Harmonic Analysis for Real Spherical Spaces

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Abstract   We give an introduction to basic harmonic analysis and representation theory for homogeneous spaces $Z = G/H$ attached to a real reductive Lie group G. A special emphasis is made to the case where Z is real spherical.

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1   Introduction

It is a well-known fact that in the decomposition of the coordinate ring of a reductive complex algebraic group G each irreducible algebraic representation occurs with multiplicity exactly equal to its dimension. In fact,

$$\mathbb{C}[G] = \oplus_{\lambda \text{ irrep}}\mathbb{C}[G]^{\lambda} \simeq \oplus_{\lambda \text{ irrep}}V_{\lambda} \otimes V_{\lambda}^{*} \qquad (1.1)$$

as a left×right representation of $G \times G$ (the sum is over equivalence classes of irreducible representations). This result is closely related (via the so-called Weyl's unitary trick) to the Peter–Weyl theorem for compact Lie groups, which exhibits a similar decomposition of the Hilbert space $L^2(G)$.

For $Z = G/H$, a homogeneous space of an algebraic subgroup $H \subset G$, it immediately follows that

$$\mathbb{C}[Z] \simeq \oplus_{\lambda \text{ irrep}}V_{\lambda} \otimes V_{\lambda}^{*H}, \qquad (1.2)$$

where $V_{\lambda}^{*H} \subset V_{\lambda}^{*}$ denotes the space of H-fixed linear forms on V_{λ}. In particular, the multiplicity of an equivalence class λ in $\mathbb{C}[Z]$ is exactly the dimension of V_{λ}^{*H}. Again an analogous decomposition holds for the L^2-Hilbert space of a homogeneous space of a compact Lie group, and it represents the epitome of *harmonic analysis* on such spaces.

When it comes to homogeneous spaces of non-compact reductive Lie groups the harmonic analysis is however much more complicated. A major complication comes already from the

fact that irreducible representations are infinite dimensional in general. The decomposition of $L^2(G)$, the so-called *Plancherel formula*, for a reductive real Lie group was the life time achievement of Harish–Chandra, and the generalization to homogeneous spaces of G is still far from being understood. However, recent investigations (see [13, 15]) suggest that if the homogeneous space G/H is *real spherical*, then such a decomposition may be reachable.

It is the aim of these notes to explain some of the background for this development for readers who are not well acquainted with the representation theory of real reductive Lie groups. In particular, we will emphasize the *geometric* properties of real spherical spaces that are relevant. The main results are taken from [3] and [15, 20].

Let us now be a bit more specific about the contents of this survey. In general, for a homogeneous space $Z = G/H$ with G-invariant measure one has an abstract Plancherel formula for the left regular representation of G on $L^2(Z)$:

$$L^2(Z) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{M}_\pi \, d\mu(\pi). \tag{1.3}$$

Here $\widehat{G}$ is the unitary dual of G, the set of equivalence classes of irreducible unitary representations of G. This set carries a natural topology, and μ is a Radon measure on $\widehat{G}$, the so-called *Plancherel measure of Z*. In case G is compact, we recall that $\widehat{G}$ is discrete and taking G-finite vectors in (1.3) yields (1.2). The multiplicity space $\mathcal{M}_\pi$ corresponds to $(V_\lambda^*)^H$ in (1.2), but in general it is not $(\mathcal{H}_\pi^*)^H$ or a subspace thereof. It is however contained in an enlarged dual $(\mathcal{H}_\pi^{-\infty})^H$, the space of H-invariant distribution vectors of the representation $(\pi, \mathcal{H}_\pi)$. To be more precise, $\mathcal{H}_\pi^{-\infty}$ is the strong dual of the space of smooth vectors $\mathcal{H}_\pi^\infty$ of the representation $(\pi, \mathcal{H}_\pi)$. As a topological vector space $\mathcal{H}_\pi^\infty$ is a so-called Fréchet space.

This article starts with a survey on topological vector spaces with an emphasis on Fréchet spaces, and continues in Section 3 with a review of group representations on topological vector spaces. In particular we explain smooth vectors. The representation of our concern is the left regular representation of G on $L^2(Z)$. The characterization of its smooth vectors $L^2(Z)^\infty$ is tied to volume growth on the homogeneous space Z. This is the topic of Section 4. The main tool is the invariant Sobolev lemma of Bernstein [3] of which we include a proof. From Section 5 on we are more specific about our group G and request that is real reductive. We review the basic representation theory of infinite dimensional representations, i.e., the theory of Harish–Chandra modules and their smooth completions. Beginning with Section 6 we consider homogeneous spaces $Z = G/H$ for $H < G$ a closed subgroup. Our first topic is of generalized matrix coefficients for a representation (π, E): attached to a smooth vector $v \in E^\infty$ and $\eta \in (E^{-\infty})^H$ we form the function $m_{v,\eta}(g) = \eta(\pi(g^{-1})v)$ which descends to a smooth function on $Z = G/H$. This gives us a large supply of smooth functions on Z. Smooth Frobenius reciprocity asserts that $(E^{-\infty})^H$ parametrizes $\mathrm{Hom}_G(E^\infty, C^\infty(Z))$, i.e., the space of continuous G-equivariant embeddings of E^∞ into $C^\infty(Z)$.

In Section 7 we introduce homogeneous real spherical spaces and review the local structure theorem (LST). The LST allows us to attach certain geometric invariants to a real spherical space, the real rank and a conjugacy class of a parabolic subgroup Q of G, called Z-adapted. These geometric invariants are tied to representation theory as follows: one is led to consider spherical pairs (V, η), that is, V is a Harish–Chandra module and η is a non-zero continuous

H-invariant linear functional on V^∞, the unique smooth moderate growth completion of V. The spherical subrepresentation theorem then asserts that irreducible spherical pairs admit embedding in induced modules for the parabolic Q. In addition there are sharp bounds for the dimension of the space $(V^{-\infty})^H$; in particular, the multiplicity spaces $\mathcal{M}_\pi$ from above are finite dimensional for a real spherical space.

In Section 8 we give a short introduction to direct integrals of Hilbert spaces and explain the abstract Plancherel decomposition (1.3). The main goal of L^2-harmonic analysis on Z is the determination of the Plancherel measure μ. The first step is to determine the support $\operatorname{supp}\mu \subset \widehat{G}$, a bit more precisely one should determine all spherical pairs (V_π, η) where V_π is the Harish–Chandra module of $\mathcal{H}_\pi$, $\eta \in \mathcal{M}_\pi$ and $\pi \in \operatorname{supp}(\mu)$. Those pairs are called *tempered*.

It was realized by Bernstein, in a very general setup, that the determination of all tempered real spherical pairs is tied to a certain Schwartz space $\mathcal{C}(Z)$ of smooth rapidly decreasing functions on Z. In general the definition of $\mathcal{C}(Z)$ is based on the volume growth of Z. For a real spherical space this growth can be exactly determined via the polar decomposition of Z. This yields us very explicit families of semi-norms which determine the topology on the Fréchet space $\mathcal{C}(Z)$. All that is explained in Section 9.

Our survey ends with a growth bound for generalized matrix coefficients for a tempered spherical pair (V, η). Let us mention that this bound is the starting bound for the characterization of the tempered spectrum of Z in terms of the relative discrete spectrum of certain satellites of Z, called boundary degenerations (see [12]). As for harmonic analysis on real spherical spaces this is the current state of the art.

For p-adic spherical spaces the theory is further developed and a Plancherel theorem is established for an interesting class of spherical spaces, termed wave-front. This is the work of Sakellaridis and Venkatesh, who in particular developed new geometric concepts for Plancherel theory [22]. Presently it appears that part of their geometric ideas can be adapted to real spherical spaces, but the analytic aspects of the discrete spectrum are not parallel. It is an exciting topic for future research.

2 Topological Vector Spaces

2.1 Generalities

Let K be a field and E be a K-vector space. This means that we have two structures:

(i) scalar multiplication:

$$\operatorname{sc} : K \times E \to E, \quad (\lambda, x) \mapsto \lambda x,$$

(ii) addition:

$$\operatorname{add} : E \times E \to E, \quad (x, y) \mapsto x + y.$$

This gives a category $\mathbf{Vect}_K$ with morphisms the K-linear maps $T : E_1 \to E_2$. Next we want to explain

$$\boxed{\text{Topological linear algebra = functional analysis}}$$

The easiest vector space is $E = K$. We call K a topological field if it is endowed with a topology such that for $E = K$ the operations sc and add are continuous, and, in addition,

the inversion on $K^\times = K\backslash\{0\}$ is continuous. The important examples are the local fields $K = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_q$ and $\mathbb{F}_q(X)$.

From now on we assume that K is a topological field. A *topological vector space* over K is a K-vector space E which is endowed with a topology such that the structure operations sc and add become continuous. This gives us a new category $\mathbf{TopVect_K}$ with morphisms the continuous K-linear maps.

The theory is sensitive on the nature of the topological field. For example there is no notion of convexity if $\mathrm{char}\,K > 0$. From now on we assume that $K = \mathbb{C}$.

We request that E is separated (Hausdorff) which is equivalent to the fact that $\{0\}$ is closed. This already brings us a variety of problems.

Let E be a TVS (topological vector space) and $F \subset E$ a subspace. Then:

• F endowed with its subspace topology is a TVS. Furthermore, F is separated.

• E/F endowed with the quotient topology is a TVS. However, E/F is separated iff F is closed.

So we need to be careful with quotients, especially because the most interesting morphisms of our category rarely have closed images. Note that non-separated spaces typically have little meaning.

Also we request from now on that our TVS are *separable*, that is there exists a countable dense subset.

2.2   Examples of TVS

2.2.1   Hilbert Spaces

Since all (separable) Hilbert spaces admit a countable orthonormal basis they are unitarily isomorphic to either $\mathbb{C}^n$ or

$$\ell^2(\mathbb{N}_0) = \left\{ x = (x_n)_{n \in \mathbb{N}_0} \,\Big|\, x_n \in \mathbb{C},\ \langle x, x \rangle := \sum_{n=0}^{\infty} |x_n|^2 < \infty \right\}.$$

If M is a manifold endowed with a positive Radon measure μ_M, then $L^2(M, \mu_M)$ is a Hilbert space. Often it is quite difficult, if not impossible to write down a concrete orthonormal basis for $L^2(M, \mu_M)$. For example, if $M = \mathbb{R}$ and μ is the Lebesgue measure, you will already find this a quite challenging task. One classical solution is given by the Hermite functions $H_n(x) = \mathrm{e}^{-x^2/2} h_n(x)$ with h_n the Hermite-polynomials.

Later our concern will be with $E = L^2(G/H)$ where G/H is a unimodular real spherical space.

In the context of Hilbert spaces we can already see that morphisms of Hilbert spaces generically have non-closed image. Consider

$$T : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0), \quad (x_n)_{n \in \mathbb{N}_0} \mapsto \left(\frac{1}{n+1} x_n \right)_{n \in \mathbb{N}_0}.$$

Then $y^k := (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, 0, 0, \ldots) \in \mathrm{im}\, T$ for every $k \in \mathbb{N}$. Moreover, the $(y^k)_{k \in \mathbb{N}}$ form a Cauchy-sequence with limit $y = \lim y^k \notin \mathrm{im}\, T$.

2.2.2   Banach Spaces

A vector space E endowed with a norm p is called a *Banach space* if (E, p) is a complete topological space. Typical examples are $\ell^k(\mathbb{N})$ for $1 \le k \le \infty$ and, more generally, $L^k(M)$ for

any measure space M. Others are

$$C_b(M) := \{ f \in C(M) \mid f \text{ bounded} \}, \quad C_0(M) := \Big\{ f \in C(M) \,\Big|\, \lim_{m \to \infty} f(m) = 0 \Big\}$$

for a locally compact space M, with the uniform norm

$$p(f) := \sup_{m \in M} |f(m)|.$$

Here $C(M)$ denotes the space of continuous functions, and the condition of vanishing limit at infinity means that $\{ m \in M \mid |f(m)| \geq c \}$ is compact for all $c > 0$.

2.2.3 Fréchet Spaces

Contrary to what you might expect it is not the class of Banach spaces or Hilbert spaces which is most important for the discussion of infinite dimensional representations but another more flexible class which we introduce next.

Let E be a vector space. A function $p : E \to \mathbb{R}_{\geq 0}$ is called a *semi-norm* provided that

- $p(\lambda x) = |\lambda| p(x)$ for all $x \in E$ and $\lambda \in \mathbb{C}$.
- $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$.

Observe that we do not request $p(x) = 0 \Rightarrow x = 0$. For example let M be a locally compact space and $E = C(M)$. For a compact subset $K \subset M$, set

$$p(f) := \sup_{m \in K} |f(m)|, \quad f \in E.$$

Then p is a norm if and only if $K = M$.

A semi-normed space (E, p) is a typically non-separated TVS. However, it features a natural completion E_p which is a Banach space. Concretely E_p can be constructed in two steps: i) Note that $F := \{ p = 0 \}$ is a closed subspace of E and $\widetilde{E} := E/F$ is separated. Furthermore, p induces a norm $\widetilde{p}$ on $\widetilde{E}$ so that we obtain a normed space $(\widetilde{E}, \widetilde{p})$; ii) The completion of the normed space $(\widetilde{E}, \widetilde{p})$ then is the desired Banach space E_p.

In our example $E = C(M)$ from above one has $E_p = C(K)$ (a consequence of the Tietze extension theorem).

Having introduced semi-normed spaces we can now define Fréchet spaces. Let E be a vector space endowed with a countable family of semi-norms $(p_n)_{n \in \mathbb{N}}$. We endow E with the coarsest topology such that the diagonal embedding

$$E \to \prod_{n \in N} (E, p_n)$$

becomes continuous (initial topology). An alternative description of this topology is given via the semi-metric

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}, \quad x, y \in E.$$

A vector space E endowed with a countable family of semi-norms $(p_n)_{n \in N}$ is a *Fréchet space* provided that (E, d) is a complete metric space. Note that E is a TVS. In applications the topology on E can be introduced from different families of semi-norms $(q_n)_{n \in \mathbb{N}}$. To get a flavour, set $q_n := p_1 + \cdots + p_n$. Then the $(q_n)_n$ induce the same topology as the $(p_n)_n$. In particular, we can always assume that the family $(p_n)_n$ is increasing.

2.2.4 Examples of Fréchet Spaces

It is clear that Banach spaces are Fréchet spaces, but the converse is not true as the following example shows.

Schwartz spaces on $\mathbb{R}^m$. For every $n \in \mathbb{N}_0$ and every smooth function $f \in C^\infty(\mathbb{R}^m)$, set

$$p_n(f) := \sup_{x \in \mathbb{R}^m} (1 + \|x\|)^n \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ |\alpha| \le n}} |\partial^\alpha f(x)|.$$

Then

$$\mathcal{S}(\mathbb{R}^m) := \{f \in C^\infty(\mathbb{R}^m) \mid \forall n \in \mathbb{N}, \ p_n(f) < \infty\}$$

is a Fréchet space, the so-called *Schwartz space* of smooth rapidly decreasing functions on $\mathbb{R}^m$.

Later on our concern will be with $\mathcal{C}(G/H)$, the Harish–Chandra Schwartz space of a real spherical space.

Exercise 2.1 (a) Show that $\mathcal{S}(\mathbb{R}^m)$ has the Heine–Borel property: *closed and bounded sets are compact.* Here a subset $B \subset E$ is called bounded provided that there exists for all $n \in \mathbb{N}$ a constant $c_n > 0$ such that $\sup_{v \in B} p_n(v) < c_n$. Hint: Show that every sequence in a closed and bounded subset has a convergent subsequence.

(b) Conclude from (a) that $\mathcal{S}(\mathbb{R}^m)$ is not a Banach space. Hint: A basic theorem of Riesz asserts that a locally compact topological vector space is finite dimensional.

Continuous functions. A further typical example is $E = C(M)$ for a manifold M with compact exhaustion $M = \bigcup_{n \in \mathbb{N}} K_n$ and associated semi-norms $p_n(f) := \sup_{m \in K_n} |f(m)|$. The resulting topology is the one of locally uniform convergence.

Exercise 2.2 Define the natural Fréchet topology on $C^\infty(M)$ for a manifold M.

2.3 Functional Analysis

Finally we come to the explanation why topological linear algebra is usually referred to as functional analysis. The theory emerged as a theory of function spaces and linear operators between them. The main examples of TVS are spaces of functions on manifolds, and many abstract notions in the theory are motivated by that. For example there is the notion of a Montel space, which is derived from Montel's theorem about families of holomorphic functions. The crowning achievement is perhaps Grothendieck's notion of nuclear Fréchet spaces which puts the Schwartz kernel theorem from (functional) analysis in a much wider and more transparent context.

We conclude with an issue of functional analysis which will concern us later on (see Problem 7.5 below). We are interested in linear differential operators, i.e., operators which in their simplest form are of the type

$$T : C^\infty(U) \to C^\infty(U), \quad f \mapsto \sum_{|\alpha| \le n} a_\alpha \partial^\alpha f,$$

where $U \subset \mathbb{R}^N$ is an open set and a_α are smooth coefficient functions. In particular we would like to know whether such an operator is surjective or at least has closed range. Here is the famous Lewy example which destroys all hopes for general results. Let $U = \mathbb{R}^3$ and

$$T = \partial_x + \mathrm{i}\partial_y - 2\mathrm{i}(x + \mathrm{i}y)\partial_t.$$

Then $T(u) = f$ implies that f is analytic in t near $t = 0$. In particular, T is neither surjective nor has closed range.

3 Group Representations

Let G be a locally compact group. For a topological vector space E, we set

$$\mathrm{GL}(E) := \{T : E \to E \mid T \text{ linear, bijective}; T, T^{-1} \text{ continuous}\}.$$

A group homomorphism $\pi : G \to \mathrm{GL}(E)$ is called a *representation*, provided that the associated action map

$$G \times E \to E, \quad (g, v) \mapsto \pi(g)v$$

is continuous.

Remark 3.1 This definition is not what you might expect. For example, if E is a normed space, then $\mathrm{Hom}(E, E)$ is normed as well; if p is the norm on E, then the operator norm is declared as

$$\|T\| := \sup_{p(v) \leq 1} p(Tv), \quad T \in \mathrm{Hom}(E, E).$$

Now we could consider $\mathrm{GL}(E) \subset \mathrm{Hom}(E, E)$ endowed with the subspace topology and request that $\pi : G \to \mathrm{GL}(E)$ is continuous. As we will see soon this is a too strong requirement which would exclude almost all interesting examples if $\dim E = \infty$.

By a *morphism* between two G-representations (π_1, E_1) and (π_2, E_2) we understand a continuous linear map $T : E_1 \to E_2$ which intertwines the G-action, i.e., $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$.

If E is a Banach, resp. Fréchet, space, then we call (π, E) a *Banach, resp. Fréchet, representation*. An application of the uniform boundedness principle then gives the following useful criterion:

Lemma 3.2 *Let E be a Banach space and $\pi : G \to \mathrm{GL}(E)$ be a group homomorphism. Then the following are equivalent:*

(1) *(π, E) is a representation.*

(2) *For all $v \in E$, the orbit map $f_v : G \to E$, $g \mapsto \pi(g)v$ is continuous.*

Proof The implication $(1) \Rightarrow (2)$ is almost clear: If we restrict the continuous action map $G \times E \to E$, to $G \times \{v\}$, we obtain the orbit maps.

For the converse implication, let p be the norm of E. We observe that continuity of f_v implies that the map $G \ni g \mapsto p(\pi(g)v) \in \mathbb{R}_{\geq 0}$ is continuous and hence locally bounded for all $v \in E$.

The uniform boundedness principle then implies that $g \mapsto \|\pi(g)\|$ is locally bounded.

Now for g' close to g and v' close to v we obtain that

$$\pi(g)v - \pi(g')v' = (\pi(g)v - \pi(g')v) + (\pi(g')v - \pi(g')v')$$
$$= (f_v(g) - f_v(g')) + \pi(g')(v - v')$$

and thus $\pi(g)v$ is close to $\pi(g')v'$. $\qquad\qquad\square$

Example 3.3 (Exercise, left regular representation) Let $G = (\mathbb{R}^n, +)$ and $E = L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

(a) Show that

$$L : G \to \mathrm{GL}(E), \quad x \mapsto L(x); \quad L(x)f := f(\cdot - x)$$

defines a group homomorphism with $\|L(x)\| = 1$ for all $x \in G$.

(b) Show that $L : G \to \mathrm{GL}(E)$ is not continuous if $\mathrm{GL}(E)$ is endowed with the operator topology.

(c) (L, E) is a representation if and only if $p < \infty$. (Hint: Use that $C_c(\mathbb{R}^n)$ is dense in E for $p < \infty$.)

The above example admits the following generalization. Let $H < G$ be a closed subgroup and form the homogeneous space $Z := G/H$. We assume that Z is *unimodular*, that is, Z carries a positive G-invariant Radon measure. This measure is then unique up to scalar (Haar measure). We write $z_0 = H$ for the standard base point of Z.

Then $E = L^p(G/H)$ with $1 \leq p < \infty$ is a Banach space and

$$L : G \to \mathrm{GL}(E); \quad L(g)f(z) = f(g^{-1}z), \quad f \in E, g \in G, z \in Z$$

defines a Banach representation.

3.1   Smooth Vectors

From now on we assume that G is a Lie group. Let (π, E) be a representation and $v \in E$. Call v *smooth* provided that the vector valued orbit map $f_v : G \to E$ is smooth. We set

$$E^\infty = \{v \in E \mid v \text{ is smooth}\}$$

and observe that E^∞ is a G-invariant subspace of E.

Exercise 3.4   Let (π, E) be a representation on a complete TVS. Consider the test functions $C_c^\infty(G)$ as an algebra under convolution:

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) dx,$$

where dx is a left Haar measure on G.

(a) Show that (π, E) gives rise to an algebra representation

$$\Pi : C_c^\infty(G) \to \mathrm{End}(E),$$

where

$$\Pi(f)v = \int_G f(x)\pi(x)v \, dx, \quad f \in C_c^\infty(G), v \in E.$$

(Hint: Use Riemann sums and the completeness of E to show that the vector valued integrals converge in E.)

(b) Suppose now that E is a Banach space. Show that:

(1) $\Pi(C_c^\infty(G))E \subset E^\infty$.

(2) $E^\infty \subset E$ is dense. (Hint: Use a Dirac sequence $(\phi_n)_n$ of G centered at $\mathbf{1}$.)

Remark 3.5   A much stronger statement than (1) is true, namely $E^\infty = \Pi(C_c^\infty(G))E$ (Theorem of Dixmier–Malliavin).

From now on we adopt the general convention that when a Lie group is denoted by an upper case Latin letter, then the corresponding lower case German letter denotes the associated Lie algebra. In particular $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G.

For $X \in \mathfrak{g}$ and $v \in E^\infty$, the limit

$$d\pi(X)v := \lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}$$

is defined and yields a Lie algebra representation

$$d\pi : \mathfrak{g} \to \mathrm{End}(E^\infty) \,.$$

We let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and extend $d\pi$ to an algebra representation of $\mathcal{U}(\mathfrak{g})$.

Here is an example for E^∞. We let $E = L^p(G/H)$ as before. Then, the local Sobolev lemma implies that

$$E^\infty = \{ f \in C^\infty(Z) \mid \forall\, u \in \mathcal{U}(\mathfrak{g}) \ dL(u)f \in E \} \,.$$

In the next section we state a version of the Sobolev lemma which takes also the geometry into account.

3.2   The Topology on Smooth Vectors

Let (π, E) be a Banach representation of G. Let p be the norm on E and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. Define for every $k \in \mathbb{N}_0$ a norm p_k on E^∞ by

$$p_k(v) := \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le k}} p(d\pi(X_1^{\alpha_1} \cdot \cdots \cdot X_n^{\alpha_n})v). \tag{3.1}$$

We refer to p_k as a k-th Sobolev norm of p.

Lemma 3.6   *Let (π, E) be a Banach representation of G. Then the following assertions hold:*

   (1) The $(p_k)_{k \in \mathbb{N}}$ define a Fréchet topology on E^∞ and the topology is independent of the choice of the particular basis $X_1, \ldots, X_n$ of $\mathfrak{g}$.

   (2) The action $G \times E^\infty \to E^\infty$ is continuous and gives rise to a Fréchet representation (π^∞, E^∞) of G.

3.3   F and SF-representations

Let (π, E) be a Fréchet representation. We say that (π, E) is an F-*representation* provided that there exists a family of semi-norms $(p_n)_{n \in \mathbb{N}}$ which induces the topology on E such that the actions

$$G \times (E, p_n) \to (E, p_n)$$

are continuous for all $n \in \mathbb{N}$.

Remark 3.7   The notion of F-representations is equivalent to the perhaps more familiar notion of moderate growth Fréchet representations (see [4], Section 2.3.1 and in particular Lemma 2.10).

   An F-representation is called an SF-*representation* if all orbit maps are smooth. By a morphism between two SF-representations (π_1, E_1) and (π_2, E_2) we understand a morphism $T : E_1 \to E_2$ such that for some fixed $k \in \mathbb{N}$ the induced linear maps $(E_1, p_{1,n}) \to (E_2, p_{2,k+n})$ are continuous for all $n \in \mathbb{N}$.

Lemma 3.8   *Let (π, E) be a Banach representation of G. Then (π^∞, E^∞) is an SF-representation w.r.t. a family of Sobolev norms.*

4　Volume Weights on G/H

Let $Z = G/H$ be a unimodular homogeneous space as before. By a *weight* on Z we understand a locally bounded function $w : Z \to \mathbb{R}_{>0}$ with the following property. For all compact subsets $\Omega \subset G$ there exists a constant $C > 0$ such that

$$w(gz) \leq Cw(z), \quad z \in Z, g \in \Omega. \tag{4.1}$$

For later reference, we note that by applying (4.1) to the compact set Ω^{-1} we obtain as well that there exists $C' > 0$ such that

$$w(gz) \geq C'w(z), \quad z \in Z, g \in \Omega. \tag{4.2}$$

By a ball $B \subset G$ we understand a compact symmetric neighborhood of $\mathbf{1}$ in G. Fix a ball B and define

$$\mathbf{v}(z) := \mathrm{vol}_Z(Bz), \quad z \in Z.$$

We refer to $\mathbf{v}$ as a *volume weight*.

Exercise 4.1　(a) Show that $\mathbf{v}$ is a weight.

(b) If B and B' are balls in G and $\mathbf{v}$ and $\mathbf{v}'$ are the associated volume weights, then $\mathbf{v}$ and $\mathbf{v}'$ are comparable, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C}\mathbf{v}(z) \leq \mathbf{v}'(z) \leq C\mathbf{v}(z), \quad z \in Z.$$

Given a homogeneous space $Z = G/H$ the volume weight $\mathbf{v}$ is closely tied to the harmonic analysis of Z. The next lemma gives a first flavor how geometry and analysis are linked.

Lemma 4.2 (Bernstein's invariant Sobolev-lemma [3])　*Let $1 \leq p < \infty$ and $k > \frac{\dim G}{p}$. Fix a ball $B \subset G$ with associated volume weight $\mathbf{v}$. Then there exists a constant $C_B > 0$ such that*

$$|f(z)| \leq C_B \mathbf{v}(z)^{-\frac{1}{p}} \|f\|_{p,k,Bz} \tag{4.3}$$

for all $z \in Z$ and all smooth functions f on Z. Here $\|\cdot\|_{p,k,Bz}$ refers to a k-th Sobolev norm of the L^p-norm $\|\cdot\|_p$ of $L^p(Z)$ restricted to Bz.

Proof　For fixed z the estimate (4.3) reduces to the classical Sobolev lemma, which is valid for $k > \frac{\dim Z}{p}$ on any smooth manifold Z, since by means of local coordinates it can be reduced to $\mathbb{R}^n$.

We first consider the special case $Z = G$, say with the left action of G. For simplicity let us assume that G is unimodular. Then the right invariance of Haar measure implies that $\mathbf{v}$ is a constant function. Furthermore, in this case the right action allows us to immediately deduce the global estimate from the local estimate at $z = e$ (where $e = \mathbf{1}$ is the identity in the group).

Next we consider the general case of a unimodular homogeneous space G/H. Let $F \in C^\infty(G)$ be the pull-back of f to G, then it follows from what was said above that

$$|F(x)| \leq C_B \|F\|_{p,k,Bx}, \quad x \in G.$$

Here the Sobolev norm is given by an L^p-integral of the derivatives of F over $Bx \subset G$, and hence in order to relate to the corresponding integral for f, it suffices to show that

$$\int_B \phi(g \cdot z)\, dg \leq C\mathbf{v}(z)^{-1} \int_{Bz} \phi(y)\, dy$$

for all $z \in Z$ and all measurable positive functions ϕ on Z, with a constant C which is independent of z and ϕ.

In order to see this we assume, as we may, that ϕ is supported on Bz. For each $y \in Bz$, we write $z = b^{-1}y$ and deduce

$$\int_B \phi(gz)\, dg = \int_{Bb^{-1}} \phi(gy)\, dg \leq \int_{B^2} \phi(gy)\, dg.$$

By averaging this estimate over $y \in Bz$ we obtain

$$\int_B \phi(gz)\, dg \leq \frac{1}{\mathbf{v}(z)} \int_{Bz} \int_{B^2} \phi(gy)\, dg\, dy$$

and with Fubini and invariance of dy we conclude

$$\int_B \phi(gz)\, dg \leq \frac{\mathrm{vol}_G B^2}{\mathbf{v}(z)} \int_{Bz} \phi(y)\, dy$$

as claimed (in the last step it was used that $\operatorname{supp}\phi \subset Bz$). $\qquad\square$

In general one would like to determine the growth behavior of the volume weight $\mathbf{v}$. For that one needs to understand the large scale geometry of the space Z. In case Z is a real spherical space we will see later that this is in fact possible.

For homogeneous spaces there is the following general criterion.

Theorem 4.3 ([18])   *Let G be a real reductive group and $H < G$ a connected subgroup of G with real algebraic Lie algebra. Then the following statements are equivalent*:

(1) *The volume weight $\mathbf{v}$ is bounded from below, i.e., there exists a $c > 0$ such that $\mathbf{v}(z) \geq c$ for all $z \in Z$.*

(2) *$\mathfrak{h}$ is reductive in $\mathfrak{g}$, i.e., $\operatorname{ad}_{\mathfrak{g}}|_{\mathfrak{h}}$ is completely reducible.*

By a *real reductive group* we understand a real Lie group G with finitely many connected components such that there is a Lie group morphism $\iota : G \to \mathrm{GL}(n, \mathbb{R})$ with:

- $\iota(G)$ is closed and stable under matrix-transposition,
- $\ker \iota$ is finite.

Exercise 4.4   Consider $G = \mathrm{SL}(2, \mathbb{R})$ and

$$H = N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

(1) Show that there is a natural identification of $Z = G/N$ with $\mathbb{R}^2 \backslash \{0\}$ under which the Haar measure on Z corresponds to the Lebesgue measure $dx \wedge dy$ on $\mathbb{R}^2 \backslash \{0\}$.

(2) For $t > 0$ and $a_t := \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$ show that

$$\mathbf{v}(a_t \cdot z_0) \asymp t^2.$$

Hint: Take a ball B which is of the form $B = B_N^t B_A B_N$ where B_N is a ball in N and B_A is a ball in the diagonal matrices.

Exercise 4.5   Use the invariant Sobolev lemma and Theorem 4.3 to deduce the following vanishing result. For G real reductive and $\mathfrak{h} < \mathfrak{g}$ algebraic and reductive in $\mathfrak{g}$ one has

$$L^p(Z)^\infty \subset C_0(Z). \tag{4.4}$$

Remark 4.6 The converse is also true, i.e., (4.4) implies $\mathfrak{h}$ is reductive in $\mathfrak{g}$, see [18].

5 Harish–Chandra Modules and Their Completions

Given a real reductive group G and a choice of morphism ι the prescription $K := \iota^{-1}(\mathrm{O}(n, \mathbb{R}))$ defines a maximal compact subgroup of G. We recall that all maximal compact subgroups of G are conjugate (Cartan's theorem). A choice of K yields an involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$, called Cartan-involution, with fixed point algebra $\mathfrak{g}^\theta = \mathfrak{k}$. For example, for our above choice of $K = \iota^{-1}(\mathrm{O}(n, \mathbb{R}))$ we would have $\theta(X) = -X^T$ if we view $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. In general we denote by $\mathfrak{s} \subset \mathfrak{g}$ the -1-eigenspace of θ and recall that the polar map

$$K \times \mathfrak{s} \to G, \quad (k, X) \mapsto k \exp(X)$$

is a diffeomorphism. In particular, K is a deformation retract of G and thus contains all topological information of G.

We denote by $\widehat{K}$ the set of equivalence classes of irreducible representations of K. Given an irreducible representation (π, E) we let $[\pi] := [(\pi, E)]$ be its equivalence class. We adopt the common abuse of notation and write π instead of $[\pi]$. Since K is compact its irreducible representations are finite dimensional, and they are essentially parametrized by their highest weights (there are some issues here if K is not connected).

Theorem 5.1 (Harish–Chandra)   *Let (π, E) be a unitary irreducible representation of G. Let*

$$V := E^{\mathrm{K-fin}} := \{v \in E \mid \mathrm{span}_{\mathbb{C}}\{\pi(K)v\} \text{ is finite dimensional}\} \,.$$

Then the following assertions hold:

  (1) *V is dense in E,*
  (2) *$V \subset E^\infty$ and V is $\mathfrak{g}$-stable,*
  (3) *V is an irreducible module for $\mathcal{U}(\mathfrak{g})$,*
  (4) *For all $\tau \in \widehat{K}$, one has*

$$\dim \mathrm{Hom}_K(\tau, V) \leq \dim \tau < \infty.$$

In other words to an irreducible unitary G-representation (π, E) we associate a module V for $\mathcal{U}(\mathfrak{g})$ (and K) which is irreducible (and hence of countable dimension) and which is some sort of skeleton of (π, E). Historically this result was the beginning of the algebraization of representations of real reductive groups. Here are the suitable definitions.

Definition 5.2   *A $(\mathfrak{g}, K)$-module V is a vector space endowed with two actions:*

$$\mathcal{U}(\mathfrak{g}) \times V \to V, \quad (u, v) \mapsto u \cdot v$$
$$K \times V \to V, \quad (k, v) \mapsto k \cdot v$$

such that the following conditions are satisfied:

   • *The K-action is algebraic, i.e., for all $v \in V$ the space $V_v := \mathrm{span}\{K \cdot v\}$ is finite dimensional and the action of K on V_v is continuous.*

   • *The two actions are compatible, i.e.,*

$$k \cdot (u \cdot v) = (\mathrm{Ad}(k)u) \cdot (k \cdot v), \quad k \in K, u \in \mathcal{U}(\mathfrak{g}), v \in V.$$

- *The derived action of K coincides with the $\mathcal{U}(\mathfrak{g})$-action when restricted to $\mathcal{U}(\mathfrak{k})$:*

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot v = X \cdot v, \quad X \in \mathfrak{k}.$$

Example 5.3 For every Banach representation (π, E) of G, the space

$$V := \{v \in E^\infty \mid v \text{ is } K\text{-finite}\}$$

is a $(\mathfrak{g}, K)$-module. Observe that smoothness of K-finite vectors is not automatic if the module is not irreducible.

Definition 5.4 *A $(\mathfrak{g}, K)$-module V is called a Harish–Chandra-module provided that*

(1) *V is admissible, that is, for all $\tau \in \widehat{K}$ one has*

$$\dim \operatorname{Hom}_K(\tau, V) < \infty. \tag{5.1}$$

(2) *V has finite length as a $\mathcal{U}(\mathfrak{g})$-module, that is, there is a sequence of $(\mathfrak{g}, K)$-submodules*

$$V_0 = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_m = V$$

such that V_{i+1}/V_i is an irreducible $\mathcal{U}(\mathfrak{g})$-module for all $0 \leq i \leq m-1$.

Having this notion, Theorem 5.1 asserts in particular, that the K-finite vectors of an irreducible unitary representation form a Harish–Chandra module.

Remark 5.5 (Rough classification of irreducible Harish–Chandra modules) Let us denote by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Recall that the Harish–Chandra isomorphism describes $\mathcal{Z}(\mathfrak{g})$ as $S(\mathfrak{c})^W$ where $\mathfrak{c} < \mathfrak{g}_\mathbb{C}$ is a Cartan subalgebra and W the associated Weyl-group.

We say that a $(\mathfrak{g}, K)$-module V *admits an infinitesimal character*, provided that there exists a character $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ such that $z \cdot v = \chi(z)v$ for all $z \in \mathcal{Z}(\mathfrak{g})$. It follows from Dixmier's version of Schur's lemma (see [23, Lemma 0.5.1]) that every irreducible $\mathcal{U}(\mathfrak{g})$-module V admits an infinitesimal character χ_V. Write now $\mathcal{HC}_{\mathrm{irr}}$ for the isomorphism classes of irreducible Harish–Chandra modules. Then another theorem of Harish–Chandra asserts that the map

$$\mathcal{HC}_{\mathrm{irr}} \to \widehat{\mathcal{Z}(\mathfrak{g})}, \quad V \mapsto \chi_V$$

has finite fibers.

Definition 5.6 *By a globalization of a Harish–Chandra module V we understand a G-representation (π, E) modelled on a complete TVS E, such that $E^{\mathrm{K-fin}} \simeq_{(\mathfrak{g}, K)} V$.*

Globalizations of Harish–Chandra modules always exist. This is a consequence of the so-called Casselman subrepresentation theorem which we review next.

5.1   The Subrepresentation Theorem

By a (real) parabolic subalgebra $\mathfrak{p} < \mathfrak{g}$ we understand a subalgebra such that $\mathfrak{p}_\mathbb{C}$ is a parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$. A parabolic subgroup P of G is the normalizer of a parabolic subalgebra $\mathfrak{p}$.

Our concern here is with minimal parabolic subalgebras of $\mathfrak{g}$ and we recall their standard construction. Recall the Cartan involution θ of $\mathfrak{g}$ and its eigenspace decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subspace. In this context we record the Lie-theoretic generalization of the spectral theorem for real symmetric matrices: all maximal abelian subspaces of $\mathfrak{s}$ are conjugate under K. Fix such a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and let $\mathfrak{m} := \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$ be the

centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Note that $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{m}$. Simultaneous diagonalization of $\mathrm{ad}_\mathfrak{g}|_\mathfrak{a}$ then yields the root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha,$$

where $\Sigma \subset \mathfrak{a}^* \backslash \{0\}$ and

$$\mathfrak{g}^\alpha = \{ Y \in \mathfrak{g} \mid (\forall X \in \mathfrak{a}) \; [X, Y] = \alpha(X)Y \} \neq \{0\}.$$

Observe that all root spaces $\mathfrak{g}^\alpha$ are modules for $\mathfrak{a} \oplus \mathfrak{m}$ which are typically not one-dimensional. However, Σ is a (possibly not reduced) root system and we let Σ^+ be a choice of positive system. Associated to Σ^+ is the nilpotent subalgebra $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$.

On the group level we define subgroups of G by $A := \exp(\mathfrak{a})$, $M := Z_K(A)$ and $N := \exp(\mathfrak{n})$. We recall the Iwasawa decomposition of G which asserts that the map

$$K \times A \times N \to G, \quad (k, a, n) \mapsto kan$$

is a diffeomorphism. It is of course not a homomorphism of groups, but the restriction to $(M \times A) \ltimes N$ is, and the image $P := MAN$ defines a minimal parabolic subgroup of G, to which all other minimal parabolic subgroups are conjugate. The Iwasawa-decomposition implies that $M = P \cap K$ and $G = KP$.

Example 5.7 (a) For $G = \mathrm{GL}(n, \mathbb{R})$ one can take for P the upper triangular matrices in G. Then N are the upper triangular unipotent matrices, $A = \mathrm{diag}(n, \mathbb{R}_{>0})$, and $M = \mathrm{diag}(n, \{\pm 1\})$.

(b) Given $n \in \mathbb{N}$ we let $G = \mathrm{SO}_e(1, n)$ be the connected component of the invariance group of the quadratic form $q(x) = x_0^2 - x_1^2 - \cdots - x_n^2$ on $\mathbb{R}^{n+1}$. Then $K = \mathrm{SO}(n, \mathbb{R})$ (standard embedding in the lower right corner) is a maximal compact subgroup of G. Furthermore, one can take

$$A = \left\{ \begin{pmatrix} \cosh t & & \sinh t \\ & \mathbf{1} & \\ \sinh t & & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\} \simeq (\mathbb{R}_{>0}, \cdot),$$

$$M = \left\{ \begin{pmatrix} 1 & & \\ & m & \\ & & 1 \end{pmatrix} \mid m \in \mathrm{SO}(n-1, \mathbb{R}) \right\} \simeq \mathrm{SO}(n-1, \mathbb{R}),$$

$$N = \left\{ \begin{pmatrix} 1 + \frac{1}{2}\|v\|^2 & v & -\frac{1}{2}\|v\|^2 \\ v^t & \mathbf{1} & -v^t \\ \frac{1}{2}\|v\|^2 & v & 1 - \frac{1}{2}\|v\|^2 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\} \simeq (\mathbb{R}^{n-1}, +).$$

Next let $\sigma : P \to \mathrm{GL}(W)$ be a finite dimensional representation of P. Attached to σ is the smooth induced representation, defined by

$$(\mathrm{Ind}_P^G \sigma)^\infty := \{ f : G \to W \mid f \text{ smooth}, f(gp) = \sigma(p)^{-1} f(g) \}.$$

Notice that there is a natural left action L of G on $(\mathrm{Ind}_P^G \sigma)^\infty$ given by

$$L(g)f(x) := f(g^{-1}x), \quad g, x \in G, f \in (\mathrm{Ind}_P^G \sigma)^\infty.$$

Next we topologize $(\operatorname{Ind}_P^G \sigma)^\infty$. Set

$$(\operatorname{Ind}_M^K \sigma)^\infty := \{f : K \to W \mid f \text{ smooth}, f(km) = \sigma(m)^{-1} f(k)\}.$$

Let $\|\cdot\|$ be any norm on the finite dimensional space W and observe that

$$p(f) := \sup_{k \in K} \|f(k)\|$$

defines a norm on $(\operatorname{Ind}_M^K \sigma)^\infty$ as K is compact. The left regular representation of K on $(\operatorname{Ind}_M^K \sigma)^\infty$ is continuous and a family of Sobolev norms $(p_k)_{k \in \mathbb{N}}$ then yields an SF-structure on the K-module $(\operatorname{Ind}_M^K \sigma)^\infty$.

Moreover, since $G = KP$ with $M = K \cap P$, the restriction map

$$\operatorname{Res} : (\operatorname{Ind}_P^G \sigma)^\infty \to (\operatorname{Ind}_M^K \sigma)^\infty, \quad f \mapsto f|_K$$

is a K-equivariant bijection. Via this map we transport the Fréchet structure of $(\operatorname{Ind}_M^K \sigma)^\infty$ to $(\operatorname{Ind}_P^G \sigma)^\infty$. It is not too difficult to show that:

Lemma 5.8 $(L, (\operatorname{Ind}_P^G \sigma)^\infty)$ *is an* SF-*representation of* G.

We write $\operatorname{Ind}_P^G \sigma$ for the space of K-finite vectors of $(\operatorname{Ind}_P^G \sigma)^\infty$.

Lemma 5.9 $\operatorname{Ind}_P^G \sigma$ *is a Harish–Chandra module.*

Proof (Sketch) The map Res restricts to an isomorphism between $\operatorname{Ind}_P^G \sigma$ and the induced representation space $\operatorname{Ind}_M^K \sigma$ of K-finite functions in $(\operatorname{Ind}_M^K \sigma)^\infty$. It follows that $\operatorname{Ind}_P^G \sigma$ is admissible. Next use a Jordan–Hölder series of W to reduce to the fact that σ is irreducible, and in particular that $\sigma|_N$ is trivial. Then it is not too hard too see that $\operatorname{Ind}_P^G \sigma$ admits an infinitesimal character. The assertion now follows from the general fact that an admissible module which admits an infinitesimal character has finite length.

This brings us finally to the

Theorem 5.10 (Casselman subrepresentation theorem [6]) *Let* V *be a Harish–Chandra module. Then there exists a finite dimensional representation* (σ, W) *of* P *and a* $(\mathfrak{g}, K)$-*embedding*

$$V \hookrightarrow \operatorname{Ind}_P^G \sigma.$$

Remark 5.11 (a) The subrepresentation theorem implies in particular that every Harish–Chandra module admits a globalization, even a Hilbert globalization. In fact, if we complete $\operatorname{Ind}_P^G \sigma$ in $L^2(K, W)$ we obtain a Hilbert globalization of $\operatorname{Ind}_P^G \sigma$. The closure of V in $L^2(K, W)$ then yields a Hilbert globalization of V. Other Banach globalizations are obtained by taking the closures in $L^p(K, W)$.

(b) Let $\mathfrak{n}$ be the Lie algebra of N, the unipotent radical of P. The subrepresentation theorem is then a consequence of a non-vanishing result for Harish–Chandra modules $V \neq \{0\}$:

$$V/\mathfrak{n}V \neq \{0\} \tag{5.2}$$

combined with a variant of Frobenius reciprocity (see [23, Th. 3.8.3 and Lemma 4.2.2]).

5.2 The Casselman–Wallach Globalization Theorem

Let V be a Harish–Chandra module, p a norm on the vector space V and V_p the Banach completion of (V, p). According to [4] a norm p is called G-*continuous* provided the infinitesimal

action of $\mathfrak{g}$ on V exponentiates to a Banach representation of G on V_p. With this terminology the globalization theorem can be phrased as follows:

Theorem 5.12 (Casselman–Wallach) *Let V be a Harish–Chandra module and p, q two G-continuous norms on V. Then*

$$V_p^\infty \simeq_{\mathrm{SF-rep}} V_q^\infty.$$

In particular, each Harish–Chandra module admits a unique SF*-globalization (up to isomorphism).*

Proof   See [24, Ch. 11] or [4].

Remark 5.13   An equivalent way to phrase the globalization theorem is as follows (see [4]). Let $V = \bigoplus_{\tau \in \widehat{K}} V[\tau]$ be the K-isotypical decomposition of V. For every $\tau \in \widehat{K}$, let $p_\tau := p|_{V[\tau]}$ and $q_\tau := q|_{V[\tau]}$. For every $\tau \in \widehat{K}$ let $|\tau|$ be the Cartan–Killing norm of the highest weight which parametrizes τ. Then there exists a constant $N \in \mathbb{N}$ such that

$$p_\tau \leq (1 + |\tau|)^N q_\tau, \quad \tau \in \widehat{K}, \tag{5.3}$$

that is, G-continuous norms are polynomially comparable on K-types.

6   Generalized Matrix Coefficients

6.1   Matrix Coefficients on Groups

Let G be a locally compact group and (π, E_π) be a unitary representation on some Hilbert space $(E_\pi, \langle \cdot, \cdot \rangle)$. For all $v, w \in E_\pi$ we form the *matrix-coefficient*

$$m_{v,w}^\pi(g) := \langle \pi(g)v, w \rangle, \quad g \in G.$$

Note that matrix-coefficients are continuous functions on G.

When G is compact the Peter–Weyl theorem asserts that the space of matrix coefficients is uniformly dense in $C(G)$, and hence also in $L^2(G)$. As a consequence one obtains that $L^2(G)$ is the Hilbert space direct sum of its isotypic components $L^2(G)^\pi$, and that $(v, w) \mapsto m_{v,w}^\pi$ provides a $G \times G$ equivariant isomorphism $E_\pi \otimes E_\pi \to L^2(G)^\pi$.

In particular, the density in $C(G)$ implies that the matrix coefficients separate points on G (and vice versa, by the Stone–Weierstrass theorem). This result holds more generally without assuming compactness of G. Let $\widehat{G}$ denote the set of equivalence classes of unitary irreducible representations of G.

Theorem 6.1 (Gelfand–Raikov) *Let G be a locally compact group. Then the functions*

$$\{m_{v,w}^\pi \mid \pi \in \widehat{G}, v, w \in E_\pi\}$$

separate points on G.

Phrased loosely, the theorem says that the geometry of G is determined by its matrix coefficients.

Let us now be more restrictive on the group G.

Theorem 6.2 (Howe–Moore) *Let G be a non-compact Lie group with simple Lie algebra and compact center. Let $\pi \in \widehat{G} \backslash \{\mathbf{1}\}$. Then, for all $v, w \in E_\pi$,*

$$m_{v,w}^\pi \in C_0(G) = \left\{ f \in C(G) \,\Big|\, \lim_{g \to \infty} f(g) = 0 \right\}.$$

This result has, at first glance, a quite frustrating corollary for "ordinary" matrix coefficients on homogeneous spaces $Z = G/H$. Recall from (1.2) that for compact groups the Peter–Weyl theorem implies the G-equivariant decomposition

$$L^2(G/H) = \widehat{\bigoplus}_{\pi \in \widehat{G}} E_\pi \otimes E_\pi^H$$

so that E_π^H controls the embeddings of π in $L^2(Z)$ (this could also be derived from the Frobenius reciprocity theorem). However,

Corollary 6.3 *Let G be as in Theorem 6.2 and let $H < G$ be a closed and non-compact subgroup of G. Then for all $\pi \in \widehat{G}\backslash\{\mathbf{1}\}$ one has*

$$E_\pi^H := \{v \in E \mid \forall h \in H, \pi(h)v = v\} = \{0\}.$$

Proof Exercise.

6.2 Generalized Matrix Coefficients

Here we let G be a Lie group and $H < G$ be a closed subgroup. We have already seen that for a Banach representation (π, E) of G the chances that $E^H \neq \{0\}$ are slim. Instead of looking at E^H we look at a larger space.

Let $E^{-\infty}$ be the strong dual of the Fréchet space E^∞. We refer to $E^{-\infty}$ as the space of distribution vectors of the representation (π, E). Then for every $\eta \in (E^{-\infty})^H$ and $v \in E^\infty$ we form the *generalized matrix coefficient*

$$m_{v,\eta}(z) = \eta(\pi(g)^{-1}v), \quad z = gH, g \in G,$$

which is a smooth function on Z. Let us see in an example that in spite of Corollary 6.3 we can obtain a huge supply of functions in that way.

Example 6.4 We assume that Z is unimodular and consider the left regular representation L of G on $E = L^p(Z)$ for $1 \leq p < \infty$. We have already seen that there is a continuous inclusion $\iota : E^\infty \hookrightarrow C^\infty(Z)$. Since evaluation at the base point

$$\delta_{z_0} : C^\infty(Z) \to \mathbb{C}, \quad f \mapsto f(z_0)$$

is a continuous H-invariant functional, the composition $\eta := \delta_{z_0} \circ \iota$ defines an element of $(E^{-\infty})^H$. Let now $f \in E^\infty$ be arbitrary.

Then we get for $z = gH \in Z$ that

$$m_{f,\eta}(z) = \eta(L(g)^{-1}f) = \eta(f(g\cdot)) = f(z),$$

that is, we recover f as a generalized matrix coefficient.

The argument in the preceding example can be formalized into a statement of Frobenius reciprocity. Recall that for any Banach representation (π, E) we equipped E^∞ with a Fréchet topology. We can then consider the vector space $\mathrm{Hom}_G^{\mathrm{cont}}(E^\infty, C^\infty(G/H))$ of continuous intertwiners $E^\infty \to C^\infty(G/H)$.

Lemma 6.5 (Smooth Frobenius reciprocity) *Let (π, E) be a Banach representation of a Lie group G and let $H < G$ be a closed subgroup. Then*

$$\mathrm{Hom}_G^{\mathrm{cont}}(E^\infty, C^\infty(G/H)) \simeq (E^{-\infty})^H . \tag{6.1}$$

Proof   Let $S \in \operatorname{Hom}_G^{\operatorname{cont}}(E^\infty, C^\infty(G/H))$ and $\delta_{z_0} : C^\infty(G/H) \to \mathbb{C}$ be the point evaluation at z_0 as before. As δ_{z_0} is continuous and H-invariant, we obtain via $\eta := \delta_{z_0} \circ S$ a continuous H-invariant functional on E^∞, that is, $\eta \in (E^{-\infty})^H$.

Conversely, let $\eta \in (E^{-\infty})^H$ be given. Then

$$S_\eta : E^\infty \to C^\infty(G/H), \quad v \mapsto m_{v,\eta}$$

is G-equivariant and continuous. The calculation in the example shows that if $\eta = \delta_{z_0} \circ S$, then $S_\eta = S$.

This result has a useful dual version. For a Banach representation (π, E) we denote by (π', E') the dual representation on the dual Banach space E', and recall that E is called reflexive when the natural inclusion $E \to E''$ is an isomorphism.

Lemma 6.6 (Dual smooth Frobenius reciprocity)   *Let $Z = G/H$ and let (π, E) be a reflexive Banach representation. For each $\eta \in (E^{-\infty})^H$ and $f \in C_c^\infty(Z)$ the linear form*

$$v \mapsto \int_Z m_{v,\eta}(z) f(z)\, dz \tag{6.2}$$

on E^∞ extends continuously to E and defines an element $T_\eta(f) \in (E')^\infty$. The map $\eta \mapsto T_\eta$ then provides a linear isomorphism

$$(E^{-\infty})^H \simeq \operatorname{Hom}_G^{\operatorname{cont}}\left(C_c^\infty(G/H), (E')^\infty\right). \tag{6.3}$$

The extension $T_\eta f$ of (6.2) is the *Fourier transform* of f at (π, η).

Proof   The proof is essentially just by dualizing, but as it involves some non-trivial functional analysis we provide the details.

Recall that the strong dual of $C^\infty(Z)$ equals $\mathcal{D}_c(Z)$, the space of compactly supported distributions. We let $\mathcal{M}_c^\infty(Z) \subset \mathcal{D}_c(Z)$ denote the subspace of compactly supported smooth measures, and note that $\mathcal{M}_c^\infty(Z) \simeq C_c^\infty(Z)$ via a non-canonical isomorphism; for example if Z is unimodular and dz is the invariant measure, then such an isomorphism would be given by

$$C_c^\infty(Z) \to \mathcal{M}_c^\infty(Z), \quad f \mapsto f \cdot dz,$$

but of course any positive measure in the Lebesgue measure class will do.

Let $\eta \in (E^{-\infty})^H$ and define $S = S_\eta \in \operatorname{Hom}_G^{\operatorname{cont}}(E^\infty, C^\infty(Z))$ as in Lemma 6.5 by $S(v) = m_{v,\eta}$. The dual morphism $S' : \mathcal{D}_c(Z) \to E^{-\infty}$ is then continuous and G-equivariant. We claim that $S'(\mathcal{M}_c^\infty(Z)) \subset (E')^\infty$ and that

$$S'\big|_{\mathcal{M}_c^\infty(Z)} : \mathcal{M}_c^\infty(Z) \to (E')^\infty \tag{6.4}$$

is continuous.

To see that, let E^{-m} be the strong dual of (E, p_m) with $m \in \mathbb{N}_0$ (see (3.1)). Note that $E^{-\infty}$ equals the increasing union $E^{-\infty} = \bigcup_{m \in \mathbb{N}} E^{-m}$, and that each E^{-m} is a G-invariant Banach subspace of $E^{-\infty}$ which is continuously included into $E^{-\infty}$. On the other hand fix a compact set $C \subset Z$ and denote by $\mathcal{M}_C(Z)$ the space of finite Borel measures with support in C. Note that $\mathcal{M}_C(Z)$ is a Banach space which is continuously included in $\mathcal{D}_c(Z)$. Therefore we obtain with

$$S'\big|_{\mathcal{M}_C(Z)} : \mathcal{M}_C(Z) \to E^{-\infty} = \bigcup_{n \in \mathbb{N}} E^{-n}$$

a continuous linear map and we can apply the Grothendieck factorization theorem (cf. [8], Ch. 4. Sect. 5., Th. 1): we obtain an m such that $S'(\mathcal{M}_C(Z)) \subset E^{-m}$, slightly more precisely, the map $S'|_{\mathcal{M}_C(Z)} : \mathcal{M}_C(Z) \to E^{-m}$ is continuous. Now note that $\mathcal{M}_c(Z)$ can be obtained as the linear span

$$\mathcal{M}_c(Z) = \mathrm{span}_{\mathbb{C}}\{L(g)\mathcal{M}_C(Z) \mid g \in G\}$$

by covering the support of any given compactly supported measure with finitely many G-translates of the interior of C, and then applying an associated partition of unity. Hence the fact that E^{-m} is G-invariant implies that we obtain a continuous G-equivariant linear map:

$$T_{\mathcal{M}} := S'|_{\mathcal{M}_c(Z)} : \mathcal{M}_c(Z) \to E^{-m}\,.$$

On the other hand, taking smooth vectors of the G-modules $\mathcal{M}_c(Z)$ and E^{-m} yields that $(\mathcal{M}_c(Z))^{\infty} = \mathcal{M}_c^{\infty}(Z)$ and $(E^{-m})^{\infty} = (E')^{\infty}$. The claim (6.4) follows with $T_\eta := T_{\mathcal{M}}^{\infty}$ and the continuous linear G-morphism

$$T_\eta : \mathcal{M}_c^{\infty}(Z) \to (E')^{\infty}, \quad \mu \mapsto \int_Z \pi(g)\eta \; d\mu(gH).$$

In particular, we obtain the map T_η as a continuous extension of (6.2). It is easily seen that $\eta \mapsto T_\eta$ is injective.

Conversely, let $T : \mathcal{M}_c^{\infty}(Z) \to (E')^{\infty}$ be a G-morphism. This G-morphism yields a G-morphism $T_0 : \mathcal{M}_c^{\infty}(Z) \to E'$. By the definition of the topology of $\mathcal{M}_c^{\infty}(Z)$ we obtain for every compact set $C \subset Z$ a $k = k(C) \in \mathbb{N}$ such that $T_0|_{\mathcal{M}_C^{\infty}(Z)}$ extends to a continuous linear map $T_{C,k} : \mathcal{M}_C^k(Z) \to E'$. As T_0 was G-equivariant we obtain with partition of unity argument as above a G-morphism $T_k : \mathcal{M}_c^k(Z) \to E'$. Since E is reflexive we obtain a G-morphism

$$(T_k)' : E \to \mathcal{D}^k(Z)$$

by dualizing. Taking smooth vectors finally provides a continuous homomorphism

$$S : E^{\infty} \to \mathcal{D}^k(Z)^{\infty} = C^{\infty}(Z)$$

which by Lemma 6.5 equals S_η for some $\eta \in (E^{-\infty})^H$. It follows from this construction that $T = T_\eta$, and thus the proof is complete. $\qquad\square$

7 Real Spherical Spaces

We assume now that G is a connected reductive real algebraic group, that is, there exists a complex reductive group $G_{\mathbb{C}}$ attached to $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that G is the analytic subgroup of $G_{\mathbb{C}}$ which is associated to the subalgebra $\mathfrak{g} < \mathfrak{g}_{\mathbb{C}}$.

Furthermore, we let $H < G$ be an algebraic subgroup, that is, there is a complex algebraic subgroup $H_{\mathbb{C}} < G_{\mathbb{C}}$ such that $H_{\mathbb{C}} \cap G = H$. This way we obtain a G-equivariant embedding

$$Z = G/H \hookrightarrow Z_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}},$$

$$gH \mapsto gH_{\mathbb{C}},$$

Let now $P < G$ be a minimal parabolic subgroup. We say that Z is real spherical provided that the P-action on Z admits open orbits. Equivalently Z is real spherical if there exists a minimal parabolic subalgebra $\mathfrak{p} < \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{h}, \tag{7.1}$$

where the sum is not necessarily direct. If (7.1) is satisfied, then we refer to $(\mathfrak{g}, \mathfrak{h})$ as a real spherical pair.

Remark 7.1 In case both $\mathfrak{g}$ and $\mathfrak{h}$ are complex, then a real spherical pair $(\mathfrak{g}, \mathfrak{h})$ is simply called spherical. Spherical pairs with $\mathfrak{h}$ reductive where classified by Krämer ([17] for $\mathfrak{g}$ simple) and by Brion and Mikityuk ([5, 21] for $\mathfrak{g}$ semi-simple). It is easy to see that a real pair $(\mathfrak{g}, \mathfrak{h})$ is real spherical if its complexification $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is spherical, such pairs are said to be *absolutely spherical*. The converse is not true, there exist many real spherical pairs which are not absolutely spherical.

If $\mathfrak{n} \triangleleft \mathfrak{p}$ is the nilradical of $\mathfrak{p}$, then a real spherical subalgebra $\mathfrak{h}$ must satisfy the *dimension bound*

$$\dim \mathfrak{h} \geq \dim \mathfrak{n}.$$

The dimension bound is a decisive tool in the classification of (complex) spherical pairs, but due to the presence of large Levi-factors in real semi-simple Lie algebras the dimension bound can become rather weak for real pairs. For example if $\mathfrak{g} = \mathfrak{so}(1, n)$, then $\dim \mathfrak{n} = n - 1$ is very small. For that reason it might be a bit surprising that it is still possible to classify reductive real spherical subalgebras (see [10, 11]).

7.1 Local Structure Theorem

If L is a real reductive group, then we denote by $L_{\mathrm{n}} \triangleleft L$ the connected normal subgroup which corresponds to the sum of all non-compact factors of $\mathfrak{l}$.

Theorem 7.2 (Local structure theorem [12])   *Let $Z = G/H$ be a real spherical space and $P < G$ be a minimal parabolic subgroup such that PH is open. Then there exists a unique parabolic subgroup $Q \supset P$ with Levi-decomposition $Q = L \ltimes U$ such that*

  (1) $PH = QH$,

  (2) $Q \cap H = L \cap H$,

  (3) $L_{\mathrm{n}} \subset L \cap H$,

  (4) $QH/H \simeq U \times L/L \cap H$.

  A parabolic Q as above will be called Z-adapted.

7.2 Spherical Representation Theory

Let V be a Harish–Chandra module and V^{∞} its unique SF-completion. We say that V is *spherical* provided that $(V^{-\infty})^H \neq 0$. If $0 \neq \eta \in (V^{-\infty})^H$, then we refer to (V, η) as a *spherical pair*.

The Levi-decomposition $Q = L \ltimes U$ defines us in particular an opposed parabolic subgroup $\overline{Q} = L\overline{U}$ to Q. Furthermore, we let $G = KAN$ be an Iwasawa decomposition such that $A \subset L$ and $N \subset Q$.

Theorem 7.3 ([20])   *The following assertions hold for a real spherical space*:

  (1) (Spherical subrepresentation theorem) *Every irreducible spherical Harish–Chandra module V admits a $(\mathfrak{g}, K)$-embedding*

$$V \hookrightarrow \mathrm{Ind}_{\overline{Q}}^{G} \sigma$$

where σ is an irreducible finite dimensional representation of $\overline{Q}$ which is trivial on $\overline{U}$ and $\mathrm{Hom}_{\mathfrak{l} \cap \mathfrak{h}, L \cap H \cap K}(\sigma, \mathbb{C}) \neq \{0\}$.

(2) (Dimension bound) *Let* $\overline{\mathfrak{q}}_1 := \mathfrak{l} \cap \mathfrak{h} + \overline{\mathfrak{u}}$. *Then*

$$\dim(V^{-\infty})^H \leq \dim(V/\overline{\mathfrak{q}}_1 V)^{L \cap H \cap K} < \infty.$$

Remark 7.4 (a) The subrepresentation theorem for symmetric spaces (all of which are real spherical) was proved by Delorme in [7]. We would like to point out that the proof of (1) in [20] does not rely on the subrepresentation theorems of Casselman and Delorme.

(b) Since $\overline{\mathfrak{n}} \subset \overline{\mathfrak{q}}_1$ the second inequality in (2) follows immediately from the Casselman–Osborne lemma ([23, Prop. 3.7.1 with Cor. 3.7.2]) according to which the Jacquet module $J(V) := V/\overline{\mathfrak{n}}V$ is finite dimensional. The first inequality in (2) is generically sharp. A non-effective bound with a different proof was obtained previously in [16]. For symmetric spaces, the finite dimensionality was established in [2].

Proof We sketch a proof of Theorem 7.3 (2) for $G = \mathrm{SL}(2, \mathbb{R})$, based on the same ideas which enter the general proof. For simplicity we assume also that $\mathfrak{h}$ is one-dimensional. In this case $\mathfrak{q} = \mathfrak{p}$ and the bound amounts just to

$$\dim(V^{-\infty})^H \leq \dim V/\overline{\mathfrak{n}}V.$$

Let

$$E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and set

$$\mathfrak{n} = \mathbb{R}E, \quad \overline{\mathfrak{n}} = \mathbb{R}F, \quad \mathfrak{a} = \mathbb{R}Y.$$

Then $\mathfrak{p} = \mathfrak{a} + \mathfrak{n}$ is a minimal parabolic subalgebra of $\mathfrak{g}$. We assume $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Hence $\mathfrak{h} = \mathbb{R}U$ where $U = F + cY + dE$ for some $c, d \in \mathbb{R}$.

Let $\mathfrak{a}^- \subset \mathfrak{g}$ denote the closed half line $\mathbb{R}_{\leq 0}Y$, and consider its open neighborhoods $\mathfrak{a}_\epsilon^- := (-\infty, \epsilon)Y$ for each $\epsilon > 0$. For $t \in \mathbb{R}$ set $a_t := \exp(tY)$. Now if we choose the maximal compact subalgebra to be $\mathfrak{k} = \mathbb{R}(E - F)$, then

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathrm{Ad}(a_t)\mathfrak{h} \tag{7.2}$$

except if $\mathrm{e}^{-4t} = -d$. It will be convenient for some $\epsilon > 0$ to have (7.2) for all $tY \in \mathfrak{a}_\epsilon^-$, and this we accomplish by choosing instead $\mathfrak{k} = \mathbb{R}\,\mathrm{Ad}(a_s)(E - F)$ for some sufficiently large $s \in \mathbb{R}$.

Now comes a piece of information which we use as a black box, the existence of convergent asymptotic expansions of generalized matrix coefficients on the compression cone[1] (see [20, Sect. 5] and [13, Sect. 6]). When (7.2) is valid as above one can show that there is $0 < \epsilon' < \epsilon$, a finite set of leading exponents $\mathcal{E} = \mathcal{E}(V) \subset \mathbb{C}$, and a number $N = N(V) \in \mathbb{N}$ such that for all $v \in V$ one has absolutely convergent expansions:

$$m_{v,\eta}(a_t) = \sum_{\lambda \in \mathcal{E}} \sum_{n \in \mathbb{N}_0} \sum_{0 \leq k \leq N} \mathrm{e}^{t(\lambda + 2n)} t^k c_{k,n,\lambda}, \quad -\infty < t < \epsilon' \tag{7.3}$$

with coefficients $c_{k,n,\lambda} \in \mathbb{C}$, depending linearly on v.

To the Harish–Chandra module V we associate the finite-dimensional space $J(V) = V/\overline{\mathfrak{n}}V$ (see Remark 7.4). Since $\overline{\mathfrak{n}}$ is an ideal of $\overline{\mathfrak{p}} = \mathfrak{a} + \overline{\mathfrak{n}}$, we see that $J(V)$ is a module for $\overline{\mathfrak{p}}$. The

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[1] See Subsection 9.1 for the definition of the compression cone.

$\mathfrak{a}$-spectrum of $J(V)$ is closely tied to the set of exponents, $\mathcal{E}(V)$, introduced above (see [13, Sect. 7] and the reasoning below). On the side we remark here that $J(V)$ is two-dimensional for a generic irreducible Harish–Chandra module V. In order to avoid heavy notation involving matrix exponential functions with Jordan blocks we assume here that $J(V)$ is one-dimensional. Let $v_0 \in V$ be such that $\overline{v}_0 := v_0 + \overline{\mathfrak{n}}V$ generates $J(V)$. As we assume that $J(V)$ is one-dimensional we have $Y\overline{v}_0 = -\lambda\overline{v}_0$ for some $\lambda \in \mathbb{C}$.

Since $J(V)$ is one-dimensional we have to show that $(V^{-\infty})^H$ is at most one-dimensional. For $\eta \in (V^{-\infty})^H$ and $v \in V$ we set

$$f_v(t) = m_{v,\eta}(a_t) := \eta(a_t^{-1}v)$$

with the simplified notation $f := f_v$ when $v = v_0$. The one-dimensionality of $(V^{-\infty})^H$ will be established by showing that the number

$$c_{-\infty} = \lim_{t \to -\infty} e^{-\lambda t}f(t) \tag{7.4}$$

exists and determines η uniquely.

Let further $w \in V$ be such that $Yv_0 = -\lambda v_0 + Fw$ and set $u := Fw$. Let $r(t) := -f_u(t)$. We observe that f satisfies the first order differential equation

$$f'(t) = \lambda f(t) + r(t), \quad t \in \mathbb{R}$$

which features the general solution

$$f(t) = e^{\lambda t}\left(f(0) + \int_0^t e^{-s\lambda}r(s)\,ds\right). \tag{7.5}$$

We note that there is an a priori bound

$$|f_v(t)| \leq C_v e^{\Lambda t}, \quad t \leq 0 \tag{7.6}$$

for some $\Lambda \in \mathbb{R}$ and $C_v > 0$. This is a simple consequence of the fact that η is a distribution vector (see [19, (3.2) and (3.7)]), but it also follows from the asymptotic expansion (7.3). The key observation is now that for $u \in \overline{\mathfrak{n}}V$ one then has an improved bound

$$|f_u(t)| \leq C'_u e^{(\Lambda+2)t}, \quad t \leq 0, u \in \overline{\mathfrak{n}}V. \tag{7.7}$$

In fact write $u = Fw$ and note that

$$f_u(t) = \eta(a_t^{-1}Fw) = e^{2t}\eta(Fa_t^{-1}w) = e^{2t}\eta((U - cY - dE)a_t^{-1}w)$$
$$= -e^{2t}\eta((cY + dE)a_t^{-1}w) = -ce^{2t}f_{Yw}(t) - de^{4t}f_{Ew}(t)$$

and thus (7.7) follows from (7.6).

It is now a matter of simple combination of (7.5), (7.6) and (7.7) to deduce that (7.6) must hold with $\Lambda = \operatorname{Re}\lambda$. Then we also have (7.7) with that value of Λ, and the existence of (7.4) follows from (7.5). Moreover we can write

$$f(t) = e^{\lambda t}\left(c_{-\infty} + \int_{-\infty}^t e^{-s\lambda}r(s)\,ds\right). \tag{7.8}$$

Observe that $c_{-\infty}$ depends linearly on η. Finally if $c_{-\infty} = 0$, then (7.8) implies that $|f(t)| \leq Ce^{(\operatorname{Re}\lambda+2)t}$, which together with (7.7) allows us to replace Λ by $\Lambda+2$. Iterating we can achieve Λ as large as we wish and thus conclude $|f(t)| \ll e^{tN}$ for all $t \leq 0$ and $N > 0$. Hence

it follows from (7.3) that $f \equiv 0$. Since we may assume that v_0 belongs to a K-type and KAH has open interior on G we get that $m_{v_0,\eta} \equiv 0$ and thus $\eta = 0$. This proves that $\eta \mapsto c_{-\infty}$ is injective. $\qquad\square$

Problem 7.5 (Comparison conjectures) Let V be a Harish–Chandra module and V^∞ its unique SF-completion. Furthermore, let $\mathfrak{h} < \mathfrak{g}$ be a real spherical subalgebra. Then for a generic choice of K all homology groups $H_p(V, \mathfrak{h})$ are finite dimensional, see [1, Prop. 4.2.2]. The comparison conjecture of [1] (for $\mathfrak{h} = \mathfrak{n}$ due to Casselman) asserts for all $p \geq 0$:

(1) $H_p(V^\infty, \mathfrak{h})$ is separated,

(2) $H_p(V, \mathfrak{h}) \simeq H_p(V^\infty, \mathfrak{h})$.

Notice that (1) for $p = 0$ means that $\mathfrak{h}V^\infty$ is closed in V^∞, in other words the first order partial differential operator

$$\mathfrak{h} \otimes V^\infty \to V^\infty, \quad X \otimes v \mapsto X \cdot v$$

has closed range. This special case was established in [1] in case $\mathfrak{h}$ is absolutely spherical, that is $\mathfrak{h}_{\mathbb{C}}$ is spherical in $\mathfrak{g}_{\mathbb{C}}$. An important application is presented in [13, Section 7].

8 Disintegration of Group Representations

8.1 Direct Integrals of Hilbert Spaces and the Theorem of Gelfand–Kostyuchenko

Let Λ be a topological space with countable basis and let μ be a σ-finite Borel measure on Λ.

Suppose that for each $\lambda \in \Lambda$ we are given a Hilbert space $\mathcal{H}_\lambda$ with inner product $\langle \cdot, \cdot \rangle_\lambda$. We identify $\prod_\lambda \mathcal{H}_\lambda$ with functions $s : \Lambda \to \coprod_{\lambda \in \Lambda} \mathcal{H}_\lambda$ which satisfy $s(\lambda) \in \mathcal{H}_\lambda$. We refer to the elements $s \in \prod_\lambda \mathcal{H}_\lambda$ as sections.

By a *measurable family of Hilbert spaces over* Λ we understand a subspace of sections $\mathcal{F} \subset \prod_\lambda \mathcal{H}_\lambda$ which satisfies the following axioms:

• For all $s, t \in \mathcal{F}$, the map $\lambda \mapsto \langle s(\lambda), t(\lambda) \rangle_\lambda$ is measurable.

• If $t \in \prod_\lambda \mathcal{H}_\lambda$ is a section such that $\lambda \mapsto \langle t(\lambda), s(\lambda) \rangle_\lambda$ is measurable for all $s \in \mathcal{F}$, then $t \in \mathcal{F}$.

• There exists a countable subset $(s_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ such that for each $\lambda \in \Lambda$ the values $\{s_n(\lambda) \mid n \in \mathbb{N}\}$ span a dense subspace in $\mathcal{H}_\lambda$.

Given a measurable family of Hilbert spaces over Λ and $s \in \mathcal{F}$, we call s *square integrable* provided that

$$\int_\Lambda \langle s(\lambda), s(\lambda) \rangle_\lambda \, d\mu(\lambda) < \infty \,.$$

The space of square integrable sections (modulo a.e. equality) is denoted by

$$\int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$$

and referred to as the direct integral of the measurable family $(\mathcal{H}_\lambda)_\lambda$. Given our assumptions on Λ, the direct integral is a separable Hilbert space as well.

The basic example is where all $\mathcal{H}_\lambda = \mathbb{C}$ and then the direct integral is simply $L^2(\Lambda, \mu)$.

Let E be a topological vector space and $T : E \to \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ be a continuous linear map. We say that T is *pointwise defined*, provided that there exists for each $\lambda \in \Lambda$ a continuous linear map

$$\mathrm{ev}_\lambda : E \to \mathcal{H}_\lambda, \quad \lambda \in \Lambda$$

such that for all $v \in E$,

$$T(v)(\lambda) = \mathrm{ev}_\lambda(v), \quad \text{for a.e. } \lambda \in \Lambda.$$

Example 8.1 The identity $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is not pointwise defined. The inclusion $\mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R})$ is pointwise defined.

Recall that a continuous linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ between Hilbert spaces is called Hilbert–Schmidt, provided that there exists an orthonormal basis $(v_n)_{n \in \mathbb{N}}$ of $\mathcal{H}_1$ such that $\sum_{n \in \mathbb{N}} \|T(v_n)\|^2 < \infty$.

Theorem 8.2 (Gelfand–Kostyuchenko)   *Let $T : \mathcal{H} \to \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ be a Hilbert–Schmidt operator. Then T is pointwise defined.*

Proof   [3, Th. 1.5]. Here is a sketch. Let $(v_n)_{n \in \mathbb{N}}$ be an ONB of $\mathcal{H}$. Choose for each n a representative s_n for $T(v_n)$. Then,

$$\infty > \sum_{n \in \mathbb{N}} \|T(v_n)\|^2 = \sum_{n \in \mathbb{N}} \int_{\Lambda \backslash \Lambda_1} \|s_n(\lambda)\|_\lambda^2 \, d\mu(\lambda) = \int_{\Lambda \backslash \Lambda_1} \sum_{n \in \mathbb{N}} \|s_n(\lambda)\|_\lambda^2 \, d\mu(\lambda).$$

We conclude that there exists a set of measure zero $\Lambda_0 \subset \Lambda$ such that

$$\sum_{n \in \mathbb{N}} \|s_n(\lambda)\|_\lambda^2 < \infty, \quad \lambda \in \Lambda \backslash \Lambda_0.$$

Via Cauchy–Schwartz this allows us to define

$$\mathrm{ev}_\lambda(v) := \sum_{n \in \mathbb{N}} \langle v, s_n \rangle v_n(\lambda), \quad \lambda \in \Lambda \backslash \Lambda_0, v \in \mathcal{H};$$

$$\mathrm{ev}_\lambda(v) := 0, \quad \text{otherwise.}$$

Exercise 8.3 Let M be a manifold and let μ be a measure on M which is locally comparable to the Euclidean measure, i.e., for all $m \in M$ there exists a neighborhood U of M, a diffeomorphism $\phi : U \to V$ with $V \subset \mathbb{R}^n$ open such that $\phi_* \mu = f \, dx$ for a measurable function $f : V \to \mathbb{R}$ which is bounded from above and below by positive constants. Let $L^2(M) = \int_\Lambda^\oplus \mathcal{H}_\lambda d\mu(\lambda)$. Show that the inclusion $C_c^\infty(M) \to L^2(M)$ is pointwise defined. (Hint: Consider first the case where $M = (-1, 1)^n \subset \mathbb{R}^n$ and use the fact that a Sobolev space $\mathcal{L}$ of high enough order on M gives a Hilbert–Schmidt embedding $\mathcal{L} \to L^2(M)$.)

8.2 Disintegration

Let G be a type I-Lie group (see [24, p. 333]). The precise definition is not important for us; it is sufficient to know that real reductive Lie groups and nilpotent Lie groups are all type I.

The disintegration theorem for unitary representations $(\pi, \mathcal{H})$ of a type I-group asserts that ([24, Thm. 14.10.5])

$$(\pi, \mathcal{H}) \simeq \left(\int_\Lambda^\oplus \pi_\lambda \, d\mu(\lambda), \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda) \right), \tag{8.1}$$

that is,

(1) $\mathcal{H}$ is unitarily isomorphic to a direct integral of Hilbert spaces $\int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$,

(2) $(\pi_\lambda, \mathcal{H}_\lambda)$ is a unitary irreducible representation of G for each $\lambda \in \Lambda$,

(3) under the isomorphism $\mathcal{H} \simeq \int_\Lambda^\oplus \mathcal{H}_\lambda \, d\mu(\lambda)$ the representation π "diagonalizes" to $\int_\Lambda^\oplus \pi_\lambda \, d\mu(\lambda)$.

Furthermore, we note that the measure class of μ in (8.1) is unique.

Usually one prefers to combine equivalent representations and write (8.1) in the form

$$(\pi, \mathcal{H}) \simeq \left(\int_{\widehat{G}}^{\oplus} \rho \otimes \mathrm{id} \; d\mu(\rho), \int_{\widehat{G}}^{\oplus} \mathcal{H}_\rho \otimes \mathcal{M}_\rho \; d\mu(\rho) \right), \tag{8.2}$$

where μ is a measure on the unitary dual $\widehat{G}$ and $\mathcal{M}_\rho$ is a Hilbert space which represents the multiplicity of ρ in π.

We refer to (8.1) or (8.2) as the abstract Plancherel theorem of the unitary representation $(\pi, \mathcal{H})$.

Example 8.4 Consider $G = (\mathbb{R}, +)$. Then all unitary irreducible representation are one-dimensional and

$$\mathbb{R} \to \widehat{G}, \quad \lambda \mapsto \chi_\lambda; \quad \chi_\lambda(x) = \mathrm{e}^{\mathrm{i}\lambda x}$$

identifies $\widehat{G}$ with $\mathbb{R}$. The regular representation $(L, L^2(\mathbb{R}))$ then decomposes as

$$(L, L^2(\mathbb{R})) \simeq \left(\int_{\mathbb{R}}^{\oplus} \chi_\lambda \; d\lambda, \int_{\mathbb{R}}^{\oplus} \mathbb{C}_\lambda \; d\lambda \right)$$

with the isomorphism given by the Fourier-transform.

8.3 Abstract Plancherel Theory for $L^2(Z)$

Our concern in these notes is with the unitary representation $(L, L^2(Z))$ of G for a real spherical space Z. The final objective is to establish a concrete Plancherel-theorem, that is an explicit determination of the measure class of μ together with the corresponding multiplicity Hilbert spaces $\mathcal{M}_\rho$. This however, is still an open problem.

The map

$$\mathcal{F} : L^2(Z) \to \int_{\Lambda}^{\oplus} \mathcal{H}_\lambda \; d\mu(\lambda) \tag{8.3}$$

is called the Fourier-transform. Let us try to understand the corresponding multiplicity spaces. For that we note that the Fourier inclusion $C_c^\infty(Z) \to \int_\Lambda^{\oplus} \mathcal{H}_\lambda \; d\mu(\lambda)$ is pointwise defined (see Exercise 8.3). Hence we obtain for all $\lambda \in \Lambda$ a G-morphism

$$T_\lambda : C_c^\infty(Z) \to \mathcal{H}_\lambda \,.$$

Dualizing we obtain an antilinear G-morphism

$$S_\lambda : \mathcal{H}_\lambda \to \mathcal{D}(Z)$$

with $\mathcal{D}(Z)$ the distributions. Now it follows from Lemma 6.6 that S_λ induces an antilinear G-morphism $S_\lambda^\infty : \mathcal{H}_\lambda^\infty \to C^\infty(Z)$. In particular,

$$\eta_\lambda : \mathcal{H}_\lambda^\infty \to \mathbb{C}, \quad v \mapsto \overline{S_\lambda^\infty(v)(z_0)}$$

yields a continuous H-equivariant linear functional.

To move on we need to describe the fibers of the map

$$\Phi : \Lambda \to \widehat{G}, \quad \lambda \mapsto [\pi_\lambda].$$

For that we recall from (8.2) the multiplicity Hilbert space $\mathcal{M}_\rho$ for $\rho \in \widehat{G}$. For every $\rho \in \widehat{G}$ we let $(v_j^\rho)_{1 \le j \le n_\rho}$ be an orthonormal basis of $\mathcal{M}_\rho$ where $n_\rho \in \mathbb{N}_0 \cup \{\infty\}$. Then $\Phi^{-1}(\rho) \simeq N_\rho :=$

$\{v_1^\rho, \ldots, v_{n_\rho}^\rho\}$. Moreover, outside a set of measure zero, the assignment

$$\Phi^{-1}(\rho) \ni \lambda \mapsto \eta_\lambda \in (\mathcal{H}_\lambda^{-\infty})^H$$

extends to an injective linear map $\operatorname{span}_{\mathbb{C}} N_\rho \hookrightarrow (\mathcal{H}_\lambda^{-\infty})^H$ (essentially a consequence of the fact that $C_c^\infty(Z)$ is dense in $L^2(Z)$). We recall from Theorem 7.3 that $\dim(\mathcal{H}_\rho^{-\infty})^H < \infty$ and obtain for the multiplicity version (8.2) of the abstract Plancherel-theorem that $\dim \mathcal{M}_\rho < \infty$ and, as a vector space,

$$\mathcal{M}_\rho \subset (\mathcal{H}_\rho^{-\infty})^H, \quad \rho \in \widehat{G} \backslash \widehat{G}_0,$$

where $\widehat{G}_0 = \Lambda_0$ is a subset of measure zero in $\widehat{G} = \Lambda$.

We also obtain a formula for the inverse Fourier-transform. For $\phi \in C_c^\infty(Z)$ with $\mathcal{F}(\phi) = (v_\lambda)_\lambda$ one has

$$\phi = \int_\Lambda m_{v_\lambda, \eta_\lambda} \, d\mu(\lambda) \tag{8.4}$$

understood as an identity of distributions, that is, for all $\psi \in C_c^\infty(Z)$ one has

$$\langle \phi, \psi \rangle_{L^2(Z)} = \int_\Lambda \langle m_{v_\lambda, \eta_\lambda}, \psi \rangle_{L^2(Z)} \, d\mu(\lambda).$$

Indeed, let $\mathcal{F}(\psi) = (w_\lambda)_\lambda$ and observe that $m_{v_\lambda, \eta_\lambda} = \overline{S_\lambda^\infty(v_\lambda)}$ by the definition of η_λ. Hence

$$\int_\Lambda \langle m_{v_\lambda, \eta_\lambda}, \psi \rangle_{L^2(Z)} \, d\mu(\lambda) = \int_\Lambda \langle \overline{S_\lambda(v_\lambda)}, \psi \rangle_{L^2(Z)} \, d\mu(\lambda)$$

$$= \int_\Lambda \langle v_\lambda, T_\lambda(\psi) \rangle_\lambda \, d\mu(\lambda) = \int_\Lambda \langle v_\lambda, w_\lambda \rangle_\lambda \, d\mu(\lambda)$$

which shows absolute convergence.

9 The Schwartz Space of a Real Spherical Space

9.1 The Polar Decomposition of a Real Spherical Space

Let $Z = G/H$ be a real spherical space. We let $P < G$ be a minimal parabolic such that PH is open and let $Q = LU \supset P$ be the associated Z-adapted parabolic subgroup.

Recall the connected split torus $A \subset L$ and set $A_H := A \cap H$. Set $A_Z := A/A_H$. The number

$$\operatorname{rank}_{\mathbb{R}} Z := \dim_{\mathbb{R}} \mathfrak{a}_Z$$

is an invariant of Z and called the *real rank of Z*.

Let $d := \dim_{\mathbb{R}} \mathfrak{h}$ and write $\operatorname{Gr}_d(\mathfrak{g})$ for the Grassmannian of d-dimensional subspaces of $\mathfrak{g}$. Set $\mathfrak{h}_{\lim} := \mathfrak{l} \cap \mathfrak{h} + \overline{\mathfrak{u}}$ (previously denoted $\overline{\mathfrak{q}}_1$).

We define an open cone $\mathfrak{a}_Z^{--}$ in $\mathfrak{a}_Z$ by the property: $X \in \mathfrak{a}_Z^{--}$ if and only if

$$\lim_{t \to \infty} e^{t \operatorname{ad} X} \mathfrak{h} = \mathfrak{h}_{\lim}$$

in $\operatorname{Gr}_d(\mathfrak{g})$. The closure of $\mathfrak{a}_Z^{--}$ is denoted by $\mathfrak{a}_Z^-$ and called the *compression cone*. Set $A_Z^- := \exp(\mathfrak{a}_Z^-) \subset A_Z$.

Remark 9.1 The compression cone has the following universal property: Let (π, V) be a finite dimensional irreducible representation of G which is H-semi-spherical, that is, there exists a vector $0 \neq v_H \in V$ such that $\pi(h)v = \chi(h)v$ for a character of H. Assume in addition that V

is P-semi-spherical and let $v_0 \in V$ be a lowest weight vector. Then $X \in \mathfrak{a}_Z^{--}$ if and only if for all (H, P)-semi-spherical (π, V) one has

$$\lim_{t \to \infty} [\pi(\exp(tX))v_H] = [v_0]$$

as limit in $\mathbb{P}(V)$ (see [14], Sect. 5 for more on this topic).

Notice that $A_Z = A/A_H$ can be naturally identified with a subset of Z via $aA_H \mapsto a \cdot z_0$. Let $A_{\mathbb{C}} = \exp(\mathfrak{a}_{\mathbb{C}}) < G_{\mathbb{C}}$ and define $A_{Z,\mathbb{C}} = A_{\mathbb{C}}/A_{\mathbb{C}} \cap H_{\mathbb{C}}$. We view $A_{Z,\mathbb{C}}$ as a subset of $Z_{\mathbb{C}}$. Note that $A_{Z,\mathbb{C}} \cap Z$ falls into finitely many A_Z-orbits and we let $\mathcal{W}$ be a set of representatives from $\exp(i\mathfrak{a}_Z) \cdot z_0$.

Theorem 9.2 (Polar decomposition [14]) *There exists a compact set $\Omega \subset G$ such that*

$$Z = \Omega A_Z^- \mathcal{W} \cdot z_0.$$

Moreover, there is a finite set $F \subset G$ such that the above holds for $\Omega = FK$.

The polar decomposition tells us that the large scale geometry of Z is determined by $A_Z^- \times \mathcal{W}$.

9.2 The Canonical Weights on Z

Assume now that $Z = G/H$ is unimodular. Let $\rho_{\mathfrak{u}} \in \mathfrak{a}^*$ be defined as $\rho_{\mathfrak{u}} := \frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{u}}$. Observe that Z unimodular implies that $\rho_{\mathfrak{u}}$ factors through a functional on $\mathfrak{a}_Z$ (see [15, Lemma 4.2]).

Having defined the polar decomposition we can now give sharp bounds on the volume weight.

Proposition 9.3 ([15, Prop. 4.3]) *Let $Z = \Omega A_Z^- \mathcal{W} \cdot z_0$. Then there exist constants $C_1, C_2 > 0$ such that for all $z = \omega a w \cdot z_0 \in Z$ with $\omega \in \Omega$, $a \in A_Z^-$ and $w \in \mathcal{W}$ one has*

$$C_1 \cdot a^{-2\rho_{\mathfrak{u}}} \leq \mathbf{v}(z) \leq C_2 \cdot a^{-2\rho_{\mathfrak{u}}}.$$

Together with the volume weight there is the *radial weight function* on Z which is defined as

$$\mathbf{r}(z) := \sup_{\substack{z = \omega a w \cdot z_0 \\ \omega \in \Omega, a \in A_Z^-, w \in \mathcal{W}}} \| \log a \|$$

for $\Omega \subset G$ a sufficiently large compact set. This function is a weight on Z by [15, Prop. 3.4.] Later we need the following property.

Lemma 9.4 *Let $s > \dim \mathfrak{a}_Z$. Then*

$$\int_Z (1 + \mathbf{r}(z))^{-s} \mathbf{v}(z)^{-1} \, dz < \infty.$$

Proof Let $\Gamma \subset A_Z$ be a lattice, that is $\log \Gamma \subset \mathfrak{a}_Z$ is a lattice in the vector space $\mathfrak{a}_Z$. Then

$$\sum_{a \in \Gamma} (1 + \| \log a \|)^{-s} < \infty. \tag{9.1}$$

It follows from Theorem 9.2 that we can assume (after enlarging Ω) $Z = \Omega \Gamma^- \mathcal{W} \cdot z_0$ where $\Gamma^- := \Gamma \cap A_Z^-$. With this we have

$$\int_Z (1 + \mathbf{r}(z))^{-s} \mathbf{v}(z)^{-1} \, dz \leq \sum_{y \in \Gamma^- \mathcal{W}} \int_{\Omega y \cdot z_0} (1 + \mathbf{r}(z))^{-s} \mathbf{v}(z)^{-1} \, dz.$$

By using (4.2) we find positive constants such that

$$\mathbf{r}(z) \geq c_1 \mathbf{r}(y \cdot z_0), \quad \mathbf{v}(z) \geq c_2 \mathbf{v}(y \cdot z_0)$$

for all $z \in \Omega y \cdot z_0$, and hence

$$\int_{\Omega y \cdot z_0} (1 + \mathbf{r}(z))^{-s} \mathbf{v}(z)^{-1} \, dz \leq C(1 + \mathbf{r}(y \cdot z_0))^{-s} \frac{\mathrm{vol}_Z(\Omega y \cdot z_0)}{\mathbf{v}(y \cdot z_0)}.$$

The ratio of volumes is bounded (see Exercise 4.1), and hence the lemma follows from the definition of $\mathbf{r}$ and (9.1). $\qquad\square$

9.3 Definition and Basic Properties of the Schwartz Space

Fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. For $\alpha \in \mathbb{N}_0^n$, set $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. For a test function $f \in C_c^\infty(Z)$ and $m, k \in \mathbb{N}_0$, we define norms

$$p_{m,k}(f) := \sup_{z \in Z}(1 + \mathbf{r}(z))^m \, \mathbf{v}(z)^{\frac{1}{2}} \sum_{|\alpha| \leq k} |dL(X^\alpha)f(z)|,$$

$$q_{m,k}(f) := \left(\sum_{|\alpha| \leq k} \|(1 + \mathbf{r})^m dL(X^\alpha)f\|_{L^2(Z)}^2 \right)^{\frac{1}{2}}.$$

For each $k \in \mathbb{N}_0$, we denote by $\mathcal{C}_m^k(Z)$ and $\mathcal{L}_m^{2,k}(Z)$ the completion of $C_c^\infty(Z)$ with respect to $p_{m,k}$ and $q_{m,k}$, respectively. Note that $\mathcal{L}_m^{2,k}(Z)$ is a Hilbert space, more precisely, a weighted Sobolev space.

In this context we record (see [15, Prop. 5.1]):

Lemma 9.5 *The two families $(p_{m,k})_{m,k}$ and $(q_{m,k})_{m,k}$ are equivalent, i.e., they define the same topology on $C_c^\infty(Z)$.*

Proof Let $l > \dim G/2$. Then for $f \in C_c^\infty(Z)$,

$$|f(z)| \leq C_B \mathbf{v}(z)^{-\frac{1}{2}} \|f\|_{2,l,Bz}$$

by Lemma 4.2. By applying (4.2) to the weight $\mathbf{r}$ we easily see that

$$(1 + \mathbf{r}(z))^m \|f\|_{2,l,Bz} \leq C q_{m,l}(f)$$

for all z, all $f \in C_c^\infty(Z)$, and some constant $C > 0$. Hence

$$(1 + \mathbf{r}(z))^m \, \mathbf{v}(z)^{\frac{1}{2}} |f(z)| \leq C_B C q_{m,l}(f) \tag{9.2}$$

and it follows that $p_{m,k}$ is dominated by $q_{m,k+l}$. For the converse domination we use

$$q_{m,k}(f)^2 \leq \left(\int_Z (1 + \mathbf{r}(z))^{2(m-n)} \mathbf{v}(z)^{-1} \, dz \right) \cdot p_{m+n,k}(f)^2$$

and Lemma 9.4. $\qquad\square$

This gives us in particular that

$$\mathcal{C}(Z) := \bigcap_{m,k} \mathcal{C}_m^k(Z) = \bigcap_{m,k} \mathcal{L}_m^{2,k}(Z).$$

We call $\mathcal{C}(Z)$ the *Harish–Chandra Schwartz-space* of Z. Note that

$$C_c^\infty(Z) \subset \mathcal{C}(Z) \subset C^\infty(Z) \cap L^2(Z).$$

Building on the general theory developed in [3] it was shown in [15, Prop. 5.2]:

Proposition 9.6 *Let $m > 2 \operatorname{rank}_\mathbb{R} Z$ and $l > \frac{\dim \mathfrak{g}}{2}$. Then the inclusion $\mathcal{L}_m^{2,l}(Z) \to L^2(Z)$ is Hilbert–Schmidt. In particular (by Theorem 8.2), the inclusion map $\mathcal{C}(Z) \to L^2(Z)$ composed with the Fourier transform (8.3) is pointwise defined.*

Proof Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{L}_m^{2,l}(Z)$. We must show

$$\sum_n \|\xi_n\|_{L^2(Z)}^2 < \infty.$$

By the Sobolev Lemma 4.2 the evaluation at $z \in Z$ is a continuous linear form on $\mathcal{L}_m^{2,l}(Z)$. Hence there exists for each $z \in Z$ a function $\kappa_z \in \mathcal{L}_m^{2,l}(Z)$ such that $f(z) = \langle f, \kappa_z \rangle_{m,l}$ for all f, and it follows from (9.2) that

$$q_{m,l}(\kappa_z) \leq C_B C (1 + \mathbf{r}(z))^{-m} \mathbf{v}(z)^{-\frac{1}{2}}.$$

Hence,

$$\sum_n \int_Z |\xi_n(z)|^2 \, dz = \int_Z \sum_n |\langle \xi_n, \kappa_z \rangle_{m,l}|^2 \, dz = \int_Z q_{m,l}(\kappa_z)^2 \, dz < \infty$$

by Lemma 9.4. $\qquad\square$

10   The Notion of a Tempered Representation

We recall the abstract Plancherel theorem for the regular representation:

$$L^2(G/H) \simeq \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{M}_\pi \, d\mu(\pi)$$

with the finite dimensional multiplicity spaces $\mathcal{M}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$.

We wish to give a necessary condition for a spherical pair (V, η) to contribute to this decomposition of $L^2(Z)$, i.e., there exists a unitary representation $\pi \in \mathrm{supp}(\mu)$, for which the underlying Harish–Chandra module V_π equals V, and an $\eta_\pi \in \mathcal{M}_\pi$ which is equal to η. Once the theory is developed further it is expected that the condition below is also necessary.

Definition 10.1 *Let V be a Harish–Chandra module and let $\eta \in (V^{-\infty})^H$. The spherical pair (V, η) is called* tempered *if there exists $m \in \mathbb{Z}$ such that*

$$\sup_{z \in Z} |m_{v,\eta}(z)| \sqrt{\mathbf{v}(z)} (1 + \mathbf{r}(z))^m < \infty, \quad v \in V^\infty.$$

Proposition 10.2 ([15, Prop. 5.5])   *There exists a subset $\widehat{G}_0 \subset \widehat{G}$ of μ-measure zero such that (V_π, η_π) is tempered for all $\pi \in \widehat{G} \backslash \widehat{G}_0$ and $\eta_\pi \in \mathcal{M}_\pi$.*

Proof The argument is parallel to the treatment in Subsection 8.3 with $\mathcal{L}_m^{2,k}$ instead of $C_c^\infty(Z)$. We start with a continuous morphism $\mathcal{T}_\pi : \mathcal{L}_m^{2,k} \to \mathcal{H}_\pi$ and take its dual $\mathcal{T}_\pi' : \mathcal{H}_\pi' \to (\mathcal{L}_m^{2,k})'$. Now observe that there are natural G-equivariant anti-linear isomorphisms $\mathcal{H}_\pi' \simeq \mathcal{H}_\pi$ and $(\mathcal{L}_m^{2,k})' \simeq \mathcal{L}_{-m}^{2,-k}$. Hence we obtain a linear G-morphism

$$\mathcal{S}_\pi : \mathcal{H}_\pi \to \mathcal{L}_{-m}^{2,-k} \, .$$

Taking smooth vectors then yields a G-morphism $\mathcal{S}_\pi^\infty : \mathcal{H}_\pi^\infty \to \mathcal{L}_{-m}^{2,\infty}$. In analogy to Subsection 8.3 we have that $\mathcal{S}_\pi^\infty(v) = m_{v,\eta_\pi}$ for all $v \in \mathcal{H}_\pi^\infty$.

Finally we observe that $q_{-m,k+l}$ dominates $p_{-m,k}$ by the same argument as in the proof of Lemma 9.5. $\qquad\square$

References

[1] Aizenbud, A., Gourevitch, D., Krötz, B.: Hausdorffness for Lie algebra homology of Schwartz spaces and applications to the comparison conjecture. *Math. Z.*, **283**, 979–992 and 993–994 (2016)

[2] van den Ban, E.: Asymptotic behaviour of matrix coefficients related to reductive symmetric spaces. *Nederl. Akad. Wetensch. Indag. Math.*, **49**(3), 225–249 (1987)

[3] Bernstein, J.: On the support of Plancherel measure. *J. Geom. Phys.*, **5**(4), 663–710 (1988)

[4] Bernstein, J., Krötz, B.: Smooth Fréchet globalizations of Harish–Chandra modules. *Israel J. Math.*, **199**, 45–111 (2014)

[5] Brion, M.: Classification des espaces homogènes sphériques. *Compositio Math.*, **63**(2), 189–208 (1987)

[6] Casselman, W.: Jacquet modules for real reductive groups. Proceedings of the International Congress of Mathematicians (Helsinki, 1978). pp. 557–563, Acad. Sci. Fennica, Helsinki, 1980

[7] Delorme, P.: Injection de modules sphériques pour les espaces symétriques réductifs dans certaines représentations induites. Noncommutative harmonic analysis and Lie groups (Marseille-Luminy, 1985), pp. 108–143, Lecture Notes in Math. **1243**, Springer, Berlin, 1987

[8] Grothendieck, A.: Topological Vector Spaces, Gordon and Breach, Science Publishers, New York, 1973

[9] Knop, F., Krötz, B.: Reductive group actions, arXiv: 1604.01005

[10] Knop, F., Krötz, B., Pecher, T., et al.: Classification of reductive real spherical pairs I. The simple case. arXiv: 1609.00963

[11] Knop, F., Krötz, B., Pecher, T., et al.: Classification of reductive real spherical pairs II. The semisimple case. arXiv: 1703.08048

[12] Knop, F., Krötz, B., Schlichtkrull, H.: The local structure theorem for real spherical spaces. *Compositio Mathematica*, **151**, 2145–2159 (2015)

[13] Knop, F., Krötz, B., Schlichtkrull, H.: The tempered spectrum of a real spherical space, arXiv: 1509.03429

[14] Knop, F., Krötz, B., Sayag, E., et al.: Simple compactifications and polar decomposition of homogeneous real spherical spaces. *Selecta Math., N. S.*, **21**(3), 1071–1097 (2015)

[15] Knop, F., Krötz, B., Sayag, E., et al.: Volume growth, temperedness and integrability of matrix coefficients on a real spherical space. *J. Funct. Anal.*, **271**, 12–36 (2016)

[16] Kobayashi, T., Oshima, T.: Finite multiplicity theorems for induction and restriction. *Advances in Math.*, **248**, 921–944 (2013)

[17] Krämer, M.: Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen. *Compositio math.*, **38**(2), 129–153 (1979)

[18] Krötz, B., Sayag, E., Schlichtkrull, H.: Vanishing at infinity on homogeneous spaces of reductive type. *Compositio Math.*, **152**(7), 1385–1397 (2017)

[19] Krötz, B., Sayag, E., Schlichtkrull, H.: Decay of matrix coefficients on reductive homogeneous spaces of spherical type. *Math. Z.*, **278**, 229–244 (2014)

[20] Krötz, B., Schlichtkrull, H.: Multiplicity bounds and the subrepresentation theorem for real spherical spaces. *Trans. Amer. Math. Soc.*, **368**, 2749–2762 (2016)

[21] Mikityuk, I. V.: On the integrability of Hamiltonian systems with homogeneous configuration spaces. *Math. USSR Sbornik*, **57**(2), 527–546 (1987)

[22] Sakellaridis, Y., Venkatesh, A.: Periods and harmonic analysis on spherical varieties. *Asterisque*, to appear

[23] Wallach, N.: Real Reductive Groups I, Academic Press, San Diego, 1988

[24] Wallach, N.: Real Reductive Groups II, Academic Press, San Diego, 1992