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Additive Preservers of Drazin Invertible Operators with Bounded Index

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Abstract Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on an infinite-dimensional complex or real Banach space X. Given an integer $n \geq 1$, we show that an additive surjective map Φ on $\mathcal{B}(X)$ preserves Drazin invertible operators of index non-greater than n in both directions if and only if Φ is either of the form $\Phi(T) = \alpha ATA^{-1}$ or of the form $\Phi(T) = \alpha BT^*B^{-1}$ where α is a non-zero scalar, $A : X \to X$ and $B : X^* \to X$ are two bounded invertible linear or conjugate linear operators.

Keywords Linear preserver problems, Drazin inverse, ascent, descent

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1 Introduction

Throughout this paper, X denotes an infinite-dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators acting on X.

An operator $T \in \mathcal{B}(X)$ is called *Drazin invertible* if there exist $S \in \mathcal{B}(X)$ and a non-negative integer k such that

$$T^kST = T^k, \quad STS = T \quad \text{and} \quad TS = ST. \tag{1.1}$$

Such operator S is unique, it is called the *Drazin inverse* of T and denoted by T^{D}. The *Drazin index* of T, designated by $\mathrm{i}(T)$, is the smallest non-negative integer k satisfying (1.1). The concept of Drazin inverse was introduced in [5] and it has numerous applications in matrix theory, iterative methods, singular differential equations, and Markov chains, see for instance [1, 2, 4, 15] and the references therein.

Let Λ be a subset of $\mathcal{B}(X)$. An additive map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ is said to *preserve* Λ *in both directions* if for every $T \in \mathcal{B}(X)$,

$$T \in \Lambda \quad \text{if and only if} \quad \Phi(T) \in \Lambda.$$

In the last decades, there has been a remarkable interest in the so-called linear preserver problems which concern the question of characterizing linear, or additive, maps on Banach algebras that leave invariant a certain subset. For excellent expositions on linear preserver problems, the reader is referred to [7, 8, 12, 13, 18–20] and the references therein.

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One of the most famous problems in this direction is Kaplansky's conjecture [9] asking whether bijective unital linear maps Φ, between semi-simple Banach algebras, preserving invertibility in both directions are Jordan isomorphisms (i.e. $\Phi(a^2) = \Phi(a)^2$ for all a). The problem is still open even for C^*-algebras. However, in the case of the algebra $\mathcal{B}(X)$, Jafarian and Sourour establish in [8] that every unital surjective linear map Φ on $\mathcal{B}(X)$ preserving invertibility in both directions has one of the following two forms

$$T \mapsto ATA^{-1} \quad \text{or} \quad T \mapsto AT^*A^{-1},$$

where A is a bounded linear operator between suitable spaces. It is worth mentioning that an elegant proof of Jafarian-Sourour's result was given later by Šemrl in [19].

Following [17], we call an operator $T \in \mathcal{B}(X)$ *group invertible* if it is Drazin invertible with $\mathrm{i}(T) \leq 1$. The term group refers to the fact that T and T^{D} generate an Abelian group with identity TT^{D}; naturally T^{D} is called the group inverse of T. Note also that every invertible element is group invertible. Recently, it was shown in [14] that an additive surjective map $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ preserves group invertible operators in both directions if and only if it has one of the following two forms

$$T \mapsto \alpha ATA^{-1} \quad \text{or} \quad T \mapsto \alpha BT^*B^{-1}$$

where $\alpha \in \mathbb{K}$ is non-zero, $A : X \to X$ and $B : X^* \to X$ are two bounded invertible linear or conjugate linear operators.

For an operator $T \in \mathcal{B}(X)$, write $\ker(T)$ for its kernel, $\mathrm{ran}(T)$ for its range and T^* for its adjoint on the topological dual space X^*. For each integer $n \geq 1$, let us introduce the following subset:

$$\mathcal{D}_n(X) = \{T \in \mathcal{B}(X) : T \text{ is Drazin invertible and } \mathrm{i}(T) \leq n\}.$$

Clearly, $\mathcal{D}_n(X)$ includes every invertible operator and, more generally, every group invertible operator.

For $T \in \mathcal{B}(X)$, the *hyper-kernel* and the *hyper-range* are respectively the subspaces $\mathcal{N}^\infty(T) = \bigcup_k \ker(T^k)$ and $\mathcal{R}^\infty(T) = \bigcap_k \mathrm{ran}(T^k)$.

For each integer $n \geq 1$, let

$$\mathcal{B}_n(X) = \{T \in \mathcal{B}(X) : T \text{ is Drazin invertible and } \dim \mathcal{N}^\infty(T) \leq n\}.$$

This useful subset will permit, as will be seen subsequently, to reduce the problem of additive preservers of $\mathcal{D}_n(X)$ to those of $\mathcal{B}_n(X)$. For this, we provide an important characterization of the set $\mathcal{B}_n(X)$ via elements of $\mathcal{D}_n(X)$.

The purpose of this paper is to extend the main theorem of [14] to the setting of $\mathcal{D}_n(X)$. More precisely, the following theorem states the main result of this paper:

Theorem 1.1 *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map, and let n be a positive integer. The following assertions are equivalent*:

(i) Φ *preserves $\mathcal{D}_n(X)$ in both directions*;

(ii) Φ *preserves $\mathcal{B}_n(X)$ in both directions*;

(iii) *There exist a non-zero scalar α, and either a bounded invertible linear, or conjugate linear, operator $A : X \to X$ such that*

$$\Phi(T) = \alpha ATA^{-1} \quad \textit{for all } T \in \mathcal{B}(X),$$

or, a bounded invertible linear, or conjugate linear, operator $B : X^* \to X$ *such that*

$$\Phi(T) = \alpha B T^* B^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

In the following we recapture, as an immediate consequence of the previous theorem, the main result in [14] that gives a characterization of the additive surjective maps preserving group invertible operators in both directions.

Corollary 1.2 *Let* $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *be an additive surjective map. The following assertions are equivalent*:

(i) Φ *preserves group invertible operators in both directions*;

(ii) *There exist a non-zero scalar* α, *and either a bounded invertible linear, or conjugate linear, operator* $A : X \to X$ *such that*

$$\Phi(T) = \alpha A T A^{-1} \quad \textit{for all } T \in \mathcal{B}(X),$$

or, a bounded invertible linear, or conjugate linear, operator $B : X^* \to X$ *such that*

$$\Phi(T) = \alpha B T^* B^{-1} \quad \textit{for all } T \in \mathcal{B}(X).$$

The paper is organized as follows. In the second section, we establish some useful results on finite rank perturbations of $\mathcal{B}_n(X)$, and we also show that $\mathcal{B}_n(X)$ is the topological interior of $\mathcal{D}_n(X)$. These results will be needed for proving our main result in the last section.

2 $\mathcal{B}_n(X)$ and Rank One Perturbations

Let $T \in \mathcal{B}(X)$ be non-zero. The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of T are defined respectively by

$$\mathrm{a}(T) = \inf\{k \geq 0 : \ker(T^k) = \ker(T^{k+1})\} \quad \text{and} \quad \mathrm{d}(T) = \inf\{k \geq 0 : \mathrm{ran}(T^k) = \mathrm{ran}(T^{k+1})\},$$

where the infimum over the empty subset is set to be infinite, see [16, 21]. It should be noted that for every non-negative integer m,

$$\dim \ker(T^{m+1}) \leq m \;\Rightarrow\; \mathrm{a}(T) \leq m. \tag{2.1}$$

Indeed, if $\mathrm{a}(T) > m$, there exists a vector $x \in \ker(T^{m+1}) \setminus \ker(T^m)$, and so $x, Tx, \ldots, T^m x$ are linearly independent vectors in $\ker(T^{m+1})$. From (2.1), we easily obtain that the ascent and the hyper-kernel are related by the following inequality

$$\mathrm{a}(T) \leq \dim \mathcal{N}^\infty(T). \tag{2.2}$$

Remark 2.1 Let $T \in \mathcal{B}(X)$. Then T is Drazin invertible if and only if T has finite ascent and descent, see [10, Theorem 4]. Moreover, we have in this case the following well-known assertions (see [16, Corollary 20.5 and Theorem 22.10]):

(i) $\mathrm{i}(T) = \mathrm{a}(T) = \mathrm{d}(T)$;

(ii) $X = \ker(T^k) \oplus \mathrm{ran}(T^k)$ where $k = \mathrm{i}(T)$ and the direct sum is topological;

(iii) $\mathcal{N}^\infty(T) = \ker(T^k)$, $\mathcal{R}^\infty(T) = \mathrm{ran}(T^k)$ and $T_{|\mathcal{R}^\infty(T)}$ is invertible;

(iv) 0 is a pole of T of order k when $k \geq 1$.

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *Fredholm* if both $\dim \ker(T)$ and $\mathrm{codim}\, \mathrm{ran}(T)$ are finite. For such operator T, the *index* is defined by

$$\mathrm{ind}(T) = \dim \ker(T) - \mathrm{codim}\, \mathrm{ran}(T).$$

It should be noted that an operator T is Fredholm if and only if T^k is Fredholm for any (some) integer $k \geq 1$, and in this case we have $\mathrm{ind}(T^k) = k.\mathrm{ind}(T)$ (see [16, Theorems 16.5, 16.6 and 16.12]). Consequently, it follows from Remark 2.1 (ii) that every operator in $\mathcal{B}_n(X)$ is Fredholm of index zero. We also mention that the set of Fredholm operators and the index are invariant under compact perturbations (see [16, Theorem 16.16]).

Proposition 2.2 *$\mathcal{B}_n(X)$ is an open subset of $\mathcal{D}_n(X)$.*

For proving this proposition, we need to establish the following lemma:

Lemma 2.3 *Let $T \in \mathcal{B}(X)$ be a Fredholm operator of index zero. Then*

(i) *$T \in \mathcal{D}_n(X)$ if and only if $\mathrm{a}(T) \leq n$.*

(ii) *$T \in \mathcal{B}_n(X)$ if and only if $\dim\ker(T^{n+1}) \leq n$.*

Proof (i) The direct implication is an immediate consequence of Remark 2.1 (i). The converse follows from the fact that the descent and the ascent of a Fredholm operator of index zero are equal (see [13, Lemma 2.3]).

(ii) If $T \in \mathcal{B}_n(X)$ then $\dim\ker(T^{n+1}) \leq \dim\mathcal{N}^\infty(T) \leq n$. Conversely, suppose that $\dim\ker(T^{n+1}) \leq n$, then it follows by (2.1) that $\mathrm{a}(T) \leq n$. Hence, $\dim\mathcal{N}^\infty(T) = \dim\ker(T^{n+1}) \leq n$ and, by assertion (i), $T \in \mathcal{D}_n(X)$. Thus $T \in \mathcal{B}_n(X)$, which completes the proof. □

Note that from (2.2), we can easily see that for such operator T, we have also $T \in \mathcal{B}_n(X)$ if and only if $\dim\mathcal{N}^\infty(T) \leq n$.

Proof of Proposition 2.2 Since every operator in $\mathcal{B}_n(X)$ is Fredholm of index zero, it easily follows from (2.2) and Lemma 2.3 (i) that $\mathcal{B}_n(X) \subset \mathcal{D}_n(X)$.

Let us show that $\mathcal{B}_n(X)$ is open. Let $S \in \mathcal{B}_n(X)$. Then $\dim\ker(S^{n+1}) \leq n$, and the operators S and S^{n+1} are Fredholm of index zero. Hence, it follows by [16, Theorems 16.11 and 16.17] that there exists $\eta > 0$ such that for all $T \in \mathcal{B}(X)$ with $\| T - S^{n+1} \| < \eta$, we have T is Fredholm of index zero and

$$\dim\ker(T) \leq \dim\ker(S^{n+1}) \leq n. \tag{2.3}$$

On the other hand, since the map $T \mapsto T^{n+1}$ is continuous on $\mathcal{B}(X)$, there exists $\varepsilon > 0$ such that

$$\| T^{n+1} - S^{n+1} \| < \eta \quad \text{for all } T \in \mathcal{B}(X) \quad \text{with } \| T - S \| < \varepsilon. \tag{2.4}$$

Combining (2.4) and (2.3) we obtain that T^{n+1}, and hence T, is Fredholm of index zero, and

$$\dim\ker(T^{n+1}) \leq \dim\ker(S^{n+1}) \leq n$$

for every $T \in \mathcal{B}(X)$ with $\| T - S \| < \varepsilon$. Thus, by Lemma 2.3 (ii), $T \in \mathcal{B}_n(X)$ for every $T \in \mathcal{B}(X)$ with $\| T - S \| < \varepsilon$. This shows that $\mathcal{B}_n(X)$ is open, and the proof is complete.

Let $T \in \mathcal{B}(X)$. From [6, Lemma 1.1], given a non-negative integer d, we have

$$\mathrm{a}(T) \leq d \Leftrightarrow \ker(T^m) \cap \mathrm{ran}(T^d) = \{0\} \quad \text{for some (equivalently, all) } m \geq 1, \tag{2.5}$$

and

$$\mathrm{d}(T) \leq d \Leftrightarrow \ker(T^d) + \mathrm{ran}(T^m) = X \quad \text{for some (equivalently, all) } m \geq 1. \tag{2.6}$$

In the next theorem, we provide a useful characterization that allows us to obtain the implication (i) ⇒ (ii) in Theorem 1.1.

Theorem 2.4 *Let $T \in \mathcal{B}(X)$. Then the following assertions are equivalent*:

(i) $T \in \mathcal{B}_n(X)$;

(ii) *For every $S \in \mathcal{B}(X)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S \in \mathcal{D}_n(X)$ for all non-negative number* (*equivalently rational number*) $\varepsilon < \varepsilon_0$.

Proof (i) $\Rightarrow$ (ii) follows immediately from the openness of $\mathcal{B}_n(X)$.

(ii) $\Rightarrow$ (i). We have in particular that $T \in \mathcal{D}_n(X)$. Suppose on the contrary that $T \notin \mathcal{B}_n(X)$. Then $\dim\ker(T^k) \geq n+1$ where $k = \mathrm{i}(T)$, and so there exist linearly independent vectors $\{e_i : 0 \leq i \leq n\}$ such that $Te_0 = 0$ and $Te_i = \varepsilon_i e_{i-1}$ for $1 \leq i \leq n$, where $\varepsilon_i \in \{0, 1\}$. Indeed, if $\dim\ker(T^k) < \infty$, the existence of such vectors is obvious; otherwise, we obtain $\dim\ker(T) = \infty$ because $\dim\ker(T^k) \leq k \dim\ker(T)$, see [3, Lemma 1], and so it suffices to take $e_i \in \ker(T)$, $0 \leq i \leq n$. Consider the operator $S \in \mathcal{B}(X)$ given by $Se_0 = 0$ and $Se_i = e_{i-1}$ for $1 \leq i \leq n$. For any $\varepsilon \notin \{-1, 0\}$, we have

$$(T + \varepsilon S)e_0 = 0 \quad \text{and} \quad (T + \varepsilon S)e_i = (\varepsilon_i + \varepsilon)e_{i-1} \quad \text{for } 1 \leq i \leq n,$$

and hence $(T + \varepsilon S)^n e_n = \lambda e_0 \neq 0$ where $\lambda = (\varepsilon_n + \varepsilon)\dots(\varepsilon_1 + \varepsilon)$. Therefore $e_0 \in \ker(T + \varepsilon S) \cap \mathrm{ran}(T + \varepsilon S)^n$, and consequently $\mathrm{a}(T + \varepsilon S) \geq n + 1$ by (2.5), this contradiction finishes the proof. □

For a subset $\Gamma \subseteq \mathcal{B}(X)$, we write $\mathrm{Int}(\Gamma)$ for its interior.

Corollary 2.5 *We have* $\mathrm{Int}(\mathcal{D}_n(X)) = \mathcal{B}_n(X)$.

Proof Note that $\mathcal{B}_n(X) \subseteq \mathrm{Int}(\mathcal{D}_n(X))$ because $\mathcal{B}_n(X)$ is open. Let $T \notin \mathcal{B}_n(X)$, then it follows by Theorem 2.4 that there exist an operator $S \in \mathcal{B}(X)$ and a sequence $(\varepsilon_k)_{k\geq 0}$ converging to zero such that $T + \varepsilon_k S \notin \mathcal{D}_n(X)$ for all $k \geq 0$. Consequently, $T \notin \mathrm{Int}(\mathcal{D}_n(X))$, as desired. The proof is complete. □

Let $z \in X$ and $f \in X^*$ be non-zero. We will denote, as customary, by $z \otimes f$ the rank one operator defined by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that every rank one operator in $\mathcal{B}(X)$ can be written in this form.

The following theorem gives necessary and sufficient conditions for the stability of $\mathcal{B}_n(X)$ under rank one perturbations.

Theorem 2.6 *Let $T \in \mathcal{B}_n(X)$, and let $p = n - \dim\mathcal{N}^\infty(T)$. If $z \in X$ and $f \in X^*$ then $T + z \otimes f \notin \mathcal{B}_n(X)$ if and only if the following assertions hold*:

(i) *$z = a + b$ where $a \in \mathcal{N}^\infty(T)$ and $b \in \mathcal{R}^\infty(T)$*;

(ii) *$f(T^j a) = 0$ for all $j \geq 0$*;

(iii) *$f(T_{\mathrm{o}}^{-i} b) = -\delta_{i1}$ for $1 \leq i \leq p + 1$ where $T_{\mathrm{o}} = T_{|\mathcal{R}^\infty(T)}$*.

Remark 2.7 Let $T \in \mathcal{B}(X)$ be a Fredholm operator of index zero. If $\dim\ker(T) \leq 1$, then $\dim\ker(T^k) \leq k$ for every integer $k \geq 0$, see [3, Lemma 1]. Moreover, in this case, $T \in \mathcal{B}_n(X)$ if and only if $\mathrm{a}(T) \leq n$.

To prove Theorem 2.6, we need to establish the following two lemmas.

Lemma 2.8 *Let $T \in \mathcal{B}_n(X)$ and $p = n - \dim\mathcal{N}^\infty(T)$. Let $F = z \otimes f$ be a rank one operator such that $\ker(T) \cap \ker(F) = \{0\}$. Then $T + z \otimes f \notin \mathcal{B}_n(X)$ if and only if $z \in \mathcal{R}^\infty(T)$ and $f(T_{\mathrm{o}}^{-i} z) = -\delta_{i1}$ for $1 \leq i \leq p + 1$, where $T_{\mathrm{o}} = T_{|\mathcal{R}^\infty(T)}$.*

Proof Note that since T is a Fredholm operator of index zero, then so is $T+F$. Clearly, the restrictions of F to $\ker(T+F)$ and $\ker(T)$ are injective rank one operators because $\ker(T)\cap\ker(F)=\{0\}$. Hence, it follows that $\dim\ker(T+F)\le 1$ and $\dim\ker(T)\le 1$. If we let $m=\mathrm{i}(T)$, we get by the previous remark and (2.2) that $m=\dim\ker(T^m)$.

Suppose that $T+F\notin\mathcal{B}_n(X)$. Then $\mathrm{a}(T+F)\ge n+1$ by Remark 2.7, and hence it follows from [12, Lemma 2.2] that there exist linearly independent vectors x_i, $0\le i\le n$, and an integer $0\le j\le n$ such that

$$(T+F)x_0=0,\quad (T+F)x_i=x_{i-1}\quad \text{for } 1\le i\le n,\quad \text{and}\quad f(x_i)=\delta_{ij}\quad \text{for } 0\le i\le n.$$

But, since $\ker(T)\cap\ker(F)=\{0\}$, we obtain that $j=0$. Consequently, $Tx_0=-z$ and $Tx_i=x_{i-1}$ for $1\le i\le n$. From this, we get easily that

$$T^ix_{i-1}=Tx_0=-z\quad \text{and}\quad T^{n+1-i}x_n=x_{i-1}\quad \text{for } 1\le i\le n+1.$$

Therefore $z=-T^{n+1}x_n\in\mathrm{ran}(T^{n+1})=\mathrm{ran}(T^m)$, and since $p=n-m$, we also have $x_{i-1}\in\mathrm{ran}(T^{n+1-i})=\mathrm{ran}(T^m)$ for $1\le i\le p+1$. Thus, $T_{\mathrm{o}}^{-i}z=-x_{i-1}$, and hence $f(T_{\mathrm{o}}^{-i}z)=-\delta_{i1}$ for $1\le i\le p+1$.

Conversely, suppose that $z\in\mathrm{ran}(T^m)$ and $f(T_{\mathrm{o}}^{-i}z)=-\delta_{i1}$ for $1\le i\le p+1$. Let $y_i=-T_{\mathrm{o}}^{-(i+1)}z$ for $0\le i\le p$. Clearly, we have

$$(T+F)y_0=-(T+F)T_{\mathrm{o}}^{-1}z=0\quad \text{and}\quad (T+F)y_i=Ty_i=y_{i-1}\quad \text{for } 1\le i\le p.$$

Consequently, $(T+F)^py_p=y_0\ne 0$. Hence, $y_0\in\mathrm{ran}(T+F)^p\cap\ker(T+F)$, and so $\mathrm{a}(T+F)\ge p+1$ by (2.5). If T is invertible then $p=n$ and $T+F\notin\mathcal{B}_n(X)$. Assume that T is not invertible. Since $\dim\ker(T)=1$, there is a basis $\{e_i:0\le i\le m-1\}$ of $\ker(T^m)$ such that $Te_0=0$ and $Te_i=e_{i-1}$ for $1\le i\le m-1$. Since $\ker(T)\cap\ker(f)=\{0\}$, we infer that $f(e_0)\ne 0$. Without loss of generality, we may assume that $f(e_0)=1$. Consider the complex numbers $c_0,c_1,\dots,c_{m-1}$ defined inductively by

$$\begin{aligned}
c_0&=f(T_{\mathrm{o}}^{-(p+2)}z),\\
c_1&=-c_0f(e_1)+f(T_{\mathrm{o}}^{-(p+3)}z),\\
c_2&=-c_1f(e_1)-c_0f(e_2)+f(T_{\mathrm{o}}^{-(p+4)}z),\\
&\ \ \vdots\\
c_{m-1}&=-c_{m-2}f(e_1)-\cdots-c_0f(e_{m-1})+f(T_{\mathrm{o}}^{-(p+m+1)}z).
\end{aligned}$$

This means that we have

$$\sum_{k=1}^{i}c_{i-k}f(e_{k-1})=f(T_{\mathrm{o}}^{-(p+i+1)}z)\quad \text{for } 1\le i\le m. \tag{2.7}$$

Let $y_{p+i}=\sum_{k=1}^{i}c_{i-k}e_{k-1}-T_{\mathrm{o}}^{-(p+i+1)}z$ for $1\le i\le m$. Clearly, from (2.7) we have $f(y_{p+i})=0$ for $1\le i\le m$. Furthermore, since $Te_0=0$, we get that

$$(T+F)y_{p+1}=Ty_{p+1}=T(c_0e_0-T_{\mathrm{o}}^{-(p+2)}z)=-T_{\mathrm{o}}^{-(p+1)}z=y_p,$$

and

$$(T+F)y_{p+i}=Ty_{p+i}=\sum_{k=2}^{i}c_{i-k}e_{k-2}-T_{\mathrm{o}}^{-(p+i)}z=\sum_{s=1}^{i-1}c_{i-1-s}e_{s-1}-T_{\mathrm{o}}^{-(p+i)}z=y_{p+i-1},$$

for $2 \le i \le m$. Hence, $(T+F)^m y_{p+m} = y_p$. But, since $n = p+m$ and $(T+F)^p y_p = y_0$, we obtain that $(T+F)^n y_n = y_0 \neq 0$, and so $\mathrm{a}(T+F) \ge n+1$. Thus, $T+F \notin \mathcal{B}_n(X)$. This completes the proof. □

For $T, F \in \mathcal{B}(X)$, let

$$\mathrm{M}(T,F) = \{x \in \mathcal{N}^\infty(T) : F(T^i x) = 0 \text{ for all } i \ge 0\}.$$

Note that $\mathrm{M}(T,F)$ is a T-invariant subspace of $\mathcal{N}^\infty(T) \cap \ker(F)$. In the sequel, $\tilde{T}$ and $\tilde{F}$ shall denote the operators induced by T and F on $X/\mathrm{M}(T,F)$, respectively. Also we will write $\tilde{x}$ for the class of x in $X/\mathrm{M}(T,F)$.

Lemma 2.9 *Let $T, F \in \mathcal{B}(X)$ be non-zero. The following assertions hold*:

(i) $\mathcal{N}^\infty(\tilde{T} + c\tilde{F}) = \mathcal{N}^\infty(T+cF)/\mathrm{M}(T,F)$ *for all* $c \in \mathbb{K}$;

(ii) *If* $T \in \mathcal{B}_n(X)$ *then* $\tilde{T} \in \mathcal{B}_{n-q}(X/\mathrm{M}(T,F))$, *where* $q = \dim \mathrm{M}(T,F)$, *and*

$$\mathcal{R}^\infty(\tilde{T}) = (\mathcal{R}^\infty(T) \oplus \mathrm{M}(T,F))/\mathrm{M}(T,F).$$

Proof (i) Fix an arbitrary $c \in \mathbb{K}$. We first claim that $\mathrm{M}(T,F) \subseteq \mathcal{N}^\infty(T+cF)$. Let $x \in \mathrm{M}(T,F)$ be arbitrary. Using the fact that $F(T^i x) = 0$ for all $i \ge 0$, one can easily verify that $(T+cF)^i x = T^i x$ for all $i \ge 0$. But, since $x \in \mathcal{N}^\infty(T)$, we have $T^j x = 0$ for some $j \ge 0$. Thus $(T+cF)^j x = T^j x = 0$, and so $x \in \mathcal{N}^\infty(T+cF)$.

Let $x \in X$, we have

$$\begin{aligned}\tilde{x} \in \mathcal{N}^\infty(\tilde{T}+c\tilde{F}) &\Leftrightarrow \exists k \ge 0 : (T+cF)^k x \in \mathrm{M}(T,F) \subseteq \mathcal{N}^\infty(T+cF)\\ &\Leftrightarrow x \in \mathcal{N}^\infty(T+cF)\\ &\Leftrightarrow \tilde{x} \in \mathcal{N}^\infty(T+cF)/\mathrm{M}(T,F).\end{aligned}$$

This shows that $\mathcal{N}^\infty(\tilde{T}+c\tilde{F}) = \mathcal{N}^\infty(T+cF)/\mathrm{M}(T,F)$.

(ii) Suppose that $T \in \mathcal{B}_n(X)$, and let $m = \mathrm{i}(T)$. Since $\mathrm{M}(T,F) \subseteq \ker(T^m)$, one can easily check that

$$\mathrm{ran}(\tilde{T}^m) = (\mathrm{M}(T,F) \oplus \mathrm{ran}(T^m))/\mathrm{M}(T,F). \tag{2.8}$$

Moreover, we have by the first assertion

$$\mathcal{N}^\infty(\tilde{T}) = \mathcal{N}^\infty(T)/\mathrm{M}(T,F) = \ker(T^m)/\mathrm{M}(T,F) \subseteq \ker(\tilde{T}^m).$$

Therefore, $\ker(\tilde{T}^m) = \mathcal{N}^\infty(\tilde{T}) = \ker(T^m)/\mathrm{M}(T,F)$. Hence, taking into account that $X = \ker(T^m) \oplus \mathrm{ran}(T^m)$, we get easily that $\ker(\tilde{T}^m) \oplus \mathrm{ran}(\tilde{T}^m) = X/\mathrm{M}(T,F)$. This shows that $\tilde{T}$ is Drazin invertible with $\mathrm{i}(\tilde{T}) \le m$. In particular, it follows from (2.8) that

$$\mathcal{R}^\infty(\tilde{T}) = (\mathcal{R}^\infty(T) \oplus \mathrm{M}(T,F))/\mathrm{M}(T,F).$$

Furthermore, since

$$\dim \ker(\tilde{T}^m) = \dim \ker(T^m) - \dim \mathrm{M}(T,F) \le n-q,$$

we obtain that $\tilde{T} \in \mathcal{B}_{n-q}(X/\mathrm{M}(T,F))$. The proof is complete. □

Proof of Theorem 2.6 Since $T \in \mathcal{B}_n(X)$, we have $\tilde{T} \in \mathcal{B}_{n-q}(X/\mathrm{M}(T,F))$ by the previous lemma. Consequently, T and $T+F$ are Fredholm of index zero. Let $q = \dim \mathrm{M}(T,F)$, it follows by the remark after Lemma 2.3 that

$$\begin{aligned} T+F\notin\mathcal{B}_n(X) &\Leftrightarrow \dim\mathcal{N}^\infty(T+F)\geq n+1\\ &\Leftrightarrow \dim\mathcal{N}^\infty(\tilde{T}+\tilde{F})\geq n-q+1\\ &\Leftrightarrow \tilde{T}+\tilde{F}\notin\mathcal{B}_{n-q}(X/\mathrm{M}(T,F)). \end{aligned}$$

Moreover, since in this case $\tilde{F}\neq 0$, we get easily that $z\notin\mathrm{M}(T,F)$, and for every $x\in X$, we have

$$\begin{aligned} \tilde{x}\in\ker(\tilde{T})\cap\ker(\tilde{F}) &\Leftrightarrow Tx\in\mathrm{M}(T,F)\quad\text{and}\quad Fx=f(x)z\in\mathrm{M}(T,F)\\ &\Leftrightarrow Tx\in\mathrm{M}(T,F)\quad\text{and}\quad f(x)=0\\ &\Leftrightarrow x\in\mathrm{M}(T,F). \end{aligned}$$

Thus $\ker(\tilde{T})\cap\ker(\tilde{F})=\{0\}$. Therefore, using Lemma 2.8 and the fact that $n-q-\dim\mathcal{N}^\infty(\tilde{T})=p$, we obtain that

$$\tilde{T}+\tilde{F}\notin\mathcal{B}_{n-q}(X/\mathrm{M}(T,F))\Leftrightarrow\begin{cases}\tilde{z}\in\mathcal{R}^\infty(\tilde{T}),\\ \tilde{f}((\tilde{T}_{|\mathcal{R}^\infty(\tilde{T})})^{-i}\tilde{z})=-\delta_{i1}\quad\text{for } 1\leq i\leq p+1.\end{cases}$$

Let $z=a+b$ where $a\in\mathcal{N}^\infty(T)$ and $b\in\mathcal{R}^\infty(T)$. Then $\tilde{z}=\tilde{a}+\tilde{b}$ and, by Lemma 2.9, $\tilde{a}\in\mathcal{N}^\infty(\tilde{T})$ and $\tilde{b}\in\mathcal{R}^\infty(\tilde{T})$. Therefore, since $\tilde{T}$ is Drazin invertible, we have $\mathcal{N}^\infty(\tilde{T})\cap\mathcal{R}^\infty(\tilde{T})=\{0\}$, and so

$$\begin{aligned} \tilde{T}+\tilde{F}\notin\mathcal{B}_{n-q}(X/\mathrm{M}(T,F)) &\Leftrightarrow\begin{cases}\tilde{a}=0,\\ \tilde{f}((\tilde{T}_{|\mathcal{R}^\infty(\tilde{T})})^{-i}\tilde{b})=-\delta_{i1}\quad\text{for } 1\leq i\leq p+1\end{cases}\\ &\Leftrightarrow\begin{cases}a\in\mathrm{M}(T,F),\\ f(T_{\circ}^{-i}b)=-\delta_{i1}\quad\text{for } 1\leq i\leq p+1.\end{cases} \end{aligned}$$

This completes the proof.

The following proposition, which is interesting in itself, will play a crucial role in the sequel.

Proposition 2.10 *Let $F\in\mathcal{B}(X)$ be non-zero. Then the following assertions hold*:

(i) *There exists an invertible operator $T\in\mathcal{B}(X)$ such that $T+F\notin\mathcal{B}_n(X)$*;

(ii) *If $\dim\mathrm{ran}(F)\geq 2$ then there exists an invertible operator $T\in\mathcal{B}(X)$ such that $T+F\notin\mathcal{B}_n(X)$ and $T-F\notin\mathcal{B}_n(X)$.*

Proof First, suppose that $\mathrm{ran}(F)$ has infinite-dimension. Then $\mathrm{codim}\ker(F)=\infty$. Let x_i, $0\leq i\leq 2n+1$, be linearly independent vectors that generate a subspace having trivial intersection with $\ker(F)$. It follows that the vectors Fx_i, $0\leq i\leq 2n+1$, are linearly independent. Write

$$X=\mathrm{span}\{x_i:0\leq i\leq 2n+1\}\oplus Y=\mathrm{Span}\{Fx_i:0\leq i\leq 2n+1\}\oplus Z,$$

where Y,Z are two closed subspaces and $Y=F^{-1}Z$. Then there exists an invertible operator $T\in\mathcal{B}(X)$ such that $TY=Z$ and $Tx_i=(-1)^iFx_i$ for $0\leq i\leq 2n+1$. Clearly, $x_{2i+1}\in\ker(T+F)$ and $x_{2i}\in\ker(T-F)$ for $0\leq i\leq n$. Hence, $\dim\ker(T\pm F)>n$, and so $T\pm F\notin\mathcal{B}_n(X)$. This establishes the assertions (i) and (ii).

Now suppose that F is finite-rank, and let $r=\min\{\dim\mathrm{ran}(F),2\}$. Since $\ker(F)$ is infinite-dimensional and $r\leq\dim\mathrm{ran}(F)$, there exist vectors $y_j\in X$, $0\leq j\leq r-1$, such that the set $\{Fy_j : 0 \le j \le r-1\}$ is linearly independent. Perturbing y_j by suitable elements of $\ker(F)$ we may assume that the vectors $\{y_j, Fy_j : 0 \le j \le r-1\}$ are linearly independent. Consider also arbitrary vectors $z_{i,j} \in \ker(F)$, $0 \le j \le r-1$ and $1 \le i \le n$, such that the set $\{z_{i,j}, y_j, Fy_j : 0 \le j \le r-1,\ 1 \le i \le n\}$ is linearly independent. Let $T \in \mathcal{B}(X)$ be an arbitrary invertible operator satisfying

$$\begin{cases} Tz_{i,j} = z_{i-1,j} & \text{for } 0 \le j \le r-1 \text{ and } 2 \le i \le n, \\ Tz_{1,j} = y_j \quad \text{and} \quad Ty_j = (-1)^{j+1}Fy_j & \text{for } 0 \le j \le r-1. \end{cases}$$

It follows that

$$\begin{cases} (T \pm F)z_{i,j} = Tz_{i,j} = z_{i-1,j} & \text{for } 0 \le j \le r-1 \text{ and } 2 \le i \le n, \\ (T \pm F)z_{1,j} = Tz_{1,j} = y_j & \text{for } 0 \le j \le r-1. \end{cases}$$

From this, we get easily that $(T \pm F)^n z_{n,j} = y_j$ for $0 \le j \le r-1$. Hence, if $r = 1$ then $y_0 \in \operatorname{ran}(T+F)^n \cap \ker(T+F)$, and so $\mathrm{a}(T+F) > n$ and $T+F \notin \mathcal{B}_n(X)$. While, if $r = 2$ then $y_0 \in \operatorname{ran}(T+F)^n \cap \ker(T+F)$ and $y_1 \in \operatorname{ran}(T-F)^n \cap \ker(T-F)$, and hence $\mathrm{a}(T \pm F) > n$ and $T \pm F \notin \mathcal{B}_n(X)$. This completes the proof. □

In the following theorem we give a useful characterization of rank one operators in terms of elements of $\mathcal{B}_n(X)$.

Theorem 2.11 *Let $F \in \mathcal{B}(X)$ be non-zero. The following assertions are equivalent*:

(i) *F is rank one*;

(ii) *For every $T \in \mathcal{B}_n(X)$*, $\mathrm{Card}\{\lambda \in \mathbb{Q} : T + \lambda F \notin \mathcal{B}_n(X)\} \le 1$.

Proof (i) ⇒ (ii). Write $F = z \otimes f$ with $z \in X$ and $f \in X^*$. Let $T \in \mathcal{B}_n(X)$. Then $z = a + b$ where $a \in \mathcal{N}^\infty(T)$ and $b \in \mathcal{R}^\infty(T)$. If there exists $\alpha \in \mathbb{Q}$ such that $T + \alpha z \otimes f \notin \mathcal{B}_n(X)$, then it follows by Theorem 2.6 that $\alpha f(T_{\mathrm{o}}^{-1}b) = -1$ where $T_{\mathrm{o}} = T_{|\mathcal{R}^\infty(T)}$. This shows that $T + \lambda F \in \mathcal{B}_n(X)$ for all rational number $\lambda \ne \alpha$.

(ii) ⇒ (i) follows immediately from the second assertion of Proposition 2.10. The proof is finished. □

Let $T = T_1 \oplus T_2$ be a diagonal operator with respect to a given decomposition of X. One can easily show that T is Drazin invertible if and only if T_1 and T_2 are Drazin invertible, and in this case $\mathrm{i}(T) = \max\{\mathrm{i}(T_1), \mathrm{i}(T_2)\}$.

The following example shows that Theorem 2.11 does not hold if we replace $\mathcal{B}_n(X)$ by $\mathcal{D}_n(X)$. Note that a nilpotent operator T of order k is Drazin invertible of index k.

Example 2.12 Let $Y \subset X$ be a subspace of dimension $n+1$, and write $X = Y \oplus Z$ where Z is a closed subspace. With respect to an arbitrary basis of Y, consider the operators $T, F \in \mathcal{B}(X)$ given by

$$T = (J_n \oplus 0) \oplus I \quad \text{and} \quad F = E_{n+1,n} \oplus 0,$$

where J_n is the $n \times n$ nilpotent matrix with 1 in the diagonal directly below the main diagonal and 0 elsewhere, and $E_{n+1,n}$ is the $(n+1) \times (n+1)$ matrix whose unique non-zero entry is 1 in position $(n+1, n)$. Clearly, F is rank one and $T \in \mathcal{D}_n(X)$. However, the matrix $(J_n \oplus 0) + \alpha E_{n+1,n}$ is nilpotent of order $n+1$, and so $T + \alpha F \notin \mathcal{D}_n(X)$ for every non-zero $\alpha \in \mathbb{C}$.

3 Proof of the Main Result

Combining Proposition 2.10 with Theorem 2.11, we obtain:

Corollary 3.1 *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map. If Φ preserves $\mathcal{B}_n(X)$ in both directions then*

(i) Φ *is injective*;

(ii) Φ *preserves the set of rank one operators in both directions.*

Proof (i) Suppose on the contrary that there exists $F \neq 0$ such that $\Phi(F) = 0$. Then by Proposition 2.10, there is an invertible operator $T \in \mathcal{B}(X)$ such that $T + F \notin \mathcal{B}_n(X)$. But, $\Phi(T + F) = \Phi(T) \in \mathcal{B}_n(X)$, the desired contradiction.

(ii) Let $F \in \mathcal{B}(X)$ be non-zero. Then $\Phi(F) \neq 0$. Note that every additive map is $\mathbb{Q}$-linear. Hence, since Φ is bijective and preserves $\mathcal{B}_n(X)$ in both directions, it follows by Theorem 2.11 that

$$\begin{aligned}\dim \operatorname{ran}(F) = 1 &\Leftrightarrow \forall T \in \mathcal{B}_n(X) \ : \ \operatorname{Card}\{\lambda \in \mathbb{Q} : T + \lambda F \notin \mathcal{B}_n(X)\} \leq 1 \\ &\Leftrightarrow \forall S \in \mathcal{B}_n(X) \ : \ \operatorname{Card}\{\lambda \in \mathbb{Q} : S + \lambda \Phi(F) \notin \mathcal{B}_n(X)\} \leq 1 \\ &\Leftrightarrow \dim \operatorname{ran}(\Phi(F)) = 1.\end{aligned}$$

This completes the proof. □

Definition 3.2 *Let $T, S \in \mathcal{B}(X)$. We will write $T \sim S$ if the following equivalence holds*

$$T + F \in \mathcal{B}_n(X) \ \Leftrightarrow \ S + F \in \mathcal{B}_n(X)$$

for every finite rank operator $F \in \mathcal{B}(X)$.

The following assertions follow immediately from the previous definition:

(i) $(\sim)$ defines an equivalence relation on $\mathcal{B}(X)$.

(ii) If $T \sim S$ then $T + F \sim S + F$ for all finite rank operator $F \in \mathcal{B}(X)$.

(iii) If $T \sim S$ then $T \in \mathcal{B}_n(X)$ if and only if $S \in \mathcal{B}_n(X)$.

Proposition 3.3 *Let $T, S \in \mathcal{B}(X)$ be two invertible operators. If $T \sim S$ then $T = S$.*

Before proving this proposition, we need to establish two auxiliary lemmas.

For an operator $T \in \mathcal{B}(X)$ and $x \in X$, we write

$$\mathrm{M}_x(T) = \{f \in X^* : T + x \otimes f \notin \mathcal{B}_n(X)\}.$$

Note that if $S \in \mathcal{B}(X)$ is an operator such that $T \sim S$, then $\mathrm{M}_x(T) = \mathrm{M}_x(S)$.

Remark 3.4 Let $T \in \mathcal{B}_n(X)$, $p = n - \dim \mathcal{N}^\infty(T)$, and $x = a + b$ where $a \in \mathcal{N}^\infty(T)$ and $b \in \mathcal{R}^\infty(T)$. The following assertions follow immediately from Theorem 2.6:

(i) $f \in \mathrm{M}_x(T)$ if and only if $f(T^j a) = 0$ for $j \geq 0$, and $f(T_\mathrm{o}^{-i} b) = -\delta_{i1}$ for $1 \leq i \leq p+1$ where $T_\mathrm{o} = T_{|\mathcal{R}^\infty(T)}$;

(ii) $\mathrm{M}_a(T) = \emptyset$ and $\mathrm{M}_x(T) \subseteq \mathrm{M}_b(T)$.

Lemma 3.5 *Let $T \in \mathcal{B}_n(X)$, $p = n - \dim \mathcal{N}^\infty(T)$, $x \in \mathcal{R}^\infty(T)$, and $T_\mathrm{o} = T_{|\mathcal{R}^\infty(T)}$. Then*

$$\mathrm{M}_x(T) \neq \emptyset \ \Leftrightarrow \ \{T_\mathrm{o}^{-i} x : 1 \leq i \leq p+1\} \text{ is linearly independent.}$$

Moreover, in this case we have $\mathrm{M}_y(T) \neq \emptyset$ for all $y \in \{T_\mathrm{o}^i x : i \in \mathbb{Z}\}$.

Proof Suppose that $\mathrm{M}_x(T)$ is not empty, and let $f \in \mathrm{M}_x(T)$. Let α_i, $1 \le i \le p+1$, be scalars such that $\alpha_1 T_o^{-1}x + \cdots + \alpha_{p+1}T_o^{-(p+1)}x = 0$. The fact that $f(T_o^{-i}x) = -\delta_{i1}$ for $1 \le i \le p+1$ implies that $\alpha_1 = 0$. Hence, $\alpha_2 T_o^{-2}x + \cdots + \alpha_{p+1}T_o^{-(p+1)}x = 0$, and so $\alpha_2 T_o^{-1}x + \cdots + \alpha_{p+1}T_o^{-p}x = 0$. Using again f, we get that $\alpha_2 = 0$. By repeating the same argument, we obtain that $\alpha_3 = \cdots = \alpha_{p+1} = 0$, and hence $\{T_o^{-i}x : 1 \le i \le p+1\}$ is linearly independent

Conversely, suppose that $\{T_o^{-i}x : 1 \le k \le p+1\}$ is linearly independent. Then the existence of a linear form $g \in X^*$ satisfying $g(T_o^{-i}x) = -\delta_{i1}$ for $1 \le i \le p+1$ is obvious. Thus $g \in \mathrm{M}_x(T)$.

Now, let $y = T_o^i x$ where i is an arbitrary integer. Since also the vectors $T_o^{-k}x$, $1 \le k \le p+1$, are linearly independent, we get that $\mathrm{M}_y(T)$ is not empty. This finishes the proof. □

For a subset $M \subseteq X$, we denote by $M^{\perp} = \{f \in X^* : M \subseteq \ker(f)\}$ its annihilator.

Lemma 3.6 *Let $T \in \mathcal{B}(X)$ be invertible, and let $S \in \mathcal{B}(X)$ be such that $T \sim S$. If there exists a vector x such that $\{x, Tx, \dots, T^{2n}x\}$ is linearly independent then S is invertible and $Ty = Sy$ for all $y \in \mathrm{Span}\{T^i x : i \in \mathbb{Z}\}$.*

Proof First, since $T \in \mathcal{B}_n(X)$ and $T \sim S$, we have $S \in \mathcal{B}_n(X)$. Suppose on the contrary that S is not invertible. Let $x = a + b$ where $a \in \mathcal{N}^\infty(S)$ and $b \in \mathcal{R}^\infty(S)$, and let $S_o = S_{|\mathcal{R}^\infty(S)}$ and $p = n - \dim \mathcal{N}^\infty(S)$. Since $\mathrm{M}_x(T) = \mathrm{M}_x(S)$ is not empty, then so is $\mathrm{M}_b(T) = \mathrm{M}_b(S)$. Hence, by the previous lemma, the sets $\{T^{-i}b : 1 \le i \le n+1\}$ and $\{S_o^{-i}b : 1 \le i \le p+1\}$ are linearly independent. But, as $p < n$, there is $1 \le k \le n+1$ such that $\{T^{-k}b, S_o^{-i}b : 1 \le i \le p+1\}$ is a linearly independent set, and so there exists a linear form $\varphi \in X^*$ such that $\varphi(T^{-k}b) = 1$ and $\varphi(S_o^{-i}b) = -\delta_{i1}$ for $1 \le i \le p+1$. Thus, $\varphi \notin \mathrm{M}_b(T)$ and $\varphi \in \mathrm{M}_b(S)$; a contradiction.

Note that since $T^i x$ satisfies the same hypothesis as x for all $i \in \mathbb{Z}$, it suffices to show that $ST^{-(n+1)}x = T^{-n}x$. Let $y \in \{T^i x, S^i x : i \in \mathbb{Z}\}$. The fact that $\mathrm{M}_x(T) = \mathrm{M}_x(S)$ is not empty asserts that $\mathrm{M}_{T^i x}(T)$ and $\mathrm{M}_{S^i x}(S)$ are not empty for all $i \in \mathbb{Z}$. So that $\mathrm{M}_y(T) = \mathrm{M}_y(S)$ is not empty. Let $f \in \mathrm{M}_y(T)$, and consider an arbitrary $g \in \{T^{-j}y : 2 \le j \le n+1\}^{\perp}$. Put $h = g + (g(T^{-1}y) + 1)f$. Clearly, we have

$$h(T^{-1}y) = -1 \quad \text{and} \quad h(T^{-i}y) = 0 \quad \text{for } 2 \le i \le n+1,$$

and so $h \in \mathrm{M}_y(T) = \mathrm{M}_y(S)$. Hence, we get that $h(S^{-1}y) = -1$ and $h(S^{-i}y) = 0$ for $2 \le i \le n+1$. Consequently, $g(S^{-1}y - T^{-1}y) = g(S^{-i}y) = 0$ for $2 \le i \le n+1$. This implies that

$$\{S^{-1}y - T^{-1}y, S^{-i}y : 2 \le i \le n+1\} \subseteq \mathrm{Span}\{T^{-j}y : 2 \le j \le n+1\}. \tag{3.1}$$

Let us show that

$$S^{-i}x - T^{-i}x \in \mathrm{Span}\{T^{-k}x : i+1 \le k \le n+1\} \quad \text{for } 1 \le i \le n+1. \tag{3.2}$$

Clearly, replacing y by x in (3.1), we get that (3.2) is satisfied for $i = 1$. Suppose that (3.2) holds for $i < n+1$. We have

$$S^{-(i+1)}x - T^{-(i+1)}x = S^{-1}(S^{-i}x - T^{-i}x) + S^{-1}T^{-i}x - T^{-1}T^{-i}x.$$

Using (3.2) and (3.1) for $y = T^{-k}x$, we get that

$$\begin{aligned} S^{-1}(S^{-i}x - T^{-i}x) &\in \mathrm{Span}\{S^{-1}T^{-k}x : i+1 \le k \le n+1\} \\ &\subseteq \mathrm{Span}\{T^{-(j+k)}x : 1 \le j \le n+1,\ i+1 \le k \le n+1\} \end{aligned}$$

$$\subseteq \mathrm{Span}\{T^{-p}x : i+2 \le p \le 2n+2\}.$$

Moreover, the formula (3.1) for $y = T^{-i}x$ asserts that

$$S^{-1}T^{-i}x - T^{-1}T^{-i}x \in \mathrm{Span}\{T^{-(j+i)}x : 2 \le j \le n+1\} \subseteq \mathrm{Span}\{T^{-p}x : i+2 \le p \le 2n+2\}.$$

Thus, $S^{-(i+1)}x - T^{-(i+1)}x \in \mathrm{Span}\{T^{-p}x : i+2 \le p \le 2n+2\}$. On the other hand, replacing y by x in (3.1), we recover that $S^{-(i+1)}x$ is a linear combination of $T^{-j}x$, $2 \le j \le n+1$, and hence so is $S^{-(i+1)}x - T^{-(i+1)}x$. Therefore,

$$\begin{aligned} S^{-(i+1)}x - T^{-(i+1)}x &\in \mathrm{Span}\{T^{-p}x : i+2 \le p \le 2n+2\} \cap \mathrm{Span}\{T^{-j}x : 2 \le j \le n+1\} \\ &\subseteq \mathrm{Span}\{T^{-k}x : i+2 \le k \le n+1\}, \end{aligned}$$

which establishes (3.2). Hence, $S^{-(n+1)}x = T^{-(n+1)}x$ and $S^{-n}x = T^{-n}x + \beta T^{-(n+1)}x$ for some $\beta \in \mathbb{C}$. Moreover, it follows from (3.1) with $y = S^{-n}x$ that there exist complex numbers $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{C}$ such that $\alpha_1 = 1$ and

$$S^{-1}S^{-n}x = \sum_{j=1}^{n+1} \alpha_j T^{-j} S^{-n}x.$$

Therefore,

$$\begin{aligned} S^{-(n+1)}x &= \sum_{j=1}^{n+1} \alpha_j T^{-j}(T^{-n}x + \beta T^{-(n+1)}x) \\ &= \sum_{j=1}^{n+1} \alpha_j (T^{-(n+j)}x + \beta T^{-(n+1+j)}x) \\ &= T^{-(n+1)}x + (\alpha_1\beta + \alpha_2)T^{-(n+2)}x + \cdots + (\alpha_n\beta + \alpha_{n+1})T^{-(2n+1)}x \\ &\quad + \alpha_{n+1}\beta T^{-(2n+2)}x. \end{aligned}$$

Since $S^{-(n+1)}x = T^{-(n+1)}x$ and $\{T^{-(n+1)}x, \ldots, T^{-(2n+2)}x\}$ is a linearly independent set, we infer that

$$\beta + \alpha_2 = \alpha_2\beta + \alpha_3 = \cdots = \alpha_n\beta + \alpha_{n+1} = \alpha_{n+1}\beta = 0.$$

So that $\beta = -\alpha_2$ and $\alpha_i = \alpha_2^{i-1}$ for $2 \le i \le n+1$. But, as $\alpha_{n+1}\beta = -\alpha_2^{n+1} = 0$, we obtain that $\beta = \alpha_i = 0$ for $2 \le i \le n+1$. Thus, $S^{-n}x = T^{-n}x$. Finally, we have $ST^{-(n+1)}x = SS^{-(n+1)}x = S^{-n}x = T^{-n}x$, as desired, and the proof is finished. □

Proof of Proposition 3.3 Notice first that for a finite codimensional subspace Y of X, it is an elementary fact that

$$\dim(Y \cap T^{-1}Y \cap \cdots \cap T^{-2n}Y) = \infty.$$

Let $x_0 \in X$ be non-zero, and let us show that $Tx_0 = Sx_0$. Let Y be a complement of $\mathrm{Span}\{x_0, Tx_0, \ldots, T^{2n}x_0\}$. Then $Y \cap T^{-1}Y \cap \cdots \cap T^{-2n}Y$ contains a non-zero vector x_1, and the sum

$$\mathrm{Span}\{x_0, Tx_0, \ldots, T^{2n}x_0\} + \mathrm{Span}\{x_1, Tx_1, \ldots, T^{2n}x_1\}$$

is direct. Repeating the same argument, we get the existence of non-zero vectors $x_2, \ldots, x_{2n} \in X$ such that the sum of the subspaces

$$Z_i = \mathrm{Span}\{x_i, Tx_i, \ldots, T^{2n}x_i\}, \quad 0 \le i \le 2n,$$

is direct. Let $f_0, \dots, f_{2n-1} \in X^*$ be such that $f_i \in Z_j^{\perp}$ for $i \neq j$, and $f_0(x_0) = f_i(Tx_i) = 1$ for $1 \leq i \leq 2n-1$. Consider also the operators $H, R \in \mathcal{B}(X)$ defined by

$$H = T + \sum_{i=1}^{2n} Tx_i \otimes f_{i-1} \quad \text{and} \quad R = S + \sum_{i=1}^{2n} Tx_i \otimes f_{i-1}.$$

Clearly, we have $H \sim R$. Note also that

$$I + \sum_{i=1}^{2n} x_i \otimes f_{i-1} = \prod_{i=1}^{2n} (I + x_i \otimes f_{i-1}),$$

and since $f_{i-1}(x_i) = 0$ for $1 \leq i \leq 2n$, we obtain that this operator is invertible. Therefore, H is invertible. Furthermore, one can easily verify that $H^k x_0 = v_{k-1} + Tx_k$ for $1 \leq k \leq 2n$, where $v_{k-1} \in Z_0 \oplus \cdots \oplus Z_{k-1}$. Consequently, the vectors $x_0, \dots, H^{2n}x_0$ are linearly independent. Thus, $Hx_0 = Rx_0$ by Lemma 3.6. But, we have also $Hx_0 = Tx_0 + Tx_1$ and $Rx_0 = Sx_0 + Tx_1$. Hence, $Tx_0 = Sx_0$. This completes the proof.

Proposition 3.7 *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map. If Φ preserves $\mathcal{B}_n(X)$ in both directions then there exists a non-zero $\alpha \in \mathbb{K}$ such that $\Phi(I) = \alpha I$.*

Before proving this proposition, we need the following technical lemma:

Lemma 3.8 *Let $U \in \mathcal{B}(X)$ be an invertible operator, and let $F \in \mathcal{B}(X)$ be such that* $\dim \operatorname{ran}(F) \leq n$. *If $UF = F$ then $U + F \in \mathcal{B}_n(X)$.*

Proof Note that $U + F$ is a Fredholm operator of index zero, and hence, by Lemma 2.3, it suffices to show that $\dim \ker((U+F)^{n+1}) \leq n$. Under the assumption $UF = F$, one can easily get that $(U+F)^{n+1} = U^{n+1} + FV$ for some $V \in \mathcal{B}(X)$. Hence, it follows that $\ker((U+F)^{n+1}) \subseteq U^{-(n+1)}\operatorname{ran}(F)$. Thus $\dim \ker((U+F)^{n+1}) \leq n$, as desired. □

Proof of Proposition 3.7 Set $S = \Phi(I)$, note that $S \in \mathcal{B}_n(X)$ because $I \in \mathcal{B}_n(X)$. We begin by proving that $S + F \in \mathcal{B}_n(X)$ for every $F \in \mathcal{B}(X)$ such that $\dim \operatorname{ran}(F) \leq n$. Let $F \in \mathcal{B}(X)$ with $\dim \operatorname{ran}(F) \leq n$. By Corollary 3.1, Φ is bijective and preserves rank one operators in both directions, it follows that there exists $K \in \mathcal{B}(X)$ such that $\dim \operatorname{ran}(K) \leq n$ and $\Phi(K) = F$. Thus, we get by the previous lemma that $I + K$, and hence $S + F = \Phi(I + K)$, belongs to $\mathcal{B}_n(X)$.

Now, suppose on the contrary that S is not scalar multiple of the identity. Then there exists $y_1 \in X$ such that the vectors y_1 and Sy_1 are linearly independent. Since $S \in \mathcal{B}_n(X)$, $\operatorname{ran}(S)$ is an infinite-dimensional subspace, and hence there exist vectors $y_i \in X$, $2 \leq i \leq n$, such that $\{y_1, Sy_i : 1 \leq i \leq n\}$ is a linearly independent set. Consider the linear forms $g_i \in X^*$, $1 \leq i \leq n$, given by

$$g_i(y_1) = 0 \quad \text{and} \quad g_i(Sy_j) = -\delta_{ij} \quad \text{for } 1 \leq i, j \leq n.$$

Let $F = \sum_{i=1}^{n} S^2 y_i \otimes g_i$. Clearly, we have $(S+F)Sy_j = 0$ for $1 \leq j \leq n$, and $(S+F)y_1 = Sy_1$, and so $Sy_1 \in \ker(S+F)^2$. Consequently, $\dim \ker(S+F)^2 \geq n+1$, and hence $S + F \notin \mathcal{B}_n(X)$. This leads to a contradiction because $\dim \operatorname{ran}(F) \leq n$. The proof is complete.

To give the proof of the main result, it remains only to establish the following two lemmas.

Lemma 3.9 *Let $x \in X$, $f \in X^*$, and let $x_1, \dots, x_n \in \ker(f)$ be linearly independent vectors.*

Then $f(x) = -1$ if and only if there exist $f_1, \dots, f_n \in X^*$ such that

$$I + x \otimes f + x_1 \otimes f_1 + \cdots + x_n \otimes f_n \notin \mathcal{B}_n(X).$$

Proof Let $S = I + x \otimes f$. Suppose that $f(x) = -1$. Then the vectors $\{x, x_1, \dots, x_n\}$ are linearly independent, and hence there exist linear forms $f_1, \dots, f_n \in X^*$ such that

$$f_i(x) = 0 \quad \text{and} \quad f_i(x_j) = -\delta_{ij} \quad \text{for } 1 \le i, j \le n.$$

If we let $F = x_1 \otimes f_1 + \cdots + x_n \otimes f_n$ we get easily that $Fx = 0$ and $Fx_i = -x_i$ for $1 \le i \le n$. So that $(S + F)x_i = (S + F)x = 0$ for $1 \le i \le n$. Thus, $\dim \ker(S + F) \ge n + 1$, which implies that $S + F \notin \mathcal{B}_n(X)$.

Conversely, suppose that $f(x) \ne -1$. Let $f_1, \dots, f_n \in X^*$ be arbitrary, and let $K = x_1 \otimes f_1 + \cdots + x_n \otimes f_n$. Then, S is invertible and $\dim \operatorname{ran}(K) \le n$. Furthermore, we have $SK = K$, and so $S + K \in \mathcal{B}_n(X)$ by the previous lemma. This completes the proof. □

Let τ be a field automorphism of $\mathbb{K}$. An additive map $A : X \to Y$ defined between two Banach spaces will be called *τ-semi linear* if $A(\lambda x) = \tau(\lambda)Ax$ holds for all $x \in X$ and $\lambda \in \mathbb{K}$. Moreover, we say simply that A is *conjugate linear* when τ is the complex conjugation. Notice that if A is non-zero and bounded, then τ is continuous, and consequently, τ is either the identity or the complex conjugation, see [11, Theorem 14.4.2 and Lemma 14.5.1]. Moreover, in this case, the adjoint operator $A^* : Y^* \to X^*$, defined by

$$A^*(g) = \tau^{-1} \circ g \circ A \quad \text{for all } g \in Y^*,$$

is again τ-semi linear.

For a bounded linear operator T on X, we have

$$T \in \mathcal{B}_n(X) \quad \text{if and only if} \quad T^* \in \mathcal{B}_n(X^*).$$

Indeed, it is well known that T is Drazin invertible if and only if T^* is Drazin invertible, and in this case $\mathrm{i}(T) = \mathrm{i}(T^*)$, see [16]. So, using [21, Theorem IV.8.4] we get easily the desired equivalence.

Lemma 3.10 *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map. If Φ preserves $\mathcal{B}_n(X)$ in both directions, then there exists a non-zero $\alpha \in \mathbb{K}$ such that $\Phi(I) = \alpha I$, and either*

(i) *there exists an invertible bounded linear, or conjugate linear, operator $A : X \to X$ such that $\Phi(F) = \alpha AFA^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$, or*

(ii) *there exists an invertible bounded linear, or conjugate linear, operator $B : X^* \to X$ such that $\Phi(F) = \alpha BF^*B^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$. In this case, X is reflexive.*

Proof The existence of a non-zero scalar α such that $\Phi(I) = \alpha I$ is ensured by Proposition 3.7. Clearly, we can assume without loss of generality that $\Phi(I) = I$. Since Φ is bijective and preserves the set of rank one operators in both directions, then by [18, Theorems 3.1 and 3.3], there exist a ring automorphism $\tau : \mathbb{K} \to \mathbb{K}$ and either two bijective τ-semi linear mappings $A : X \to X$ and $C : X^* \to X^*$ such that

$$\Phi(x \otimes f) = Ax \otimes Cf \quad \text{for all } x \in X \text{ and } f \in X^*, \tag{3.3}$$

or two bijective τ-semi linear mappings $B : X^* \to X$ and $D : X \to X^*$ such that

$$\Phi(x \otimes f) = Bf \otimes Dx \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.4}$$

Suppose that Φ satisfies (3.3), and let us show that

$$C(f)(Ax) = \tau(f(x)) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.5}$$

Clearly, it suffices to establish that for all $x \in X$ and $f \in X^*$, $f(x) = -1$ if and only if $C(f)(Ax) = -1$. Let $x \in X$ and $f \in X^*$. Consider arbitrary linearly independent vectors z_i, $1 \leq i \leq n$, in $\ker(f) \cap \ker(C(f)A)$. Then, it follows from Lemma 3.9 that

$$\begin{aligned} f(x) = -1 &\Leftrightarrow \exists \{g_i\}_{i=1}^n \subseteq X^* : I + x \otimes f + \sum_{i=1}^n z_i \otimes g_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \exists \{g_i\}_{i=1}^n \subseteq X^* : I + Ax \otimes Cf + \sum_{i=1}^n Az_i \otimes Cg_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow C(f)(Ax) = -1. \end{aligned}$$

Thus, Equation (3.5) holds, and arguing as in [18, p. 252], we get that τ, A, C are continuous, τ is the identity or the complex conjugation, and $C = (A^{-1})^*$. Therefore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\Phi(x \otimes f)u = \tau(fA^{-1}u)Ax = A(f(A^{-1}u)x) = A(x \otimes f)A^{-1}u.$$

Thus, $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X^*$; that is, $\Phi(F) = AFA^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$.

Now, suppose that Φ satisfies (3.4), and let us show that

$$D(x)(Bf) = \tau(f(x)) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{3.6}$$

Let $x \in X$ and $f \in X^*$. Let $h_1, \ldots, h_n \in X^*$ be linearly independent linear forms such that $h_i(x) = (D(x)B)(h_i) = 0$ for $1 \leq i \leq n$. Then, using the fact that D is bijective, it follows from Lemma 3.9 that

$$\begin{aligned} D(x)(Bf) = -1 &\Leftrightarrow \exists \{u_i\}_{i=1}^n \subseteq X : I + Bf \otimes Dx + \sum_{i=1}^n Bh_i \otimes Du_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \exists \{u_i\}_{i=1}^n \subseteq X : I + x \otimes f + \sum_{i=1}^n u_i \otimes h_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \exists \{u_i\}_{i=1}^n \subseteq X : I + f \otimes \mathrm{J}x + \sum_{i=1}^n h_i \otimes \mathrm{J}u_i \notin \mathcal{B}_n(X^*) \\ &\Leftrightarrow f(x) = -1, \end{aligned}$$

where $\mathrm{J} : X \to X^{**}$ is the natural embedding. Thus, Equation (3.6) holds, and arguing as in [18, p. 252], we get that τ, B, D are continuous, τ is the identity or the complex conjugation, and $D = (B^{-1})^*\mathrm{J}$. But, since D and $(B^{-1})^*$ are bijective, J is bijective and X is reflexive. Furthermore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\begin{aligned} \Phi(x \otimes f)u &= (Bf \otimes (B^{-1})^*\mathrm{J}(x))u = (B^{-1})^*\mathrm{J}(x)(u) \cdot Bf \\ &= \tau(\mathrm{J}(x)(B^{-1}u)) \cdot Bf = B(\mathrm{J}(x)(B^{-1}u)f) \\ &= B(f \otimes \mathrm{J}(x))B^{-1}u = B(x \otimes f)^*B^{-1}u. \end{aligned}$$

Thus, $\Phi(x \otimes f) = B(x \otimes f)^* B^{-1}$ for all $x \in X$ and $f \in X^*$. Hence, $\Phi(F) = BF^* B^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$. This finishes the proof. □

With these results at hand, we are now ready to prove our main result.

Proof of Theorem 1.1 (i) ⇒ (ii). First, note that Φ is $\mathbb{Q}$-linear. Hence, using the fact that Φ is surjective, it follows by Theorem 2.4 that for every $T \in \mathcal{B}(X)$,

$$\begin{aligned} T \in \mathcal{B}_n(X) &\Leftrightarrow \forall S, \exists \varepsilon_0 > 0 : \{T + \varepsilon S : \varepsilon \in \mathbb{Q} \text{ and } 0 \leq \varepsilon < \varepsilon_0\} \subseteq \mathcal{D}_n(X) \\ &\Leftrightarrow \forall S, \exists \varepsilon_0 > 0 : \{\Phi(T) + \varepsilon \Phi(S) : \varepsilon \in \mathbb{Q} \text{ and } 0 \leq \varepsilon < \varepsilon_0\} \subseteq \mathcal{D}_n(X) \\ &\Leftrightarrow \forall R, \exists \varepsilon_0 > 0 : \{\Phi(T) + \varepsilon R : \varepsilon \in \mathbb{Q} \text{ and } 0 \leq \varepsilon < \varepsilon_0\} \subseteq \mathcal{D}_n(X) \\ &\Leftrightarrow \Phi(T) \in \mathcal{B}_n(X). \end{aligned}$$

Thus Φ preserves $\mathcal{B}_n(X)$ in both directions.

(ii) ⇒ (iii). Suppose that Φ preserves $\mathcal{B}_n(X)$ in both directions. It follows that Φ takes one of the two forms in Lemma 3.10.

Assume that $\Phi(F) = \alpha AFA^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$. Let

$$\Psi(T) = \alpha^{-1} A^{-1} \Phi(T) A \quad \text{for all } T \in \mathcal{B}(X).$$

Clearly, Ψ satisfies the same properties as Φ. Furthermore, $\Psi(I) = I$ and $\Psi(F) = F$ for all finite rank operator $F \in \mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$, and choose an arbitrary rational number λ such that $T - \lambda$ and $\Psi(T) - \lambda$ are invertible. For every finite rank operator $F \in \mathcal{B}(X)$, we have

$$T - \lambda + F \in \mathcal{B}_n(X) \;\Leftrightarrow\; \Psi(T - \lambda) + F \in \mathcal{B}_n(X).$$

Hence, we get by Proposition 3.3 that $\Psi(T) = T$. This shows that $\Phi(T) = \alpha ATA^{-1}$ for all $T \in \mathcal{B}(X)$.

Now suppose that $\Phi(F) = \alpha BF^* B^{-1}$ for all finite rank operator $F \in \mathcal{B}(X)$. Then Lemma 3.10 ensures that X is reflexive. By considering

$$\Gamma(T) = \alpha^{-1} \mathrm{J}^{-1} (B^{-1} \Phi(T) B)^* \mathrm{J} \quad \text{for all } T \in \mathcal{B}(X),$$

we get in a similar way that $\Gamma(T) = T$ for all $T \in \mathcal{B}(X)$. Thus, $\Phi(T) = \alpha BT^* B^{-1}$ for all $T \in \mathcal{B}(X)$, as desired.

(iii) ⇒ (i) is obvious.

We close this paper by the following remark:

Remark 3.11 Let X and Y be infinite-dimensional Banach spaces over $\mathbb{K}$. Theorem 1.1 can be formulated without any change for additive surjective mappings $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ preserving Drazin invertible operators of index non-greater than n in both directions.

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