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On the Existence, Uniqueness and Stability of Solutions for Semi-linear Generalized Elasticity Equation with General Damping Term

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Abstract In this paper, we consider a semi-linear generalized hyperbolic boundary value problem associated to the linear elastic equations with general damping term and nonlinearities of variable exponent type. Under suitable conditions, local and global existence theorems are proved. The uniqueness of the solution have been gotten by eliminating some hypotheses that have been imposed by other authors for different particular problems. We show that any solution with nontrivial initial datum becomes stable.

Keywords Generalized semi-linear elasticity equation, nonlinear internal stabilization, generalized Lebesgue space, Sobolev spaces with variable exponents

MR(2010) Subject Classification 58J45, 35L53, 35L71, 46E30, 46E35

1 Introduction

In [9], Lions considered a semi-linear boundary value problem associated to the Laplace operator with Neumann boundary condition:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + |u|^{\nu} u = f, \quad \text{in } \Omega \times (0,T),\ \nu = p-2, \\ u = 0 \quad \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

Using the compactness method and Faedo-Galerkin techniques, the existence of a weak solution has been proved. Assuming that the condition $\nu \leq \frac{2}{n-2}$ holds, then, it follows the uniqueness and the regularity of the solution.

In [11], Rahmoune and Benabderrahmane considered a semi-linear hyperbolic boundary value problem governed by partial differential equations that describe the evolution of linear elastic materials with Dirichlet and Neumann boundary conditions as follows:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div} \sigma(u) + |u|^{\nu} u = f, \quad \text{in } \Omega \times (0,T), \\ \sigma(u) = F(\varepsilon(u)), \quad \text{in } \Omega \times (0,T), \\ u = g \quad \text{on } \Gamma_1 \times (0,T), \quad \sigma(u)\eta = 0 \quad \text{on } \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega. \end{cases}$$

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With special known theorems they established the local existence result. They showed that for a suitable initial datum, the uniqueness and the regularity of solution have been gotten by eliminating some hypotheses on the number ν that have been imposed by other authors for different particular problems.

In this work, a semi-linear generalized hyperbolic boundary value problem associated to the linear elastic equations with general damping term, dissipative term and nonlinearities of variable exponent type is considered:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div} \sigma(u) + |u|^{\nu(x)}u + g(u') = f, \quad \text{in } \Omega \times (0,T), \\ \sigma(u) = F(\varepsilon(u)), \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \Gamma_1 \times (0,T), \quad \sigma(u)\eta = 0 \quad \text{on } \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

u, f and $\sigma(u)$ represent the displacement field, the density of volume forces and the tensor of constraints, respectively. div denotes the divergence operator of the tensor valued functions and $\sigma = (\sigma_{ij})$, i, $j = 1, 2, \ldots, n$ stands for the stress tensor field. The latter is obtained from the displacement field by the constitutive law of linear elasticity defined by the second equation in (1.1). F is a linear elastic constitutive law, and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the linearized strain tensor. The equation (1.1), without the nonlinear term $|u|^{\nu(x)}u + g(u')$, describes the evolution of linear elastic materials, with the initial and mixed boundary conditions on Σ_i, $i = 1, 2$. Assume certain hypotheses on the data functions. Then, by using Faedo-Galerkin techniques combined with compactness and monotonous methods, we will prove the existence of a weak solution. Our main goal is, without taking into account the conditions on the continuous function $\nu(x)$, to prove the uniqueness of the solution. Ω is an open and bounded domain in $\mathbb{R}^n$. Recall that the boundary Γ of Ω is assumed to be regular and is composed of two relatively closed parts: Γ_1, Γ_2, with mutually disjoint relatively open interiors. We pose $\Sigma_i = \Gamma_i \times (0,T)$, $i = 1, 2$, where T is a finite real number. To simplify the writing one will put $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, and we do not indicate explicitly the dependence of the functions u and σ by report to $x \in \Omega$ and $t \in (0,T)$. Let η be the unit outward normal vector on Γ. Here and throughout this work, the summation convention over repeated indices is used. In this paper, our aim is to extend the results of [9, 11], and other's established in bounded domains to a general problem as in (1.1).

The present paper is organized as follows. Before the main results, in Section 2 we introduce the function spaces of Sobolev type with variable exponents. In Section 3 we present to the proof of the weak solution to the initial-boundary value problem (1.1). The weak solution are obtained as the limit of the sequence of Galerkin's approximations. Our object is to obtain, without taking into account the condition on $\nu(x)$, the uniqueness of the solution. In section 4, the global existence in time and the stability of solution are established of the problem (1.1) in the framework of the Lebesgue space with variable exponents.

2 Preliminaries for Function Spaces

In this section we list and recall some well-known results and facts from the theory of the Sobolev spaces with variable exponent (for the details see [2, 4–6, 8]).

Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary Γ and assume that meas $(\Gamma) > 0$, $\mu > 0$, $p(\cdot)$ is a continuous measurable function on $\overline{\Omega}$ such that

$$2 < p_- \leq p(x) \leq p_+ < \infty, \tag{2.1}$$

where

$$p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), \quad p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

We also assume that p satisfies the following Zhikov–Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \quad \text{for all } x,\, y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0. \tag{2.2}$$

Let $p : \Omega \to [1, \infty]$ be a measurable function. We denote by $L^{p(\cdot)}(\Omega)$ the set of measurable functions u on Ω such that

$$A_{p(\cdot)}(u) = \int_{\{x\in\Omega|p(x)<\infty\}} |u(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{\{x\in\Omega|p(x)=\infty\}} |u(x)| < \infty.$$

The variable-exponent space $L^{p(\cdot)}$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

is a Banach space.

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussed the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces by Kovacik and Rakosnik in [8].

Let us list some properties of the spaces $L^{p(\cdot)}(\Omega)$ which will be used in the study of the problem (1.1).

- It follows directly from the definition of the norm that

$$\min(\|u\|^{p_-}_{p(\cdot)}, \|u\|^{p_+}_{p(\cdot)}) \leq A_{p(\cdot)}(u) \leq \max(\|u\|^{p_-}_{p(\cdot)}, \|u\|^{p_+}_{p(\cdot)}).$$

- The following generalized Hölder inequality

$$\int_\Omega |u(x)v(x)|dx \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{p(x)}\|v\|_{p'(x)} \leq 2\|u\|_{p(x)}\|v\|_{p'(x)}$$

holds, for all $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{p'(\cdot)}(\Omega)$ with $p(x) \in (1, \infty)$, $p'(x) = \frac{p(x)}{p(x)-1}$.

- If the condition (2.2) is fulfilled, and Ω has a finite measure and p, q are variable exponents so that $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

- If $p : \Omega \to [1, +\infty)$ is a measurable function and $p_* > \operatorname{ess\,sup}_{\{x\in\Omega\}} p(x)$ with $p_* \leq \frac{2n}{n-2}$, then the embedding $H^1_0(\Omega) = W^{1,2}_0(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

2.1 Mathematical Assumptions

In the study of mechanical problem involving elastic materials, we assume that the operator $F : \Omega \times \mathcal{S}_n \to \mathcal{S}_n$ satisfies the following conditions:

$$\begin{cases} \text{(a)}\ \exists\, m > 0;\ (F(x,\varepsilon),\varepsilon) \geq m\|\varepsilon\|^2, \quad \forall\varepsilon \in \mathcal{S}_n \ \text{ a.e. } x \in \Omega; \\ \text{(b)}\ (F(x,\varepsilon),\tau) = (F(x,\tau),\varepsilon), \quad \forall\varepsilon,\tau \in \mathcal{S}_n \ \text{ a.e. } x \in \Omega; \\ \text{(c)}\ \text{For any } \varepsilon \in \mathcal{S}_n, \quad x \to F(x,\varepsilon) \text{ is measurable on } \Omega, \end{cases} \tag{2.3}$$

where $\mathcal{S}_n$ will denote the space of second-order symmetric tensors on $\mathbb{R}^n$.

Let $g : \mathbb{R} \to \mathbb{R}$ be an monotonous continuous function as $g(0) = 0$ and $\sigma(\cdot)$ be a continuous measurable function on $\overline{\Omega}$ such that the following inequalities hold:

$$\begin{cases} xg(x) \geq d_0|x|^{\sigma(x)}, \quad \forall x \in \mathbb{R}, \\ |g(x)| \leq d_1|x| + d_2|x|^{\sigma(x)-1}, \quad \forall x \in \mathbb{R},\, d_i \geq 0, \\ 2 < \sigma_- \leq \sigma(x) \leq \sigma_+ \leq p(x) \leq p_+ < \infty. \end{cases} \tag{2.4}$$

And we assume that the given data f, u_0 and u_1 verify

$$f \in L^2(Q), \tag{2.5}$$

$$u_0 \in V \cap L^{p(x)}(\Omega), \quad p(x) = \nu(x) + 2, \tag{2.6}$$

$$u_1 \in L^2(\Omega). \tag{2.7}$$

For convenience, we set:

$$\frac{1}{p(x)}\|v(x,t))\|^{p(x)}_{L^{p(x)}(\Omega)} = \int_\Omega \frac{1}{p(x)}|v(t)|^{p(x)}dx,$$

also we denote the norm and scalar product in $L^2(\Omega)$ by $\|v\|_{L^2(\Omega)} = |v| = (\int_\Omega |v|^2dx)^{\frac{1}{2}}$ and $(.,.)$ respectively. C and c denotes a general positives constants, which may be different in different estimates.

3 Main Result

In this section, we start with a local existence and uniqueness of the solution for (1.1).

3.1 Local Existence Result

Theorem 3.1 *For every $T > 0$ and every initial data u_0, u_1 satisfying* (2.5)–(2.7)*, under the assumptions* (2.1)–(2.4) *there exists a unique u which solves the problem* (1.1) *such that*

$$u \in L^\infty(0, T; V \cap L^{p(x)}(\Omega)), \quad p(x) = \nu(x) + 2, \tag{3.1}$$

$$g(u) \cdot u \in L^1(0, T; L^1(\Omega)), \tag{3.2}$$

$$u' \in L^\infty(0, T; L^2(\Omega)). \tag{3.3}$$

Proof We shall prove the existence by means of the Faedo-Galerkin approximation scheme. For every $j \geq 1$, let $V_m = \text{span}\{w_1, w_2, \ldots, w_m\}$, where $\{w_j\}$ is one of the orthogonal complete system of eigenfunctions in $V \cap L^{p(x)}(\Omega)$. Construct the approximate solutions of problem (1.1)

$$u_m(t) = \sum_{i=1}^{m} K_{jm}(t)w_i, \quad m = 1, 2, \ldots \tag{3.4}$$

solving the system

$$(u''_m(t), w_j) + a(u_m, w_j) + (|u_m|^{\nu(x)}u_m, w_j) + (g(u'_m), w_j) = (f, w_j), \quad 1 \leq j \leq m, \tag{3.5}$$

which is a nonlinear system of ordinary differential equations and will be completed by the following initial conditions

$$u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_{im}w_i \to u_0, \quad \text{when } m \to \infty \text{ in } V \cap L^{p(x)}(\Omega), \tag{3.6}$$

$$u'_m(0) = u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \to u, \quad \text{when } m \to \infty \text{ in } L^2(\Omega). \tag{3.7}$$

As the family $\{w_1, w_2, \ldots, w_m\}$ is linearly independent, by virtue of the theory of ordinary differential equations we can get a unique local solution u_m extended to a maximal interval $(0, T_m)$, having the following regularity

$$u_m(t) \in L^2(0, t_m; V_m), \quad u'_m(t) \in L^2(0, t_m; V_m).$$

A priori, the time interval $(0, T)$ depends on m and thereafter we shall prove that t_m does not depend on m based on the following a priori estimates.

First we set

$$\|u\|_1^2 = a(u, u) = \int_\Omega F(\varepsilon(u))\varepsilon(u)dx. \tag{3.8}$$

Then, using (2.3) and Korn's inequality it can be shown that $\|u\|_1$ is a norm on V equivalent to the norm $\|u\|$ on $H^1(\Omega)$. Multiplying the equation (3.5) by $K'_{jm}(t)$ and performing the summation over $j = 1$ to m, yields

$$(u''_m(t), u'_m(t)) + a(u_m(t), u'_m(t)) + (|u_m|^{\nu(x)} u_m(t), u'_m(t)) + (g(u'_m), u'_m(t)) = (f, u'_m(t)). \tag{3.9}$$

On the other hand

$$\begin{aligned}\frac{d}{dt} a(u_m(t), u_m(t)) &= (F(\varepsilon(u_m(t))), \varepsilon(u'_m(t))) + (F(\varepsilon(u'_m(t))), \varepsilon(u_m(t))) \\ &= a(u_m(t), u'_m(t)) + a(u'_m(t), u_m(t)).\end{aligned}$$

Then, using (2.3) (b), we obtain

$$2a(u_m(t), u'_m(t)) = \frac{d}{dt} a(u_m(t), u_m(t)) = \frac{d}{dt}\|u_m(t)\|_1^2, \tag{3.10}$$

also

$$\frac{1}{2}\frac{d}{dt}|u'_m(t)|^2 = (u''_m(t), u'_m(t)); \tag{3.11}$$

$$\frac{1}{p(x)}\frac{d}{dt}\|u_m(x,t))\|_{L^{p(x)}(\Omega)}^{p(x)} = (|u_m|^{\nu(x)} u_m(t), u'_m(t)), \quad p(x) = \nu(x) + 2. \tag{3.12}$$

Then, according to (3.10)–(3.12) by the Cauchy–Schwarz's inequality, from (3.9) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u'_m(t)|^2 + C_1\|u_m(t)\|^2) + \frac{1}{p(x)}\frac{d}{dt}\|u_m(x,t))\|_{L^{p(x)}(\Omega)}^{p(x)} + \int_\Omega g(u'_m(t))u'_m(t)dx \\ &\quad \le |f(s)||u'_m(s)|.\end{aligned} \tag{3.13}$$

Integrating on $(0, t)$ and applying Young inequality we deduce

$$\begin{aligned}&\frac{1}{2}(|u'_m(t)|^2 + C_1\|u_m(t)\|^2) + \frac{1}{p(x)}\|u_m(t)\|_{L^{p(x)}(\Omega)}^{p(x)} + \int_0^t \int_\Omega g(u'_m(s))u'_m(s)dxds \\ &\quad \le \frac{1}{2}|u_{1m}|^2 + \frac{1}{2}C_1\|u_{0m}\|^2 + \frac{1}{p(x)}\|u_{0m}\|_{L^{p(x)}(\Omega)}^{p(x)} \\ &\qquad + \frac{1}{2}\int_0^t |f(s)|^2 ds + \frac{1}{2}\int_0^t |u'_m(s)|^2 ds.\end{aligned} \tag{3.14}$$

Since

$$\frac{1}{2}|u_{1m}| + \frac{1}{2}\|u_{0m}\|^2 + \frac{1}{p(x)}\|u_{0m}\|^{p(x)}_{L^{p(x)}(\Omega)} + \frac{1}{2}\int_0^t |f(s)|^2 ds \leq C, \quad \forall m \in \mathbb{N}^*.$$

Hence it follows from (3.14) and Gronwall's inequality that

$$|u'_m(t)| \leq C_T. \tag{3.15}$$

Therefore, (3.14) gives

$$\|u_m(t)\|^{p(x)}_{L^{p(x)}(\Omega)} + \|u_m(t)\|^2 + \int_0^t \int_\Omega g(u'_m(s)) \cdot u'_m(s) dx ds \leq C_T \tag{3.16}$$

for every $m \geq 1$, and $C_T > 0$ is independent of m. Thus, we obtain

$$\begin{cases} (u_m) \text{ is a bounded sequence in } L^\infty(0,T; V \cap L^{p(x)}(\Omega)), \\ (u'_m) \text{ is a bounded sequence in } L^\infty(0,T; L^2(\Omega)), \\ g(u'_m)u'_m \text{ is a bounded sequence in } L^1(0,T; L^1(\Omega)). \end{cases} \tag{3.17}$$

Lemma 3.2 *There exists a constant $K > 0$ such that*

$$\|g(u'_m(t))\|_{L^{\frac{\sigma(x)}{\sigma(x)-1}}(\Omega\times[0,T])} \leq K,$$

for all $m \in \mathbb{N}$.

Proof We exploit Hölder's and Young's inequalities from (2.4),

$$\begin{aligned} \int_0^T \int_\Omega |g(u'_m)|^{\frac{\sigma(x)}{\sigma(x)-1}} dx dt &= \int_0^T \int_\Omega |g(u'_m)| |g(u'_m)|^{\frac{1}{\sigma(x)-1}} dx dt \\ &\leq \int_0^T \int_\Omega |g(u'_m(t))| (d_1|u'_m(t)| + d_2|u'_m(t)|^{\sigma(x)-1})^{\frac{1}{\sigma(x)-1}} dx dt \\ &\leq C \int_0^T \int_\Omega |g(u'_m(t))| (|u'_m(t)|^{\frac{1}{\sigma(x)-1}} + |u'_m(t)|) dx dt \\ &= C \int_0^T \int_\Omega |g(u'_m(t))| |u'_m(t)|^{\frac{1}{\sigma(x)-1}} dx dt \\ &\quad + C \int_0^T \int_\Omega |g(u'_m(t))| |u'_m(t)| dx dt \\ &\leq \frac{\sigma_+ - 1}{\sigma_+} \int_0^T \int_\Omega |g(u'_m)|^{\frac{\sigma(x)}{\sigma(x)-1}} dx dt \\ &\quad + C(\sigma_+, \sigma_-) \int_0^T \int_\Omega |u'_m(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dx dt \\ &\quad + C \int_0^T \int_\Omega |g(u'_m(t))| |u'_m(t)| dx dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\sigma_+} \int_0^T \int_\Omega |g(u'_m(t))|^{\frac{\sigma(x)}{\sigma(x)-1}} dx dt &\leq C(\sigma_+, \sigma_-) \int_0^T \int_\Omega |u'_m(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dx dt \\ &\quad + C \int_0^T \int_\Omega |g(u'_m(t))| |u'_m(t)| dx dt \\ &\leq C \int_0^T |u'_m(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dt \end{aligned}$$

$$+ C\int_0^T \int_\Omega |g(u_m'(t))||u_m'(t)|dxdt,$$

which yields, by the estimate (3.17),

$$\int_0^T \int_\Omega |g(u_m'(t))|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt \le K. \qquad \square$$

From (3.17) and Lemma 3.2 there exists a subsequence (u_μ) of (u_m) such that

$$\begin{cases} u_\mu \to u \text{ weak star in } L^\infty(0,T;V \cap L^{p(x)}(\Omega)), \\ u_\mu' \to u' \text{ weak star in } L^2(0,T;L^2(\Omega)), \\ g(u_\mu') \to \chi \text{ weak star in } L^{\frac{\sigma(x)}{\sigma(x)-1}}(\Omega \times (0,T)), \\ -\operatorname{div} F(\varepsilon(u_\mu(t))) \to \kappa \text{ weak star in } L^2(0,T;H^{-1}(\Omega)) \end{cases} \tag{3.18}$$

From (3.17), it is obtained that the sequences (u_m), (u_m') are bounded in $L^2(0,T;V) \subset L^2(0,T;L^2(\Omega)) = L^2(Q)$, $L^2(Q)$, respectively. Then, in particular, (u_m) is a bounded sequence in $H^1(Q)$. It is known, see [9], that the injection of $H^1(Q)$ in $L^2(Q)$ is compact. Then, from (3.18) we have

$$u_\mu \to u \quad \text{in } L^2(Q) \text{ strongly.} \tag{3.19}$$

Setting $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, $p(x) = \nu(x) + 2$, using (3.17) we have that $(|u_m|^{\nu(x)}u_m)$ is a bounded sequence in $L^\infty(0,T;L^{p'(x)}(\Omega))$. Therefore

$$|u_\mu|^{\nu(x)}u_\mu \to |u|^{\nu(x)}u \ \text{ in } L^\infty(0,T;L^{p'(x)}\ (\Omega)) \text{ weak star.} \tag{3.20}$$

Because the operator $-\operatorname{div} F(\varepsilon(\cdot)) : H_0^1(\Omega)$ to $H^{-1}(\Omega)$ is bounded, monotone, and hemicontinuous, then we have

$$-\operatorname{div} F(\varepsilon(u_m(t))) \text{ is bounded in } L^\infty(0,T;H^{-1}(\Omega))$$

as $m \to \infty$. Using the standard monotonicity argument as in [9, 10, 12], we can, thus, suppose that

$$-\operatorname{div} F(\varepsilon(u_\mu(t))) \to -\operatorname{div} F(\varepsilon(u(t))) \text{ in } L^\infty(0,T;H^{-1}(\Omega)) \text{ weak star.} \tag{3.21}$$

similarly by using the result in Lemma 3.2 and the estimate (3.17)

$$g(u_\mu') \to g(u') \text{ in } L^{\frac{\sigma(x)}{\sigma(x)-1}}(0,T;L^{\frac{\sigma(x)}{\sigma(x)-1}}(\Omega)) \text{ weak star.} \tag{3.22}$$

Let j be fixed and $\mu > j$. Then, by (3.5) we have

$$(u_\mu''(t), w_j) + a(u_\mu, w_j) + (|u_\mu|^{\nu(x)}u_\mu, w_j) + (g(u_\mu'), w_j) = (f, w_j). \tag{3.23}$$

Therefore, (3.18), (3.19), (3.20), (3.23) and (3.22) implies

$$\begin{cases} a(u_\mu, w_j) \to a(u, w_j) \text{ in } L^\infty(0,T) \text{ weak star,} \\ (u_\mu', w_j) \to (u', w_j) \text{ in } L^\infty(0,T) \text{ weak star,} \\ (u_\mu''(t), w_j) \to (u''(t), w_j) \text{ in } \mathcal{D}'(0,T), \\ (|u_\mu|^{\nu(x)}u_\mu, w_j) \to (|u|^{\nu(x)}u, w_j) \text{ in } L^\infty(0,T) \text{ weak star,} \\ (g(u_\mu'), w_j) \to (g(u'), w_j) \text{ in } L^\infty(0,T) \text{ weak star.} \end{cases} \tag{3.24}$$

Then (3.23) takes the form

$$(u'', w_j) + a(u, w_j) + (|u|^{\nu(x)}u, w_j) + (g(u'), w_j) = (f, w_j).$$

Finally, be using the density of V_m in $V \cap L^{p(x)}(\Omega)$ we obtain

$$(u'', v) + a(u, v) + (|u|^{\nu(x)}u, v) + (g(u'), v) = (f, v), \quad \forall v \in V \cap L^{p(x)}(\Omega). \tag{3.25}$$

Then u satisfies (1.1). From (3.18) we have

$$u_\mu(0) \to u(0) \text{ weakly in } L^2(\Omega).$$

Then, using (3.6) we deduce in particular that

$$u_\mu(0) = u_{0\mu} \to u_0 \quad \text{in } V \cap L^{p(x)}(\Omega).$$

Thus, the first initial condition in (1.1) is obtained. On the other hand, by using (3.24)

$$(u''_\mu(t), w_j) \to (u''(t), w_j) \quad \text{in } L^\infty(0, T) \text{ weak star}.$$

Hence $(u'_\mu(0), w_j) \to (u'(0), w_j)$. Since $(u'_\mu(0), w_j) \to (u_1, w_j)$, we have $(u'(0), w_j) = (u_1, w_j)$, $\forall j$. Then the second initial condition in (1.1) is satisfied. □

3.2 Uniqueness

Many authors, for some particular problems, when $\nu(x) = \nu$ is a constant number, have showed the uniqueness of the solution basing on the condition $\nu \le \frac{2}{n-2}$. In this subsection the uniqueness of the solution will be proved without any condition on $\nu(x)$.

Theorem 3.3 *Let the conditions of Theorem* 3.1 *hold and in addition*

$$\nu(x) \le \nu_+ \le \frac{2k}{n-2}, \quad k \in \mathbb{N}^*, \ (n \ne 2;\ \nu_+ < \infty \textit{ if } n = 2). \tag{3.26}$$

Then, the solution u obtained in Theorem 3.1 *is unique.*

Proof Let u, v be two solutions of problem (1.1), to the sense of the Theorem 3.1. Setting $w = u - v$, since F is linear we have

$$w'' - \operatorname{div} F(\varepsilon(w)) + (|u|^{\nu(x)}u - |v|^{\nu(x)}v) + (g(u') - g(v')) = 0, \quad \text{in } Q, \tag{3.27}$$

$$w(0) = w'(0) = 0, \quad \text{in } \Omega, \tag{3.28}$$

$$w = 0 \quad \text{on } \Sigma_1, \quad \sigma(w)\eta = 0 \quad \text{on } \Sigma_2, \tag{3.29}$$

$$w \in L^\infty(0, T; V \cap L^{p(x)}(\Omega)), \quad p(x) = \nu(x) + 2, \tag{3.30}$$

$$w' \in L^\infty(0, T; L^2(\Omega)). \tag{3.31}$$

Multiplying the equation (3.27) by w' and integrating on Ω. Then, by using Green's formula together with the conditions (3.28), (3.29), we obtain

$$\frac{1}{2}\frac{d}{dt}|w'(t)|^2 + a(w(t), w'(t)) + (g(u') - g(v'), w'(t)) = \int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w'dx. \tag{3.32}$$

Then by (2.3) (b), we have

$$\begin{aligned} a(w(t), w'(t)) &= \frac{d}{dt}a(w(t), w(t)) - \int_\Omega \frac{d}{dt}(F(\varepsilon(w)))\varepsilon(w)dx \\ &= C_1\frac{d}{dt}\|w\|^2 - \int_\Omega F(\varepsilon(w'))\varepsilon(w)dx \end{aligned}$$

$$= C_1 \frac{d}{dt}\|w\|^2 - a(w(t), w'(t)).$$

In this case (3.32) takes the form

$$\frac{1}{2}\frac{d}{dt}(|w'(t)|^2 + C_1\|w\|^2) + (g(u') - g(v'), w'(t)) = \int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w'dx. \tag{3.33}$$

Also, we have

$$\left|\int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w'dx\right| \leq (\nu_+ + 1)\int_\Omega \sup(|u|^{\nu(x)}, |v|^{\nu(x)})|w||w'|dx.$$

Next, by using the Hölder inequality we have

$$\left|\int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w'dx\right| \leq C_2(\||u|^{\nu(x)}\|_{L^n(\Omega)} + \||v|^{\nu(x)}\|_{L^n(\Omega)})\|w(t)\|_{L^q(\Omega)}|w'(t)|,$$

where $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$. Also, by referring to [1] we have

$$\|v\|_{L^{kq}(\Omega)} = \||v|^k\|^{\frac{1}{k}}_{L^q(\Omega)} \quad \forall k, q \in \mathbb{N}^*. \tag{3.34}$$

Therefore by (3.34) $\|v\|^{\nu(x)}_{L^{kq}(\Omega)} = \||v|^k\|^{\frac{\nu(x)}{k}}_{L^q(\Omega)}$, for all $\nu(x) \in \mathbb{R}$, using (3.26) we have $\nu(x)n \leq \nu_+ n \leq kq$. Then, this conditions implies that

$$\begin{aligned}\||v|^{\nu(x)}\|_{L^n(\Omega)} &\leq \|v\|^{\nu(x)}_{L^{\nu(x)n}(\Omega)} \leq \|v\|^{\nu(x)}_{L^{\nu_+ n}(\Omega)} \leq \|v\|^{\nu(x)}_{L^{kq}(\Omega)} = \||v|^k\|^{\frac{\nu(x)}{k}}_{L^q(\Omega)} \\ &\leq \||v|^k\|^{\frac{\nu(x)}{k}} \leq C\|v\|^{\nu(x)},\end{aligned}$$

which implies by the estimate (3.1) and as $H^1_0(\Omega) \subset L^q(\Omega)$ that

$$\left|\int_\Omega (|v|^{p(x)-2}v - |u|^{p(x)-2}u)w'dx\right| \leq C(\|u\|^{\nu(x)} + \|v\|^{\nu(x)})\|w(t)\|_{H^1_0(\Omega)}|w'(t)| \leq C_4\|wt\||w'|.$$

Then, by Young inequality from (3.33) we deduce

$$\frac{1}{2}\frac{d}{dt}(|w'(t)|^2 + C_1\|w(t)\|^2) \leq \frac{1}{2}C_4(|w'(t)|^2 + \|w(t)\|^2). \tag{3.35}$$

Integrating equation (3.35) together with the initial conditions (3.28), we use Gronwall's inequality to find $w = 0$. □

Corollary 3.4 *Assume that the conditions of Theorem* 3.1 *hold. Then, for all* $\nu(x) \in \mathbb{R}$ *the solution* u *found to Theorem* 3.1 *is unique.*

Proof For all $n > 2$, set

$$k = \mathrm{Ent}\left(\frac{\nu_+(n-2)}{2}\right) + 1,$$

where $\mathrm{Ent}(x)$ denotes the integer part of x. Then, we have

$$\nu(x) \leq \nu_+ \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^*, \ (n \neq 2; \ \nu_+ < \infty \text{ if } n = 2).$$

Thus, using Theorem 3.3, there exists a unique solution satisfying (3.1)–(3.3). □

4 Global Existence and Nonlinear Internal Stabilization

In this section, we discuss the global existence and the stability property of the unique weak solution u of the problem (1.1). To this aim, we define the modified energy function corresponding to the unique solution by the formula

$$E(t)=\frac{1}{2}|u_t(t)|^2+\frac{1}{2}\|u(t)\|_1^2+\frac{1}{p(x)}\|u(t)\|_{L^{p(x)}(\Omega)}^{p(x)}, \quad t\in\mathbb{R}^+. \tag{4.1}$$

The goal of this note is to get the stability of the system considered under the appropriate conditions on the functions g. Suppose that for the continuous functions $p(x)$, $p'(x)\geq 1$ and for the positive constants C_1, C_2, C_3, C_4 the following statements hold:

$$C_1|x|^{p(x)}\leq |g(x)|\leq C_2|x|^{\frac{1}{p(x)}}, \quad \text{if } |x|\leq 1, \tag{4.2}$$

$$C_3|x|\leq |g(x)|, \quad \text{if } |x|>1, \tag{4.3}$$

$$|g(x)|\leq C_4|x|^{p'(x)}, \quad \text{if } |x|>1 \text{ and } n\geq 3. \tag{4.4}$$

The next lemma shows that our functional energy (4.1) is a nonincreasing function along the trajectory of solution of (1.1).

Lemma 4.1 *The energy $E:\mathbb{R}^+\to\mathbb{R}^+$ is a nonincreasing function for $t\geq 0$ and*

$$E'(t)=-\int_\Omega u_t g(u_t)dx\leq 0. \tag{4.5}$$

Proof For all $0\leq S<T<\infty$, multiplying the equation of (1.1) by u_t and integrating over Ω, using integrating by parts and summing up the product results, we get

$$E(t)-E(0)=-\int_0^t\int_\Omega u_t g(u_t)dxds, \quad \text{for } t\geq 0. \tag{4.6}$$

Being the primitive of an integrable function, $E(t)$ is absolutely continuous and the equality (4.5) is satisfied. □

4.1 Global Existence

Theorem 4.2 *Let the assumptions of Theorem* 3.1 *hold. Then the solution u to problem* (1.1) *verifies the following estimates*

$$u\in C(\mathbb{R}^+, V\cap L^{p(x)}(\Omega)), \quad u'\in C(\mathbb{R}^+, L^2(\Omega)).$$

Proof Under the hypotheses of Theorem 3.1, $(u,u')\in (V\cap L^{p(x)}(\Omega))\times L^2(\Omega)$ on $[0,T)$. Then by the identity (4.5) we have

$$\frac{1}{2}|u_t(t)|^2+\frac{1}{2}\|u(t)\|_1^2+\frac{1}{p(x)}\|u(t)\|_{L^{p(x)}(\Omega)}^{p(x)}\leq E(0), \quad \forall t\geq 0$$

bounded independently of t. □

4.2 Stability of Solution

Theorem 4.3 *Supposes that* (4.2)–(4.4) *hold. Then the solution of the problem* (1.1) *verifies for positive constants c and ϖ the estimates*:

$$E(t)\leq ct^{\frac{-2}{p_+-1}}, \quad \forall t\in\mathbb{R}^+ \text{ if } p_+>1, \tag{4.7}$$

and

$$E(t)\leq E(0)\,\mathrm{e}^{(1-\varpi t)}, \quad \forall t\in\mathbb{R}^+ \text{ if } p_+ = 1.$$

Here, the constant c *depends on the initial energy* $E(0)$*, the constant* ϖ *does not depend of* $E(0)$.

First, we shall give some lemmas which will be used for the proof of Theorem 4.3.

Lemma 4.4 (see [7, Theorem 8.1]) *Let* $E:\mathbb{R}^+\to\mathbb{R}^+$ *be a nonincreasing function verifying for two constants* $\alpha\geq 0$ *and* $T>0$ *the estimates*:

$$\int_t^{\infty} E^{\alpha+1}(s)ds \leq TE^{\alpha}(0)E(t), \quad \forall t\in\mathbb{R}^+.$$

Then

$$E(t)\leq E(0)\Big(\frac{T+\alpha t}{T+\alpha T}\Big)^{\frac{-1}{\alpha}}, \quad \forall t\in\mathbb{R}^+ \ \text{if}\ \alpha>0$$

and

$$E(t)\leq E(0)\mathrm{e}^{1-\frac{1}{T}t}, \quad \forall t\in\mathbb{R}^+ \ \text{if}\ \alpha=0.$$

Lemma 4.5 *For all* $0\leq S<T<\infty$ *we have the estimate*

$$\begin{aligned}2\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \leq &-\Big[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\Big]_S^T\\ &+\frac{p(x)-1}{2}\int_S^T E^{\frac{p(x)-3}{2}}(t)E'(t)\int_\Omega u_t u dxdt\\ &+\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2-ug(u_t))dxdt. \end{aligned}\tag{4.8}$$

Proof First, note that $\int_\Omega u_{tt}udx=\frac{d}{dt}\int_\Omega u_t u dx-\int_\Omega (u_t)^2dx$, then

$$\begin{aligned}0&=\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega u(u_{tt}-\operatorname{div}\sigma(u)+|u|^{\nu(x)}u(t)+g(u_t))dxdt\\ &=\Big[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\Big]_S^T-\frac{p(x)-1}{2}\int_S^T E^{\frac{p(x)-3}{2}}(t)E'(t)\int_\Omega uu_t dxdt\\ &\quad+\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ((-u\cdot\operatorname{div}\sigma(u))+|u|^{p(x)}+ug(u_t)-(u_t)^2)dxdt.\end{aligned}\tag{4.9}$$

By using the definition of the energy (4.1) we have

$$\int_\Omega (-u\operatorname{div}\sigma(u)+|u|^{p(x)})dx\geq 2E(t)-\int_\Omega (u_t)^2dx.\tag{4.10}$$

By substitution (4.10) in (4.9) it gives

$$\begin{aligned}0\geq &\Big[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\Big]_S^T-\frac{p(x)-1}{2}\int_S^T E^{\frac{p(x)-3}{2}}(t)E'(t)\int_\Omega uu_t dxdt\\ &+\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2E(t)-(u_t)^2+ug(u_t)-(u_t)^2)dxdt.\end{aligned}$$

Then

$$\begin{aligned}0\geq &\Big[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\Big]_S^T-\frac{p(x)-1}{2}\int_S^T E^{\frac{p(x)-3}{2}}(t)E'(t)\int_\Omega uu_t dxdt\\ &+2\int_S^T E^{\frac{p(x)+1}{2}}(t)dt-\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2-ug(u_t))dxdt,\end{aligned}$$

deriving (4.8). □

Lemma 4.6 *The energy E verifies the estimate*

$$2\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le cE^{\frac{p(x)+1}{2}}(S) + \int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2 - ug(u_t))dxdt \tag{4.11}$$

for all $0 \le S < T < \infty$, where c design, from this lemma, a positive constant independent of $E(0)$, S and of T.

Proof The boundary condition and assumptions (2.3) imply

$$\int_\Omega -u\text{div}\,\sigma(u)dx = C_1\int_\Omega \|u\|^2 dx \ge c\int_\Omega |u|^2 dx. \tag{4.12}$$

From (4.12), (4.1) and Young inequality, we have

$$\begin{aligned}\left|E^{\frac{p(x)-1}{2}}(t)\int_\Omega uu_t dx\right| &\le cE^{\frac{p(x)-1}{2}}(t)\int_\Omega ((u)^2 + (u_t)^2)dx\\ &\le cE^{\frac{p(x)-1}{2}}(t)\int_\Omega (-u\text{div}\,\sigma(u) + (u_t)^2)dx\\ &\le cE^{\frac{p(x)-1}{2}}(t)E(t) = cE^{\frac{p(x)+1}{2}}(t).\end{aligned}$$

Therefore

$$\left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega uu_t dx\right]_S^T \le cE^{\frac{p(x)+1}{2}}(S).$$

On the other hand,

$$\begin{aligned}&\left|\frac{p(x)-1}{2}\int_S^T E^{\frac{p(x)-3}{2}}(t)E'(t)\int_\Omega uu_t dxdt\right|\\ &\le c\int_S^T E^{\frac{p(x)-3}{2}}(t)(-E'(t))E(t)dt\\ &= cE^{\frac{p(x)+1}{2}}(S) - cE^{\frac{p(x)+1}{2}}(T) \le cE^{\frac{p(x)+1}{2}}(S).\end{aligned}$$

One replaces these two estimates in (4.8) to find (4.11). □

Lemma 4.7 *For all $0 \le S < T < \infty$ and all $\epsilon > 0$:*

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (u_t)^2 dxdt \le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(s) + cE^{\frac{p(x)+1}{2}}(S). \tag{4.13}$$

Proof For $t \in \mathbb{R}^+$ fixed, we have

$$\int_\Omega (u_t)^2 dx = \int_{|u_t|\le 1}(u_t)^2 dx + \int_{|u_t|>1}(u_t)^2 dx.$$

Using the Hölder inequality we get

$$\int_\Omega (u_t)^2 dx \le c\left(\int_{|u_t|\le 1}|u_t|^{p(x)+1}dx\right)^{\frac{2}{p(x)+1}} + \int_{|u_t|>1}(u_t)^2 dx.$$

By virtue of (4.2), (4.3) and (4.5) we observe that

$$\begin{aligned}\int_\Omega (u_t)^2 dx &\le c\left(\int_{|u_t|\le 1}|u_t|^{p(x)}\ |u_t|dx\right)^{\frac{2}{p(x)+1}} + \int_{|u_t|>1} u_t u_t dx\\ &\le c\left(\int_{|u_t|\le 1}|u_t g(u_t)|dx\right)^{\frac{2}{p(x)+1}} + c\int_{|u_t|>1}|u_t g(u_t)|dx\end{aligned}$$

$$= c\Big(\int_{|u_t|\le 1} u_t g(u_t)dx\Big)^{\frac{2}{p(x)+1}} + c\int_{|u_t|>1} u_t g(u_t)dx$$
$$\le c(-E'(t))^{\frac{2}{p(x)+1}} - cE'(t).$$

Therefore,

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (u_t)^2 dxdt \le c\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E'(t))^{\frac{2}{p(x)+1}}dt - c\int_S^T E^{\frac{p(x)-1}{2}}(t)E'(t)dt.$$

Using Young inequality, we yield

$$\begin{aligned} c\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E'(t))^{\frac{2}{p(x)+1}}dt &\le c\frac{p(x)-1}{p(x)+1}\int_S^T E^{\frac{p(x)-1}{2}\frac{p(x)+1}{p(x)-1}}(t)dt \\ &\quad + c\frac{2}{p(x)+1}\int_S^T (-E'(t))^{\frac{2}{p(x)+1}\frac{p(x)+1}{2}}dt \\ &\le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt - c(\epsilon)\int_S^T E'(t)dt \\ &\le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(S). \end{aligned}$$

Combining the last two inequalities, we find

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (u_t)^2 dxdt \le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(S) + cE^{\frac{p(x)+1}{2}}(S).$$

Thus (4.13) holds. □

Lemma 4.8 *For all* $0 \le S < T < \infty$ *and all* $\epsilon > 0:$

$$\Big|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt\Big| \le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(S). \tag{4.14}$$

Proof By applying the generalized young inequality, for all $\epsilon' > 0$ we have

$$\Big|\int_{|u_t|\le 1} ug(u_t)dx\Big| \le \epsilon'\int_{|u_t|\le 1} u^2 dx + c(\epsilon')\int_{|u_t|\le 1} g^2(u_t)dx$$

then from (4.2) and (4.12) we get

$$\begin{aligned} \Big|\int_{|u_t|\le 1} ug(u_t)dx\Big| &\le \epsilon'\int_{|u_t|\le 1} -u\mathrm{div}\,\sigma(u)dx + c(\epsilon')\int_{|u_t|\le 1} g^2(u_t)dx \\ &\le 2\epsilon' E(t) + c(\epsilon')\Big(\int_{|u_t|\le 1}|g(u_t)|^{p(x)+1}dx\Big)^{\frac{2}{p(x)+1}} \\ &= 2\epsilon' E(t) + c(\epsilon')\Big(\int_{|u_t|\le 1}|g(u_t)|^{p(x)}\ |g(u_t)|dx\Big)^{\frac{2}{p(x)+1}} \\ &\le 2\epsilon' E(t) + cc(\epsilon')\Big(\int_{|u_t|\le 1}|g(u_t)||u_t|dx\Big)^{\frac{2}{p(x)+1}} \\ &= 2\epsilon' E(t) + cc(\epsilon')\Big(\int_{|u_t|\le 1} u_t g(u_t)dx\Big)^{\frac{2}{p(x)+1}} \\ &= 2\epsilon' E(t) + cc(\epsilon')(-E(t))^{\frac{2}{p(x)+1}}. \end{aligned}$$

Therefore,

$$\left|\int_{|u_t|\le 1} ug(u_t)dx\right| \le 2\epsilon' E(t) + cc(\epsilon')(-E(t))^{\frac{2}{p(x)+1}}. \tag{4.15}$$

For all $p'(x) \ge 1$, and all $n > 2$, we put $k = \mathrm{Ent}(\frac{(p'_+ +1)(n-2)}{2n}) + 1$, where the notation $\mathrm{Ent}(x)$ designates the integer part of real x, and therefore k must verify the condition

$$p'(x) + 1 \le p'_+ + 1 \le \frac{2nk}{n-2} \le kq, \quad k \in \mathbb{N}^*,\ n \ne 2.$$

By referring to (3.23) we have the following inequalities:

$$\|v\|_{L^{p'(x)+1}(\Omega)} \le \|v\|_{L^{kq}(\Omega)} = \||v|^k\|^{\frac{1}{k}}_{L^q(\Omega)} \le c\|v\|_{L^q(\Omega)} \le c\|v\|_{H^1(\Omega)}.$$

Consequently

$$\left(\int_{|u_t|>1} |u|^{p'(x)+1}dx\right)^{\frac{1}{p'(x)+1}} \le c\|u\|_{H^1(\Omega)} \le CE(t)^{\frac{1}{2}}.$$

From (4.4) we have

$$\begin{aligned}\left(\int_{|u_t|>1} |g(u_t)|^{\frac{p'(x)+1}{p'(x)}} dx\right)^{\frac{p'(x)}{p'(x)+1}} &= \left(\int_{|u_t|>1} |g(u_t)||g(u_t)|^{\frac{1}{p'(x)}} dx\right)^{\frac{p'(x)}{p'(x)+1}}\\ &\le C\left(\int_{|u_t|>1} |u_t g(u_t)|dx\right)^{\frac{p'(x)}{p'(x)+1}}\\ &\le c(-E'(t))^{\frac{p'(x)}{p'(x)+1}},\end{aligned}$$

which implies

$$\left|\int_{|u_t|>1} ug(u_t)dx\right| \le cE(t)^{\frac{1}{2}}(-E'(t))^{\frac{p'(x)}{p'(x)+1}}. \tag{4.16}$$

Then from (4.15) and (4.16) we arrives to

$$\begin{aligned}\left|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt\right| &\le 2\epsilon' \int_S^T E^{\frac{p(x)-1}{2}}(t)E(t)dt\\ &\quad + cc(\epsilon')\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E'(t))^{\frac{2}{p(x)+1}}dt\\ &\quad + c\int_S^T E^{\frac{p(x)-1}{2}}(t)E(t)^{\frac{1}{2}}(-E'(t))^{\frac{p'(x)}{p'(x)+1}}dt\end{aligned}$$

or

$$\begin{aligned}\left|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt\right| &\le 2\epsilon' \int_S^T E^{\frac{p(x)+1}{2}}(t)dt\\ &\quad + cc(\epsilon')\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E'(t))^{\frac{2}{p(x)+1}}dt\\ &\quad + c\int_S^T E^{\frac{p(x)}{2}}(t)(-E'(t))^{\frac{p'(x)}{p'(x)+1}}dt.\end{aligned}$$

Using the fact that $\frac{2}{p(x)+1} + \frac{p(x)-1}{p(x)+1} = 1$, by the Young inequality we see

$$cc(\epsilon')\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E'(t))^{\frac{2}{p(x)+1}}dt \le \epsilon' \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon')\int_S^T (-E'(t))dt. \tag{4.17}$$

In the same way, since $\frac{p'(x)}{p'(x)+1}+\frac{1}{p'(x)+1}=1$ we have

$$c\int_S^T E^{\frac{p(x)}{2}}(t)(-E'(t))^{\frac{p'(x)}{p'(x)+1}}dt \le c\int_S^T E(t)^{\frac{p(x)(p'(x)+1)}{2}}dt + c\int_S^T(-E'(t))dt. \tag{4.18}$$

Combine (4.17) with (4.18) to get

$$\begin{aligned}
&\Big|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt\Big| \\
&\le 2\epsilon'\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + \epsilon'\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \\
&\quad + c(\epsilon')\int_S^T(-E'(t))dt + c\int_S^T E(t)^{\frac{p(x)(p'(x)+1)}{2}}dt + c\int_S^T(-E'(t))dt \\
&= 3\epsilon'\int_S^T E^{\frac{p(x)+1}{2}}(t)dt - c(\epsilon')\int_S^T E'(t)dt + c\int_S^T E(t)^{\frac{p(x)(p'(x)+1)}{2}}dt.
\end{aligned} \tag{4.19}$$

As E nonincreasing and as $p(x)(p'(x)+1)\ge p(x)+1$, then

$$\int_S^T E(t)^{\frac{p(x)(p'(x)+1)}{2}}dt \le c\int_S^T E^{\frac{p(x)+1}{2}}(t)dt. \tag{4.20}$$

Thus, it follows from (4.19) and (4.20) that

$$\Big|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt\Big| \le \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(s).$$

This is (4.14). □

Lemma 4.9 *For all $0\le S<T<\infty$ we have the estimate*

$$\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le c(1+E^{\frac{p(x)-1}{2}}(0))E(s),\quad 0\le S\le T<\infty. \tag{4.21}$$

Proof Choosing $\epsilon=\frac{1}{3}$ in (4.13) and in (4.14) it finds

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega 2u_t^2dxdt \le \frac{2}{3}\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s) \tag{4.22}$$

and

$$-\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt \le \frac{1}{3}\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s). \tag{4.23}$$

Therefore, by addition of (4.22) and (4.23) it comes

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega(2u_t^2-ug(u_t))dxdt \le \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s). \tag{4.24}$$

Using in (4.11) the inequality (4.24) we find that

$$2\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le cE^{\frac{p(x)+1}{2}}(s) + \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s).$$

Therefore,

$$\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le c(1+E^{\frac{p(x)-1}{2}}(s))E(s) \le c(1+E^{\frac{p(x)-1}{2}}(0))E(s),\quad 0\le S\le T<\infty. \quad □$$

The Lemmas 4.1 and 4.9 imply that $E : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing function and verify the inequality

$$\int_t^\infty E^{\frac{p_+ + 1}{2}}(s)ds \leq \int_t^\infty E^{\frac{p(x)+1}{2}}(s)ds \leq cE(t), \quad \forall t \in \mathbb{R}^+. \tag{4.25}$$

The applications of the well-known Lemma 4.4 and (4.25) yield the estimates (4.7) and (4.3) and we complete the proof of Theorem 4.3.

Example 4.10 Consider the following function

$$F(\varepsilon(u)) = 2\varepsilon(u) - \mathrm{Trace}(\varepsilon(u))I,$$

where I denotes the identity operator and Trace denotes the trace operator.

Then, the problem (1.1), without the condition $\sigma(u)\eta = 0$ on Σ_2, is reduced to the following problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^{\nu(x)}u + g(u') = f, & \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \Sigma, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{P}$$

Since F is linear and satisfies the assumption (2.3). Then, Theorems 3.1, 3.3, 4.2 and 4.3 are verified for the problem (P), which gives an importance to this general problem.

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