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Solutions of the Generalized Lennard-Jones System

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Abstract In this paper, we study solution structures of the following generalized Lennard-Jones system in $\mathbb{R}^n$,

$$\ddot{x} = \left(-\frac{\alpha}{|x|^{\alpha+2}} + \frac{\beta}{|x|^{\beta+2}} \right) x,$$

with $0 < \alpha < \beta$. Considering periodic solutions with zero angular momentum, we prove that the corresponding problem degenerates to 1-dimensional and possesses infinitely many periodic solutions which must be oscillating line solutions or constant solutions. Considering solutions with non-zero angular momentum, we compute Morse indices of the circular solutions first, and then apply the mountain pass theorem to show the existence of non-circular solutions with non-zero topological degrees. We further prove that besides circular solutions the system possesses in fact countably many periodic solutions with arbitrarily large topological degree, infinitely many quasi-periodic solutions, and infinitely many asymptotic solutions.

Keywords Generalized Lennard-Jones system, mountain pass solutions, periodic solutions, quasi-periodic solutions, asymptotic solutions

MR(2010) Subject Classification 34C25, 58E05, 70H12, 92E20

1 Introduction

Periodic solutions of the generalized Lennard-Jones Hamiltonian system (1.2) have been extensively studied using variational methods since 1980s (see [1, 4, 5, 7–9, 16, 18, 21], and the

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references therein). Found by Lennard-Jones [13] in 1924, the Lennard-Jones potential is one of the commonly used potentials in the molecular dynamics. For the N-body case with $N \geq 3$, we refer to interesting works of [5, 7, 8], and [21].

In the 2-body case, Lennard-Jones potential describes the interaction between two atoms or molecules. The interaction consists with two parts, repulsion and attraction. It is defined by

$$V_{\mathrm{LJ}} = d\left[\left(\frac{\sigma}{|y_1 - y_2|}\right)^{12} - \left(\frac{\sigma}{|y_1 - y_2|}\right)^{6}\right],$$

where $d/4$ is the depth of the potential well, σ is the finite distance at which the inter-particle potential disappears, and $|y_1 - y_2|$ is the distance between the two particles. The first term $(\frac{\sigma}{|y_1-y_2|})^{12}$ is the repulsive term describing Pauli repulsion; and the second term $-(\frac{\sigma}{|y_1-y_2|})^{6}$ is the attraction in the long range describing the van der Waals force ([18] for accurate quantum chemistry computations).

We re-scale V_{LJ} by the unit mass, unit length and fix one particle at the origin of $\mathbb{R}^n$ with $n \geq 2$, and consider the following generalized Lennard-Jones potential as in [4, 16] and [18],

$$U_{\mathrm{LJ}}(y) = \frac{b}{|y|^{\beta}} - \frac{a}{|y|^{\alpha}}, \tag{1.1}$$

where $0 < \alpha < \beta$, a and $b > 0$ are fixed constants, $|y|$ denotes the norm of $y \in \mathbb{R}^n$. The corresponding equation system of the motion is given by

$$y_{ss} = -\nabla U_{\mathrm{LJ}}(y) = \left(-\frac{a\alpha}{|y|^{\alpha+2}} + \frac{b\beta}{|y|^{\beta+2}}\right)y, \tag{1.2}$$

where y_{ss} is the second order derivative of $y = y(s)$ with respect to s. We introduce $\lambda = (\frac{b}{a})^{\frac{1}{\beta-\alpha}}$ and $\mu = \sqrt{\lambda^{2+\beta}/b}$ and let $y = \lambda x$, $s = \mu t$. Then (1.2) is reduced to

$$\ddot{x} = -\nabla U(x) = \left(-\frac{\alpha}{|x|^{\alpha+2}} + \frac{\beta}{|x|^{\beta+2}}\right)x, \tag{1.3}$$

where $\ddot{x}$ is the second order derivative of $x = x(t)$ with respect to t and $U(x) = \frac{1}{|x|^{\beta}} - \frac{1}{|x|^{\alpha}}$.

After the pioneer work [19] of Rabinowitz in 1978, many contributions have been devoted to the singular Hamiltonian and Lagrangian systems via the variational method. To set up the variational structure, in this paper, for $\tau > 0$, let $X_\tau = W^{1,2}(S_\tau, \mathbb{R}^n)$ with $S_\tau = \mathbb{R}/(\tau\mathbb{Z})$, equipped with the usual $W^{1,2}$-norm $\|\cdot\|_1$ given by

$$\|x\|_1 = \left(\int_0^{\tau} (|\dot{x}|^2 + |x|^2)dt\right)^{\frac{1}{2}},$$

for $x \in X_\tau$. We denote the usual L^2-norm by $\|x\|_{L^2} = (\int_0^{\tau} |x|^2 dt)^{\frac{1}{2}}$. The action functional on X_τ corresponding to the system (1.3) is defined by

$$f_\tau(x) = \int_0^{\tau} \left(\frac{1}{2}|\dot{x}|^2 - U(x)\right)dt, \quad \forall\, x \in X_\tau. \tag{1.4}$$

Then it is well known that critical points of f_τ on X_τ correspond to τ-periodic solutions of (1.3).

The S_τ-action on every $x \in X_\tau$ is defined by $\theta \cdot x(t) = x(\theta + t)$ as usual for $\theta \in S_\tau$. The $\mathbb{Z}_2$-action on every $x \in X_\tau$ is defined by $0 * x(t) = x(t)$ and $1 * x(t) = x(\tau - t)$ as usual for 0 and $1 \in \mathbb{Z}_2$. The $O(n)$-action on every $x \in X_\tau$ is defined by $(Mx)(t) = Mx(t)$ for $M \in O(n)$. Note that $f_\tau(x)$ is invariant under these group actions, i.e., $f_\tau(\theta \cdot x) = f_\tau(1 * x) = f_\tau(Mx) = f_\tau(x)$ holds for any $\theta \in S_\tau$, $M \in O(n)$ and $x \in X_\tau$.

In [9] of 1988, Coti Zelati used Morse index theory to study a family of Lagrangian systems with effective-like potential V satisfying Gordon's strong force condition ([10]) and that the set $Z_v \equiv \{x \in \mathbb{R}^n \mid \nabla V(x) = 0\}$ is finite and consists of only non-degenerate critical points. In [9] he proved that there exists a constant $\tau_0 > 0$ such that the system possesses a non-constant τ-periodic solution for every $\tau \geq \tau_0$. Note that the system (1.3) satisfies his conditions except that on Z_v. In [22] of 1990, Solimini studied periodic solutions of the forced singular systems, where Theorem 3 in [22] with $h = 0$ includes (1.3) and proved the existence of τ-periodic solutions for every $\tau > 0$. In [23] of 1993, Terracini proved the existence and multiplicity of the periodic solutions of repulsive systems in $\mathbb{R}^n$ with the singular set $\mathcal{K}_c$ containing at least two points. In Ambrosetti and Coti Zelati [1, Chapter 9] a family of Lagrangian systems which is more general than (1.2) with the strong force condition, i.e., $\beta \geq 2$ there, was studied by mountain pass theorem and the existence of at least one τ-periodic solution of (1.2) was proved for every $\tau > 0$. In [7, 8] of 2004, Corbera et al. studied the 2 and 3-body problems of the Lennard-Jones system with $\alpha = 6$, $\beta = 12$, $a = 2$, and $b = 1$, and specially its constant and circular solutions and central configurations. In [4] of 2011, Bărbosu et al. studied the system (1.2) in $\mathbb{R}^2$ with $2 < \alpha < \beta$ on the intersections of the level surfaces of the conserved Hamiltonian energy and angular momentum. They described the behavior of constant solutions, circular solutions, line periodic solutions and other non-periodic solutions by analyzing the flow defined by the system (1.2). In [21] of 2014, Sbano and Southall proved the existence of τ-periodic solutions of the N-body problems of the Lennard-Jones system for sufficiently large $\tau > 0$ when the potential satisfies the strong force condition $2 < \alpha < \beta$. In the strong force case, they also proved that no τ-periodic solution exists when the period τ is too small. In [16] of 2015, Llibre and Long studied the system (1.2) in $\mathbb{R}^n$. Besides the constant solutions they characterized the circular solutions for $0 < \alpha < \beta$, $a > 0$, and $b \in \mathbb{R}$, and proved that $\tau/2$-antiperiodic solutions of (1.2) in $\mathbb{R}^n$ do not exist when $\tau > 0$ is too small for $2 < \alpha < \beta$ and $a, b > 0$.

This paper is devoted to the understanding of the solution structure of the system (1.3) in $\mathbb{R}^n$ for all $0 < \alpha < \beta$, including specially the case of weak forces in the sense of Gordon [10].

In Section 2, we notice that if $x(t)$ is a classical solution of (1.3) on the open interval (t_1, t_2), $x(t)$ must be a classical solution on the closed interval $[t_1, t_2]$ and each solution of the system (1.3) must be contained in a 2-dimensional sub-plane $P(x)$ of $\mathbb{R}^n$ passing through the origin such that $x \in P(x)$ holds always. Therefore, we restrict our study from $\mathbb{R}^n$ to $\mathbb{R}^2$ in Sections 3 to 5. We can also obtain $\tau_0 > 0$ such that no periodic solutions exist when $\tau < \tau_0$.

In Section 3, by computing the Morse indices and nullities of circular solutions of (1.3) mentioned in [16], when $1 < \alpha < \beta$ we obtain an explicit constant $\tau_1^\# > 0$ such that when $\tau > \tau_1^\#$ the action functional f_τ at one of the circular solutions which we denote by x_2 possesses Morse index $i(x_2) = 0$ and nullity $\nu(x_2) = 1$, which is produced by the S^1-invariance of f_τ. All the other circular solutions of (1.3) possess Morse indices at least 5 when $\tau \geq \tau_1^\#$. Therefore the S^1-orbit $S_\tau \cdot x_2$ of x_2 forms a strictly local minimal non-degenerate critical manifold of f_τ in Λ_*. Here, Λ_* consists of all τ-periodic curves x in $W^{1,2}$ satisfying $\deg(x, 0) \neq 0$ and $x(t) \neq 0$ for all $t \in S_\tau$. Therefore in Theorem 3.8 below, for $\tau > \tau_1^\#$ we can apply the mountain pass theorem at $S_\tau \cdot x_2$ and obtain a τ-periodic solution $x = x(t)$ of (1.3), which satisfies $f_\tau(x) > f_\tau(x_2)$ and possesses Morse index $i(x) \le 1$. Consequently x is not a circular solution of (1.3).

When the angular momentum $c = 0$, the system (1.3) degenerates to a 1-dimensional problem. In Theorem 4.1 below we study the family of 1-dimensional solutions of (1.3) which we call *oscillating line solutions*. Such a periodic solution always oscillates periodically on a ray emanating from the origin $0 \in \mathbb{R}^2$ but never touches the origin 0. Specially we prove that there exists a constant $\tau_{\rm os} > 0$ such that the period τ of any oscillating line solution must satisfy $\tau \ge \tau_{\rm os}$. Since many classical books such as [2] and [15] have already provided the framework for central force potential, we only list related results in Section 4.

Note that our above results already show that when $\tau > 0$ is small enough, the τ-periodic solution of the system (1.3) found by Ambrosetti and Coti Zelati in Theorem 9.1 of [1] must be constant solutions (also Remark 9.5 of [1]). Note that the system (1.3) with the potential function $U(x)$ satisfies the conditions of Theorem 3 of [22] of Solimini in 1990 with $h \equiv 0$. Thus for the period $\tau > 0$ small enough, Theorem 3 with $h = 0$ in [22] yields also only constant solutions of the system (1.3).

In Section 5, we study the solution structure of the system (1.3) when the angular momentum $c \ne 0$. By fixing the angular momentum, we obtain that under certain conditions the system possesses at least countably many periodic solutions with arbitrarily large topological degree and infinitely many quasi-periodic solutions.

Furthermore, we also prove the existence of the so called *asymptotic solution* $x = x(t)$ of (1.3), which is asymptotic to two rays $L_\pm(x)$ starting from the origin in the plane $P(x)$ as $t \to \pm\infty$ respectively. The minimal angle covered by $x(t)$ between $L_+(x)$ and $L_-(x)$ is called the asymptotic angle of the solution x. Because the system (1.3) is rotational invariant in the plane $P(x)$, the asymptotic property of x is determined by its asymptotic angle and its minimal distance from the origin $\min_{t\in\mathbb{R}} |x(t)|$.

In this paper, we use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ to denote the sets of the positive integers, integers, real numbers, positive real numbers and complex numbers respectively.

2 Non-collisions and Planar Motions

Let $x = x(t)$ be a classical solution of the system (1.3). For all t in the domain of x, the Hamiltonian energy $H(x, \dot{x})$ is defined by

$$H(x, \dot{x}) = \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{|x(t)|^\beta} - \frac{1}{|x(t)|^\alpha}. \tag{2.1}$$

It is well known that the Hamiltonian energy $H(x, \dot{x})$ is conserved if $x = x(t)$ is a solution of (1.3). By this property, we have the following proposition.

Proposition 2.1 (i) *Suppose $x \in C^2((t_1, t_2), \mathbb{R}^n)$, $x(t) \ne 0$ for all $t \in (t_1, t_2)$ and satisfies* (1.3) *on (t_1, t_2). Then $x(t)$ can be extended to a function in $C^2([t_1, t_2], \mathbb{R}^n)$ such that $x(t) \ne 0$ for all $t \in [t_1, t_2]$ and satisfies* (1.3) *on $[t_1, t_2]$. Consequently, any solution $x(t)$ satisfying* (1.3) *on an open interval can be extended to all $t \in \mathbb{R}$.*

(ii) *There exists a 2-dimensional sub-plane $P(x) \subset \mathbb{R}^n$ passing through the origin* 0 *such that $x(t) \in P(x)$ holds for all $t \in \mathbb{R}$.*

Proof (i) Since $x(t)$ is a classical solution of (1.3) on (t_1, t_2), Hamiltonian energy $H(x(t), \dot{x}(t))$

is a constant on (t_1, t_2), i.e., for $t \in (t_1, t_2)$,

$$\begin{aligned} \text{const.} \equiv H(x(t), \dot{x}(t)) &= \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{|x(t)|^\beta} - \frac{1}{|x(t)|^\alpha} \\ &\geq \frac{1}{|x(t)|^\beta} - \frac{1}{|x(t)|^\alpha}. \end{aligned} \tag{2.2}$$

If $\liminf_{t\to t_1+} |x(t)| = 0$, the right side of (2.2) approaches $+\infty$ along a subsequence. It contradicts the conservation of Hamiltonian and thus there exists $\delta > 0$ such that $|x(t)| > \delta$ for all $t \in (t_1, \frac{1}{2}(t_1 + t_2)]$. Since the right side of (1.3) is bounded on $\{x \in \mathbb{R}^n : |x| \geq \delta\}$, we obtain that $x(t)$ is Cauchy as $t \to t_1+$ and thus $\lim_{t\to t_1+} x(t) \neq 0$ exists. Similarly, $\lim_{t\to t_2-} x(t) \neq 0$ also exists. A standard continuation argument allows $x(t)$ to be extended to a solution $x(t)$ for all $t \in \mathbb{R}$.

(ii) The motion is contained in the plane $P(x)$ which is spanned by initial conditions $x(0)$ and $\dot{x}(0)$ in central force field. □

By Proposition 2.1, we restrict our attention to $x \in C^2(\mathbb{R}, \mathbb{R}^2)$ in the rest of this paper. Define the angular momentum of the system (1.3) as

$$c(x) = x(t) \wedge \dot{x}(t). \tag{2.3}$$

When $2 < \alpha < \beta$, Sbano and Southall proved in [21, Propositions 3.4 and 5.13] that for the tied homotopy class (Definition 5.6 of [21]) there exists a small $\tau_0 > 0$ such that the Lennard-Jones system of N-bodies possesses no τ-periodic solution for $\tau \in (0, \tau_0)$. Their method of [21] also works when $0 < \alpha < \beta$. By estimating the minimal period of periodic solutions of (1.3) directly, we also obtain a similar result which is stated in Proposition 2.2. We omit the proof here and refer readers to [14] for details.

Proposition 2.2 *There exists a $\tau_0 > 0$ such that the system* (1.3) *possesses no τ-periodic solutions with non-zero topological degree when $0 < \tau < \tau_0$.*

3 Variational Properties of Circular Solutions and the Mountain Pass Solutions

3.1 Existence of the Circular Solution

In [16, Proposition 5], Llibre and Long gave a characterization of $\tau/2$-antiperiodic circular solutions of the system (1.3). In order to simplify the computations of the Morse indices of the action functional at these circular solutions, we give an equivalent and slightly modified way to characterize them by introducing a function $p(r)$ for $r > 0$ instead of using the function $\varphi_\tau(r)$ in [16]. This method gives more explicit representations for $r_i(\tau)$ with $i = 0, 1, 2$ than [16] and simplifies the Morse index computation.

For $\tau > 0$, $r > 0$ and $\hat{t} \in [0, \tau]$, we write a circular motion as

$$x = x_{\pm,r,\hat{t}}(t) = \left(r\cos\frac{2\pi(t-\hat{t})}{\tau}, \pm r\sin\frac{2\pi(t-\hat{t})}{\tau} \right).$$

By direct computation, we have

$$\ddot{x}(t) = -\frac{4\pi^2}{\tau^2}x(t). \tag{3.1}$$

Comparing (3.1) with (1.3) yields that $x = x_{\pm,r,\hat{t}}(t)$ is a τ-periodic solution of (1.3) if and only if $r > 0$ is a solution of the equation $p(r) = \frac{4\pi^2}{\tau^2}$, where we define

$$p(r) = \frac{\alpha}{r^{\alpha+2}} - \frac{\beta}{r^{\beta+2}}, \quad \forall\, r > 0. \tag{3.2}$$

$p(r)$ possesses a unique maximal point at

$$\hat{r} = \left(\frac{\beta(\beta+2)}{\alpha(\alpha+2)}\right)^{\frac{1}{\beta-\alpha}} \tag{3.3}$$

satisfying $p'(\hat{r}) = 0$. Thus $p'(r) > 0$ for $0 < r < \hat{r}$ and $p'(r) < 0$ for $\hat{r} < r < +\infty$.

As a remark, note that for the function $\varphi_\tau(r) = -\frac{4\pi^2}{\tau^2} r^{\beta+2} + \alpha r^{\beta-\alpha} - \beta$ with $\tau > 0$ and $r > 0$ defined in [16, Proposition 5], we have $p(r) = \varphi_\tau(r) r^{-\beta-2} + \frac{4\pi^2}{\tau^2}$. Thus $r > 0$ is a solution of the equation

$$p(r) = \frac{\alpha}{r^{\alpha+2}} - \frac{\beta}{r^{\beta+2}} = \frac{4\pi^2}{\tau^2}, \tag{3.4}$$

if and only if r is a root of $\varphi_\tau(r) = 0$, which is used in [16].

Let $\tau = \hat{\tau}$ be the unique positive solution of the equation $p(\hat{r}) = \frac{4\pi^2}{\tau^2}$. We obtain

$$\hat{\tau} = 2\pi\sqrt{\frac{\beta+2}{\alpha(\beta-\alpha)}}\,\hat{r}^{(\alpha+2)/2} = 2\pi\sqrt{\frac{\beta+2}{\alpha(\beta-\alpha)}}\left(\frac{\beta(\beta+2)}{\alpha(\alpha+2)}\right)^{\frac{\alpha+2}{2(\beta-\alpha)}}. \tag{3.5}$$

With these preparations, we can give the following equivalent version of [16, Proposition 5].

Proposition 3.1 ([16, Proposition 5]) *If $0 < \alpha < \beta$, then the following results hold.*

(i) *When $\tau > \hat{\tau}$, the equation* (3.4) *possesses two solutions $r = r_i(\tau)$ with $i = 1, 2$ satisfying $0 < r_1(\tau) < \hat{r} < r_2(\tau)$ such that the system* (1.3) *possesses following $\tau/2$-antiperiodic circular solutions centered at the origin given by*

$$x_{\pm,r_i(\tau),t_i}(t) = \left(r_i(\tau)\cos\frac{2\pi(t-t_i)}{\tau}, \pm r_i(\tau)\sin\frac{2\pi(t-t_i)}{\tau}\right),$$

for $t_i \in [0,\tau]$ with $i = 1$ and 2.

(ii) *For $\tau = \hat{\tau}$, the equation* (3.4) *possesses precisely one solution $r = \hat{r}$ such that the system* (1.3) *possesses following $\tau/2$-antiperiodic circular solution centered at the origin given by*

$$x_{\pm,\hat{r},t_0}(t) = \left(\hat{r}\cos\frac{2\pi(t-t_0)}{\tau}, \pm\hat{r}\sin\frac{2\pi(t-t_0)}{\tau}\right)$$

with $t_0 \in [0,\tau]$.

(iii) *For $0 < \tau < \hat{\tau}$, the equation* (3.4) *possesses no positive solution and the system* (1.3) *possesses no periodic circular solution centered at the origin.*

Remark 3.2 For $\tau \geq \hat{\tau}$, comparing the constants $r_1(\tau)$ and $r_2(\tau)$ defined in Proposition 3.1 with the constant

$$r_0(\tau) = \left(\frac{\tau\alpha(\beta-\alpha)}{4\pi^2(\beta+2)}\right)^{\frac{1}{\alpha+2}}$$

defined by (9) in [16], we have

$$0 < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta-\alpha}} < r_1(\tau) \leq \hat{r} \leq r_0(\tau) \leq r_2(\tau) < \infty. \tag{3.6}$$

Note that the above third to the fifth inequalities become equalities simultaneously when $\tau = \hat{\tau}$.

Note also that in the rest of this section, we always assume $\tau \geq \hat{\tau}$ as in Proposition 3.1.

3.2 Morse Indices of Circular Solutions

For $(\rho, \sigma) \in W^{1,2}(S_\tau, \mathbb{R}^2)$, we write the Fourier expansions of ρ and σ as $\rho = \sum_{k\geq 0}(\rho_k \mathrm{e}^{\mathrm{i}\frac{2k\pi t}{\tau}} + \bar{\rho}_k \mathrm{e}^{-\mathrm{i}\frac{2k\pi t}{\tau}})$ and $\sigma = \sum_{k\geq 0}(\sigma_k \mathrm{e}^{\mathrm{i}\frac{2k\pi t}{\tau}} + \bar{\sigma}_k \mathrm{e}^{-\mathrm{i}\frac{2k\pi t}{\tau}})$ with ρ_k and $\sigma_k \in \mathbb{C}$ and $k \geq 0$. Then we have

$$W^{1,2}(S_\tau, \mathbb{R}^2) = \left\{ (\rho, \sigma) \,\middle|\, \sum_{k\geq 0}(1+k^2)(|\rho_k|^2 + |\sigma_k|^2) < \infty \right\}.$$

Next we calculate the Morse indices of the functional f_τ at the circular solutions using the polar coordinates. By $x = r\mathrm{e}^{\mathrm{i}\theta}$ and $\dot{x} = (\dot{r} + \mathrm{i}r\dot{\theta})\mathrm{e}^{\mathrm{i}\theta}$, the action functional (1.4) can be written as

$$f_\tau(r, \theta) = \int_0^\tau \left(\frac{1}{2}(|\dot{r}|^2 + r^2\dot{\theta}^2) - r^{-\beta} + r^{-\alpha} \right) dt. \tag{3.7}$$

Then the first variation of $f_\tau(r, \theta)$ at (r, θ) with respect to (ρ, σ) is given by

$$\langle f_\tau'(r, \theta), (\rho, \sigma) \rangle = \int_0^\tau (\dot{r}\dot{\rho} + r^2\dot{\theta}\dot{\sigma} + (r\dot{\theta}^2 + \beta r^{-\beta-1} - \alpha r^{-\alpha-1})\rho) dt. \tag{3.8}$$

The second variation of f_τ at (r, θ) with respect to (ρ, σ) and (ξ, ς) is given by

$$\begin{aligned}\langle f_\tau''(r, \theta)(\xi, \varsigma), (\rho, \sigma) \rangle = \int_0^\tau &(\dot{\xi}\dot{\rho} + 2r\dot{\theta}(\xi\dot{\sigma} + \rho\dot{\varsigma}) + r^2\dot{\sigma}\dot{\varsigma} \\ &+ (\dot{\theta}^2 - \beta(\beta+1)r^{-\beta-2} + \alpha(\alpha+1)r^{-\alpha-2})\rho\xi) dt.\end{aligned} \tag{3.9}$$

We let $\mu = \frac{2\pi}{\tau} = \dot{\theta}$ and simplify $\langle f_\tau''(r, \theta)(\rho, \sigma), (\rho, \sigma) \rangle$ to

$$\begin{aligned}&\langle f_\tau''(r, \theta)(\rho, \sigma), (\rho, \sigma) \rangle \\ &= \int_0^\tau (\dot{\rho}^2 + r^2\dot{\sigma}^2 + 4r\dot{\theta}\rho\dot{\sigma} + (\dot{\theta}^2 - \beta(\beta+1)r^{-\beta-2} + \alpha(\alpha+1)r^{-\alpha-2})\rho^2) dt \\ &= \tau(\mu^2 - \beta(\beta+1)r^{-\beta-2} + \alpha(\alpha+1)r^{-\alpha-2})\rho_0^2 + 2\tau\sum_{k=1}^{+\infty}\{r^2\mu^2k^2|\sigma_k|^2 \\ &\quad + 4r\mu^2 k \mathrm{Im}(\rho_k\bar{\sigma}_k) + (\mu^2(k^2+1) - \beta(\beta+1)r^{-\beta-2} + \alpha(\alpha+1)r^{-\alpha-2})|\rho_k|^2\} \\ &= \tau\mu^2 A_0(r)\rho_0^2 + 2\tau\mu^2\sum_{k=1}^{+\infty}\{|kr\sigma_k - 2\mathrm{i}\rho_k|^2 + A_k(r)|\rho_k|^2\},\end{aligned} \tag{3.10}$$

where $A_k(r) = k^2 - 4 + A_0(r)$ and $A_0(r) = 1 - \mu^{-2}(\beta(\beta+1)r^{-\beta-2} - \alpha(\alpha+1)r^{-\alpha-2})$. By (3.4), $A_0(r)$ can be rewritten as

$$A_0(r) = \beta + 2 + \frac{\alpha(\beta-\alpha)}{\beta r^{\alpha-\beta} - \alpha}, \tag{3.11}$$

and for $k \geq 1$, $A_k(r)$ can be rewritten as

$$A_k(r) = k^2 - 2 + \beta + \frac{\alpha(\beta-\alpha)}{\beta r^{\alpha-\beta} - \alpha}. \tag{3.12}$$

Lemma 3.3 (i) $A_0(r) > 0$ *if and only if*

$$r > \hat{r}_2 = \left(\frac{\beta(\beta+2)}{\alpha(\alpha+2)} \right)^{\frac{1}{\beta-\alpha}}. \tag{3.13}$$

(ii) *When $1<\alpha<\beta$ and $k\geq 1$, $A_k(r)>0$ if and only if*

$$r>\hat{r}_k\equiv\left(\frac{\beta(\beta+k^2-2)}{\alpha(\alpha+k^2-2)}\right)^{\frac{1}{\beta-\alpha}},\tag{3.14}$$

where $\hat{r}_k$ decreases when k increases and $\lim_{k\to\infty}\hat{r}_k=(\frac{\beta}{\alpha})^{\frac{1}{\beta-\alpha}}$.

This lemma can be obtained by direct computation. We omit it here.

Let $E_k=\{(\xi e^{i\frac{2k\pi t}{\tau}}+\bar{\xi}e^{-i\frac{2k\pi t}{\tau}},\varsigma e^{i\frac{2k\pi t}{\tau}}+\bar{\varsigma}e^{-i\frac{2k\pi t}{\tau}})\,|\,\xi,\varsigma\in\mathbb{C}\}$ for all integer $k\geq 0$. Then these E_ks are mutually $W^{1,2}$-orthogonal to each other, span $W^{1,2}(S_\tau,\mathbb{R}^2)$, and satisfy $\dim_{\mathbb{R}}E_0=2$ and $\dim_{\mathbb{R}}E_k=4$ for $k\geq 1$.

We decompose E_0 as $E_0=E_0^1\oplus E_0^2$ where $E_0^1=\mathrm{span}\{(1,0)\}$ and $E_0^2=\mathrm{span}\{(0,1)\}$. For $k\geq 1$ and $r>0$, we decompose each subspace E_k as $E_k=E_k^1(r)\oplus E_k^2(r)$ where

$$E_k^1(r)=\{(0,\varsigma e^{i\frac{2k\pi t}{\tau}}+\bar{\varsigma}e^{-i\frac{2k\pi t}{\tau}})\in E_k\},$$
$$E_k^2(r)=\left\{(\xi e^{i\frac{2k\pi t}{\tau}}+\bar{\xi}e^{-i\frac{2k\pi t}{\tau}},\varsigma e^{i\frac{2k\pi t}{\tau}}+\bar{\varsigma}e^{-i\frac{2k\pi t}{\tau}})\in E_k\,\middle|\,\varsigma=\frac{2i}{kr}\xi\in\mathbb{C}\right\},$$

and then $\dim_{\mathbb{R}}E_k^1(r)=\dim_{\mathbb{R}}E_k^2(r)=2$.

Lemma 3.4 (i) *For $k\neq j$, E_k and E_j are $f''_\tau(r,\theta)$-orthogonal. For given $k\geq 0$ and $r>0$, $E_k^1(r)$ and $E_k^2(r)$ are $f''_\tau(r,\theta)$-orthogonal.*

(ii) *$E_0^2\subset\ker f''_\tau(r,\theta)$ holds.*

(iii) *If the radius of the circular solution r satisfies $r>\hat{r}_2$, $f''_\tau(r,\theta)$ is positive definite on E_0^1; if $r<\hat{r}_2$, $f''_\tau(r,\theta)$ is negative definite on E_0^1; if $r=\hat{r}_2$, $E_0\subset\ker f''_\tau(r,\theta)$.*

(iv) *When $1<\alpha<\beta$ and $k\geq 1$, if $r>\hat{r}_k$, $f''_\tau(r,\theta)$ is positive definite on E_k; if $r<\hat{r}_k$, $f''_\tau(r,\theta)$ is positive definite on $E_k^1(r)$ and is negative definite on $E_k^2(r)$; if $r=\hat{r}_k$, $f''_\tau(r,\theta)$ is positive definite on $E_k^1(r)$ and $E_k^2(r)\subset\ker f''_\tau(r,\theta)$.*

Proof By (3.9), one can directly verify that $\langle f''_\tau(r,\theta)(\xi_k,\varsigma_k),(\xi_j,\varsigma_j)\rangle=0$ where $(\xi_k,\varsigma_k)\in E_k$, $(\xi_j,\varsigma_j)\in E_j$ with $k\neq j$. Similarly, $E_k^1(r)$ and $E_k^2(r)$ are $f''_\tau(r,\theta)$-orthogonal by (3.9). This yields (i) of this lemma.

Since E_0 is $f''_\tau(r,\theta)$-orthogonal to E_k for $k>0$, we have $\langle f''_\tau(r,\theta)(0,1),(\xi_k,\varsigma_k)\rangle=0$. Additionally, $\langle f''_\tau(r,\theta)(0,1),(1,0)\rangle=0$ and $\langle f''_\tau(r,\theta)(0,1),(0,1)\rangle=0$. Then we obtain (ii) of this lemma. Note that $E_0^2\subset\ker f''_\tau(r,\theta)$ corresponds to the degeneracy caused by the S_τ-symmetry of $f_\tau(x)$.

Suppose $(\rho,\sigma)=(\rho_0,\sigma_0)\in E_0$ where $\rho_0,\sigma_0\in\mathbb{R}$. (3.10) can be simplified to

$$\langle f''_\tau(r,\theta)(\rho_0,\sigma_0),(\rho_0,\sigma_0)\rangle=\tau\mu^2A_0(r)\rho_0^2.\tag{3.15}$$

If $A_0(r)>0$, $f_\tau(r,\theta)$ is positive definite on $E_0^1\subset E_0$; if $A_0(r)<0$, $f_\tau(r,\theta)$ is negative definite on $E_0^1\subset E_0$; if $A_0(r)=0$, $E_0\subset\ker f''_\tau(r,\theta)$. Applying (i) of Lemma 3.3, we obtain (iii) of this lemma.

Suppose $(\rho,\sigma)=(\rho_k e^{i\frac{2\pi kt}{\tau}}+\bar{\rho}_k e^{-i\frac{2\pi kt}{\tau}},\sigma_k e^{i\frac{2\pi kt}{\tau}}+\bar{\sigma}_k e^{-i\frac{2\pi kt}{\tau}})\in E_k$ where $\rho_k,\sigma_k\in\mathbb{C}$. (3.10) can be simplified to

$$\langle f''_\tau(r,\theta)(\rho,\sigma),(\rho,\sigma)\rangle=2\tau\mu^2\{|kr\sigma_k-2i\rho_k|^2+A_k(r)|\rho_k|^2\}.\tag{3.16}$$

If $A_k(r)>0$ for $k>0$, $f''_\tau(r,\theta)$ is positive definite on the whole subspace E_k. If $A_k(r)<0$, $f''_\tau(r,\theta)$ is positive definite on $E_k^1(r)$ and negative definite on $E_k^2(r)$. If $A_k(r)=0$, $f''_\tau(r,\theta)$ is positive definite on the subspace $E_k^1(r)$ and $E_k^2(r) \subset \ker f''_\tau(r,\theta)$. Applying (ii) of Lemma 3.3, we obtain (iv) of this lemma. $\square$

Because the functional f_τ is $O(2)$-invariant, the Morse indices at the circular solutions in the same $O(2)$-orbit are the same. Thus it suffices to compute the Morse indices for the following circular solutions of (1.3) in Proposition 3.1,

$$x_i(t) = \left(r_i(\tau)\cos\frac{2\pi t}{\tau}, r_i(\tau)\sin\frac{2\pi t}{\tau} \right),$$

for $i = 0, 1$ and 2 with $r_0(\tau) = \hat{r}$ in (ii), and $r_1(\tau)$ and $r_2(\tau)$ being given in (i) of Proposition 3.1.

Denote the Morse index and nullity of f_τ at x_i by $i(x_i)$ and $\nu(x_i)$ for $i = 1, 2$ respectively. For $k \geq 1$ and $1 < \alpha < \beta$, we define $\tau_k^\#$ by

$$\tau_k^\# = \frac{2\pi}{\sqrt{\beta-\alpha}} \left(\frac{\beta^{\alpha+2}(\beta+k^2-2)^{\beta+2}}{\alpha^{\beta+2}(\alpha+k^2-2)^{\alpha+2}} \right)^{\frac{1}{2(\beta-\alpha)}}. \tag{3.17}$$

One can directly verify that $\tau_1^\# > \tau_2^\# = \hat{\tau}$ and $\tau_j^\# > \tau_k^\#$ if $j > k \geq 2$.

Proposition 3.5 *When $\tau > \hat{\tau}$ and $\beta > \alpha > 1$, the following conclusions on the Morse index $i(x_2)$ and nullity $\nu(x_2)$ of f_τ at x_2 hold.*

(i) *When $\tau > \tau_1^\#$, the Morse index $i(x_2)$ and nullity $\nu(x_2)$ satisfy*

$$i(x_2) = 0 \quad \textit{and} \quad \nu(x_2) = 1. \tag{3.18}$$

(ii) *When $\tau = \tau_1^\#$, the Morse index $i(x_2)$ and nullity $\nu(x_2)$ satisfy*

$$i(x_2) = 0 \quad \textit{and} \quad \nu(x_2) = 3. \tag{3.19}$$

(iii) *When $\hat{\tau} < \tau < \tau_1^\#$, the Morse index $i(x_2)$ and nullity $\nu(x_2)$ satisfy*

$$i(x_2) = 2 \quad \textit{and} \quad \nu(x_2) = 1. \tag{3.20}$$

Proof By (3.6) and (3.14), we have $r_2 > \hat{r} = \hat{r}_2 > \hat{r}_k$ for all $k > 2$. Then by Lemma 3.4, $f''_\tau(r_2,\theta)$ is positive definite on $E_0^1 \oplus (\bigoplus_{k\geq 2} E_k)$ and $E_0^2 \subset \ker f''_\tau(r_2,\theta)$.

By $p(r) = \alpha r^{-\alpha-2} - \beta r^{-\beta-2}$, we know that $r_2 > \hat{r}_1$ if and only if

$$p(\hat{r}_1) = \alpha\hat{r}_1^{-\alpha-2} - \beta\hat{r}_1^{-\beta-2} > p(r_2) = \frac{4\pi^2}{\tau^2}. \tag{3.21}$$

By solving the inequality (3.21), we obtain that $r_2 > \hat{r}_1$ is equivalent to

$$\tau > \tau_1^\# \equiv \frac{2\pi}{\sqrt{\beta-\alpha}} \left(\frac{\beta^{\alpha+2}(\beta-1)^{\beta+2}}{\alpha^{\beta+2}(\alpha-1)^{\alpha+2}} \right)^{\frac{1}{2(\beta-\alpha)}}, \tag{3.22}$$

when $\beta > \alpha > 1$.

By Lemma 3.4, if $\tau > \tau_1^\#$, $f''_\tau(r_2,\theta)$ is positive definite on $E_0^1 \oplus (\bigoplus_{k\geq 1} E_k)$ and $\ker f''_\tau(r_2,\theta) = E_0^2$. Thus, the Morse index and the nullity of $f_\tau(x_2)$ satisfy $i(x_2) = 0$ and $\nu(x_2) = 1$. These yield (i) of this proposition.

By Lemma 3.4, if $\tau = \tau_1^\#$, $A_1(r_2) = 0$. This yields $f''_\tau(r_2,\theta)$ is positive definite on $E_0^1 \oplus E_1^1(r_2) \oplus (\bigoplus_{k\geq 2} E_k)$ and $\ker f''_\tau(r_2,\theta) = E_0^2 \oplus E_1^2(r_2)$. Thus, the Morse index and the nullity of $f_\tau(x_2)$ satisfy $i(x_2) = 0$ and $\nu(x_2) = 3$. These yield (ii) of this proposition.

By Lemma 3.4, if $\hat{\tau} < \tau < \tau_1^\#$, $f''_\tau(r_2,\theta)$ is still positive definite on $E_0^1 \oplus E_1^1(r_2) \oplus (\bigoplus_{k\geq 2} E_k)$, is negative definite on $E_1^2(r_2)$ and $\ker f''_\tau(r_2,\theta) = E_0^2$. Thus, the Morse index and the nullity of $f_\tau(x_2)$ satisfy $i(x_2) = 2$ and $\nu(x_2) = 1$. These yield (iii) of this proposition. $\square$

Proposition 3.6 *When $\tau > \hat{\tau}$ and $\beta > \alpha > 1$, then the following two conclusions on the Morse index $i(x_1)$ and nullity $\nu(x_1)$ of f_τ at x_1 hold.*

(i) *When $\tau \in (\tau_j^{\#}, \tau_{j+1}^{\#})$ for some $j \geq 2$, the Morse index $i(x_1)$ and nullity $\nu(x_1)$ satisfy*

$$i(x_1) = 2j + 1 \geq 5 \quad \textit{and} \quad \nu(x_1) = 1. \tag{3.23}$$

(ii) *When $\tau = \tau_j^{\#}$ for some $j > 2$, the Morse index $i(x_1)$ and nullity $\nu(x_1)$ satisfy*

$$i(x_1) = 2j - 1 \geq 5 \quad \textit{and} \quad \nu(x_1) = 3. \tag{3.24}$$

Proof By (3.6) and (3.14), we have $r_1 < \hat{r} = \hat{r}_2 < \hat{r}_1$ when $\beta > \alpha > 1$. By Lemma 3.4, $f_\tau''(x_1)$ is negative definite on $E_0^1 \oplus E_1^2(r_1) \oplus E_2^2(r_1)$.

When $j \geq 2$, $A_k(r_1) < 0$ for all $k \leq j$ and $A_k(r_1) > 0$ for all $k \geq j+1$ is equivalent to $\hat{r}_{j+1} < r_1 < \hat{r}_j$. This is equivalent to $p(\hat{r}_{j+1}) < \frac{4\pi^2}{\tau^2} = p(r_1) < p(\hat{r}_j)$. By direct computation, we obtain $\hat{r}_{j+1} < r_1 < \hat{r}_j$ if and only if

$$\tau_j^{\#} < \tau < \tau_{j+1}^{\#}, \tag{3.25}$$

where $\tau_j^{\#}$ is defined in (3.17).

By Lemma 3.4, if $\tau_j^{\#} < \tau < \tau_{j+1}^{\#}$ for $j \geq 2$, $f_\tau''(r_1, \theta)$ is negative definite on $E_0^1 \oplus (\bigoplus_{k=1}^{j} E_k^2(r_1))$, is positive definite on $(\bigoplus_{k=1}^{j} E_k^1(r_1)) \oplus (\bigoplus_{k \geq j+1} E_j)$ and $\ker f_\tau''(r_1, \theta) = E_0^2$. Therefore, we obtain Morse index and the nullity of $f_\tau(x_1)$ satisfy $i(x_1) = 2j+1$ and $\nu(x_1) = 1$. Then we obtain (i) of this lemma.

By Lemma 3.4, if $\tau = \tau_j^{\#}$ for some $j > 2$, the kernel of $f_\tau''(r_1, \theta)$ is $E_0^2 \oplus E_j^2(r_1)$. Furthermore, $f_\tau''(r_1, \theta)$ is negative definite $E_0^1 \oplus (\bigoplus_{k=1}^{j-1} E_k^2(r_1))$ and is positive definite on $(\bigoplus_{k=1}^{j} E_k^1(r_1)) \oplus (\bigoplus_{k \geq j+1} E_j)$. These yield the Morse index and the nullity of $f_\tau(x_1)$ satisfy $i(x_1) = 2j - 1$ and $\nu(x_1) = 3$. Then we obtain (ii) of this lemma. □

Remark 3.7 For the case of $j = 2$ in (ii) of Proposition 3.6, i.e., when $\tau = \hat{\tau} = \tau_2^{\#}$, we have $x_1 = x_2 = x_{\pm, \hat{r}, t_0}$ where $x_{\pm, \hat{r}, t_0}$ is defined in Proposition 3.1. We can proceed a similarly calculation as in (ii) of Proposition 3.6 and obtain that the Morse index $i(x_{\pm, \hat{r}, t_0})$ and nullity $\nu(x_{\pm, \hat{r}, t_0})$ satisfy

$$i(x_{\pm, \hat{r}, t_0}) = 3, \quad \text{and} \quad \nu(x_{\pm, \hat{r}, t_0}) = 3.$$

3.3 Application of the Mountain Pass Theorem

This subsection is devoted to the proof of the existence of a mountain pass solution of the system (1.3). We define the subset Λ_* of $X_\tau = W^{1,2}(S_\tau, \mathbb{R}^2)$ by

$$\Lambda_* = \{x \in X_\tau \mid x(t) \notin \mathcal{K}_c, \forall\, t \in [0, \tau] \text{ and } \deg(x, 0) \neq 0\},$$

where $\mathcal{K}_c = \{0\}$ is the singular set of the system (1.3), and $\deg(x, 0)$ is the winding number of $x = x(t)$ with respect to the origin. We use the convention $\deg(x, 0) > 0$ if x winds counter-clockwise with respect to the origin, and $\deg(x, 0) < 0$ if x winds clockwise with respect to the origin.

Theorem 3.8 *For every $1 < \alpha < \beta$ and*

$$\tau > \tau_1^{\#} = \left(\frac{\beta^{\alpha+2}(\beta - 1)^{\beta+2}}{\alpha^{\beta+2}(\alpha - 1)^{\alpha+2}} \right)^{\frac{1}{2(\beta-\alpha)}},$$

which is defined in (3.17)*, there exists a smooth* τ*-periodic non-circular solution* $x = x(t)$ *of the system* (1.3) *with* $\deg(x,0) = 1$.

Proof Fix a $\tau > \tau_1^{\#}$. Let $x_2 = x_{+,r_2(\tau),0}(t)$ be a circular solution with $|x_2(t)| = r_2(\tau)$ given by (i) of Proposition 3.1. We continue the proof in six steps.

Step 1 Properties of f_τ near $S_\tau \cdot x_2$.

By (i) of Proposition 3.5, $S_\tau \cdot x_2$ forms a strict local minimal non-degenerate critical manifold of f_τ in X_τ by the condition $\tau > \tau_1^{\#}$. We let $x_{-,2} = x_{-,r_2(\tau),0}(t)$ defined in (i) of Proposition 3.1 Since x_2 and $x_{-,2}$ have different topological degrees 1 and -1 respectively, $S_\tau \cdot x_2 \cap S_\tau \cdot x_{-,2} = \emptyset$ holds. Here we have

$$\begin{aligned} f_\tau(x_2) &= \int_0^\tau \left(\frac{1}{2}|\dot{x}_2(t)|^2 + \frac{1}{|x_2(t)|^\alpha} - \frac{1}{|x_2(t)|^\beta} \right) dt \\ &> \int_0^\tau \left(\frac{1}{|x_2(t)|^\alpha} - \frac{1}{|x_2(t)|^\beta} \right) dt \\ &= \frac{\tau}{r_2(\tau)^\alpha} \left(1 - \frac{1}{r_2(\tau)^{\beta-\alpha}} \right). \end{aligned} \tag{3.26}$$

By (i) of Proposition 3.1, (3.3) and (3.6), we obtain

$$r_2(\tau) > \hat{r} = \left(\frac{\beta(\beta+2)}{\alpha(\alpha+2)} \right)^{\frac{1}{\beta-\alpha}} > 1.$$

Therefore the right hand side of (3.26) is positive and we obtain

$$f_\tau(x_2) > 0. \tag{3.27}$$

For any $\lambda > 0$, let $\mathcal{N}_\lambda(S_\tau \cdot x_2) = \{x \in X_\tau \,|\, \mathrm{dist}(x, S_\tau \cdot x_2) < \lambda\}$, where we let $\mathrm{dist}(x, A) = \inf_{y \in A} \|x - y\|_1$ for any subset $A \subset X_\tau$.

For the given $\tau > \tau_1^{\#}$, we have $A_k(r_2) > 0$ for all $k \geq 0$ because of Lemma 3.3. By (3.15) and (3.16), there exists a constant $C_1 > 0$ such that for any $(\rho, \sigma) \in E_0^1 \oplus E_1(r_2) \oplus E_2(r_2)$ satisfying $\|(\rho,\sigma)\|_1 = 1$, $\langle f_\tau''(x_2)(\rho,\sigma), (\rho,\sigma) \rangle \geq C_1$ holds.

For $k \geq 3$, suppose $(\rho, \sigma) \in E_k(r_2)$ with $\rho = \xi \mathrm{e}^{\mathrm{i}\frac{2k\pi t}{\tau}} + \bar{\xi}\mathrm{e}^{-\mathrm{i}\frac{2k\pi t}{\tau}}$, $\sigma = \varsigma \mathrm{e}^{\mathrm{i}\frac{2k\pi t}{\tau}} + \bar{\varsigma}\mathrm{e}^{-\mathrm{i}\frac{2k\pi t}{\tau}}$ and $\|(\rho,\sigma)\|_1 = 1$. Note that

$$\frac{1}{2}k^2 r_2^2 |\varsigma|^2 + 8|\xi|^2 - 4kr_2 \mathrm{Im}\bar{\xi}\varsigma \geq 0.$$

By (3.16), the facts $r_2(\tau) > 1$ and $A_1(r_2) > 0$ and direct computations, we have

$$\begin{aligned} \langle f_\tau''(x_2)(\rho,\sigma), (\rho,\sigma) \rangle &= 2\tau\mu^2 \{ |kr_2\varsigma - 2\mathrm{i}\xi|^2 + A_k(r)|\xi|^2 \} \\ &\geq 2\tau\mu^2 \left\{ \frac{1}{2}k^2 r_2^2 |\varsigma|^2 - 4|\xi|^2 + A_1(r_2)|\xi|^2 + (k^2-1)|\xi|^2 \right\} \\ &> \tau\mu^2 \{ k^2|\varsigma|^2 + (k^2-5)|\xi|^2 \} \\ &> \frac{\tau\mu^2(k^2-5)}{2(k^2+1)} \{ 2(k^2+1)(|\xi|^2 + |\varsigma|^2) \} \\ &\geq \frac{2\tau\mu^2}{10}. \end{aligned}$$

Due to the $f_\tau''(x_2)$-orthogonality among $E_k(r_2)$, there exists $C_2 > 0$ such that

$$\langle f_\tau''(x_2)(\rho,\sigma), (\rho,\sigma) \rangle \geq C_2 \|(\rho,\sigma)\|_1^2, \quad \forall (\rho,\sigma) \in Y \equiv E_0^1 \oplus \left(\bigoplus_{k \geq 1} E_k(r_2) \right).$$

Due to the invariance of f_τ under S_τ action, $f'_\tau(x_2)=0$, and the above positivity of $f''_\tau(x_2)$ in Y which is transversal to the orbit $S_\tau\cdot x_2$, we obtain some $\lambda>0$ sufficiently small and an $\eta>0$ such that

$$f_\tau(x)>f_\tau(x_2),\quad \forall\, x\in\mathcal{N}_\lambda(S_\tau\cdot x_2)\setminus S_\tau\cdot x_2, \tag{3.28}$$

$$f_\tau(x)\ge f_\tau(x_2)+\eta,\quad \forall\, x\in\partial\mathcal{N}_\lambda(S_\tau\cdot x_2). \tag{3.29}$$

Step 2 The mountain pass value of f_τ.

For $\epsilon>0$ we define $y_\epsilon=\epsilon x_2\in\Lambda_*$. Because

$$0=\langle f'_\tau(x_2),x_2\rangle=\int_0^\tau\left(|\dot{x}_2(t)|^2-\frac{\alpha}{|x_2(t)|^\alpha}+\frac{\beta}{|x_2(t)|^\beta}\right)dt,$$

we obtain

$$\begin{aligned}
f_\tau(y_\epsilon)&=\int_0^\tau\left(\frac{1}{2}|\dot{y}_\epsilon(t)|^2+\frac{1}{|y_\epsilon(t)|^\alpha}-\frac{1}{|y_\epsilon(t)|^\beta}\right)dt\\
&=\int_0^\tau\left(\frac{\epsilon^2}{2}\left(\frac{\alpha}{|x_2(t)|^\alpha}-\frac{\beta}{|x_2(t)|^\beta}\right)+\frac{1}{\epsilon^\alpha|x_2(t)|^\alpha}-\frac{1}{\epsilon^\beta|x_2(t)|^\beta}\right)dt\\
&=\tau\left(\frac{\epsilon^2}{2}\left(\frac{\alpha}{r_2(\tau)^\alpha}-\frac{\beta}{r_2(\tau)^\beta}\right)+\frac{1}{\epsilon^\alpha r_2(\tau)^\alpha}-\frac{1}{\epsilon^\beta r_2(\tau)^\beta}\right).
\end{aligned}$$

Because $1<\alpha<\beta$, we have

$$\frac{1}{\epsilon^\alpha r_2(\tau)^\alpha}-\frac{1}{\epsilon^\beta r_2(\tau)^\beta}\to-\infty,\quad \text{as } \epsilon\to 0.$$

Therefore we can fix an $\epsilon_0\in(0,\bar{r})$ sufficiently small such that

$$f_\tau(y_{\epsilon_0})<0. \tag{3.30}$$

We define the path set Γ by

$$\Gamma=\{\gamma\in C([0,1],\Lambda_*)\,|\,\gamma(0)=x_2,\ \gamma(1)=y_{\epsilon_0}\}. \tag{3.31}$$

Let $\gamma(s)=(1-s)x_2+sy_{\epsilon_0}=(1-s+s\epsilon_0)x_2$ for every $s\in[0,1]$. Then $\gamma(s)\in\Lambda_*$ for $s\in[0,1]$, $\gamma(0)=x_2$, and $\gamma(1)=\epsilon_0x_2=y_{\epsilon_0}$. Thus $\gamma\in\Gamma$ and $\Gamma\ne\emptyset$.

Now we define

$$\kappa=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}f_\tau(\gamma(s)). \tag{3.32}$$

Here by (3.27) and (3.29), we have

$$\kappa\ge f_\tau(x_2)+\eta>f_\tau(x_2)>0. \tag{3.33}$$

Step 3 f_τ satisfies the (PS) condition on Λ_* at every f_τ-level $\kappa\ne 0$.

As usual, a sequence $\{\hat{x}_n\}_{n\ge1}\subset\Lambda_*$ satisfying $f_\tau(\hat{x}_n)\to\kappa$ and $\|f'_\tau(\hat{x}_n)\|_1\to 0$ as $n\to+\infty$ is called a Palais–Smale (PS) sequence in Λ_*. The functional $f_\tau(x)$ is called to satisfy the Palais–Smale condition (PS) in Λ_*, if every (PS) sequence $\{\hat{x}_n\}_{n\ge1}\subset\Lambda_*$ possesses a convergent subsequence in Λ_*.

Note first that because the L-J potential function U in (1.3) satisfies the conditions (V2) and (V3) on p. 210 of [9], the first part of the proof of on [9, Lemma 3, pp. 211–212] can be applied here for us to get that every (PS) sequence $\{\hat{x}_n\}_{n\ge1}\subset\Lambda_*$ is bounded, and thus contains a subsequence, which is still denoted by $\{\hat{x}_n\}_{n\geq 1}$ below, which converges to some $\hat{x}$ weakly in $W^{1,2}(S_\tau, \mathbb{R}^2)$ and strongly in $C(S_\tau, \mathbb{R}^2)$.

If $\hat{x}(t) = 0$ for all t, then $\sup_{t\in S_\tau} |\hat{x}_n(t)| \to 0$ as $n \to \infty$ which contradicts the boundedness of $f_\tau(\hat{x}_n)$ and the boundedness $\|\hat{x}_n\|_1$. Therefore there exists a $t_0 \in S_\tau$ such that $\hat{x}(t_0) \neq 0$.

Now we assume $\{t \in S_\tau \mid \hat{x}(t) = 0\} \neq \emptyset$. By the existence of the above t_0 the set $\{t \in S_\tau \mid \hat{x}(t) \neq 0\}$ is a non-empty open subset of S_τ. Consequently there exists a non-trivial open interval (t_1, t_2) with $t_1 < t_2$ such that $t_0 \in (t_1, t_2)$, $\hat{x}(t) \neq 0$ for all $t \in (t_1, t_2)$, and $\hat{x}(t_1) = \hat{x}(t_2) = 0$.

For any compact subinterval $I \subset (t_1, t_2)$, let $\psi \in W_0^{1,2}(I, \mathbb{R}^2)$, and then ψ can be extended trivially to a function in X_τ, i.e., set $\psi(t) = 0$ for all $t \in S_\tau \setminus I$. Then because $\{\hat{x}_n\}$ is a (PS) sequence and converges to $\hat{x}$ weakly in X_τ and strongly in $C(S_\tau, \mathbb{R}^2)$, we obtain

$$\int_I (\dot{\hat{x}} \cdot \dot{\psi} - \nabla U(\hat{x}) \cdot \psi) dt = 0.$$

Then $\hat{x}$ is a classical solution of (1.3) on I by the theory of calculus of variations.

Consequently $\hat{x}$ is a classical solution of (1.3) on the open interval (t_1, t_2). But then it is a classical solution of (1.3) on $[t_1, t_2]$ by Proposition 2.1. This contradicts the assumption $\hat{x}(t_1) = \hat{x}(t_2) = 0$. Therefore the set $\{t \in S_\tau \mid \hat{x}(t) = 0\}$ must be empty and $\hat{x}$ is a classical solution of (1.3) on S_τ.

Now by the part (ii) of the proof of Lemma 3 on [9, p. 213], $\{x_n\}$ converges to $\hat{x}$ strongly in Λ_*, i.e., the (PS) condition holds.

Step 4 The existence of mountain pass solution when $\beta \geq 2$ and $1 < \alpha < \beta$.

When $\beta \geq 2$ and $1 < \alpha < \beta$, the potential $U(x)$ satisfies the strong force condition ([10, 21]). If x satisfies $f_\tau(x) = \kappa$, we have that $x \in \Lambda_*$. Note that by the strong force condition, the classical deformation flow ([6, 20]) pushes elements in Λ_* into Λ_* ([3, Proposition 1.17]). Therefore κ is a critical value of f_τ. Let $x = \hat{x}(t)$ be a critical point of f_τ corresponding to κ, which is a classical solution of (1.3) in Λ_* by Step 3.

Step 5 The existence of mountain pass solution when $1 < \alpha < \beta < 2$.

When $1 < \alpha < \beta < 2$, the potential $U(x)$ does not satisfy the strong force condition any more. We further require $\epsilon_0 < 1$ given by (3.30). Inspired by [3], we define the perturbed potential by

$$U_\delta(x) = U(x) + g(|x|)\frac{\delta}{|x|^4} = \frac{1}{|x|^\beta} - \frac{1}{|x|^\alpha} + g(|x|)\frac{\delta}{|x|^4}, \tag{3.34}$$

where $0 < \delta < 1$ and $g(r) \in C^2([0, +\infty), \mathbb{R})$ such that $g(r) = 1$ if $r \leq \frac{\epsilon_0}{4}$, $g(r) = 0$ if $r \geq \frac{\epsilon_0}{2}$ and $g'(r) < 0$ if $r \in (\frac{\epsilon_0}{4}, \frac{\epsilon_0}{2})$. Note that $U_\delta(x) \in C^2(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ satisfies the strong force condition for any $0 < \delta < 1$ and $\lim_{\delta\to 0} U_\delta(x) = U(x)$ for $x \in \mathbb{R}^2 \setminus \{0\}$.

For x satisfying $0 < |x| < \frac{\epsilon_0}{2}$, we have

$$0 < U(x_{\frac{\epsilon_0}{2}}) < U(x) < U_{\delta_1}(x) < U_{\delta_2}(x) < U_1(x), \tag{3.35}$$

where $|x_{\frac{\epsilon_0}{2}}| = \frac{\epsilon_0}{2}$ and $0 < \delta_1 < \delta_2 < 1$. The first and the second inequalities of (3.35) hold because $U(x)$ is monotonically decreasing in $|x|$ when $0 < |x| < \frac{\epsilon_0}{2} < 1$ and $U(x) = 0$ if $|x| = 1$. The third to the fifth inequalities hold because of definition of $U_\delta(x)$.

We calculate $(\nabla U_\delta(x), x)$ and obtain

$$(\nabla U_\delta(x), x) = (\nabla U(x), x) + g'(|x|)\frac{\delta}{|x|^3} - g(|x|)\frac{4\delta}{|x|^4}. \tag{3.36}$$

For x satisfying $0 < |x| < \frac{\epsilon_0}{2}$, we have

$$(\nabla U_1(x), x) < (\nabla U_{\delta_2}(x), x) < (\nabla U_{\delta_1}(x), x) < (\nabla U(x), x) < 0, \tag{3.37}$$

where $0 < \delta_1 < \delta_2 < 1$. The first to the third inequality of (3.37) hold because $g'(r) < 0$ when $r \in (\frac{\epsilon_0}{4}, \frac{\epsilon_0}{2})$ and $g'(r) = 0$, $g(r) = 1$ when $r \in (0, \frac{\epsilon_0}{4})$. The last inequality holds because $(\nabla U(x), x) = -\frac{\beta}{|x|^\beta} + \frac{\alpha}{|x|^\alpha} < 0$ when $|x| < \frac{\epsilon_0}{2}$.

When $|x| \geq \frac{\epsilon_0}{2}$, we have $g(|x|) = 0$, $U_\delta(x) = U(x)$ and $(\nabla U_\delta(x), x) = (\nabla U(x), x)$ for every $0 < \delta < 1$. By $(\nabla U(x), x) \to 0$, $U(x) \to 0$ as $|x| \to \infty$, there exists an N_0 sufficiently large such that $0 < |(\nabla U(x), x)| < 1$ and $0 < |U(x)| < 1$ when $|x| > N_0$. When $\frac{\epsilon_0}{2} \leq |x| \leq N_0$, both $U(x)$ and $(\nabla U(x), x)$ are continuous. This yields that there exist c_i, where $i = 1, 2, 3, 4$, such that $c_1 \leq U(x) \leq c_2$ and $c_3 \leq (\nabla U(x), x) \leq c_4$. Then for every $0 < \delta < 1$, we have

$$\min\{c_1, -1\} \leq U_\delta(x) = U(x) \leq \max\{c_2, 1\}, \tag{3.38}$$

$$\min\{c_3, -1\} \leq (\nabla U_\delta(x), x) = (\nabla U(x), x) \leq \max\{c_4, 1\}, \tag{3.39}$$

when $|x| \geq \frac{\epsilon_0}{2}$. Note that c_i and ϵ are independent of δ where $i = 1, 2, 3, 4$.

The perturbed action functional is defined by

$$f_{\tau,\delta}(x) = \int_0^\tau \left(\frac{1}{2}|\dot{x}|^2 - U_\delta(x)\right)dt. \tag{3.40}$$

Then $f_{\tau,\delta} \in C^2(\Lambda_*, \mathbb{R})$ and $f_\tau(x) = f_{\tau,\delta}(x)$ if $\min_{t\in S_\tau} |x(t)| \geq \frac{\epsilon_0}{2}$. Since $|x_2(t)| > |y_{\epsilon_0}(t)| = \epsilon_0 > \frac{\epsilon_0}{2}$ for all $t \in S_\tau$, both the Morse index and the nullity of the action functional $f_{\tau,\delta}(x)$ at the circular solution x_2 are the same as those of $f_\tau(x)$ at x_2, and $f_{\tau,\delta}(y_{\epsilon_0}) = f_\tau(y_{\epsilon_0}) < f_\tau(x_2) = f_{\tau,\delta}(x_2)$. Then the discussion of f_τ in Step 1 to Step 4 are still valid for $f_{\tau,\delta}$.

The critical value $\kappa(\delta)$ is defined by

$$\kappa(\delta) = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} f_{\tau,\delta}(\gamma(s)), \tag{3.41}$$

where Γ is still defined by $\Gamma = \{\gamma \in C([0,1], \Lambda_*) \,|\, \gamma(0) = x_2,\ \gamma(1) = y_{\epsilon_0}\}$. By Step 1 to Step 4, for each δ sufficiently small there exists an $x_\delta \in \Lambda_*$ satisfying $f_{\tau,\delta}(x_\delta) = \kappa(\delta)$ such that x_δ is a critical point of $f_{\tau,\delta}$.

We define $\bar{\kappa} \equiv \max_{s\in[0,1]} f_{\tau,\delta}(sx_2 + (1-s)y_{\epsilon_0})$ and $\underline{\kappa} \equiv f_{\tau,\delta}(x_2) > 0$ where both $\underline{\kappa}$ and $\bar{\kappa}$ are independent of δ by the definition of x_2, y_{ϵ_0} and $U_\delta(x)$. (3.41) indicates that

$$0 < \underline{\kappa} < \kappa(\delta) \leq \bar{\kappa}. \tag{3.42}$$

Note that $|\dot{x}_\delta(t)|^2 \geq 0$ and (3.42) yield $-\int_0^\tau U_\delta(x_\delta)dt \leq \bar{\kappa}$. Since x_δ is a critical point of $f_{\tau,\delta}(x)$, we have that $\langle f'_{\tau,\delta}(x_\delta), x_\delta\rangle = 0$, i.e, $\int_0^\tau |\dot{x}_\delta|^2 dt = \int_0^\tau (\nabla U_\delta(x_\delta), x_\delta)dt$. This yields

$$\begin{aligned} f_{\tau,\delta}(x_\delta) &= \int_0^\tau \left(\frac{1}{2}|\dot{x}_\delta|^2 - U_\delta(x_\delta)\right)dt \\ &= \frac{1}{2}\int_0^\tau ((\nabla U_\delta(x_\delta), x_\delta) - 2U_\delta(x_\delta))dt \\ &= \frac{1}{2}(\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3), \end{aligned} \tag{3.43}$$

where we define

$$\mathrm{I}_1 \equiv \int_{\{t\in S_\tau;|x_\delta(t)|\geq\frac{\epsilon_0}{2}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt;$$

$$\mathrm{I}_2 \equiv \int_{\{t\in S_\tau;\frac{\epsilon_0}{4}<|x_\delta(t)|<\frac{\epsilon_0}{2}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt;$$

$$\mathrm{I}_3 \equiv \int_{\{t\in S_\tau;|x_\delta(t)|\leq\frac{\epsilon_0}{4}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt.$$

By (3.38) and (3.39), we have $(\nabla U_\delta(x),x)\leq \max\{c_4,1\}$ and $U_\delta(x)\geq\min\{c_1,-1\}$ when $|x(t)|\in[\frac{\epsilon_0}{2},+\infty)$. Then

$$\mathrm{I}_1 = \int_{\{t\in S_\tau;|x_\delta(t)|\geq\frac{\epsilon_0}{2}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt < C_1, \tag{3.44}$$

where $C_1\equiv\tau(\max\{c_4,1\}+\max\{-2c_1,2\})>0$ is a constant independently of δ. By (3.35) and (3.37), $(\nabla U_\delta(x),x)<0$ and $U_\delta(x)>U(x_{\frac{\epsilon_0}{2}})>0$ when $|x(t)|\in(\frac{\epsilon_0}{4},\frac{\epsilon_0}{2})$. Then

$$\mathrm{I}_2 = \int_{\{t\in S_\tau;\frac{\epsilon_0}{4}<|x_\delta(t)|<\frac{\epsilon_0}{2}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt < 0. \tag{3.45}$$

By (3.42), we have that I_3 has a lower bound independently of δ namely $\mathrm{I}_3\geq 2\underline{\kappa}-C_1$.

$$\begin{aligned}
\mathrm{I}_3 &= \int_{\{t\in S_\tau;|x_\delta(t)|\leq\frac{\epsilon_0}{4}\}} ((\nabla U_\delta(x_\delta),x_\delta)-2U_\delta(x_\delta))dt \\
&= \int_{\{t\in S_\tau;|x_\delta(t)|\leq\frac{\epsilon_0}{4}\}} \left(-\frac{\beta}{|x|^\beta}+\frac{\alpha}{|x|^\alpha}-\frac{4\delta}{|x|^4}-\frac{2}{|x|^\beta}+\frac{2}{|x|^\alpha}-\frac{2\delta}{|x|^4}\right)dt \\
&= (\alpha+2)\int_{\{t\in S_\tau;|x_\delta(t)|\leq\frac{\epsilon_0}{4}\}} \left(-\frac{\beta+2}{\alpha+2}\frac{1}{|x|^\beta}+\frac{1}{|x|^\alpha}-\frac{6\delta}{\alpha+2}\frac{1}{|x|^4}\right)dt \\
&\leq (\alpha+2)\int_{\{t\in S_\tau;|x_\delta(t)|\leq\frac{\epsilon_0}{4}\}} -U_\delta(x_\delta)dt.
\end{aligned} \tag{3.46}$$

It yields that

$$\int_{\{t\in S_\tau;|x(t)|\leq\frac{\epsilon_0}{4}\}} -U_\delta(x_\delta)dt \geq \frac{2\underline{\kappa}-C_1}{\alpha+2}. \tag{3.47}$$

By (3.35), $U_\delta(x)\leq U_1(x)$ when $\frac{\epsilon_0}{4}\leq|x|\leq\frac{\epsilon_0}{2}$. Since $U_1(x)$ is monotonically decreasing along the radial direction when $\frac{\epsilon_0}{4}<|x|<\frac{\epsilon_0}{2}$, we have $U_\delta(x)\leq U_1(x)\leq U_1(x_{\frac{\epsilon_0}{4}})$ where $|x_{\frac{\epsilon_0}{4}}|=\frac{\epsilon_0}{4}$. When $\frac{\epsilon_0}{2}<|x|<\infty$, by (3.38), $U_\delta(x)=U(x)\leq\max\{c_2,1\}$ holds. Then we have that $U_\delta(x)\leq C_2\equiv\max\{c_2,1,U_1(x_{\frac{\epsilon_0}{4}})\}$ and $C_2>0$ independently of δ when $|x|\geq\frac{\epsilon_0}{4}$. Then

$$\int_{\{t\in S_\tau;|x_\delta(t)|\geq\frac{\epsilon_0}{4}\}} -U_\delta(x_\delta)dt \geq -\tau C_2. \tag{3.48}$$

By (3.48) and (3.49),

$$\int_0^\tau -U_\delta(x_\delta)dt \geq -\tau C_2+\frac{2\underline{\kappa}-C_1}{\alpha+2} \tag{3.49}$$

independently of δ.

By (3.42) and (3.49), $\|\dot{x_\delta}\|_{L^2}$ has an upper bound independently of δ. Note that $\|x_\delta\|_{L^2}$ must be bounded independently of δ. Since for any $0<t_1<t_2<\tau$,

$$(t_2-t_1)\int_{t_1}^{t_2}|\dot{x}_\delta(t)|^2dt \geq \left(\int_{t_1}^{t_2}\dot{x}_\delta(t)dt\right)^2 = |x_\delta(t_2)-x_\delta(t_1)|^2,$$

we have

$$|\max_{t\in S_\tau} x_\delta(t) - \min_{t\in S_\tau} x_\delta(t)| \le \sqrt{\tau}\|\dot{x_\delta}\|_{L^2}. \tag{3.50}$$

If there exists a sequence $\{\delta_i\}$ satisfying $\delta_i \to 0$ as $i \to +\infty$ such that $\|x_{\delta_i}\|_{L^2} \to \infty$ as $i \to \infty$. Then by (3.50) and the boundedness of $\|\dot{x}_\delta\|_{L^2}$, we obtain $|x_{\delta_i}(t)| \to \infty$ as $i \to \infty$ uniformly for all $t \in S_\tau$. This yields that both $U_\delta(x_\delta)$ and $(U'_\delta(x_\delta), x_\delta)$ tends to 0 as $\delta \to 0$ uniformly for $t \in [0,\tau]$. By (3.43), when $\delta \to 0$, we have

$$f_{\tau,\delta}(x_\delta) = \frac{1}{2}\int_0^\tau ((\nabla U_\delta(x_\delta), x_\delta) - 2U_\delta(x_\delta))dt \to 0. \tag{3.51}$$

This contradicts (3.42).

When δ is sufficiently small, $\|x_\delta\|_1$ are bounded independently of δ. Then we can choose a sequence $\{x_{\delta_i}\}_{i=1}^\infty$ such that $\{x_{\delta_i}\}_{i=1}^\infty$ converges weakly in $W^{1,2}(S_\tau,\mathbb{R}^2)$ and strongly in $C(S_\tau,\mathbb{R}^2)$ to some $\hat{x} \in \overline{\Lambda_*}$ as $\delta_i \to 0$.

Assume $\{t \in S_\tau \mid \hat{x}(t) = 0\} \ne \emptyset$. There exists at least a $t_0 \in S_\tau$ such that $\hat{x}(t_0) \ne 0$. If not, for any ϵ sufficiently small there exists a $\delta' > 0$ such that $\max_{t\in S_\tau} |x_\delta(t)| < \epsilon$ when $0 < \delta < \delta'$. Then $\min_{t\in S_\tau} U_\delta(x_\delta(t)) \to +\infty$ as $\delta \to 0$. This contradicts that $-\int_0^\tau U_\delta(x_\delta)dt$ is lower bounded independently of δ given by (3.49). Thus there must exists a non-trivial open interval $(t_0 - \bar{\epsilon}, t_0 + \bar{\epsilon})$ such that $\hat{x}(t) \ne 0$ when $t \in (t_0 - \bar{\epsilon}, t_0 + \bar{\epsilon})$ and $\hat{x}(t_0 - \bar{\epsilon}) = \hat{x}(t_0 + \bar{\epsilon}) = 0$. We proceed the computations as in Step 3 and obtain that $\hat{x}$ is a classical solution of (1.3) on the open interval $(t_0 - \bar{\epsilon}, t_0 + \bar{\epsilon})$. By Proposition 2.1, $\hat{x}$ is a classical solution of (1.3) on $[t_0 - \bar{\epsilon}, t_0 + \bar{\epsilon}]$ which yields a contradiction. Then $\{t \in S_\tau \mid \hat{x}(t) = 0\} = \emptyset$ and $\hat{x} \in \Lambda_*$ is a classical solution of the system (1.3).

Step 6 Morse index and the topological degree of the mountain pass solution.

Because $f_\tau(\hat{x}) = \kappa > f_\tau(x_2) = f_\tau(x_{-,2})$, we obtain $\hat{x} \notin (S_\tau \cdot x_2) \cup (S_\tau \cdot x_{-,2})$. On the other hand, by results of Hofer in [11] or Tian in [24], the Morse index $i(\hat{x})$ of f_τ at $\hat{x}$ satisfies

$$i(\hat{x}) \le 1. \tag{3.52}$$

By Proposition 3.6, the Morse indices of f_τ at every solution of (1.3) in $S_\tau \cdot x_1$ is at least 5. Therefore $\hat{x}$ cannot be a circular solution of (1.3).

Note that $\deg(\hat{x}, 0) = 1$ holds, because $\hat{x}$ is in the connected component of Λ_* containing x_2 and $\deg(x, 0)$ is locally constant in Λ_*. This completes the proof. □

Remark 3.9 (i) The condition $\alpha > 1$ is used only in the definition of $\tau_1^{\#}$, which is not used in the above proof of the (PS) condition.

(ii) In this section, we have understood the variational properties of the circular solutions and used the mountain pass theorem to find one non-circular solution with topological degree 1. These variational properties will be useful in the later studies of the N-body problem of the Lennard-Jones potential. The circular solutions and the mountain pass solutions are certainly special cases of the solutions found in Section 5 below.

4 Zero Angular Momentum Solutions

We introduce the polar coordinates by setting

$$x(t) = r(t)e^{i\theta(t)}, \tag{4.1}$$

and the system (1.3) is rewritten as

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 - \frac{\beta}{r^{\beta+1}} + \frac{\alpha}{r^{\alpha+1}} = 0, \\ c = r^2\dot{\theta}, \end{cases} \tag{4.2}$$

where c is the angular momentum of the solution $x(t)$ of (1.3). When $c = 0$, the system degenerates to a 1-dimensional problem and the topological degree of the solution is zero; when $c \neq 0$, the solution is planar and the topological degree of the τ-periodic solutions are not zero. In this section, we study the degenerate case. The study on the non-zero angular momentum case is discussed in the next section.

If the angular momentum $c = 0$, (4.2) degenerates to

$$\ddot{r} - \frac{\beta}{r^{\beta+1}} + \frac{\alpha}{r^{\alpha+1}} = 0. \tag{4.3}$$

We define the potential $U(x)$ in one dimension by

$$V(r) = \frac{1}{r^\beta} - \frac{1}{r^\alpha},$$

where $r \in \mathbb{R}^+$. $V(r)$ attains its global minima at

$$\bar{r} = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{\beta-\alpha}}.$$

Let $r_{h,\min}$ and $r_{h,\max}$ be the two solutions of $V(r) - h = 0$ where $V(\bar{r}) < h < 0$ and $r_{h,\min} < \bar{r} < r_{h,\max}$. Then we can state following standard results of the one dimension system of (4.3). By $x(t) = r(t)\mathrm{e}^{\mathrm{i}\theta_0}$ for some fixed θ_0, the solution of (4.3) is equivalent to the solution of (1.3).

Theorem 4.1 (i-1) *Suppose $0 < \alpha < \beta$. Then for every $h \in (V(\bar{r}), 0)$ and $\theta_0 \in [0, 2\pi)$, the given energy problem*

$$\begin{cases} \ddot{x}(t) = -\nabla U(x(t)), \\ H(x(t), \dot{x}(t)) = h, \\ x(\tau) = x(0), \quad \dot{x}(\tau) = \dot{x}(0), \end{cases} \tag{4.4}$$

always possesses a periodic oscillating line solution (τ, x) such that $x(t) = r(t)\mathrm{e}^{\mathrm{i}\theta_0}$ with $r = r(t)$ being a periodic solution of (4.3) *possessing energy h, and the period $\tau = \tau(h)$ is given by*

$$\tau(h) = 2\int_{r_{h,\min}}^{r_{h,\max}} \frac{1}{\sqrt{2(h - V(r))}} dr, \tag{4.5}$$

which is finite, positive, and depends on h continuously.

(i-2) $\lim_{h\to 0^-} \tau(h) = +\infty$ *holds.*

(i-3) $\lim_{h\to V(\bar{r})^+} \tau(h) = \frac{2\pi}{\sqrt{V''(\bar{r})}} = \frac{2\pi}{\sqrt{\beta(\beta-\alpha)}}\left(\frac{\beta}{\alpha}\right)^{\frac{\beta+2}{2(\beta-\alpha)}}$ *holds.*

(i-4) *There exists a constant $\tau_{\mathrm{os}} = \tau_{\mathrm{os}}(\alpha, \beta) \in (0, \frac{2\pi}{\sqrt{V''(\bar{r})}}]$ such that $\tau(h) \geq \tau_{\mathrm{os}}$ for any $h \in [V(\bar{r}), 0)$, i.e., $\tau_{\mathrm{os}} > 0$ is the maximal value such that there exists no τ-periodic oscillating line solution for every $\tau \in [0, \tau_{\mathrm{os}})$.*

(i-5) *For any given $\tau \geq \tau_{\mathrm{os}}$ and $\theta_0 \in [0, 2\pi)$, there exists a τ-periodic oscillating line solution $x(t) = r(t)\mathrm{e}^{\mathrm{i}\theta_0}$ possessing energy h such that $\tau(h) = \tau$.*

(ii) *For $h = V(\bar{r})$ and $\theta_0 \in [0, 2\pi)$, $x(t) = \bar{r}\mathrm{e}^{\mathrm{i}\theta_0}$ for $t \in \mathbb{R}$ is a constant solution x_c of* (1.3).

(iii) *For* $h < V(\bar{r})$ *and* $\theta_0 \in [0, 2\pi)$, *the system* (1.3) *possesses no solution.*

(iv) *For* $h \geq 0$ *and* $\theta_0 \in [0, 2\pi)$, $x(t) = r(t)e^{i\theta_0}$ *for* $t \in \mathbb{R}$ *must be a non-periodic line solution of the system* (1.3) *and the system* (1.3) *possesses no periodic oscillating line solutions.*

Since the system degenerates to a 1-dimensional problem, we can get these results by direct computations. Readers are referred to [2] and [15] for outlines of proof. The detailed proof can be found in [14].

Remark 4.2 We also calculate the Morse index $i(x_c)$ and nullity $\nu(x_c)$ of f_τ at the constant solution x_c of (1.3) viewed as a τ-periodic solution and obtain

$$i(x_c) = 1 + 2\varrho_- \quad \text{and} \quad \nu(x_c) = \begin{cases} 1, & \text{if } \frac{\tau}{2\pi}\sqrt{\frac{\beta(\beta-\alpha)}{\bar{r}^{\beta+2}}} \notin \mathbb{N}, \\ 3, & \text{if } \frac{\tau}{2\pi}\sqrt{\frac{\beta(\beta-\alpha)}{\bar{r}^{\beta+2}}} \in \mathbb{N}. \end{cases} \tag{4.6}$$

where

$$\varrho_- = \varrho_-(x_c) = \max\left\{ j \in \mathbb{N} \cup \{0\} \,\middle|\, j < \frac{\tau}{2\pi}\sqrt{\frac{\beta(\beta-\alpha)}{\bar{r}^{\beta+2}}} \right\}. \tag{4.7}$$

We refer readers to [14] for details of the proof.

Remark 4.3 (i) Using the method of Remark A.9 and 1° of Lemma A.10 of [17], it can be proved that $0 < \tau_{\rm os} < \frac{2\pi}{\sqrt{V''(\bar{r})}}$ and there exists $\delta > 0$ such that $\tau(h)$ is strictly decreasing for $h \in (V(\bar{r}), V(\bar{r}) + \delta)$. Let $h_{\rm os} = V(\bar{r}) + \delta$. We tend to believe that $\tau(h_{\rm os}) = \tau_{\rm os}$ holds, and also that $\tau(h)$ is strictly increasing for $h \in [h_{\rm os}, 0)$. Namely, $\tau(h)$ changes monotonicity only once on $(V(\bar{r}), 0)$.

(ii) Note that for every $\tau > 0$, Ambrosetti and Coti Zelati found a τ-periodic solution of (1.3) with $\beta \geq 2$ in Theorem 9.1 of [1] via the mountain pass theorem. Proposition 2.2 and (i-4) of Theorem 4.1 show that when τ is sufficiently small, this τ-periodic solution of (1.3) found in [1] must be a constant solution (also [1, Remark 9.5]). By the same reason, when $\tau \in (0, \tau_c)$, Theorem 3 with $h = 0$ in [22] yields also only constant solutions of the system (1.3).

(iii) When $\tau > \frac{2\pi}{\sqrt{\beta(\beta-\alpha)}}(\frac{\beta}{\alpha})^{\frac{\beta+2}{2(\beta-\alpha)}}$, by our Remark 4.2 a constant solution x possesses Morse index $i(x) \geq 3$ and thus cannot be a mountain pass solution. Then the τ-periodic solution of (1.3) found in [1] must be a non-constant solution. It is interesting to understand how this non-constant solution behaves, whether it is still an oscillating line solution, and how the transition happens from constant solutions to non-constant solutions as τ increases.

5 Non-zero Angular Momentum Solutions

Since every solution $x = x(t)$ of the system (1.3) is free from collision with the origin, $r = |x(t)|$ is never 0 if it is not initially. In this section, we always assume $c \neq 0$. By (4.2), we may justify the change of the variable from t to θ as

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{c}{r^2}\frac{d}{d\theta} \tag{5.1}$$

and obtain

$$\frac{c}{r^2}\frac{d}{d\theta}\left(\frac{c}{r^2}\frac{d}{d\theta}r\right) - \frac{c^2}{r^3} - \frac{\beta}{r^{\beta+1}} + \frac{\alpha}{r^{\alpha+1}} = 0. \tag{5.2}$$

Let $u = \frac{1}{r} \neq 0$ and $u_\theta = -\frac{1}{r^2} r_\theta$, and (5.2) is written as

$$-c^2 u^2 \frac{d}{d\theta} u_\theta - c^2 u^3 - \beta u^{\beta+1} + \alpha u^{\alpha+1} = 0,$$

i.e.,

$$u_{\theta\theta} + u + \frac{1}{c^2}(\beta u^{\beta-1} - \alpha u^{\alpha-1}) = 0. \tag{5.3}$$

We define the effective potential $F(u,c)$ by

$$F(u,c) = \frac{1}{2}u^2 + \frac{1}{c^2}u^\beta - \frac{1}{c^2}u^\alpha,$$

where $u > 0$ and c is the angular momentum defined in (4.2). In rest of this section, when the angular momentum c is fixed, we write $F(u)$ instead of $F(u,c)$, write $F'(u)$ instead of $\frac{\partial F}{\partial u}$ and write $F''(u)$ instead of $\frac{\partial^2 F}{\partial u^2}$, if it does not cause any confusion.

For any fixed c, the Hamiltonian function of the system (5.3) is

$$\begin{aligned}\bar{H}(u,u_\theta) &= \frac{1}{2}|u_\theta|^2 + F(u,c)\\ &= \frac{1}{2}|u_\theta|^2 + \frac{1}{2}u^2 + \frac{1}{c^2}u^\beta - \frac{1}{c^2}u^\alpha.\end{aligned} \tag{5.4}$$

The Hamiltonian energy of (2.1) are related with the Hamiltonian energy of (5.3) by

$$H(x,\dot{x}) = c^2 \bar{H}(u,u_\theta). \tag{5.5}$$

Proposition 5.1 *Suppose $F(u)$ attains its critical value at $u_0 > 0$ for some given angular momentum c. The system* (5.3) *possesses a constant solution $u(\theta) \equiv u_0$ for all $\theta \in \mathbb{R}$. Correspondingly, the system* (1.3) *possesses a circular solution*

$$x(t) = \frac{1}{u_0}(\cos\omega t, \sin\omega t), \tag{5.6}$$

where $\omega = cu_0^2$.

Proof Since $u_{\theta\theta} = -F'(u_0) = 0$, $u(\theta) \equiv u_0$ for all $\theta \in \mathbb{R}$ is a constant solution of the system (5.3). Because the angular momentum c satisfies $c = |x|^2\dot{\theta}$, we have $\dot{\theta} = cu_0^2$. Thus the period of $x(t)$ is $\tau = \frac{2\pi}{cu_0^2}$. Since $|x| = \frac{1}{u_0}$, $x(t)$ is a circular periodic solution given by

$$x(t) = \frac{1}{u_0}(\cos\omega t, \sin\omega t),$$

where $\omega = cu_0^2$. □

If u is a solution of (5.3), its Hamiltonian energy h satisfies $h = \bar{H}(u,u_\theta)$. This yields that

$$d\theta = \pm\frac{du}{\sqrt{2(h - F(u,c))}}. \tag{5.7}$$

Integrating both sides of (5.7), let

$$\theta(h,c) = \int_{u_1}^{u_2} \frac{du}{\sqrt{2(h - F(u,c))}}, \tag{5.8}$$

where u_1 and u_2 are two different solutions of $F(u,c) = h$ satisfying $F(u,c) < h$ when $u_1 < u < u_2$.

Proposition 5.2 *For given angular momentum c and $h = \bar{H}(u, u_\theta)$, suppose $u_1 < u < u_2$ satisfy $F(u, c) < h = F(u_1, c) = F(u_2, c)$, then there exists a periodic solution $u(\theta)$ with energy h, period $2\theta(h, c)$, $\min u(\theta) = u_1$, and $\max u(\theta) = u_2$. If $\frac{\theta(h,c)}{\pi}$ is rational, $u(\theta)$ corresponds to a periodic solution $x(t)$ of (1.3); if $\frac{\theta(h,c)}{\pi}$ is irrational, $u(\theta)$ corresponds to a quasi-periodic solution $x(t)$ of (1.3) with two frequencies at the ratio $\frac{\theta(h,c)}{\pi}$. More precisely, if $u(\theta)$ is a periodic solution of (5.3) and $\frac{\theta(h,c)}{\pi} = \frac{q}{p}$ with $(p, q) = 1$, the corresponding configuration path of $x(t)$ in $\mathbb{R}^2$ possesses the topological degree q with respect to the origin and achieves the maximal distance to the origin p times.*

Proof For such given c and h, from the standard ODE theory, the solution $u(\theta)$ with $u(0) = u_1$, $u_\theta(0) = 0$ is periodic and satisfies $\min u(\theta) = u_1$ and $\max u(\theta) = u_2$, with period $2\theta(h, c)$.

We express the configuration path $x(t) \in \mathbb{R}^2$ of a solution to (1.2) in the form of the parametric equation with the parameter θ as

$$x(\theta) = \frac{1}{u(\theta)}(\cos\theta, \sin\theta). \tag{5.9}$$

The motion of $u(\theta)$ is oscillating between u_1 and u_2. The motion of $x(t)$ is in the region of the annulus whose the outer radius $|x| = \frac{1}{u_1}$ and the inner radius are $|x| = \frac{1}{u_2}$. The half period $\theta(h, c)$ of $u(\theta)$ is the angle of x in $\mathbb{R}^2$ between consecutive $\frac{1}{u_2}$ and $\frac{1}{u_1}$. If $2\theta(h, c)$ dependents rationally on 2π, i.e.,

$$\frac{q}{p} \cdot \pi = \theta(h, c) = \int_{u_1}^{u_2} \frac{du}{\sqrt{2(h - F(u, c))}}, \tag{5.10}$$

where p and q are co-prime positive integers, $x(\theta)$ is a closed path of the system (1.3) in $\mathbb{R}^2$ where p is the number of times of $|x(\theta)|$ realizing $\frac{1}{u_1}$ or $\frac{1}{u_2}$ in one period and q is the topological degree of the orbit $x(\theta)$ with respect to the origin in one period.

If $\theta(h, c)$ satisfies

$$w \cdot \pi = \theta(h, c) = \int_{u_1}^{u_2} \frac{du}{\sqrt{2(h - F(u, c))}}, \tag{5.11}$$

where w is irrational, then $x(t)$ is a quasi-periodic solution and orbit of x is dense in the annulus of $\mathbb{R}^2$. □

Remark 5.3 Let $u(\theta)$ be a periodic solution of (5.3) with $\bar{H}(u, u_\theta) = h$, whose period is given by $2\theta(h, c)$. Let $\theta(t)$ be the solution of

$$\dot{\theta} = cu(\theta)^2, \quad \theta(0) = 0.$$

Clearly

$$\theta(t + T) = \theta(t) + 2\theta(h, c), \quad \forall t \in \mathbb{R}, \quad \text{where } T = \min\{t > 0 \mid \theta(t) = 2\theta(h, c)\} > 0.$$

From (5.1) and the definition of $u(\theta)$, if we identify $\mathbb{R}^2$ with $\mathbb{C}$,

$$x(t) = \frac{1}{u(\theta(t))} e^{i\theta(t)}, \quad \dot{x}(t) = c(iu(\theta(t)) - u_\theta(\theta(t)))e^{i\theta(t)} \tag{5.12}$$

is a solution of (1.3). The orbits of its all phase rotations $(e^{i\theta_0} x(t), e^{\theta_0}\dot{x}(t))$ foliate the invariant torus $\mathbb{T}(h, c) = \Psi(\mathbb{T}^2)$ of (1.3), where $\mathbb{T}^2$ is the standard 2-dimensional torus with period 2π

and $\Psi : \mathbb{T}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ is given by

$$\Psi(s_1, s_2) = \left(\frac{1}{u(\theta(\frac{T}{2\pi}s_1))}, c\left(\mathrm{i}u\left(\theta\left(\frac{T}{2\pi}s_1\right)\right) - u_\theta\left(\theta\left(\frac{T}{2\pi}s_1\right)\right)\right)\right)\mathrm{e}^{\mathrm{i}(s_2+\tilde{\theta}(\frac{T}{2\pi}s_1))},$$

where $\tilde{\theta}(t) = \theta(t) - \frac{2\theta(h,c)}{T}t$ is T-periodic. In terms of Ψ we can write

$$(x(t), \dot{x}(t)) = \Psi\left(\frac{2\pi}{T}t, \frac{2\theta(h,c)}{T}t\right).$$

If $\frac{\theta(h,c)}{\pi}$ is irrational, this solution is quasi-periodic with two frequencies at the ratio $\frac{\theta(h,c)}{\pi}$.

Finally we note that the invariant torus $\mathbb{T}(h,c)$ is actually the intersection of a level surface of the energy and a level surface of the angular momentum of (1.3). We refer readers to [4] for discussions on solutions of (1.3) on such surfaces with constant Hamiltonian energy and constant angular momentum.

Corollary 5.4 *For given c, if the range of $\theta(c,h)$ has a non-empty interior, then the system* (5.3) *possesses infinitely many periodic solutions. Correspondingly, the system* (1.3) *possesses at least countably many periodic solutions with arbitrarily large topological degree and uncountably many quasi-periodic solutions with constant angular momentum.*

Proof For a given constant c, if the range of $\theta(h,c)$ contains an open interval I, there are countably many rational numbers $\frac{q}{p} \in I$ satisfying q and p are co-prime integers, i.e., $(q,p) = 1$, such that (5.10) holds. Different such fractions $\frac{q}{p}$ correspond to different solutions of (5.3). By Proposition 5.2, (1.3) possesses at least countably many distinct periodic solutions with distinct energy. Furthermore, q can be arbitrarily large in the interval I. This indicates that the degree of the solution of (1.3) can be arbitrarily large.

Similarly, if the range of $\theta(h,c)$ contains an open interval I, there are uncountably many irrational numbers $w \in I$ such that (5.11) holds. This implies that the system (1.3) possesses uncountably many quasi-periodic solutions. □

Proposition 5.5 *If there exists some $u_h > 0$ such that $h = F(u_h) > 0$, $F'(u_h) > 0$, and $F(u) < h$ for all $u \in (0, u_h)$, then there exists a solution $u = u(\theta)$ of* (5.3) *satisfying $u(0) = u_h$ and $\bar{H}(u, u_\theta) = h$. Furthermore, the solution $u(\theta)$ exists only on the open interval $(-\theta_0, \theta_0)$ for some $\theta_0 > 0$ such that $u(\theta)$ converges to 0 when θ converges to $\pm\theta_0$. This solution $u = u(\theta)$ corresponds to an asymptotic solution $x = x(t)$ of the system* (1.3) *given by* (5.12) *with asymptotic angle $2\theta_0$ and $\min_{t\in\mathbb{R}} |x(t)| = \frac{1}{u_h}$.*

Proof Extend $F(u)$ as an even function. Then $h = F(u_h) = F(-u_h) > 0$, $F'(u_h) > 0$, $F'(-u_h) < 0$ and $F(u) < h$ for all $u \in (-u_h, u_h)$. Note first that the energy equation $\bar{H}(u, u_\theta) = h > 0$ defines a closed curve in the (u, u_θ) space $\mathcal{U}$ around the origin $(0,0) \in \mathcal{U}$ if we allow $u \in \mathbb{R}$, and this curve defines a periodic solution $u = u(\theta)$ of (5.3). Without loss of generality, we suppose $u(0) = u_h > 0$. By (5.4), this curve is symmetric about the u-axis and $u(\theta) = u(-\theta)$ and $u_\theta(-\theta) = -u_\theta(\theta)$ hold for all $\theta > 0$. Thus there exists a unique $\theta_0 > 0$ such that $u(\pm\theta_0) = 0$.

Whenever $u(\theta) > 0$, (5.9) defines a function $x = x(\theta)$. Using the function $\theta = \theta(t)$ defined in Remark 5.3, we obtain a solution $x(t) = x(\theta(t))$ of (1.3) given by (5.12) with $\lim_{t\to\pm\infty} \theta(t) = \pm\theta_0$. Correspondingly, $x = x(t)$ satisfies $\min_{t\in\mathbb{R}} |x(t)| = \frac{1}{u_h}$ and $\lim_{\theta\to\pm\theta_0} = |x(\theta)| = \infty$. Therefore, the asymptotic angle of $x(t)$ is $2\theta_0$. □

5.1 The Solutions When $0<\alpha<2$

When $0<\alpha<2$ and $0<\alpha<\beta$, $F(u,c)=\frac{1}{c^2}u^\beta+\frac{1}{2}u^2-\frac{1}{c^2}u^\alpha$. For any given c satisfying $0<c<+\infty$, $F(u)=0$ has exact one positive root denoted by u_c. And $F'(u)=0$ has only one positive root denoted by $u_{\min}$ which is the local minima of $F(u)$. Furthermore, $0<u_{\min}<u_c$ and $F'(u_c)>0$. When $0<u<u_{\min}$, $F'(u)<0$ and when $u>u_{\min}$, $F'(u)>0$ and $\lim_{u\to\infty}F(u)=\infty$.

Proposition 5.6 *When $0<\alpha<2$, $0<\alpha<\beta$ and $c\neq 0$,*

(i) *if $h=F(u_{\min})$, the system* (5.3) *possesses one constant solution $u(\theta)\equiv u_{\min}$ where $u_{\min}$ is the unique global minimizer of $F(u)$. Correspondingly, the system* (1.3) *possesses one circular solution*

$$x(t)=\frac{1}{u_{\min}}(\cos\omega t,\sin\omega t),$$

where $\omega=cu_{\min}^2$;

(ii) *any solution of* (1.3) *with energy $h>0$ is an asymptotic solution.*

Proof (i) Since $F'(u_{\min})=0$, we can apply Proposition 5.1 and obtain the results directly.

(ii) Suppose that $u_h>u_c$ satisfying $F(u_h)=h>0$. By the monotonicity of $F(u)$, $F'(u_h)>0$ and $F(u)<F(u_h)$ when $0<u<u_h$. Then we can apply Proposition 5.5 and obtain the results directly. □

By Proposition 5.6, the Hamiltonian energy h of (5.3) satisfies $F(u_{\min})\le h<0$ is a necessary condition for the solution $x(t)$ of (1.3) to be globally bounded when $0<\alpha<2$ and $0<\alpha<\beta$.

Theorem 5.7 *When $0<\alpha<2$, $0<\alpha<\beta$, $\alpha\neq 1$, $|c|$ is sufficiently large and $h\in(F(u_{\min}),0)$, all solutions $u(\theta)$ of* (5.3) *are periodic in θ. They correspond to at least countably many periodic solutions of* (1.3) *with arbitrarily large topological degree and uncountably many quasi-periodic solutions.*

Proof When $0<\alpha<2$ and $0<\alpha<\beta$, for any given $c\neq 0$, $F(u)=0$ has only one positive root u_c.

$$F(u_c)=\frac{1}{c^2}u_c^\beta+\frac{1}{2}u_c^2-\frac{1}{c^2}u_c^\alpha=0. \tag{5.13}$$

Because $F(1)>0$, we have $0<u_c<1$. We rewrite (5.13) as

$$\frac{1}{2}u_c^{2-\alpha}=\frac{1}{c^2}-\frac{1}{c^2}u_c^{\beta-\alpha}. \tag{5.14}$$

We obtain $\lim_{c\to\infty}u_c=0$. Since $0<u_c<1$ for any given $c\neq 0$, we have

$$0<u_c<\left(\frac{c^2}{2}\right)^{\frac{1}{\alpha-2}}.$$

Furthermore, by (5.14), when $|c|$ tends to infinity,

$$u_c=\left(\frac{c^2}{2}\right)^{\frac{1}{\alpha-2}}-o\left(\frac{c^2}{2}\right)^{\frac{1}{\alpha-2}}. \tag{5.15}$$

When $h=0$, we calculate the $\lim_{|c|\to\infty}\theta(0,c)$ as

$$\lim_{|c|\to\infty}\theta(0,c)=\lim_{|c|\to\infty}\int_0^{u_c}\frac{du}{\sqrt{2(0-F(u,c))}}$$

$$= \lim_{|c|\to\infty} \frac{1}{\sqrt{2}} \int_0^{u_c} \frac{du}{\sqrt{-\frac{1}{c^2}u^\beta - \frac{1}{2}u^2 + \frac{1}{c^2}u^\alpha}}. \tag{5.16}$$

We change the variable u to s and u_c to s_c by following two equalities,

$$u = \left(\frac{c^2}{2}\right)^{\frac{1}{\alpha-2}} s^{\frac{1}{2-\alpha}} \quad \text{and} \quad u_c = \left(\frac{c^2}{2}\right)^{\frac{1}{\alpha-2}} s_c^{\frac{1}{2-\alpha}}. \tag{5.17}$$

By (5.15) and (5.17), $0 < s_c < 1$ for any given c and $\lim_{c\to\infty} s_c = 1$. Plugging (5.17) into (5.16), we have

$$\lim_{|c|\to\infty} \theta(0,c) = \frac{1}{\sqrt{2}(2-\alpha)} \lim_{c\to\infty} \int_0^{s_c} \frac{ds}{\sqrt{-c^{\frac{2(\beta-\alpha)}{\alpha-2}} 2^{\frac{\beta-2}{2-\alpha}} s^{\frac{\beta+2-2\alpha}{2-\alpha}} - \frac{1}{2}s^2 + \frac{1}{2}s}}. \tag{5.18}$$

To simplify the notations, let $A_c \equiv c^{\frac{2(\beta-\alpha)}{\alpha-2}} 2^{\frac{\beta-2}{2-\alpha}} > 0$ satisfying $\lim_{c\to\infty} A_c = 0$. Since u_c is the only positive root of $F(u) = 0$, s_c satisfies that

$$-A_c s_c^{\frac{\beta+2-2\alpha}{2-\alpha}} - \frac{1}{2}s_c^2 + \frac{1}{2}s_c = 0. \tag{5.19}$$

Let $s = s_c t$ and $t \in [0,1]$. Then (5.18) can be written as

$$\lim_{c\to\infty} \theta(0,c) = \frac{1}{\sqrt{2}(2-\alpha)} \lim_{c\to\infty} \int_0^1 \frac{s_c dt}{\sqrt{-A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}} - \frac{1}{2}s_c^2 t^2 + \frac{1}{2}s_c t}}. \tag{5.20}$$

By (5.18), when $t = 0$ or 1,

$$-\left(\frac{s_c^2}{2} - \frac{1}{4}\right)t^2 - A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}} + \left(\frac{s_c}{2} - \frac{1}{4}\right)t = 0. \tag{5.21}$$

When s_c is sufficiently close to 1, $-(\frac{s_c^2}{2} - \frac{1}{4}) < 0$, $-A_c s_c^{\frac{\beta+2-2\alpha}{2-\alpha}} < 0$ and $\frac{s_c}{2} - \frac{1}{4} > 0$. By the generalized Descartes' rule of signs in [12], we know that $t = 1$ is the only positive root of (5.21). Additionally,

$$-\left(\frac{s_c^2}{2} - \frac{1}{4}\right)t^2 - A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}} + \left(\frac{s_c}{2} - \frac{1}{4}\right)t > 0, \tag{5.22}$$

when $t > 0$ is sufficiently small. Therefore, (5.22) holds for all $t \in (0,1)$. Then (5.22) implies that

$$\frac{1}{2}s_c t - \frac{1}{2}s_c^2 t^2 - A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}} > \frac{1}{4}(t - t^2) > 0, \tag{5.23}$$

for all $t \in (0,1)$. When $\frac{1}{\sqrt{2}} < s_c \le 1$ and $A_c > 0$,

$$\frac{s_c}{\sqrt{\frac{1}{2}s_c t - \frac{1}{2}s_c^2 t^2 - A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}}}} < \frac{1}{\sqrt{\frac{1}{4}t - \frac{1}{4}t^2}} \tag{5.24}$$

holds for all $t \in (0,1)$. Furthermore, the right hand of (5.24) is integrable. By the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{c\to\infty} \theta(0,c) &= \frac{1}{\sqrt{2}(2-\alpha)} \lim_{c\to\infty} \int_0^1 \frac{s_c dt}{\sqrt{-A_c(s_c t)^{\frac{\beta+2-2\alpha}{2-\alpha}} - \frac{1}{2}s_c^2 t^2 + \frac{1}{2}s_c t}} \\ &= \frac{1}{\sqrt{2}(2-\alpha)} \int_0^1 \frac{dt}{\sqrt{\frac{1}{2}t - \frac{1}{2}t^2}} = \frac{\pi}{2-\alpha}. \end{aligned} \tag{5.25}$$

When $0 < \alpha < 2$, $u_{\min} \neq 0$ is the solution of $F'(u) = 0$. By $\lim_{c\to\infty} u_c = 0$ and $0 < u_{\min} < u_c$, we obtain $\lim_{c\to\infty} u_{\min} = 0$. When h tends to $F(u_{\min})$, for any given $c \neq 0$, the half period of the system (5.3) can be computed as

$$\lim_{h\to F(u_{\min})} \theta(h,c) = \frac{\pi}{\sqrt{F''(u_{\min})}} \tag{5.26}$$

and it is well defined because $F(u_{\min})$ is a non-degenerate local minima. Therefore we obtain

$$\begin{aligned} \lim_{c\to\infty} \lim_{h\to F(u_{\min})} \theta(h,c) &= \lim_{c\to\infty} \frac{\pi}{\sqrt{F''(u_{\min})}} \\ &= \lim_{c\to\infty} \frac{c\pi}{\sqrt{\beta(\beta-1)u_{\min}^{\beta-2} + c^2 - \alpha(\alpha-1)u_{\min}^{\alpha-2}}}. \end{aligned} \tag{5.27}$$

For any given $c \neq 0$, $u_{\min}$ satisfies $F'(u_{\min}) = 0$. We plug $\alpha u_{\min}^{\alpha-2} = \beta u_{\min}^{\beta-2} + c^2$ into (5.27) and obtain

$$\begin{aligned} \lim_{c\to\infty} \lim_{h\to F(u_{\min})} \theta(h,c) &= \lim_{c\to\infty} \frac{c\pi}{\sqrt{\beta(\beta-\alpha)u_{\min}^{\beta-2} + (2-\alpha)c^2}} \\ &= \frac{\pi}{\sqrt{2-\alpha}}. \end{aligned} \tag{5.28}$$

If $\alpha \neq 1$, by the two results (5.25) and (5.28), we have

$$\lim_{c\to\infty} \theta(0,c) \neq \lim_{c\to\infty} \lim_{h\to F(u_{\min})} \theta(h,c). \tag{5.29}$$

Thus, when $|c|$ is sufficiently large, we still have

$$\theta(0,c) \neq \lim_{h\to F(u_{\min})} \theta(h,c). \tag{5.30}$$

Therefore, when $|c|$ is sufficiently large, the range of $\theta(h,c)$ denoted by $I_{c,\infty}$ contains a non-empty interval. By Proposition 5.2 and Corollary 5.4, for any given $|c|$ large enough, the system (1.3) possesses infinitely many periodic solutions with arbitrarily large topological degree and infinitely many quasi-periodic solutions. □

Before proving Theorem 5.9, we prove Lemma 5.8 as a preparation. We use $B(\cdot,\cdot)$ to denote the Beta function.

Lemma 5.8 *If* $0 < \alpha < 2$ *and* $0 < \alpha < \beta$,

$$\frac{1}{\sqrt{2}(\beta-\alpha)} B\left(\frac{2-\alpha}{2(\beta-\alpha)}, \frac{1}{2}\right) = \frac{\pi}{\sqrt{\beta(\beta-\alpha)}} \left(\frac{\beta}{\alpha}\right)^{\frac{\beta-2}{2(\beta-\alpha)}} \tag{5.31}$$

holds on a zero measure set $\mathcal{B}$ *of* α *and* β *in* $\mathbb{R}^2$.

Proof (5.31) is equivalent to

$$B\left(\frac{2-\alpha}{2(\beta-\alpha)}, \frac{1}{2}\right) = \sqrt{2}\pi\sqrt{\frac{\beta}{\alpha} - 1}\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha-2}{2(\beta-\alpha)}}. \tag{5.32}$$

We define

$$p = \frac{2-\alpha}{2(\beta-\alpha)} \quad \text{and} \quad q = \frac{\beta}{\alpha}. \tag{5.33}$$

(5.32) can be rewritten as

$$B\left(p, \frac{1}{2}\right) = \sqrt{2}\pi\sqrt{q-1}q^{-p}. \tag{5.34}$$

Define

$$\varphi = \sqrt{q-1}q^{-p}.$$

For any fixed p, $\frac{1}{\sqrt{2}\pi}B(p, \frac{1}{2})$ is a constant. Take the derivative of φ with respect to q and obtain

$$\varphi_q = \frac{(1-2p)q + 2p}{2q^{p+1}\sqrt{q-1}}.$$

If $1 - 2p = 0$, i.e., $\beta = 2$, it yields $\varphi_q = \frac{2p}{2q^{p+1}\sqrt{q-1}} \neq 0$ when $0 < \alpha < 2$. When $\beta = 2$, we have $p = \frac{1}{2}$, $q = \frac{2}{\alpha}$ and $\varphi = \sqrt{1 - \frac{\alpha}{2}}$. Therefore, (5.34) holds when $\alpha = 1$ and $\beta = 2$.

When $\beta \neq 2$, $\varphi_q = 0$ if and only if

$$q = \frac{2p}{2p-1}. \tag{5.35}$$

Plugging (5.33) into (5.35), we obtain

$$\frac{\beta}{\alpha} = \frac{2-\alpha}{2-\beta}. \tag{5.36}$$

Since $0 < \alpha < 2$ and $0 < \alpha < \beta$, (5.36) holds when

$$0 < \alpha < 1 \quad \text{and} \quad \beta = 2 - \alpha. \tag{5.37}$$

Above all, (5.31) holds at most on a zero measure set. We define $\mathcal{B}$ as the zero measure set such that (5.31) holds. □

Theorem 5.9 *When* $0 < \alpha < 2$, $0 < \alpha < \beta$, $\alpha, \beta \notin \mathcal{B}$, $|c|$ *is sufficiently small and* $h \in (F(u_{\min}), 0)$, *all solutions of* (5.3) *are periodic in* θ. *They correspond to at least countably many periodic solutions of* (1.3) *with arbitrarily large topological degree and uncountably many quasi-periodic solutions.*

Proof When $|c|$ is sufficiently small, it happens that

$$\lim_{|c|\to 0}\theta(0,c) = \lim_{|c|\to 0}\lim_{h\to F(u_{\min})}\theta(h,c) = 0. \tag{5.38}$$

To distinguish the two values of $\lim_{c\to 0}\theta(0,c)$ and $\lim_{c\to 0}\lim_{h\to F(u_{\min})}\theta(h,c)$ when $|c|$ is sufficiently small, we change the system (5.3) to

$$c^2u_{\theta\theta} + c^2u + \beta u^{\beta-1} - \alpha u^{\alpha-1} = 0.$$

Setting $\frac{d}{d\tau} = c\frac{d}{d\theta}$, we have

$$u_{\tau\tau} + c^2u + \beta u^{\beta-1} - \alpha u^{\alpha-1} = 0. \tag{5.39}$$

We define the effective potential of (5.39) by

$$G(u,c) \equiv \frac{c^2}{2}u^2 + u^\beta - u^\alpha.$$

For simplicity, we often use $G(u)$ to denote $G(u,c)$, use $G'(u)$ to denote $\frac{\partial G}{\partial u}$, and use $G''(u)$ to denote $\frac{\partial^2 G}{\partial u^2}$ when c is given.

For any given $c \neq 0$, $G(u) = 0$ has only one positive root denoted by u_c and $G'(u) = 0$ has only one positive root denoted by $u_{\min}$. Since $G(1) > 0$ when $c \neq 0$, we have $0 < u_c < 1$. From $G(u_c) = 0$, we have $1 - u_c^{\beta-\alpha} = \frac{c^2}{2} u_c^{2-\alpha} \to 0$ as $c \to 0$. Then we have $\lim_{c\to 0} u_c = 1$.

The corresponding Hamiltonian energy of (5.39) is

$$h = \frac{1}{2}|u_\tau|^2 + G(u, c). \tag{5.40}$$

For any given sufficiently small $|c|$ and $h \in (G(u_{\min}), 0)$, the half period $\tau(h, c)$ of (5.39) is given by

$$\tau(h, c) = \int_{u_1}^{u_2} \frac{du}{\sqrt{2(h - G(u, c))}}, \tag{5.41}$$

where u_1 and u_2 are two consecutive different roots of $G(u, c) - h = 0$ where $u_1 < u_2$.

In this proof, the calculations proceed similarly as in the proof of Theorem 5.7. We consider $\tau(h, c)$ when $h = 0$ and c tends to 0.

$$\begin{aligned}
\lim_{c\to 0} \tau(0, c) &= \lim_{c\to 0} \int_0^{u_c} \frac{du}{\sqrt{2(0 - G(u, c))}} \\
&= \frac{1}{\sqrt{2}} \lim_{c\to 0} \int_0^{u_c} \frac{du}{\sqrt{-u^\beta - \frac{c^2}{2}u^2 + u^\alpha}} \\
&= \frac{1}{\sqrt{2}} \int_0^1 \frac{du}{\sqrt{u^\alpha - u^\beta}} \\
&= \frac{1}{\sqrt{2}(\beta - \alpha)} B\left(\frac{2-\alpha}{2(\beta-\alpha)}, \frac{1}{2}\right).
\end{aligned} \tag{5.42}$$

When h tends to $G(u_{\min})$ and c tends to 0, we have

$$\begin{aligned}
\lim_{c\to 0} \lim_{h\to G''(u_{\min})} \tau(h, c) &= \lim_{c\to 0} \frac{\pi}{\sqrt{G''(u_{\min})}} \\
&= \lim_{c\to 0} \frac{\pi}{\sqrt{\beta(\beta-1)u_{\min}^{\beta-2} + c^2 - \alpha(\alpha-1)u_{\min}^{\alpha-2}}}.
\end{aligned} \tag{5.43}$$

Since $G'(u_{\min}) = c^2 u_{\min} + \beta u_{\min}^{\beta-1} - \alpha u_{\min}^{\alpha-1} = 0$ and $0 < u_{\min} < u_c < 1$, from $1 - \frac{\beta}{\alpha} u_{\min}^{\beta-\alpha} = \frac{c^2}{\alpha} u_{\min}^{2-\alpha} \to 0$ as $c \to 0$, we have

$$\lim_{c\to 0} u_{\min} = \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\beta-\alpha}}. \tag{5.44}$$

Plugging (5.44) into (5.43), we obtain

$$\begin{aligned}
\lim_{c\to 0} \lim_{h\to G''(u_{\min})} \tau(h, c) &= \frac{\pi}{\sqrt{\beta(\beta-1)(\frac{\alpha}{\beta})^{\frac{\beta-2}{\beta-\alpha}} - \alpha(\alpha-1)(\frac{\alpha}{\beta})^{\frac{\alpha-2}{\beta-\alpha}}}} \\
&= \frac{\pi}{\sqrt{\beta(\beta-\alpha)}} \left(\frac{\beta}{\alpha}\right)^{\frac{\beta-2}{2(\beta-\alpha)}}.
\end{aligned} \tag{5.45}$$

By Lemma 5.8, if $\alpha, \beta \notin \mathcal{B}$, we obtain

$$\lim_{c\to 0} \tau(0, c) \neq \lim_{c\to 0} \lim_{h\to G''(u_{\min})} \tau(h, c).$$

Therefore, when $|c|$ is sufficiently small

$$\tau(0,c) \neq \lim_{h\to G''(u_{\min})} \tau(h,c). \tag{5.46}$$

The range of $\tau(h,c)$ denoted by $I_{c,0}$ contains a nontrivial open interval when $|c|$ is sufficiently small. By Proposition 5.2 and Corollary 5.4, when $|c|$ is sufficiently small, the system (1.3) possesses infinitely many periodic solutions with arbitrarily large topological degree and infinitely many quasi-periodic solutions. □

5.2 The Solutions When $2 = \alpha < \beta$

When $\alpha = 2$ and $0 < \alpha < \beta$, $F(u,c)$ can be written in descending order as

$$F(u,c) = \frac{1}{c^2}u^\beta + \left(\frac{1}{2} - \frac{1}{c^2}\right)u^2.$$

Proposition 5.10 *If $2 = \alpha < \beta$ and $|c| \geq \sqrt{2}$, the system* (1.3) *possesses only asymptotic solutions.*

Proof When $\alpha = 2$ and $|c| \geq \sqrt{2}$, we have $\frac{1}{2} > \frac{1}{c^2}$ and $F(u)$ is a monotonically increasing function and $F(u) > 0$ for $u \in (0, +\infty)$. By Proposition 5.5, we can conclude that the system (1.3) possesses only asymptotic solutions. □

In the rest of this section, we assume that $0 < |c| < \sqrt{2}$ and $F(u)$ attains its unique local minima at $u_{\min}$. When $0 < u < u_{\min}$, $F(u)$ is monotonically increasing, and when $u > u_{\min}$, $F(u)$ is monotonically decreasing.

Theorem 5.11 *Let $2 = \alpha < \beta$ and $h = \bar{H}(u, u_\theta)$. If the given c satisfies $0 < |c| < \sqrt{2}$,*

(i) *for $h > 0$, the system* (1.3) *possesses only asymptotic solutions*;

(ii) *for $h = F(u_{\min})$ of* (5.3)*, the system* (5.3) *possesses a constant solution. Correspondingly, the system* (1.3) *possesses a circular solution*;

(iii) *for $F(u_{\min}) < h < 0$, all solutions of* (5.3) *are periodic in θ. They correspond to at least countably many periodic solutions of* (1.3) *with arbitrarily large topological degree and uncountably many quasi-periodic solutions.*

Proof (i) When the Hamiltonian energy $h > 0$, $F(u) - h = 0$ has only one positive root $u_h > 0$ and $F'(u_h) > 0$. By the monotonicity of $F(u)$, $F(u) < F(u_h)$ when $0 < u < u_h$. Then we can apply Proposition 5.5 and obtain infinitely many asymptotic solutions.

(ii) Since $F(u)$ attains its local minima at $u_{\min}$, we can obtain a constant solution $u(\theta) \equiv u_{\min}$ for all $\theta \in \mathbb{R}$. By Proposition 5.1, we obtain the circular solution of (1.3) as

$$x(t) = \frac{1}{u_{\min}}(\cos\omega t, \sin\omega t),$$

where $\omega = cu_{\min}^2$.

(iii) If $0 < |c| < \sqrt{2}$, i.e., $\frac{1}{2} < \frac{1}{c^2}$, $u_{\min} = (\frac{2-c^2}{\beta})^{\frac{1}{\beta-2}}$ and $u_c = (\frac{2-c^2}{2})^{\frac{1}{\beta-2}}$. We have half period of (5.3) satisfies

$$\lim_{h\to F(u_{\min})} \theta(h,c) = \frac{\pi}{\sqrt{F''(u_{\min})}} = \frac{\pi c}{\sqrt{\beta(\beta-1)u_{\min}^{\beta-2} + c^2 - 2}} = \frac{c\pi}{\sqrt{(\beta-2)(2-c^2)}}. \tag{5.47}$$

When $h = 0$, we have

$$\theta(0,c) = \int_0^{u_c} \frac{du}{\sqrt{2(0 - F(u))}} = \int_0^{u_c} \frac{cdu}{\sqrt{(2-c^2)u^2 - 2u^\beta}}. \tag{5.48}$$

We let $u = u_c t$ and obtain

$$\theta(0,c) = \int_0^1 \frac{\sqrt{2}cu_c dt}{\sqrt{(2-c^2)(u_c t)^2 - 2(u_c t)^\beta}} \geq \int_0^1 \frac{cdt}{\sqrt{(2-c^2)t^2}} = +\infty. \tag{5.49}$$

For any given c satisfying $0 < |c| < \sqrt{2}$, (5.47) and (5.49) yield the range of $\theta(h,c)$ denoted by I_c satisfies

$$\left(\frac{c\pi}{\sqrt{(\beta-2)(2-c^2)}}, +\infty\right) \subset I_c. \tag{5.50}$$

By the Proposition 5.2 and Corollary 5.4, for any given c satisfying $0 < |c| < \sqrt{2}$, the system (1.3) possesses at least countably many periodic solutions with arbitrarily large topological degree and uncountably many quasi-periodic solutions. □

5.3 The Solutions When $2 < \alpha < \beta$

When $2 < \alpha < \beta$, we rewrite $F(u,c)$ in descending order of u as

$$F(u,c) = \frac{1}{c^2}u^\beta - \frac{1}{c^2}u^\alpha + \frac{1}{2}u^2.$$

Lemma 5.12 *If $2 < \alpha < \beta$ and c satisfies*

$$0 < |c| < \left(\frac{\alpha(\alpha-2)}{\beta(\beta-2)}\right)^{\frac{\alpha-2}{2(\beta-\alpha)}} \sqrt{\frac{\alpha(\beta-\alpha)}{\beta-2}}, \tag{5.51}$$

$F(u)$ has one local minima attained at $u_{\min}$ and one local maxima attained at $u_{\max}$. When $0 < u < u_{\max}$, $F(u)$ is monotonically increasing; when $u_{\max} < u < u_{\min}$, $F(u)$ is monotonically decreasing; when $u > u_{\min}$, $F(u)$ is monotonically increasing. If (5.51) does not hold, $F(u)$ is a monotonically increasing function.

Proof The derivative of $F(u)$ is

$$F'(u) = \frac{\beta}{c^2}u^{\beta-1} - \frac{\alpha}{c^2}u^{\alpha-1} + u = u\left(\frac{\beta}{c^2}u^{\beta-2} - \frac{\alpha}{c^2}u^{\alpha-2} + 1\right).$$

$F'(u) < 0$ for some $u > 0$ if and only if there exist some u such that $c^2 < \alpha u^{\alpha-2} - \beta u^{\beta-2}$ holds. We define $g(u)$ as

$$g(u) \equiv \alpha u^{\alpha-2} - \beta u^{\beta-2}. \tag{5.52}$$

$g(u)$ attains its global maxima at $u_{g,\max} = (\frac{\alpha(\alpha-2)}{\beta(\beta-2)})^{\frac{1}{\beta-\alpha}}$. Therefore, when

$$0 < |c| < \sqrt{g(u_{g,\max})} = \left(\frac{\alpha(\alpha-2)}{\beta(\beta-2)}\right)^{\frac{\alpha-2}{2(\beta-\alpha)}} \sqrt{\frac{\alpha(\beta-\alpha)}{\beta-2}},$$

$g(u) = c^2$ has two roots $u_{\max}$ and $u_{\min}$ where $u_{\max} < u_{g,\max} < u_{\min}$. When $u_{\max} < u < u_{\min}$, we have $g(u) > c^2$. This yields that $F'(u) < 0$. Then when $0 < u < u_{\max}$, $F(u)$ is monotonically increasing; when $u_{\max} < u < u_{\min}$, $F(u)$ is monotonically decreasing; when $u > u_{\min}$, $F(u)$ is monotonically increasing. Furthermore, $F(u)$ has one local maxima at $u_{\max}$ and one local minima at $u_{\min}$. □

Proposition 5.13 *If $2 < \alpha < \beta$ and*

$$|c| > \left(\frac{\alpha(\alpha-2)}{\beta(\beta-2)}\right)^{\frac{\alpha-2}{2(\beta-\alpha)}} \sqrt{\frac{\alpha(\beta-\alpha)}{\beta-2}},$$

the system (1.3) *possesses only asymptotic solutions.*

Proof If $|c| > \sqrt{g(u_{g,\max})}$, it yields $F'(u) > 0$ and $F(u)$ is a monotonically increasing function for all $u > 0$. By Proposition 5.5, the system (1.3) possesses only asymptotic solutions. $\square$

Theorem 5.14 *If* $2 < \alpha < \beta$ *and*

$$|c| = \left(\frac{\alpha(\alpha-2)}{\beta(\beta-2)}\right)^{\frac{\alpha-2}{2(\beta-\alpha)}}\sqrt{\frac{\alpha(\beta-\alpha)}{\beta-2}}, \tag{5.53}$$

the system (5.3) *possesses a constant solution. Correspondingly, the system* (1.3) *possesses a circular solution. Furthermore, the system* (1.3) *possesses infinitely many asymptotic solutions.*

Proof When (5.53) holds, $F'(u) = 0$ if $u = u_{g,\max}$ and $F'(u) > 0$ when $u \neq u_{g,\max}$. Then the system (5.3) a constant solution $u(\theta) \equiv u_{g,\max}$ for all $\theta \in \mathbb{R}$. By Proposition 5.1, we obtain the circular solution of (1.3) as

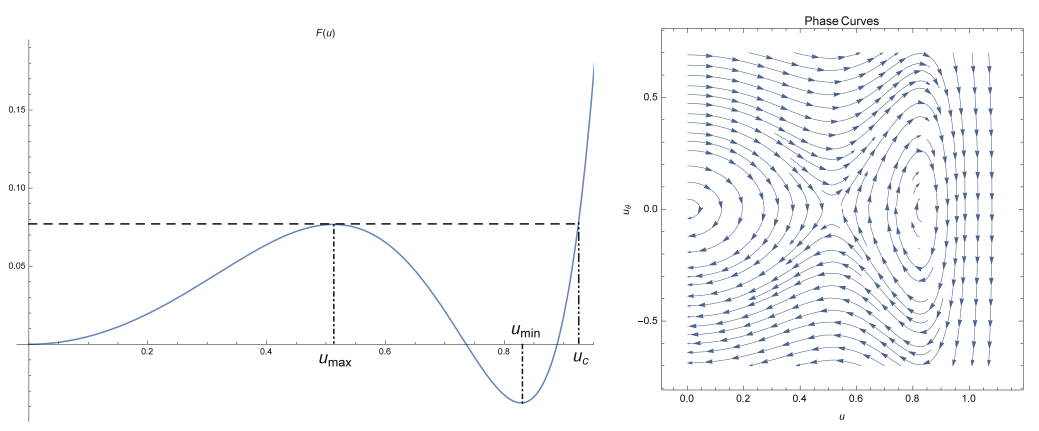
$$x(t) = \frac{1}{u_{g,\max}}(\cos\omega t, \sin\omega t) \tag{5.54}$$

where $\omega = cu_{g,\max}^2$.

If $u \neq u_{g,\max}$, $F'(u) > 0$ and the condition of Proposition 5.5 is satisfied. The system (1.3) possesses only asymptotic solutions. $\square$

In the rest of this subsection, we always assume (5.51) holds.

If the given c satisfies (5.51), there exists only one u_c satisfying $u_c > u_{\max}$ such that $F(u_c) - F(u_{\max}) = 0$ holds because of the monotonicity of $F(u)$ discussed in Lemma 5.12. Again by Lemma 5.12, we have $u_c \geq u_{\min}$ and $F'(u_c) > 0$.



(a) (b)

Figure 1 Figure (a) shows the graph of $F(u) = 1.6u^{10} - 1.6u^5 + \frac{1}{2}u^2$ where we have $\alpha = 5$, $\beta = 10$, and $c^2 = \frac{5}{8}$. Figure (b) shows the phase curves of (5.3) with the effective potential $F(u) = 1.6u^{10} - 1.6u^5 + \frac{1}{2}u^2$. $u_{\max}$ of $F(u)$ in (a) corresponds to the saddle point $(u_{\max}, 0)$ in (b) and $u_{\min}$ of $F(u)$ corresponds to the center point $(u_{\min}, 0)$ in (b). The arrows show the directions of flows at (u, u_θ). Along each flow line, (u, u_θ) possesses the same Hamiltonian energy.

Theorem 5.15 *When* $2 < \alpha < \beta$, $h = \bar{H}(u, u_\theta)$ *and* c *satisfies* (5.51), *every solution of* (1.3) *satisfying one of the following conditions must be an asymptotic solution*:

(i) *the Hamiltonian energy* h *satisfies* $h > F(u_{\max})$, *or*

(ii) *the Hamiltonian energy h satisfies* $0 < h < F(u_{\max})$ *and* $0 < u(0) < u_{\max}$.

Proof When condition (i) holds, we suppose $F(u_h) = h$. By the monotonicity of $F(u)$, we have that $F'(u_h) > 0$ and $F(u) < h$ for all $u \in (0, u_h)$. Then we can apply Proposition 5.5 and conclude that the solution is an asymptotic solution of (1.3) on each Hamiltonian energy surface.

If condition (ii) holds, $F(u)$ is in the monotonically increasing interval. If we define u_h by $F(u_h) = h$ and $0 < u_h < u_{\max}$, then $h > F(u)$ when $u \in (0, u_h)$. We also can apply Proposition 5.5 to obtain that the solution is an asymptotic solution of (1.3) on each Hamiltonian energy surface. □

For any given c satisfying (5.51), when the Hamiltonian energy of (5.3) h satisfying $F(u_{\min}) \leq h \leq F(u_{\max})$ and $u_{\max} < u < u_c$, there may exist the periodic solutions or quasi-periodic solutions. When $F(u_{\min}) < 0$, the figure of $F(u)$ is shown in Figure 5.1.

Theorem 5.16 *For* $2 < \alpha < \beta$, $h = \bar{H}(u, u_\theta)$ *and any given* c *satisfies* (5.51), *the followings hold.*

(i) *If* $h = F(u_{\min})$ *or* $h = F(u_{\max})$, *the system* (5.3) *possesses one constant solution. The constant solution corresponds to a circular solution of system* (1.3);

(ii) *if* $F(u_{\min}) < h < F(u_{\max})$ *and* $u_{\max} < u < u_c$, *all solutions of* (5.3) *are periodic in* θ. *They correspond to at least countably many periodic solutions of* (1.3) *with arbitrarily large topological degree and uncountably many quasi-periodic solutions*;

(iii) *there exists an orbit which is homoclinic to* $(u_{\max}, 0)$ *on the Hamiltonian energy surface* $h = F(u_{\max})$.

Proof Since $F'(u) = 0$ when $u = u_{\min}$ (or $u = u_{\max}$), we apply the Proposition 5.1 and obtain the constant solution of (5.3) and the corresponding circular solution of (1.3):

$$x(t) = \frac{1}{u}(\cos \omega t, \sin \omega t),$$

where $u = u_{\min}$ (or $u = u_{\max}$) and $\omega = cu_{\min}^2$ (or $\omega = cu_{\max}^2$).

When c satisfies (5.51), $F(u) - F(u_{\max}) = 0$ has two roots $u_{\max}$ and u_c. According to Theorem 5.14, we consider this theorem in the invariant subset $\{(u, u_\theta) : h < F(u_{\max}), u \in (u_{\max}, u_c)\}$ in the phase space of (5.3).

For any given c satisfies (5.51), when h tends to $F(u_{\min})$, we calculate $\lim_{h \to F(u_{\min})} \theta(h, c)$ as

$$\lim_{h \to F(u_{\min})} \theta(h, c) = \frac{\pi}{\sqrt{F''(u_{\min})}} = \frac{\pi c}{\sqrt{\beta(\beta - 1)u_{\min}^{\beta-2} + c^2 - \alpha(\alpha - 1)u_{\min}^{\alpha-2}}}. \tag{5.55}$$

Since $F(u)$ attains a non-degenerate local minima at $u_{\min}$, $F'(u_{\min}) = \frac{\beta}{c^2} u_{\min}^{\beta-1} - \frac{\alpha}{c^2} u_{\min}^{\alpha-1} + u_{\min} = 0$ and $F''(u_{\min}) > 0$. Then $\lim_{h \to F(u_{\min})} \theta(h, c)$ is finite.

Next, we calculate $\lim_{h \to F(u_{\max})} \theta(h, c)$. Since $F'(u_{\max}) = 0$, $u(\theta) \equiv u_{\max}$ for all $\theta \in \mathbb{R}$ is a constant solution of the system (5.3). One may verify that $F''(u_{\max}) < 0$. Therefore $u = u_{\max}$ is a saddle equilibrium point in the phase space.

We define $\Gamma_h = \{(u, u_\theta) \mid H(u, u_\theta) = h\}$ is the level set of the energy of (5.3). When $h = F(u_{\max})$, the connect component of $\Gamma_{F(u_{\max})} \backslash (u_{\max}, 0)$ containing $(u_c, 0)$ does not contain any equilibrium and $(u_{\max}, 0)$ is its only boundary point. Since $u = u_{\max}$ is an saddle equilibrium point in the phase space, the orbit of initial value problem

$$u_{\theta\theta} = -F''(u); \tag{5.56}$$

$$u_\theta(0) = 0, \quad u(0) = u_c \tag{5.57}$$

is homoclinic to $(u_{\max}, 0)$, i.e.,

$$\lim_{\theta\to+\infty} u(\theta) = u_{\max} \quad \text{and} \quad \lim_{\theta\to-\infty} u(\theta) = u_{\max}.$$

Therefore, the periodic $\theta(F(u_{\max}), c)$ of the initial value problem (5.56)–(5.57)satisfies

$$\lim_{h\to F(u_{\max})} \theta(h, c) = +\infty. \tag{5.58}$$

By (5.55), (5.58) and the continuity of $\theta(h, c)$, the range of $\theta(h, c)$ contains the open interval $(\lim_{h\to F(u_{\min})} \theta(h, c), +\infty)$ for any fixed c satisfies (5.51). By Proposition 5.2 and Corollary 5.4, the system (1.3) possesses at least countably many periodic solutions with arbitrarily large topological degree and uncountably many quasi-periodic solutions. □

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