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Super Weak Compactness and Uniform Eberlein Compacta

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Abstract We prove that a topological space is uniform Eberlein compact iff it is homeomorphic to a super weakly compact subset C of a Banach space such that the closed convex hull $\overline{\mathrm{co}}C$ of C is super weakly compact. We also show that a Banach space X is super weakly compactly generated iff the dual unit ball B_{X^*} of X^* in its weak star topology is affinely homeomorphic to a super weakly compactly convex subset of a Banach space.

Keywords Banach space, uniform Eberlein compactas, super weak compactness

MR(2010) Subject Classification 46B20, 46B50

1 Introduction

A topological space K is called uniform Eberlein compact (UEC) if K is homeomorphic to a weakly compact set in a super-reflexive space endowed with its weak topology (recall that a Banach space is super-reflexive if and only if it admits an equivalent uniformly convex norm [14] or equivalently the space admits an equivalent norm with power type modulus of convexity [21]). If the super-reflexive space in this definition is replaced with a general Banach space, then we speak of Eberlein compact (EC) introduced by Amir and Lindenstrauss [1]. The notion of a uniform Eberlein compact space was introduced by Benyamini and Starbird [6] and further studied by Argyros, Benyamini, Fabian, Farmaki, Rudin, etc (see e.g. [2, 5, 15]). They [5] showed that the class of uniform Eberlein compact spaces has some interesting stability

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properties such as: The continuous Hausdorff image of a UEC space is also UEC, and for any UEC space K the closed subspace of K is also UEC.

Recently a new class of compactness in Banach spaces, super weak compactness, has been introduced and studied by many authors. See, for instance, [8–10, 17, 22, 23]. Recall that a subset K of a Banach space X is said to be super weakly compact (SWC) if $K^{\mathcal{U}}$ is a weakly compact subset of $X^{\mathcal{U}}$ for any free ultrafiliter $\mathcal{U}$ on $\mathbb{N}$ (see [8, Definition 3.1] and [23, Definition 2.3]). And a Banach space X called super weakly compact generated (super WCG) if there is an SWC convex set $K \subset X$ such that X is the closed linear span of K, i.e., $\overline{\mathrm{span}} K = X$. The class of SWC sets lies strictly between compact sets and weakly compact sets and shares many good properties of those classes. For example, the SWC set is the exactly localized setting of super-reflexive Banach spaces, since a Banach space X is super-reflexive if and only if every bounded weakly closed subset of the space X is SWC. Some determining conditions on a subset C in a general Banach space X so that C is SWC have been given in [9, 10, 17, 22, 23]. In particular, every weakly compact subset in the space $L_1[0, 1]$ is SWC.

The purpose of this note is to study the following three closed related topics:

(1) The representation of SWC set in $c_0(\Gamma)$ for some set Γ.

(2) The relationship of super weak compactness and uniform Eberlein compacta.

(3) The characterization of super WCG spaces.

In the first topic, we show that the well-known representation theorem of weakly compact sets in $c_0(\Gamma)$ due to Amir and Lindenstrauss [1] is still valid for SWC sets.

Theorem 1.1 *Assume that K is an* SWC *subset of a Banach space X. Then K is affinely homeomorphic to an* SWC *subset of $c_0(\Gamma)$ for some set Γ.*

In Section 3, we focus on the second topic mentioned above and prove:

Theorem 1.2 *Let K be a compact Hausdorff space. Then K is uniform Eberlein compact if and only if it is homeomorphic to an* SWC *subset B of a Banach space such that the closed convex hull $\overline{\mathrm{co}} B$ of B is* SWC.

Let us review previous work related to this topic. First Benyamini and Stabird [6] introduced a property for a bounded subset of $c_0(\Gamma)$, called "*l_2-type property*" by us in this paper (see Definition 3.1 below), to study UEC spaces. They proved that a topological space is UEC iff it is homeomorphic to some bounded subset of $c_0(\Gamma)$ with such a property. In this present paper we will prove that every weakly closed bounded subset of $c_0(\Gamma)$ with such a property is SWC and so is its closed convex hull. On the other hand, recently Raja [22] introduced the finite-slice-index property for a bounded closed convex set in Banach spaces and proved that every bounded closed convex set with such a property is UEC. This, in fact, gives that every SWC convex set of Banach spaces is UEC since the two notions of super weak compactness and the finite-slice-index property are equivalent for a bounded closed convex subset of Banach spaces (see [10, 23]). But it needs to mention that there exist many examples of UEC in Banach spaces which are not SWC. See, for example, Fabian et al.'s example [18, Theorem 12].

Based on Benyamini and Stabird's characterization of UEC and Raja's result, we complete the proof of Theorem 3.8. This result provides a new criterion for recognizing UEC spaces.

In Section 4 we give a characterization of super WCG spaces as follows.

Theorem 1.3 *A Banach space X is super* WCG *if and only if the unit ball B_{X^*} of X^* in its weak star topology is affinely homeomorphic to an* SWC *convex subset of a Banach space.*

For an abstract set Γ, $c_0(\Gamma)$ will denote the Banach space of all scalar valued functions f on Γ which vanish at infinity (i.e., for which $\{\gamma \in \Gamma : \ |f(\gamma)| > \varepsilon\}$ is finite for every $\varepsilon > 0$) with the maximum norm $\|f\|_\infty = \max\{|f(\gamma)| : \gamma \in \Gamma\}$. The weak topology on a weakly compact subset of $c_0(\Gamma)$ is exactly the topology of point-wise convergence. By $l_p(\Gamma)$, $1 \le p < \infty$, we shall denote the space of all scalar-valued functions f on Γ satisfying $\sum_{\gamma\in\Gamma} |f(\gamma)|^p < \infty$, endowed with the norm $\|f\| = (\sum_{\gamma\in\Gamma} |f(\gamma)|^p)^{\frac{1}{p}}$. For a Banach space E, we denote by E^* its dual, and by B_{E^*} the unit ball of E^* endowed with the w^*-topology. The cardinality of a set A will be denoted by $^\sharp A$. All Banach spaces throughout the paper are supposed to be real. A standard topological argument will be used repeatedly in this paper: If M is a compact Hausdorff space and N is a Hausdorff space, then any one-to-one continuous map from M onto N is a homeomorphism.

2 The Representation of SWC Sets in $c_0(\Gamma)$

The possibility of representing every weakly compact set as a subset of $c_0(\Gamma)$ was conjectured in [11]. Later Amir and Lindenstrauss [1] proved the conjecture by applying the celebrated structure theorem of weakly compactly generated (WCG) spaces: For each WCG space there is a continuous one-to-one mapping $T : X \to c_0(\Gamma)$ for some set Γ. Recall that a Banach space X called WCG if there is a WC set $K \subset X$ such that X is the closed linear span of K.

Theorem 2.1 ([1, Theorem 1]) *Assume that K is a weakly compact subset of a Banach space. Then K is affinely homeomorphic to a weakly compact subset of $c_0(\Gamma)$ for some set Γ.*

We next show that the result is still valid for SWC sets.

Recall that for a nonempty set Ω, a family $\mathcal{U}$ of subsets of Ω is said to be free ultrafilter if i) $\emptyset \notin \mathcal{U}$, and $\cap\{U \in \mathcal{U}\} = \emptyset$; ii) $U, V \in \mathcal{U} \Rightarrow U \cap V \in \mathcal{U}$; iii) $U \in \mathcal{U}$ and $U \subset V \subset \Omega$ entail $V \in \mathcal{U}$; and (4) $U \subset \Omega \Longrightarrow$ either $U \in \mathcal{U}$, or $\Omega \backslash U \in \mathcal{U}$. Let K be a Hausdorff topological space. A map $f : \Omega \to K$ is said to be $\mathcal{U}$-convergent to some $k \in K$ if for every neighborhood U of k, we have $f^{-1}(U) \in \mathcal{U}$. In this case, we write $\lim_{\mathcal{U}} f = k$.

We will also need the definition of an ultraproduct (ultrapower) of Banach spaces. For a nonempty set Ω, let $(X_\omega : \omega \in \Omega)$ be a collection of Banach spaces. Then their ultraproduct is defined by

$$\prod_{\mathcal{U}} X_\omega = \Big(\bigoplus_{\omega\in\Omega} X_\omega\Big)_{\ell_\infty} \Big/ \Big\{(x_\omega) : \lim_{\mathcal{U}} \|x_\omega\| = 0\Big\}. \tag{2.1}$$

$\lim_{\mathcal{U}} \|x_\omega\| = 0$ means for all $\varepsilon > 0$, $\{\omega \in \Omega : \|x_\omega\| < \varepsilon\} \in \mathcal{U}$. Please note that the ultraproduct is a quotient of the ℓ_∞-sum of (X_ω), so its elements are classes of the respective equivalences relation, not the generalized sequences itself. We will use in the sequel the notations $[(x_\omega)]$ to denote the equivalence class of (x_ω). Thus, for a collection $(A_\omega \subset X_\omega : \omega \in \Omega)$ of subsets, its ultraproduct is

$$\prod_{\mathcal{U}} A_\omega = \Big\{[(x_\omega)] : (x_\omega) \in \Big(\bigoplus_{\omega\in\Omega} X_\omega\Big)_{\ell_\infty} : x_\omega \in A_\omega \text{ for } \omega \in \Omega\Big\}. \tag{2.2}$$

In particular, if $X_\omega = X$, and $A_\omega = A \subset X$ for all $\omega \in \Omega$, then we denote by $A^{\mathcal{U}} = \prod_{\mathcal{U}} A$, the $\mathcal{U}$-ultrapower of A.

Definition 2.2 ([23, Definition 2.3]) *A subset K of a Banach space X is said to be super weakly compact* (SWC) *if $K^{\mathcal{U}}$ is a weakly compact subset of $X^{\mathcal{U}}$ for any free ultrafiliter $\mathcal{U}$ on $\mathbb{N}$.*

The following result is from [8]. For the sake of completeness, we here give its proof.

Lemma 2.3 *Assume that $T : X \longrightarrow Y$ is a continuous linear operator between Banach spaces X and Y. If A is an* SWC *subset of X, then TA is also an* SWC *subset of Y.*

Proof Given an arbitrary free ultrafiliter $\mathcal{U}$ on $\mathbb{N}$. We consider the extension operator $T^{\mathcal{U}}$ of T defined by $T^{\mathcal{U}}([(x_\omega)]) = [(Tx_\omega)]$ for every $[(x_\omega)] \in X^{\mathcal{U}}$. Then $T^{\mathcal{U}}$ is a bounded linear operator from $X^{\mathcal{U}}$ to $Y^{\mathcal{U}}$ such that $T^{\mathcal{U}}(A^{\mathcal{U}}) = (TA)^{\mathcal{U}}$. By the definition of SWC sets, it follows that $A^{\mathcal{U}}$ is weakly compact in $X^{\mathcal{U}}$ and so $(TA)^{\mathcal{U}}$ is also weakly compact in $Y^{\mathcal{U}}$. Thus TA is SWC in Y.

Theorem 2.4 *Assume that K is an* SWC *subset of a Banach space X. Then K is affinely homeomorphic to an* SWC *subset of $c_0(\Gamma)$ for some set Γ.*

Proof Let $X_K = \overline{\mathrm{span}} K$. Then X_K is WCG. By Amir and Lindenstrauss theorem there is a continuous one-to-one mapping T from X_K into $c_0(\Gamma)$ for some Γ, and so the set TK is SWC in $c_0(\Gamma)$ from Lemma 2.3. Note that T is w-w-continuous from X_K to $c_0(\Gamma)$ and that the weak topology on X_K coincides with that from X. Thus T is a linear homeomorphism of K in its weak topology as a subset of X, onto the SWC subset TK of $c_0(\Gamma)$ in its weak topology.

3 The Relationship Between SWC Sets and UEC Spaces

In this section, we aim to prove Theorem 3.8. To do so, we need to make some preparation. The following property is due to Benyamini–Stabird [6], we will call it "*l_2-type*" for convenience.

Definition 3.1 *Let Γ be an arbitrary index set. A bounded subset K of $c_0(\Gamma)$ is said to have l_2-type property provided for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that*

$$^\sharp\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} < n_\varepsilon, \quad \textit{for all } k \in K. \tag{3.1}$$

As a result, this property provides a characterization of UEC due to Benyamini and Stabird [6] (see also [18, Theorem 12.16]).

Theorem 3.2 (Benyamini, Starbird [6]) *Let K be a compact Hausdorff space. Then the following conditions are equivalent.*

(i) *K is* UEC.

(ii) *K is homeomorphic to a weakly compact subset $\widetilde{K}$ of $c_0(\Gamma)$ with l_2-type property.*

The l_2-type property also plays an important role in this section and we give some fundamental properties as follows.

Example 3.3 Let Γ be an arbitrary index set. Then

i) Every bounded subset of $l_p(\Gamma)$ has l_2-type property, where $1 \le p < \infty$.

ii) Every bounded subset of Hilbet spaces has l_2-type property.

iii) The set $\mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\}$ has l_2-type property in $c_0(\Gamma)$. Here the set $\{e_\gamma : \gamma \in \Gamma\} \subset c_0(\Gamma)$ is such that the only nonzero coordinates of e_γ is 1 at the γ-th place.

Proof The statements (i) and (ii) are easy to check. To prove (iii) we first observe that if $k \in \mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\}$, there exist $\{x_j\}_{j=1}^n \subset \{\pm e_\gamma : \gamma \in \Gamma\}$ and $\{\alpha_j\}_{j=1}^n \subset [0, 1]$ with

$\sum_{j=1}^{n} \alpha_j = 1$ such that $k = \sum_{j=1}^{n} \alpha_j x_j$. Given $\varepsilon > 0$, it follows that

$$^\sharp\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} =^\sharp \{j : \alpha_j > \varepsilon\} \leq \left[\frac{1}{\varepsilon}\right] + 1, \tag{3.2}$$

where $[a]$ denotes the integral function of the number $a \in \mathbb{R}$. Therefore the set $\mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\}$ has l_2-type property in $c_0(\Gamma)$.

Proposition 3.4 *Let Γ be an arbitrary index set and $K \subset c_0(\Gamma)$ be a bounded set. Then the following conditions are equivalent.*

(i) *K has l_2-type property.*

(ii) *For every $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that*

$$K \subset M_\varepsilon \mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\} + \varepsilon B_{c_0(\Gamma)}. \tag{3.3}$$

Proof Assume that K has l_2-type property in $c_0(\Gamma)$. Given $\varepsilon > 0$, we choose $n_\varepsilon \in \mathbb{N}$ such that $^\sharp\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} < n_\varepsilon$ for all $k \in K$. Let $M = \sup\{\|k\| : k \in K\} < \infty$. Then it is easy to see

$$K \subset M_\varepsilon \overline{\mathrm{co}}\{\pm e_\gamma : \gamma \in \Gamma\} + \varepsilon B_{c_0(\Gamma)}. \tag{3.4}$$

Here $M_\varepsilon = M n_\varepsilon$.

Conversely, given $\varepsilon > 0$ we assume $M_\varepsilon > 0$ such that

$$K \subset M_\varepsilon \mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\} + \frac{\varepsilon}{2} B_{c_0(\Gamma)}. \tag{3.5}$$

Fix now any $k \in K$. Then there exist $k_1 \in \mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\}$ and $k_2 \in B_{c_0(\Gamma)}$ such that $k = M_\varepsilon k_1 + \frac{\varepsilon}{2} k_2$. Thus

$$\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} \subset \left\{\gamma \in \Gamma : |k_\varepsilon(\gamma)| > \frac{\varepsilon}{2M_\varepsilon}\right\}. \tag{3.6}$$

Since $\mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\}$ has l_2-type property from Example 3.3, for $\frac{\varepsilon}{2M_\varepsilon}$ we may choose $N_\varepsilon \in \mathbb{N}$ such that $^\sharp\{\gamma \in \Gamma : |k_\varepsilon(\gamma)| > \frac{\varepsilon}{2M_\varepsilon}\} < N_\varepsilon$. Thus

$$^\sharp\{\gamma \in \Gamma : |k(\gamma)| > \varepsilon\} < N_\varepsilon, \tag{3.7}$$

and we complete the proof.

The following is an analogue of Grothendieck's lemma for SWC sets.

Lemma 3.5 ([9, Lemma 4.5]) *A nonempty closed convex set C of a Banach space X is* SWC *(if and only) if for every $\varepsilon > 0$, there exists an* SWC *convex set C_ε in X such that $C \subset C_\varepsilon + \varepsilon B_X$.*

Theorem 3.6 *Let Γ be an arbitrary index set. Assume that the set $K \subset c_0(\Gamma)$ is weakly closed and has l_2-type property. Then K is* SWC *and its closed convex hull $\overline{\mathrm{co}}K$ is also* SWC.

Proof $l_2(\Gamma)$ is Hilbert space. Thus the closed unit ball $B_{l_2(\Gamma)}$ of $l_2(\Gamma)$ is SWC in $l_2(\Gamma)$. The identity map $j : l_2(\Gamma) \to c_0(\Gamma)$ is continuous, and so $j(B_{l_2(\Gamma)})$ is SWC in $c_0(\Gamma)$. Hence the set $\overline{\mathrm{co}}\{\pm e_\gamma : \gamma \in \Gamma\}$ is also SWC in $c_0(\Gamma)$ due to $B \subset j(B_{l_2(\Gamma)})$. Since K has l_2-type property, by Proposition 3.4, for each $\varepsilon > 0$, there is $M_\varepsilon > 0$ such that

$$K \subset M_\varepsilon \mathrm{co}\{\pm e_\gamma : \gamma \in \Gamma\} + \varepsilon B_{c_0(\Gamma)}, \tag{3.8}$$

and so

$$\overline{\mathrm{co}}K \subset M n_\varepsilon B + \varepsilon B_{c_0(\Gamma)}. \tag{3.9}$$

Since B is SWC convex set, applying Lemma 3.5, we deduce that $\overline{\text{co}}K$ is SWC. Therefore the weakly closed subset K of $\overline{\text{co}}K$ is also SWC.

The result (see also [8]) together with Theorem 3.2 immediately implies:

Corollary 3.7 *Every* UEC *is homeomorphic to an* SWC *subset of* $c_0(\Gamma)$ *for some* Γ.

Theorem 3.8 *Let K be a compact Hausdorff space. Then K is uniform Eberlein compact if and only if it is homeomorphic to an* SWC *subset B of a Banach space such that $\overline{\text{co}}B$ is* SWC.

Proof "If part". Assume that K is homeomorphic to an SWC set B under the homeomorphic map f and that $\overline{\text{co}}B$ is SWC. Thus $\overline{\text{co}}B$ is UEC due to [23, Theorem 1.3]. Since every closed subspace of UEC is still UEC, we obtain that B is again UEC. By [5, Theorem 3.1], the image $f^{-1}(B) = K$ is UEC under the continuous map f^{-1}.

"Only if part". Assume that K is UEC. Then K is homemorphic to a weakly compact set C with l_2-type property in $c_0(\Gamma)$. By Theorem 3.6, C and $\overline{\text{co}}C$ are SWC in $c_0(\Gamma)$ and we complete the proof.

Remark 3.9 Theorem 3.8 should be compared with the definition of Eberlein compact (EC) spaces introduced by Amir and Lindenstrauss: A compact topological space K is EC if and only if it is homeomorphic to a weakly compact subset B of a Banach space.

Remark 3.10 Observe that in Theorem 3.8 we have used the hypothesis of super weak compactness for the closed convex hull of some known SWC set. Actually, we do not know that if the closed convex hull of SWC is again SWC, that is, a sort of Kerin–Smulýan theorem [13, p. 434] for weakly compact sets.

4 Super WCG Spaces

The WCG spaces have been introduced by Amir and Lindenstrauss [1] and extensively studied by people (see, for instance, [1, 4, 7, 12, 17, 19, 20]). Some classical characterizations of WCG spaces have also been established. For example, Lindenstrauss [19, Theorem 3.1] showed that a Banach space is WCG if and only if the unit ball B_{X^*} of X^* in its weak star topology is affinely homeomorphic to a weakly compact subset of some Banach space (see also [24, Corollary 3.4]). In this section, we mainly aim to show that an analogue of Lindenstrauss's theorem for super WCG spaces is still valid.

We begin by recalling the following notion.

Definition 4.1 ([23]) *A Banach space X is said to be super weakly compactly generated* (*super* WCG *for short*) *if there is an* SWC *convex set $K \subset X$ such that $\overline{\text{span}}K = X$.*

Equivalently, a Banach space is super WCG if there exists a Banach space Y and a super weakly compact operator $T : Y \to X$ such that TY is dense in X. Recall that a bounded linear operator $T : E \to F$ between Banach spaces E and F is called super weakly compact if the closed hull $\overline{TB_E}$ of TB_E is SWC in F. We also know that $T : E \to F$ is super weakly compact if and only if its dual operator $T^* : F^* \to E^*$ is super weakly compact. See, for instance, [3, 22, 23]. Some more characterizations of super WCG can be found in [3, 16, 17, 23].

As examples, obviously every super reflexive space is super WCG. Another natural example of super WCG spaces is that the space $c_0(\Gamma)$ is super WCG for every set Γ (indeed, the identity map from $l_2(\Gamma)$ into $c_0(\Gamma)$ has a dense range).

The following Kerin–Smulýan theorem will be used later.

Lemma 4.2 (Kerin–Smulýan) *Let X be a Banach space and $F \in X^{**}$. If F is continuous in the w^*-topology, then $F \in X$.*

We also need a revised version of the well-known Davis–Figiel–Johnson–Pelczynski factorization lemma [12].

Lemma 4.3 *Let X be a Banach space and assume that W is a convex, symmetric,* SWC *subset of X. For each n, let $U_n = 2^n W + 2^{-n} B_X$ and denote by $\|\cdot\|_n$ the gauge of U_n. For $x \in X$, let*

$$|||x||| = \left(\sum_{n=1}^{\infty} \|x\|^2 \right)^{\frac{1}{2}}$$

and let $Y = \{x \in X : |||x||| < \infty\}$. Denote by j the inclusion map from Y into X. Then

(i) *$(Y, |||\cdot|||)$ is a reflexive Banach space and jY is a dense subset in X.*

(ii) *$j : Y \to X$ is an* SWC *operator.*

(iii) *$j^* : X^* \to Y^*$ is one-to-one and j^* is continuous with respect to the weak star topology (w^*-topology) of X^* and the weak topology (w-topology) of Y^*.*

(iv) *The $|||\cdot|||$-topology and $\|\cdot\|$-topology coincide when restricted to the set W.*

(v) *The closed unit ball B_Y in its weak topology is* UEC.

Proof The statement (i) is from the Davis–Figiel–Johnson–Pelczynski factorization lemma (see [12]). The assertions (ii) and (iv) are from [9, Lemma 4.7] and (v) is from [22, Theorem 1.3]. It remains to prove (iii). Since jY is a dense subset in X, then j^* is one-to-one. By the reflexivity of Y, it suffices to prove that j^* is w^*-w^* continuous. Indeed, let $\{x_\alpha^*\}_{\alpha \in I} \subset B_{X^*}$ be a net and it is w^*-convergently to x_0^* (obviously $x_0^* \in B_{X^*}$). This means that for each $y \in Y$ and so $jy \in X$, we have that

$$\langle j^* x_\alpha^*, y \rangle = \langle x_\alpha^*, jy \rangle \to \langle x_0^*, jy \rangle = \langle j^* x_0^*, y \rangle. \tag{4.1}$$

This shows that the net $\{j^* x_\alpha^*\}_{\alpha \in I}$ is w^*-convergently to x_0^* in Y^*, and hence j^* is w^*-w^* continuous.

Theorem 4.4 *A Banach space X is super* WCG *if and only if the unit ball B_{X^*} of X^* in its weak star topology is affinely homeomorphic to an* SWC *convex subset of a Banach space.*

Proof "If part". Suppose that K is an SWC convex set of a Banach space Y and $T : B_{X^*} \to K$ is an affine homeomorphism with respect to the w^*-topology on B_{X^*} and the w-topology on K. We define the canonical extension $\widetilde{T}$ of T is defined by $\widetilde{T}(x^*) = \|x^*\| T(\frac{x^*}{\|x^*\|})$ if $x^* \neq 0$ and $\widetilde{T}(0) = 0$. Then $\widetilde{T} : X^* \to Y$ is one-to-one linear map with $\widetilde{T}(B_{X^*}) = K$. Obviously, $\widetilde{T}$ is still continuous from the w^*-topology on B_{X^*} to the w-topology on Y. Thus $\widetilde{T}$ is a super weakly compact operator, and hence so is $\widetilde{T}^* : Y^* \to X^{**}$. This immediately implies that $\widetilde{T}^*(B_{Y^*})$ is an SWC subset of X^{**}. We next claim that $\widetilde{T}^*(B_{Y^*})$ is contained in X. By Theorem 4.2, it is enough to prove that $\widetilde{T}^* y^*$ is continuous in the w^*-topology on X^* for every $y^* \in B_{Y^*}$. Indeed, given $y^* \in B_{Y^*}$ and let $\{x_\alpha^*\}_{\alpha \in I}$ be a bounded net in X^* such that $\{x_\alpha^*\}_{\alpha \in I}$ converges to 0 in the w^*-topology on X^*. The fact that $\widetilde{T}$ is continuous from the w^*-topology on B_{X^*} to the w-topology on Y implies that $\widetilde{T} x_\alpha^*$ converges to 0 in the w-topology on Y. Thus $\langle \widetilde{T}^* y^*, x_\alpha^* \rangle = \langle y^*, \widetilde{T} x_\alpha^* \rangle \to 0$, and so $\widetilde{T}^* y^*$ is w^*-continuous. This completes our claim. Note that $\widetilde{T}^*(B_{Y^*})$ is total with respect to X^* since $\widetilde{T}$ is one-one. Recall that a subset M of the space X^{**} is called total with respect to X^* if for every $0 \neq f \in X^*$ there exists $F \in M$ such that $F(f) \neq 0$. From the Hahn–Banach theorem it follows that $\overline{\text{span}}\, \widetilde{T}^*(B_{Y^*}) = X$. Since $\widetilde{T}^*(B_{Y^*})$ is SWC in X, and so X is a super WCG space.

"Only if part". Assume that X is generated by an SWC convex subset K of X. Then the set $W = \overline{\text{co}}(K \cup -K)$ is still SWC in X from [9, Corollary 3.10]. Lemma 4.3 ensures that from the set W we obtain a reflexive Banach space Y and the inclusion map $j : Y \to X$ such that

(a) $j^* : X^* \to Y^*$ is one-to-one and is a super weakly compact operator; and

(b) j^* is w^*-w continuous with respect to the w^*-topology of X^* and w-topology of Y^*.

Note that (B_{X^*}, w^*) in the w^*-topology of X^* is a compactly Hausdorff space and that the w-topology in Y^* is Hausdorff. This fact together with (a) and (b) above implies that $j^* : (B_{X^*}, w^*) \to j^*((B_{X^*}, w^*))$ is an (affine) homeomorphism. By again (a) the set $j^*(B_{X^*}, w^*)$ is SWC in Y^*, and we complete the proof.

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