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# A Characterization for a Complete Random Normed Module to Be Mean Ergodic

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**Abstract** In this paper, we first study the mean ergodicity of random linear operators using some techniques of measure theory and $L^0$-convex analysis. Then, based on this, we give a characterization for a complete random normed module to be mean ergodic.

**Keywords** Random normed module, mean ergodic, random linear operator, the local property

**MR(2010) Subject Classification** 46H25, 47A35

## 1 Introduction

Motivated by the idea of randomizing space theory of Menger, Schweizer and Sklar from the theory of probabilistic metric space [20], Guo initiated a new approach to random functional analysis in [6, 7]. In particular, random metric spaces, random normed modules and random inner product modules have formed three basic frameworks of random functional analysis. In fact, the notion of a random normed module (briefly, an RN module) is a randomnization of that of an ordinary normed space, see [7] for further details. In this paper, an RN module is always endowed with the $(\varepsilon, \lambda)$-topology and the $(\varepsilon, \lambda)$-topology is very natural, for example, let $L^0(\mathcal{F}, R)$ be the algebra of equivalence classes of random variables from a probability space $(\Omega, \mathcal{F}, P)$ to the real scalar field $R$, then $L^0(\mathcal{F}, R)$ is an RN module and the $(\varepsilon, \lambda)$-topology on $L^0(\mathcal{F}, R)$ is exactly the topology of convergence in probability $P$. However, an RN module is not a locally convex space in general and in particular the theory of classical conjugate spaces universally fails to serve the theory of RN modules, which motivates Guo to have developed the theory of random conjugate spaces in [8]. Subsequently, the theory of RN modules together with their random conjugate spaces has obtained a deep and systematic development in the direction of functional analysis [9–14, 22–26], and it has also been applied to random linear operators [21, 27–29], Lebesgue–Bochner function spaces [3] and conditional convex risk measures [15–18].

Motivated by the work with respect to mean ergodicity in [1, 2, 5], we have recently begun to study the mean ergodic theorem under the framework of RN modules and have obtained some results. For example, in 2012, Zhang and Guo proved a mean ergodic theorem on a

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random reflexive random normed module [30]. Subsequently, Zhang studied some mean ergodic semigroups of random linear operators [27] and further gave two forms of conditional mean ergodic theorem for a type of semigroup of random linear operators [28]. The purpose of this paper is to continue to study the mean ergodicity of random linear operators by using some techniques of measure theory and $L^0$-convex analysis. Intuitively, an RN module possesses not only the random normed structure but also the structure of a probability space. It is the module structure of an RN module that makes its random normed structure and the structure of its base space behave well, which makes an RN module possess many rich and complicated stratification structures, see [13, 14] for details. In this paper, besides the analysis on stratification structures in measure theory, we use a approach of $L^0$-convex analysis to study the mean ergodicity of random linear operators. Precisely, using a fixed point approach of $L^0$-convex hull, we proved that a random linear operator $T$ under some condition is mean ergodic in the $\|\cdot\|_p$-topology if and only if $T$ is mean ergodic in the $(\varepsilon,\lambda)$-topology. Subsequently, we are devoted to establishing a characterization for a complete RN module to be mean ergodic. It should be pointed out that in this process the discussion with respect to the local property and the local mean ergodicity of a linear operator is a difficult point of this paper.

The remainder of this paper is organized as follows: in Section 2 we briefly recall some basic notion and facts; in Section 3 we study the mean ergodicity of some random linear operators and in Section 4 we give a characterization for a complete RN module to be mean ergodic.

## 2 Preliminaries

Throughout this paper, $N$ denotes the set of natural numbers, $K$ the scalar field $R$ of real numbers or $C$ of complex numbers, $(\Omega,\mathcal{F},P)$ a given probability space, $\bar{L}^0(\mathcal{F},R)$ the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $\Omega$, $L^0(\mathcal{F},K)$ the algebra of equivalence classes of $K$-valued $\mathcal{F}$-measurable random variables on $\Omega$ under the ordinary addition, scalar multiplication and multiplication operations on equivalence classes. It is well known from [4] that Proposition 2.1 below holds, which will be used in the proof of Lemma 3.1.

**Proposition 2.1** ([4]) *$\bar{L}^0(\mathcal{F},\, R)$ is a complete lattice under the ordering $\leq$: $\xi\leq\eta$ if and only if $\xi^0(\omega)\leq\eta^0(\omega)$, for $P$-almost all $\omega$ in $\Omega$ (briefly, a.s.), where $\xi^0$ and $\eta^0$ are arbitrarily chosen representatives of $\xi$ and $\eta$, respectively, and has the following three properties*:

(1) *each subset $A$ of $\bar{L}^0(\mathcal{F},R)$ has a supremum (denoted by $\bigvee A$) and an infimum (denoted by $\bigwedge A$) and there exist two sequences $\{a_n,\ n\in N\}$ and $\{b_n,\ n\in N\}$ in $A$ such that $\bigvee_{n\geq 1} a_n=\bigvee A$ and $\bigwedge_{n\geq 1} b_n=\bigwedge A$;*

(2) *if $A$ is directed (dually directed), namely for any two elements $c_1$ and $c_2$ in $A$ there exists some $c_3$ in $A$ such that $c_1\bigvee c_2\leq c_3$ $(c_1\bigvee c_2\geq c_3)$, then the present $\{a_n,\ n\in N\}$ $(\{b_n,\ n\in N\})$ can be chosen as nondecreasing (nonincreasing);*

(3) *$L^0(\mathcal{F},R)$, as a sublattice of $\bar{L}^0(\mathcal{F},R)$, is complete in the sense that each subset with an upper bound (a lower bound) has a supremum (an infimum).*

As usual, we denote $L^0_+(\mathcal{F})=\{\xi\in L^0(\mathcal{F},R)\,|\,\xi\geq 0\}$.

**Definition 2.2** ([8, 9]) *An ordered pair $(X,\|\cdot\|)$ is called an RN module over $K$ with base $(\Omega,\mathcal{F},P)$ if $X$ is a left module over over the algebra $L^0(\mathcal{F},K)$ and $\|\cdot\|$ is a mapping from $X$*

*to* $L^0_+(\mathcal{F})$ *such that the following three axioms are satisfied*:

(1) $\|\xi x\| = |\xi| \cdot \|x\|, \forall \xi \in L^0(\mathcal{F}, K)$ *and* $x \in X$;

(2) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(3) $\|x\| = 0$ *implies* $x = \theta$ *(the null vector of* $X$*)*,

*where* $\|\cdot\|$ *is called the* $L^0$*-norm on* $X$ *and* $\|x\|$ *is called the* $L^0$*-norm of a vector* $x \in X$.

Clearly, $(L^0(\mathcal{F}, K), |\cdot|)$ is an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$.

Let $(X, \|\cdot\|)$ be an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. For any positive real numbers $\varepsilon$ and $\lambda$ such that $\lambda < 1$, let

$$N_\theta(\varepsilon, \lambda) = \{x \in X \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > \lambda\},$$

then

$$\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0,\ 0 < \lambda < 1\}$$

is a local base at the null vector $\theta$ of some Hausdorff linear topology, and the linear topology is called the $(\varepsilon, \lambda)$-topology. It should be pointed out that the idea of introducing the $(\varepsilon, \lambda)$-topology is due to Schweizer and Sklar, see [20] for details. In this paper, given an RN module $(X, \|\cdot\|)$, it is always assumed that $(X, \|\cdot\|)$ is endowed with the $(\varepsilon, \lambda)$-topology. Besides, one needs to notice that a sequence $\{x_n, n \in N\}$ in $X$ converges to $x \in X$ in the $(\varepsilon, \lambda)$-topology if and only if $\{\|x_n - x\|, n \in N\}$ converges to 0 in probability $P$.

**Example 2.3** Let $X$ be a normed space over $K$ and $L^0(\mathcal{F}, X)$ the linear space of equivalence classes of $X$-valued $\mathcal{F}$-random variables on $\Omega$. The module multiplication operation $\cdot : L^0(\mathcal{F}, K) \times L^0(\mathcal{F}, X) \to L^0(\mathcal{F}, X)$ is defined by $\xi x =$ the equivalence class of $\xi^0 x^0$, where $\xi^0$ and $x^0$ are the respective arbitrarily chosen representatives of $\xi \in L^0(\mathcal{F}, K)$ and $x \in L^0(\mathcal{F}, X)$, and $(\xi^0 x^0)(\omega) = \xi^0(\omega) \cdot x^0(\omega), \forall \omega \in \Omega$. Furthermore, the mapping $\|\cdot\| : L^0(\mathcal{F}, X) \to L^0_+(\mathcal{F})$ is defined by $\|x\| =$ the equivalence class of $\|x^0\|, \forall x \in L^0(\mathcal{F}, X)$, where $x^0$ is as above. Then it is easy to see that $(L^0(\mathcal{F}, X), \|\cdot\|)$ is an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. In particular, $L^0(\mathcal{F}, K)$ is an RN module over $K$ with base $(\Omega, \mathcal{F}, P)$.

**Definition 2.4** ([19]) *Let* $(X^1, \|\cdot\|_1)$ *and* $(X^2, \|\cdot\|_2)$ *be two RN modules over* $K$ *with base* $(\Omega, \mathcal{F}, P)$. *A linear operator* $T$ *from* $X^1$ *to* $X^2$ *is called a random linear operator, and further the random linear operator* $T$ *is called a.s. bounded if there exists a* $\xi \in L^0_+(\mathcal{F})$ *such that*

$$\|Tx\|_2 \leq \xi \cdot \|x\|_1$$

*for any* $x \in X^1$. *Denote by* $B(X^1, X^2)$ *the linear space of a.s. bounded random linear operators from* $X^1$ *to* $X^2$, *define* $\|\cdot\| : B(X^1, X^2) \to L^0_+(\mathcal{F})$ *by*

$$\|T\| := \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid \|Tx\|_2 \leq \xi \cdot \|x\|_1,\ \forall x \in X^1\}$$

*for any* $T \in B(X^1, X^2)$, *then it is clear that* $(B(X^1, X^2), \|\cdot\|)$ *is an RN module over* $K$ *with base* $(\Omega, \mathcal{F}, P)$.

**Proposition 2.5** ([19]) *Let* $(X^1, \|\cdot\|_1)$ *and* $(X^2, \|\cdot\|_2)$ *be two RN modules over* $K$ *with base* $(\Omega, \mathcal{F}, P)$. *Then we have the following statements*:

(1) $T \in B(X^1, X^2)$ *if and only if* $T$ *is a continuous module homomorphism*;

(2) *If* $T \in B(X^1, X^2)$, *then* $\|T\| = \bigvee\{\|Tx\|_2 : x \in X^1$ *and* $\|x\|_1 \leq 1\}$, *where* 1 *denotes the identity element in* $L^0(\mathcal{F}, R)$.

Now let us recall the notions of an a.s. power-bounded random linear operator and a mean ergodic operator on an RN module below.

**Definition 2.6** ([30]) *Let $(X, \|\cdot\|)$ be an RN module and $T : X \to X$ an a.s. bounded random linear operator. If*

$$\bigvee_{n \in N} \|T^n\| \in L^0_+(\mathcal{F}),$$

*then $T$ is called an a.s. power-bounded random linear operator on $X$.*

**Definition 2.7** ([30]) *Let $(X, \|\cdot\|)$ be an RN module and $T : X \to X$ an a.s. bounded random linear operator. If the sequence*

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} T^i x, \ n \in N \right\}$$

*converges to some point in $X$ for any $x \in X$, then $T$ is called a mean ergodic operator on $X$. Furthermore, if any a.s. power-bounded random linear operator on $X$ is mean ergodic, then $X$ is said to be mean ergodic.*

## 3 Mean Ergodicity of Random Linear Operators

The central result of this section is Theorem 3.4, which is devoted to establishing a relation with respect to mean ergodicity of random linear operators between the $\|\cdot\|_p$-topology and the $(\varepsilon, \lambda)$-topology. What's more important is that this relation plays a key role in the proof of Theorem 4.4. Before presenting them, let us give some preliminaries below for the reader's convenience.

In the sequel of this paper, let $1 \le p \le \infty$ and $(X, \|\cdot\|)$ denote a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. Define the mapping $\|\cdot\|_p : X \to [0, +\infty]$ by

$$\|x\|_p = \begin{cases} \left[ \int_\Omega \|x\|^p dP \right]^{\frac{1}{p}}, & \text{when } 1 \le p < +\infty, \\ \text{the } P\text{-essential supremum of } \|x\|, & \text{when } p = +\infty \end{cases}$$

for any $x \in X$ and denote

$$L^p(X) = \{x \in X \mid \|x\|_p < +\infty\}.$$

Then $(L^p(X), \|\cdot\|_p)$ is a Banach space. Let $L(L^p(X))$ denote the set of bounded linear operators on $L^p(X)$ and $|\!|\!|\cdot|\!|\!|$ denote the usual norm of $L^p(X)$. Then it is clear that $(L(L^p(X)), |\!|\!|\cdot|\!|\!|)$ is also a Banach space. Furthermore, let $I$ denote the identical operator on $L^p(X)$ and $T$ a linear operator on $L^p(X)$. Then range$(I - T)$ stands for the range of the operator $I - T$, $\overline{\text{range}(I - T)}^{\varepsilon,\lambda}$ the closure of range$(I - T)$ under the $(\varepsilon, \lambda)$-topology and $\overline{\text{range}(I - T)}^{\|\cdot\|_p}$ the closure of range$(I - T)$ under the $\|\cdot\|_p$-topology.

Besides, for any $A$ in $\mathcal{F}$, $I_A$ denotes the characteristic function of $A$ and $\tilde{I}_A$ the equivalence class of $I_A$. When $\tilde{A}$ denotes the equivalence class of $A$, i.e., $\tilde{A} = \{B \in \mathcal{F} \mid P(A \triangle B) = 0\}$, we also use $I_{\tilde{A}}$ for $\tilde{I}_A$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Specifically, $[\xi < \eta]$ denotes the equivalence class of $\{\omega \in \Omega \mid \xi^0(\omega) < \eta^0(\omega)\}$, where $\xi^0$ and $\eta^0$ are arbitrarily chosen representatives of $\xi$ and $\eta$ in $\bar{L}^0(\mathcal{F}, R)$, respectively, one can similarly understood such notations as $[\xi = \eta]$ and $[\xi \neq \eta]$.

Now let us recall the notion of the local property of a linear operator. A linear operator $T : L^p(X) \to L^p(X)$ is called to have the local property if $T(\tilde{I}_A x) = \tilde{I}_A T(x)$ for any $A \in \mathcal{F}$ and $x \in L^p(X)$, where $\tilde{I}_A$ stands for the equivalence class of the characteristic function of $A$. The key is that Lemma 3.1 below holds.

**Lemma 3.1** *Let $(X, \|\cdot\|)$ be a complete RN modules over $K$ with base $(\Omega, \mathcal{F}, P)$ and $T : L^p(X) \to L^p(X)$ be a bounded linear operator having the local property such that $\sup_{m\in N} \|T^m\| < \infty$ for any given $p$ satisfying $1 \le p \le \infty$. Then $\overline{\mathrm{range}(I-T)}^{\varepsilon,\lambda} = \overline{\mathrm{range}(I-T)}^{\|\cdot\|_p}$.*

*Proof* It is easy to see that $\overline{\mathrm{range}(I-T)}^{\|\cdot\|_p} \subset \overline{\mathrm{range}(I-T)}^{\varepsilon,\lambda}$, and it is enough to prove that $\overline{\mathrm{range}(I-T)}^{\varepsilon,\lambda} \subset \overline{\mathrm{range}(I-T)}^{\|\cdot\|_p}$. For any $x \in \overline{\mathrm{range}(I-T)}^{\varepsilon,\lambda}$, then

$$d(x,\ \mathrm{range}(I-T)) = \bigwedge\{\|x-g\| \,|\, g \in \mathrm{range}(I-T)\} = \theta.$$

Now we will first prove that $\{\|x-g\| \,|\, g \in \mathrm{range}(I-T)\}$ is directed. In fact, for any $g_1,\ g_2 \in \mathrm{range}(I-T)$, let $E = [\|x-g_1\| \le \|x-g_2\|]$. Then

$$\begin{aligned}\|x-g_1\| \bigwedge \|x-g_2\| &= I_E \cdot \|x-g_1\| + (1-I_E)\cdot\|x-g_2\| \\ &= \|x-(I_E\cdot g_1 + (1-I_E)\cdot g_2)\|. \qquad (3.1)\end{aligned}$$

Since $T$ is linear and has the local property, it follows that

$$I_E \cdot g_1 + (1-I_E)\cdot g_2 \in \mathrm{range}(I-T).$$

Thus $\{\|x-g\| \,|\, g \in \mathrm{range}(I-T)\}$ is directed. Hence it follows by Proposition 2.1 that there exists a sequence $\{g_n,\ n \in N\}$ in $\mathrm{range}(I-T)$ such that

$$\|x-g_n\| \searrow \bigwedge\{\|x-g\| \,|\, g \in \mathrm{range}(I-T)\}$$

as $n \to \infty$. Denote

$$\mathrm{dist}(x,\ \mathrm{range}(I-T)) = \inf\{\|x-g\|_p \,|\, g \in \mathrm{range}(I-T)\},$$

and observe that

$$\|x-g_n\| \le \|x-g_1\|, \quad \forall n \ge 1.$$

Then, when $1 \le p < \infty$, according to Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}[\mathrm{dist}(x,\ \mathrm{range}(I-T))]^p &= [\inf\{\|x-g\|_p \,|\, g \in \mathrm{range}(I-T)\}]^p \\ &\le \lim_{n\to\infty} \int_\Omega \|x-g_n\|^p dP \\ &= \int_\Omega \Big[\bigwedge\{\|x-g\| \mid g \in \mathrm{range}(I-T)\}\Big]^p dP \\ &= 0. \qquad (3.2)\end{aligned}$$

Thus

$$\mathrm{dist}(x,\ \mathrm{range}(I-T)) = 0,$$

namely $x \in \overline{\mathrm{range}(I-T)}^{\|\cdot\|_p}$ and the desired result follows.

When $p = \infty$, one can similarly prove the result.

This completes the proof. □

Using Egorov's theorem, one can obtain Lemma 3.2 below, which will be used in Lemma 3.5, Theorem 3.6 and Theorem 4.2.

**Lemma 3.2** *If a linear operator $T : L^p(X) \to L^p(X)$ having the local property is continuous in the $\|\cdot\|_p$-topology for any given $p$ satisfying $1 \le p \le \infty$, then $T : L^p(X) \to L^p(X)$ is continuous in the $(\varepsilon, \lambda)$-topology.*

*Proof* For any $\{x_n, n \in N\} \subset L^p(X)$ and $x \in L^p(X)$ such that $x_n \to x$ as $n \to \infty$ in the $(\varepsilon, \lambda)$-topology, then, according to Egorov's theorem, for any $\varepsilon,\ \lambda > 0$ such that $\lambda < 1$, there exist some $A \in \mathcal{F}$ satisfying $P(A) < \frac{\lambda}{2}$ and some $N_1 \in N$ such that

$$\|I_{A^c} \cdot x_n - I_{A^c} \cdot x\| < \varepsilon \quad \text{on } \Omega$$

as $n > N_1$. Thus $I_{A^c} \cdot x_n \to I_{A^c} \cdot x$ as $n \to \infty$ in the $\|\cdot\|_p$-topology. Since $T : L^p(X) \to L^p(X)$ is continuous in the $\|\cdot\|_p$-topology, it follows that $T(I_{A^c} \cdot x_n) \to T(I_{A^c} \cdot x)$ as $n \to \infty$ in the $\|\cdot\|_p$-topology. According to Egorov's theorem again, for the same $\varepsilon,\ \lambda$ above, there exists some $B \in \mathcal{F}$ and $B \subset A^c$ satisfying $P(B) < \frac{\lambda}{2}$ and some $N_2 \in N$ such that

$$\|I_{B^c} \cdot T(I_{A^c} \cdot x_n) - I_{B^c} \cdot T(I_{A^c} \cdot x)\| < \varepsilon \quad \text{on } \Omega$$

as $n > N_2$. Let $N_0 = \max\{N_1, N_2\}$. Then obviously

$$\|I_{B^c} \cdot T(I_{A^c} \cdot x_n) - I_{B^c} \cdot T(I_{A^c} \cdot x)\| < \varepsilon \quad \text{on } \Omega$$

holds as $n > N_0$.

Since $T$ possesses the local property, it follows that

$$I_{B^c} \cdot T(I_{A^c} \cdot x_n) = I_{A^c \cap B^c} \cdot T(x_n)$$

and

$$I_{B^c} \cdot T(I_{A^c} \cdot x) = I_{A^c \cap B^c} \cdot T(x)$$

hold. Consequently,

$$\|T(x_n) - T(x)\| < \varepsilon \quad \text{on } A^c \cap B^c$$

as $n > N_0$, and further we can obtain that for any $\varepsilon,\ \lambda > 0$ such that $\lambda < 1$, there exists some $N_0 \in N$ such that

$$\begin{aligned} P[\|T(x_n) - T(x)\| < \varepsilon] &\ge P[A^c \cap B^c] \\ &= P[(A \cup B)^c] \\ &= 1 - P[A] - P[B] + P[A \cap B] \\ &> 1 - \lambda \end{aligned} \tag{3.3}$$

as $n > N_0$, which shows that $T : L^p(X) \to L^p(X)$ is continuous in the $(\varepsilon, \lambda)$-topology.

This completes the proof. □

**Proposition 3.3** ([12]) *$L^p(X)$ is dense in $X$ in the $(\varepsilon, \lambda)$-topology for any given $p$ satisfying $1 \le p \le \infty$.*

Based on Proposition 3.3, we can present Lemma 3.4 below.

**Lemma 3.4** *If a linear operator $T : L^p(X) \to L^p(X)$ having the local property is continuous in the $(\varepsilon, \lambda)$-topology for any given $p$ satisfying $1 \le p \le \infty$, then $T$ can be extended to a continuous module homomorphism on $X$ in the $(\varepsilon, \lambda)$-topology.*

*Proof* Since $T : L^p(X) \to L^p(X)$ is continuous in the $(\varepsilon,\lambda)$-topology and $L^p(X)$ is dense in $X$ in the $(\varepsilon,\lambda)$-topology by Proposition 3.3, it follows that for any $x$ in $X$, there exists a sequence $\{x_n, n \in N\} \subset L^p(X)$ such that $x_n \to x$ as $n \to \infty$ in the $(\varepsilon,\lambda)$-topology. Set

$$\tilde{T}(x) = \lim_{n\to\infty} T(x_n),$$

and we only need to prove that $\tilde{T}$ is linear and is continuous in the $(\varepsilon,\lambda)$-topology. Obviously,

$$\tilde{T}(x+y) = \tilde{T}(x) + \tilde{T}(y), \quad \forall x, y \in X,$$
$$\tilde{T}(kx) = k \cdot \tilde{T}(x), \quad \forall k \in K, x \in X$$

hold. Now we will prove that $\tilde{T}$ is continuous in the $(\varepsilon,\lambda)$-topology. For any sequence $\{x_m, m \in N\} \subset X$ such that $\{x_m, m \in N\}$ converges to some $x^*$ in $X$ as $m \to \infty$ in the $(\varepsilon,\lambda)$-topology, since $L^p(X)$ is dense in $X$ in the $(\varepsilon,\lambda)$-topology, it follows that there exist $m$ sequences $\{x_l^{(1)}, l \in N\}, \{x_l^{(2)}, l \in N\}, \ldots, \{x_l^{(i)}, l \in N\}, \ldots, \{x_l^{(m)}, l \in N\}$ in $L^p(X)$ such that $\{x_l^{(i)}, l \in N\}$ converges to $x_i$ as $l \to \infty$ in the $(\varepsilon,\lambda)$-topology for any $i = 1, 2, \ldots, m$. Furthermore, the above $m$ sequences form a table below:

$$\begin{array}{l}
x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, \ldots, x_l^{(1)}, \ldots \\
x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, \ldots, x_l^{(2)}, \ldots \\
x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, \ldots, x_l^{(3)}, \ldots \\
\quad \cdots \\
x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, x_4^{(m)}, \ldots, x_l^{(m)}, \ldots.
\end{array}$$

Then the diagonal elements form a new sequence $\{x_m^{(m)}, m \in N\}$ and it is easy to see that $x_m^{(m)} \to x^*$ as $m \to \infty$ in the $(\varepsilon,\lambda)$-topology. Since

$$\begin{aligned}
\|\tilde{T}(x_m) - \tilde{T}(x^*)\| &= \|\tilde{T}(x_m) - \tilde{T}(x_m^{(m)}) + \tilde{T}(x_m^{(m)}) - \tilde{T}(x^*)\| \\
&= \|\tilde{T}(x_m) - \tilde{T}(x_m^{(m)})\| + \|\tilde{T}(x_m^{(m)}) - \tilde{T}(x^*)\|,
\end{aligned}$$

it follows that $\tilde{T}$ is continuous in the $(\varepsilon,\lambda)$-topology. Moreover, $\tilde{T}$ is linear and has the local property, and one can obtain that $\tilde{T}$ is a continuous module homomorphism on $X$.

This completes the proof. □

Before giving Lemma 3.5 and Theorem 3.6, let us recall the notions of an $L^0$-convex set and an $L^0$-convex hull. Let $X$ be a complete RN module over $K$ and $A$ a subset of $X$. If

$$\{\alpha x + (1-\alpha) y \mid \alpha \in L^0_+(\mathcal{F}) \text{ and } 0 \le \alpha \le 1\} \subseteq A,$$

then $A$ is called an $L^0$-convex set of $X$. Let $B$ be a subset of $X$ and $\{A_\lambda \mid \lambda \in \Lambda\}$ all the $L^0$-convex set containing $B$ of $X$. Then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is the smallest $L^0$-convex set containing $B$ of $X$, called the $L^0$-convex hull of $B$ (denoted by $h_{L^0}(B)$). Furthermore, it is easy to see that

$$h_{L^0}(B) = \Big\{ \sum_{i=1}^n \alpha_i x_i \;\Big|\; \alpha_i \in L^0_+(\mathcal{F}), x_i \in A \text{ and } \sum_{i=1}^n \alpha_i = 1,\ n \ge 1 \Big\}.$$

**Lemma 3.5** *Let $(X, \|\cdot\|)$ be a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$ and $T$ : $L^p(X) \to L^p(X)$ be a bounded linear operator having the local property such that* $\sup_{m\in N} |||T^m|||$ $< \infty$ *for any given $p$ satisfying $1 \le p \le \infty$. If $\bar{x}$ belongs to the $(\varepsilon,\lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$ for any $x \in L^p(X)$, then $\bar{x}$ still belongs to $L^p(X)$.*

*Proof* Since $\bar{x}$ belongs to the $(\varepsilon, \lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$ for any $x \in L^p(X)$, there exists a sequence $\{S_l x,\ l \in N\}$ of $L^0$-convex combinations with the form

$$S_l x = \sum_{i=1}^{m_l} \lambda_{il} T^i x$$

($\lambda_{il} \in L^0_+(\mathcal{F})$ and $\sum_{i=1}^{m_l} \lambda_{il} = 1$ for any $l \in N$) such that $\{S_l x,\ l \in N\}$ converges to $\bar{x}$ in the $(\varepsilon, \lambda)$-topology.

According to Lemma 3.2, we have $T : L^p(X) \to L^p(X)$ is continuous in the $(\varepsilon, \lambda)$-topology; furthermore, it follows from Lemma 3.4 that $T$ can be extended to a continuous module homomorphism $\tilde{T}$ on $X$ in the $(\varepsilon, \lambda)$-topology. This implies that

$$\begin{aligned} x - S_l x &= \sum_{i=1}^{m_l} \lambda_{il}(x - T^i x) \\ &= \sum_{i=1}^{m_l} \lambda_{il}(x - \tilde{T}^i x) \\ &= \sum_{i=1}^{m_l} \lambda_{il}(I - \tilde{T})(I + \tilde{T} + \tilde{T}^2 + \cdots + \tilde{T}^{i-1})x \\ &= (I - \tilde{T})\Big[\sum_{i=1}^{m_l} \lambda_{il}(I + \tilde{T} + \tilde{T}^2 + \cdots + \tilde{T}^{i-1})x\Big] \\ &= (I - T)\Big[\sum_{i=1}^{m_l} \lambda_{il}(I + T + T^2 + \cdots + T^{i-1})x\Big] \\ &\in \text{range}(I - T) \end{aligned} \tag{3.4}$$

since one can observe that $\sum_{i=1}^{m_l} \lambda_{il}(I + T + T^2 + \cdots + T^{i-1})x$ still belongs to $L^p(X)$, where $I$ denotes the identity element in $L(L^p(X))$, so that $x - \bar{x} \in \overline{\text{range}(I - T)}^{\varepsilon,\lambda}$. Since, according to Lemma 3.1,

$$\overline{\text{range}(I - T)}^{\varepsilon,\lambda} = \overline{\text{range}(I - T)}^{\|\cdot\|_p},$$

it follows that $x - \bar{x} \in \overline{\text{range}(I - T)}^{\|\cdot\|_p} \subset L^p(X)$; furthermore, observing that $x \in L^p(X)$, which shows that $\bar{x}$ still belongs to $L^p(X)$. This completes the proof. □

Now we can present Theorem 3.6 below.

**Theorem 3.6** *Let $(X, \|\cdot\|)$ be a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$ and $T : L^p(X) \to L^p(X)$ be a bounded linear operator having the local property such that $\sup_{m \in N} \|T^m\| < \infty$ for any given $p$ satisfying $1 \le p \le \infty$. Then for any $x \in L^p(X)$, the following statements are equivalent*:

(1) $\{n^{-1} \sum_{k=1}^n T^k x,\ n \in N\}$ *converges in the $\|\cdot\|_p$-topology.*

(2) $\{n^{-1} \sum_{k=1}^n T^k x,\ n \in N\}$ *converges in the $(\varepsilon, \lambda)$-topology.*

(3) *$T$ has a fixed point in the $(\varepsilon, \lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$.*

*Proof* (1)⇒ (2) It is clear.

(2)⇒(3) Let

$$A_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

for any $n \in N$. Then there exists an $\bar{x} \in X$ such that $A_n x \to \bar{x}$ as $n \to \infty$ in the $(\varepsilon, \lambda)$-topology. Clearly, $\bar{x}$ belongs to the $(\varepsilon, \lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$, then, according to Lemma 3.5, it follows that $\bar{x} \in L^p(X)$.

It is easy to notice that $A_n x \in L^p(X)$ and

$$TA_n x - A_n x = \frac{1}{n}(T^{n+1}x - Tx)$$

for any $n \in N$, thus

$$\begin{aligned}\|TA_n x - A_n x\|_p &= \Big\|\frac{1}{n}(T^{n+1}x - Tx)\Big\|_p \\ &\le \frac{1}{n}(\|T^{n+1}x\|_p + \|Tx\|_p) \\ &\le \frac{2}{n}\Big(\sup_{m\in N} \|\!|T^m|\!\| \cdot \|x\|_p\Big) \\ &\to 0 \end{aligned} \tag{3.5}$$

as $n \to \infty$, hence $TA_n x - A_n x \to 0$ as $n \to \infty$ in the $\|\cdot\|_p$-topology. Furthermore, according to Lemma 3.2, $T$ is continuous in the $(\varepsilon, \lambda)$-topology. Consequently $T\bar{x} = \bar{x}$. Observing that $\bar{x}$, being the $(\varepsilon, \lambda)$-limit of $L^0$-convex combinations of $\{T^k x,\ k \in N\}$, is in the $(\varepsilon, \lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$.

(3)$\Rightarrow$(1) Let $\bar{x}$ be the fixed point. Since $T^k x = T^k \bar{x} + T^k(x - \bar{x}) = \bar{x} + T^k(x - \bar{x})$ for any $k \in N$, we have that

$$A_n x = \bar{x} + A_n(x - \bar{x})$$

for any $n \in N$, and it is enough to prove that $A_n(x - \bar{x})$ converges to $\theta$ as $n \to \infty$ in the $\|\cdot\|_p$-topology. Since $\bar{x}$ belongs to the $(\varepsilon, \lambda)$-closed $L^0$-convex hull of $\{T^k x,\ k \in N\}$, it follows from Lemma 3.5 that $x - \bar{x} \in \overline{\mathrm{range}(I - T)}^{\|\cdot\|_p}$. Thus for any $\varepsilon > 0$, we can choose a $y \in \mathrm{range}(I - T)$ such that $\|y - (x - \bar{x})\|_p < \varepsilon$. Since

$$A_n(I - T) = \frac{1}{n}(T - T^{n+1})$$

for any $n \in N$, it follows that $A_n x$ converges to $\theta$ as $n \to \infty$ in the $\|\cdot\|_p$-topology. Consequently, for the same $\varepsilon$ above, there exists some $N(\varepsilon) \in N$ such that

$$\begin{aligned}\|A_n(x - \bar{x})\|_p &\le \|A_n y\|_p + \|A_n(y - (x - \bar{x}))\|_p \\ &< \varepsilon + \Big(\sup_{m\in N} \|\!|T^m|\!\|\Big) \cdot \varepsilon \end{aligned} \tag{3.6}$$

as $n \ge N(\varepsilon)$, i.e. $A_n(x - \bar{x})$ converges to $\theta$ in the $\|\cdot\|_p$-topology.

This completes the proof. □

## 4 A Mean Ergodic Characterization

The central result of this section is Theorem 4.3, which establishes a characterization for a complete RN module to be mean ergodic. For the sake of clearness, the proof of Theorem 4.5 is divided into Theorem 4.1 and Theorem 4.2 below. First, we give the proof of Theorem 4.1 and it is just based on the analysis of module structure of complete RN modules that makes the Theorem 4.1 established. However, when we turn to the proof of Theorem 4.2, we are forced to deal with a difficult point with respect to the local property and the local mean ergodicity of a linear operator.

For the reader's convenience, a Banach space $B$ is said to be locally mean ergodic if any power-bounded linear operator on $B$ having the local property is mean ergodic. Clearly, if $B$ is mean ergodic, then $B$ must be locally mean ergodic.

**Theorem 4.1** *Let $(X, \|\cdot\|)$ be a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. If $L^p(X)$ is locally mean ergodic in the $\|\cdot\|_p$-topology for any given $p$ satisfying $1 \le p \le \infty$, then $X$ is mean ergodic in the $(\varepsilon, \lambda)$-topology.*

*Proof* Let $T$ be an arbitrary a.s. power-bounded random linear operator on $X$ and

$$\xi = \bigvee_{m \in N} \|T^m\|.$$

Then $\xi \in L^0_+(\mathcal{F})$. Let $E_i = [i-1 \le \xi < i]$ for each $i \in N$. Then $\{E_i,\ i \ge 1\}$ is a sequence of pairwise disjoint $\mathcal{F}$-measurable sets such that $\sum_{i=1}^{\infty} E_i = \Omega$. Let $T_i = I_{E_i} \cdot T$ for each $i \in N$. Then $T_i \in L(L^p(X))$ and $\sup_{m \in N} |||T_i^m||| < \infty$, i.e., $T_i$ is a power-bounded operator on the Banach space $L^p(X)$ for each $i \in N$. Furthermore, $T_i$ obviously has the local property for each $i \in N$ since $T$ is a module homomorphism on $X$. For any $x \in X$, let $F_j = [j-1 \le \|x\| < j]$ for each $j \in N$. Then $\{F_j,\ j \ge 1\}$ is a sequence of pairwise disjoint $\mathcal{F}$-measurable sets such that $\sum_{j=1}^{\infty} F_j = \Omega$. Let $x_j = I_{F_j} \cdot x$ for each $j \in N$. Then $x_j \in L^p(X)$ for the same $p$ above for each $j \in N$. Since $L^p(X)$ is locally mean ergodic in the $\|\cdot\|_p$-topology, it follows that for each $i \in N$, $T_i$ is always mean ergodic on $L^p(X)$. Namely, for each fixed $i, j \in N$, there exists a $y_{ij} \in L^p(X)$ such that $\frac{1}{n}\sum_{k=1}^{n} T_i^k x_j$ converges to $y_{ij}$ in the $\|\cdot\|_p$-topology as $n \to \infty$. Consequently, $\frac{1}{n}\sum_{k=1}^{n} T_i^k x_j$ converges to $y_{ij}$ in the $(\varepsilon, \lambda)$-topology as $n \to \infty$ for each $i, j \in N$. Observe that $y_{ij} = I_{E_i \cap F_j} \cdot y_{ij}$ for each $i, j \in N$ and we can suppose $P(E_i \cap F_j) > 0$ for each $i, j \in N$ (otherwise such an $E_i \cap F_j$ is automatically removed). Furthermore, since

$$\sum_{i,j=1}^{\infty} P(E_i \cap F_j) = P\bigg(\sum_{i,j=1}^{\infty} (E_i \cap F_j)\bigg) = P(\Omega) = 1,$$

we have that $\{\sum_{i=1}^{l}\sum_{j=1}^{m} y_{ij},\ (l, m) \in N \times N\}$ is a Cauchy net in the $(\varepsilon, \lambda)$-topology in $X$. Since $X$ is complete, it follows that there exists a $y$ in X such that $\{\sum_{i=1}^{l}\sum_{j=1}^{m} y_{ij},\ (l, m) \in N \times N\}$ converges to $y$ in the $(\varepsilon, \lambda)$-topology as $l, m \to \infty$. Since $T_i^k = (I_{E_i} \cdot T)^k = I_{E_i} \cdot T^k$, it follows that $\{\sum_{i=1}^{l}\sum_{j=1}^{m}(\frac{1}{n}\sum_{k=1}^{n} T_i^k x_j),\ (l, m) \in N \times N\}$ converges to $\frac{1}{n}\sum_{k=1}^{n} T^k x$ as $l, m \to \infty$ in the $(\varepsilon, \lambda)$-topology for each $n \in N$. Thus it is easy to check that $\frac{1}{n}\sum_{k=1}^{n} T^k x$ converges to $y$ as $n \to \infty$ in the $(\varepsilon, \lambda)$-topology, which shows that $X$ is mean ergodic in the $(\varepsilon, \lambda)$-topology.

This completes the proof. □

Now we can present Theorem 4.2 below.

**Theorem 4.2** *Let $(X, \|\cdot\|)$ be a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. If $X$ is mean ergodic in the $(\varepsilon, \lambda)$-topology, then $L^\infty(X)$ is locally mean ergodic in the $\|\cdot\|_\infty$-topology.*

*Proof* Let $T$ be an arbitrary power-bounded linear operator on $L^\infty(X)$ such that $T$ has the locally property. Then there exists an $M > 0$ such that $|||T^m||| \le M$ for any $m \in N$. It is clear that the linear operator $T : L^\infty(X) \to L^\infty(X)$ having the local property is continuous in the $\|\cdot\|_\infty$-topology, then, according to Lemma 3.2, it follows that $T : L^\infty(X) \to L^\infty(X)$ is also continuous in the $(\varepsilon, \lambda)$-topology. Furthermore, according to Lemma 3.4, $T$ can be extended to a continuous module homomorphism on $X$ (still denoted by $T$) in the $(\varepsilon, \lambda)$-topology. Moreover, one can observe that

$$\|T^m x\| \leq \|T^m x\|_\infty \leq M$$

for any $m \in N$ and $x \in \{x \in X \text{ and } \|x\| \leq 1\}$. Since the random closed unit ball on $X$ and the closed unit ball on $L^\infty(X)$ are the same, it follows that

$$\bigvee_{m \in N} \|T^m\| \leq M,$$

i.e., $\bigvee_{m\in N} \|T^m\| \in L_0^+(\mathcal{F})$, which shows that $T$ is an a.s. power-bounded random linear operator on $X$. Since $X$ is mean ergodic in the $(\varepsilon, \lambda)$-topology, it follows that $\{n^{-1} \sum_{k=1}^n T^k x,\ n \in N\}$ converges in the $(\varepsilon, \lambda)$-topology for any $x \in L^\infty(X) \subset X$. According to Theorem 3.6, the sequence $\{n^{-1} \sum_{k=1}^n T^k x,\ n \in N\}$ also converges in the $\|\cdot\|_\infty$-topology, which says that $L^\infty(X)$ is locally mean ergodic in the $\|\cdot\|_\infty$-topology.

This completes the proof. □

Based on Theorems 4.1 and 4.2 above, we can now present a characterization for a complete RN module to be mean ergodic below.

**Theorem 4.3** *Let $(X, \|\cdot\|)$ be a complete RN module over $K$ with base $(\Omega, \mathcal{F}, P)$. Then $X$ is mean ergodic in the $(\varepsilon, \lambda)$-topology if and only if $L^\infty(X)$ is locally mean ergodic in the $\|\cdot\|_\infty$-topology.*

**Remark 4.4** Theorem 4.3 establishes a bridge between a complete RN module $X$ and its generated Banach space $L^\infty(X)$ with respect to mean ergodicity of random linear operators, which will motivated us to adopt the properties of classical Banach spaces to study the further properties and structures of random normed modules.

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