrm{div}(\mathrm{v})\}^2 - \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial v_j}{\partial x_i}\right\}.
\end{aligned} \tag{6.4}$$

It follows from the co-area formula and Lebesgue's differential theorem, and by choosing $\psi \in C_0^\infty(B(x_0, 2r))$ and take $\mathrm{v} = (\nabla u)\psi$, thus one may notice that $\mathrm{v} \cdot n = 0$, and some computation shows that

$$\int_{g=t} |\nabla g| d\sigma \leq Ct^{1/2} \int_{g=t} |\nabla u||\nabla\psi| + C\int_{g>t} |\nabla u|^2 |\nabla\psi|^2. \tag{6.5}$$

To this end, we use the co-area formula repeatedly, and in view of convexity of Ω as well as weighted Hölder inequality we obtain that for $p > 1$,

$$\int_\Omega |g|^q dx \leq C\left\{\int_\Omega |f|^{2p} dx\right\}^{q/p} + C|\Omega|^{1-q}\left\{\int_\Omega |g| dx\right\}^q, \tag{6.6}$$

where $\frac{1}{q} = \frac{1}{p} - \frac{2}{d}$.

Finally, the desired estimate (6.2) follows from simple iteration argument, we refer the readers to [15]. □

Remark 6.2 It worths pointing out that in view of the Sobolev embedding, the $W^{1,p}$ estimate implies the uniform boundary Hölder estimate for any $0 < \eta < 1$. Let A, Ω and u satisfy the same assumption as in Lemma 6.1. Then it is not hard to see that for any $x, y \in Z(r/2)$,

$$|u(x) - u(y)| \leq C\Big(\frac{|x-y|}{r}\Big)^{\eta}\Big(\fint_{Z(r)} |u|^2\Big)^{1/2}, \tag{6.7}$$

where the constant C depends only on μ, λ, τ, η and Ω.

Next theorem provide an approximation process for the weak solution of second order elliptic equation with VMO coefficients by the constant coefficients regularity results in convex domains.

Theorem 6.3 *Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$ and $2 < p < \infty$. Let $\mathcal{L}$ be a second order elliptic operator of divergence form with real-valued, bounded, measurable coefficients. Also assume that the coefficient matrix A is symmetric and in* VMO$(\mathbb{R}^n)$. *Then there exists a unique (up to constants) $u \in W^{1,p}(\Omega)$ such that $\mathcal{L}u = \mathrm{div} f$ in Ω and $\frac{\partial u}{\partial \nu} = -f \cdot N + g$ on $\partial\Omega$. Moreover, the $W^{1,p}$ estimate*

$$\|\nabla u\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|g\|_{B^{-1/p,p}(\partial\Omega)}\} \tag{6.8}$$

holds with constant C depending only on n, p, A and Ω.

Proof The proof is essentially the same as Theorem 3.1. Letting u be a $W^{1,2}$ solution of $\mathcal{L}(u) = 0$ in $B(x_0, 8r) \cap \Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $B(x_0, 8r) \cap \partial\Omega$, by Lemma 6.1 and a real-variable perturbation argument, then for any $2 < p < \infty$, there exist a function $\theta(r)$ such that $\lim_{r\to 0} \theta(r) = 0$ and a function $v \in W^{1,p}(B(x_0, r) \cap \Omega)$ satisfying

$$\Big\{\fint_{B(x_0,r)\cap\Omega} |\nabla v|^p dx\Big\}^{\frac{1}{p}} \leq C\Big\{\fint_{B(x_0,8r)\cap\Omega} |\nabla u|^2 dx\Big\}^{\frac{1}{2}}, \tag{6.9}$$

$$\Big\{\fint_{B(x_0,r)\cap\Omega} |\nabla u - \nabla v|^2 dx\Big\}^{\frac{1}{2}} \leq \theta(r)\Big\{\fint_{B(x_0,8r)\cap\Omega} |\nabla u|^2 dx\Big\}^{\frac{1}{2}}. \tag{6.10}$$

This, combining with Lemma 2.1 and duality argument, leads to the desired estimates. □

Theorem 6.4 *Let Ω, A and $\mathcal{L}$ be the same as in Theorem 6.3. Assume $2 < p < \infty$. Let u be a weak solution of $\mathcal{L}u = 0$ in $Z(2r)$ and $\frac{\partial u}{\partial \nu} = 0$ on $S(2r)$. Then for any $x, y \in Z(r)$,*

$$|u(x) - u(y)| \leq C\Big(\frac{|x-y|}{r}\Big)^{\eta}\Big(\fint_{Z(r)} |u|^2\Big)^{1/2}, \tag{6.11}$$

where the constant C depends only on μ, λ, τ, η and Ω.

Proof The proof directly follows from Theorem 6.3 by Sobolev embedding. □

Armed with Remark 6.2 and Theorem 6.4, we are ready to investigate the uniform boundary C^{η} estimate in convex domains by using the compensated compactness argument, which originated by Avellaneda and Lin in [2] and further developed by Kenig et al. in [21].

Theorem 6.5 *Let $0 < \eta < 1$ and $Z(r)$ and $S(r)$ defined by* (1.18) *and* (1.19) *be convex. Suppose that $\mathcal{L}_\varepsilon$ be defined as in* (1.2) *with A satisfying* (1.3), (1.4), (1.5) *and* (1.6). *Suppose* *that*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } Z(2r), \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0 & \text{on } S(2r) \end{cases}$$

Then for any $x, y \in Z(r)$,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\Big(\frac{|x-y|}{r}\Big)^\eta \Big(\fint_{Z(r)} |u_\varepsilon|^2\Big)^{1/2}, \tag{6.12}$$

where the constant C depends only on μ, λ, τ, η and Ω.

Proof It was essentially proved in [21], we only point out the necessary modification here for the convenience. To show (6.12), in view of the Campanoto's characterization of Hölder spaces, it suffices for us to show that

$$\fint_{Z(r)} |u_\varepsilon - \fint_{Z(r)} u_\varepsilon|^2 \le C r^{2\eta} \fint_{Z(1)} |u_\varepsilon|^2 \quad \text{for } 0 < r < \frac{1}{4}. \tag{6.13}$$

The proof (6.13) will follows from the elegant three step compactness method. The first step is the improvement. To proceed our proof, let $\{\psi_k\}$ be a sequence of convex domain and one may notice that $\psi_k \to \psi_0$ in $|x'| < 1$ where ψ_0 is a convex domain. Assume that there exists sequences $\{\varepsilon_k\}, \{A_k\}, \{u_{\varepsilon_k}\}$ such that u_{ε_k} satisfies

$$\begin{cases} \mathcal{L}^k_{\varepsilon_k}(u_{\varepsilon_k}) = 0 & \text{in } Z_k(1), \\ \dfrac{\partial u_{\varepsilon_k}}{\partial \nu_{\varepsilon_k}} = g & \text{on } S_k(1), \end{cases} \tag{6.14}$$

where $\mathcal{L}^k_{\varepsilon_k} = -\mathrm{div}(A_k(x/\varepsilon_k)\nabla)$, $Z_k(r) = Z(r, \psi_k)$ and $S_k(r) = S(r, \psi_k)$. It is known that, as $\varepsilon_k \to 0$, the equations (6.14) uniformly converge to

$$\begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } Z(1/2, \psi_0), \\ \dfrac{\partial u_0}{\partial \nu_0} = g & \text{on } S(1/2, \psi_0), \end{cases}$$

where ψ_0 is convex and $\mathcal{L}_0$ is a second order elliptic operator with constant coefficient. Thus, it follows from Remark 6.2 that u_0 satisfies boundary Hölder estimates (6.7). This, by passing to a subsequence, combining with a limit argument will give the improvement step, we refer the reader to [21] for more details. The iteration step is exactly the same as in [21].

To this end, to utilize the blow up argument, the key step is to show the boundary Hölder estimate for second order elliptic equations with VMO coefficients in convex domains, and this was given in Theorem 6.4. Thus we complete the proof. □

Now, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3 The proof follows the same line as the proof of Theorem 1.2. We first assume that $g = 0$ and $p > 2$. Letting $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, by the interior estimate (1.15) and Theorem 6.5, for any $\eta \in (0, 1)$, we obtain that

$$\begin{aligned} |\nabla u_\varepsilon(x)| &\le C \frac{1}{\delta(x)} \Big(\fint_{B(x, c\delta(x))} |u_\varepsilon(y) - u_\varepsilon(x)|^2 dy\Big)^{1/2} \\ &\le C\Big(\frac{r}{\delta(x)}\Big)^{1-\eta} \Big(\fint_{Z(2r)} |\nabla u_\varepsilon|^2 dy\Big)^{1/2}. \end{aligned} \tag{6.15}$$

By choosing suitable η such that $1-\frac{1}{p}<\eta<1$, and integrating both sides on $Z(r)$, we have

$$\fint_{Z(r)}\left(\fint_{B(x,c\delta(x))}|\nabla u_\varepsilon|^p dy\right)dy \le C\|\nabla u_\varepsilon\|^p_{L^2(Z(2r))}. \tag{6.16}$$

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References

[1] Auscher, P., Qafsaoui, M.: Observation on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients. *Boll. Unione Mat. Ital. Sez. B Artic. Ric.*, **5**(7), 487–509 (2002)

[2] Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, **40**, 803–847 (1987)

[3] Avellaneda, M., Lin, F.: Homogenization of elliptic problems with L^p boundary data. *Appl. Math. Optim.*, **15**, 93–107 (1987)

[4] Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. ii. equations in nondivergence form. *Comm. Pure Appl. Math.*, **42**, 139–172 (1989)

[5] Avellaneda, M., Lin, F.: Homogenization of poisson's kernel and applications to boundary control. *J. Math. Pures Appl.*, **68**, 1–29 (1989)

[6] Avellaneda, M., Lin, F.: L^p bounds on singular in homogenization. *Comm. Pure Appl. Math.*, **44**, 897–910 (1991)

[7] Bensoussan, A., Lions, J. L., Papanicolaou, G. C.: Asymptotic Analysis for Periodic Structures, North Hollan, Providence, RI, 1978

[8] Byun, S.: Elliptic equations with BMO coefficients in Lipschitz domains. *Trans. Amer. Math. Soc.*, **357**(3), 1025–1046 (2005)

[9] Byun, S., Wang, L.: Elliptic equations with bmo coefficients in Reifenberg domains. *Comm. Pure Appl. Math.*, **57**(10), 1283–1310 (2004)

[10] Byun, S., Wang, L.: The conormal derivative problem for elliptic equations with BMO coefficients on Reifenberg flat domains. *Proc. London Math. Soc.*, **90**(3), 245–272 (2005)

[11] Byun, S., Wang, L.: Gradient estimates for elliptic systems in non-smooth domains. *Math. Ann.*, **341**(3), 629–650 (2008)

[12] Caffarelli, L. A., Peral, I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.*, **51**, 1–21 (1998)

[13] Dong, H. J., Kim, D.: Elliptic equations in divergence form with partially BMO coefficients. *Arch. Rational Mech. Anal.*, **196**, 25–70 (2010)

[14] Geng. J.: $W^{1,p}$ estimates for elliptic problems with neumann boundary conditions in Lipschitz domains. *Adv. Math.*, **229**, 2427–2448 (2007)

[15] Geng, J., Shen, Z.: The Neumann problem and helmholtz decomposition in convex domains. *J. Funct. Anal.*, **259**, 2147–2164 (2010)

[16] Geng, J., Shen, Z., Song, L.: Uniform $W^{1,p}$ estimates for systems of linear elasticity in a periodic medium. *J. Funct. Anal.*, **262**, 1742–1758 (2012)

[17] Giaquinta, M.: Multiple Intergrals in the Calculus of Variations and Nonlinear Elliptic Systems, **105** of Ann. of Math. Studies, Princeton Univ. Press, Princeton, 1983

[18] Jerison, D., Kenig, C.: The Neumann problem in Lipschitz domains. *Bull. Amer. Math. Soc.* (*N.S.*), **4**, 203–207 (1981)

[19] Jerison, D., Kenig, C.: The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, **130**, 161–219 (1995)

[20] Kenig, C., Lin, F., Shen, Z.: Estimates of eigenvalues and eigenfunctions in periodic homogenization. *J. Eur. Math. Soc.*, **15**, 1901–1925 (2013)

[21] Kenig, C., Lin, F., Shen, Z.: Homogenization of elliptic systems with Neumann boundary conditions. *J. Amer. Math. Soc.*, **26**, 901–937 (2013)

[22] Kenig, C., Shen, Z.: Homogenization of elliptic boundary value problems in Lipschitz domains. *Math. Ann.*, **350**, 867–917 (2011)

[23] Kenig, C., Shen, Z.: Layer potential methods for elliptic homogenization problems. *Comm. Pure Appl. Math.*, **64**, 1–44 (2011)

[24] Krylov, N. V.: Parabolic and elliptic equations with vmo coefficients. *Comm. Partial Diff. Eq.*, **32**, 453–475 (2007)

[25] Shen, Z.: The L^p boundary value problems on Lipschitz domains. *Adv. Math.*, **216**, 212–254 (2007)

[26] Shen, Z.: The $W^{1,p}$ estiamtes for elliptic homogenization problems in nonsmooth domains. *Indiana Univ. Math. J.*, **57**, 2283–2298 (2008)