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On the Darboux Integrability of the Hindmarsh–Rose Burster

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Abstract We study the Hindmarsh–Rose burster which can be described by the differential system

$$\dot{x} = y - x^3 + bx^2 + I - z, \quad \dot{y} = 1 - 5x^2 - y, \quad \dot{z} = \mu(s(x - x_0) - z),$$

where b, I, μ, s, x_0 are parameters. We characterize all its invariant algebraic surfaces and all its exponential factors for all values of the parameters. We also characterize its Darboux integrability in function of the parameters. These characterizations allow to study the global dynamics of the system when such invariant algebraic surfaces exist.

Keywords Polynomial integrability, rational integrability, Darboux polynomials, Darboux first integrals, invariant algebraic surfaces, exponential factors, Hindmarsh–Rose burster

MR(2010) Subject Classification 34C05, 34A34, 34C14

1 Introduction and Statement of Main Results

The existence of one or two independent first integrals in a differential system in dimension 3 reduces the analysis of the dynamics of this system in 1 or 2 dimensions. This justifies the study of the integrability (the existence of first integrals) of a differential system like the Hindmarsh–Rose differential system. The Darboux integrability essentially captures the elementary first integrals using the invariant algebraic surfaces. We prove that the unique invariant algebraic surface of the Hindmarsh–Rose burster differential system is $z = 0$ when the parameter $\mu = 0$, or $s = 0$ and $\mu \neq 0$ with cofactors zero and non-zero, respectively. We also have shown that the Hindmarsh–Rose burster differential system has a Darboux first integral only when the parameter $\mu = 0$. The fact that the Hindmarsh–Rose burster differential system exhibits almost no Darboux integrability shows that it is a candidate to have complex dynamics, in general not easy to study.

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One of the most studied three-dimensional ordinary differential systems that appears as a reduction of the conductance-based in the Hodgkin–Huxley model for neural spiking [9] is the Hindmarsh–Rose model [8] which can be written as

$$\begin{aligned}\dot{x} &= y - x^3 + bx^2 + I - z,\\ \dot{y} &= 1 - 5x^2 - y,\\ \dot{z} &= \mu(s(x - x_0) - z),\end{aligned} \tag{1.1}$$

where b, I, μ, s, x_0 are parameters and the dot indicates derivative with respect to the time t.

The success of system (1.1) comes, first from its simplicity since it is just a differential system in $\mathbb{R}^3$ with a polynomial nonlinearity containing only five parameters, and second its ability to qualitatively capture the three main dynamical behaviors displayed by real neurons, namely quiescence, tonic spiking and bursting. Many papers have investigated the dynamics which takes place in system (1.1) when we vary one or more of its parameters, see for instance [1, 6, 7, 10–12, 17, 19, 21–27]. Among these amount of papers, none of them study its integrability.

The existence of a first integral in a differential system defined in dimension three reduces the analysis of this system in one dimension when we fix the value of the first integral. Of course, this simplifies strongly the analysis of the dynamics of such systems. Moreover, if a system in dimension three has two independent first integrals then fixing these two first integrals we obtain the curves where the solutions live. These arguments justify the study of the integrability (the existence of first integrals) of a differential system like the Hindmarsh–Rose differential system.

The Darboux integrability essentially captures the elementary first integrals, i.e. the first integrals given by elementary functions, which are the ones that roughly speaking can be obtained by composition of exponential, trigonometric, logarithmic and polynomial functions, see for more details about the Darboux integrability Chapter 8 of [5], the references quoted there, and in special see the references [18, 20, 28].

The Darboux integrability in dimension three is based in the existence of invariant algebraic surfaces $f(x, y, z) = 0$, where $f(x, y, z)$ is a polynomial, called a Darboux polynomial. A sufficient number of such polynomials taking into account their multiplicity (through the so–called exponential factors) force the existence of first integrals.

The main aim of this paper is to study the existence of first integrals of system (1.1). The vector field X associated to system (1.1) is

$$X = (y - x^3 + bx^2 + I - z)\frac{\partial}{\partial x} + (1 - 5x^2 - y)\frac{\partial}{\partial y} + \mu(s(x - x_0) - z)\frac{\partial}{\partial z}.$$

Let $U \subset \mathbb{R}^3$ be an open set. We say that the non–constant function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a *first integral* of the vector field X on U, if $F(x(t), y(t), z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t))$ of X is defined on U. Clearly f is a first integral of X on U if and only if $Xf = 0$ on U.

The aim of this paper is to study the existence of first integrals of system (1.1) that can be described by functions of Darboux type (see (1.5)). Note that one of the main tools for studying the dynamics of the differential system (1.1) is to know the existence of first integrals for some values of the parameters b, I, μ, s, x_0. In general, for a given differential system it is difficult to determine the existence or nonexistence of first integrals.

First we study the existence of first integrals of system (1.1) given by polynomials.

A *polynomial first integral* $f = f(x, y, z)$ of system (1.1) is a polynomial in the variables x, y and z such that

$$(y - x^3 + bx^2 + I - z)\frac{\partial f}{\partial x} + (1 - 5x^2 - y)\frac{\partial f}{\partial y} + \mu(s(x - x_0) - z)\frac{\partial f}{\partial z} = 0. \tag{1.2}$$

The first main result is the following.

Theorem 1.1 *System* (1.1) *with $\mu \neq 0$ has no polynomial first integrals. If $\mu = 0$, then the unique polynomial first integrals are polynomials in the variable z.*

The proof of Theorem 1.1 is given in Section 2.

A *rational first integral* of system (1.1) is a rational function f satisfying (1.2).

Theorem 1.2 *System* (1.1) *with $\mu \neq 0$ has no rational first integrals. If $\mu = 0$, then the unique rational first integrals are rational functions in the variable z.*

The proof of Theorem 1.2 is given in Section 4.

For proving Theorem 1.2 we will use the Darboux theory of integrability. The Darboux theory of integrability in dimension 3 is based on the existence of invariant algebraic surfaces (or Darboux polynomials). For more details see [3–5, 14]. This theory is one of the best theories for studying the existence of first integrals for the polynomial differential systems.

A *Darboux polynomial* of system (1.1) is a polynomial $f \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ such that

$$(y - x^3 + bx^2 + I - z)\frac{\partial f}{\partial x} + (1 - 5x^2 - y)\frac{\partial f}{\partial y} + \mu(s(x - x_0) - z)\frac{\partial f}{\partial z} = Kf, \tag{1.3}$$

for some polynomial

$$K = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 xz + \alpha_7 y^2 + \alpha_8 yz + \alpha_9 z^2, \tag{1.4}$$

called the *cofactor* of f. Note that $f = 0$ is an *invariant algebraic surface* for the flow of system (1.1) and a *polynomial first integral* is a Darboux polynomial with zero cofactor.

Theorem 1.3 *The unique irreducible Darboux polynomial with non-zero cofactor for system* (1.1) *exists for $s = 0$ and $\mu \neq 0$, and in this case the Darboux polynomial is z and the cofactor is $-\mu$.*

The proof of Theorem 1.3 is given in Section 3.

An *exponential factor* $F(x, y, z)$ of system (1.1) is a function of the form $F = \exp(g_0/g_1) \notin \mathbb{C}$ with $g_0, g_1 \in \mathbb{C}[x, y, z]$ coprime satisfying that

$$(y - x^3 + bx^2 + I - z)\frac{\partial F}{\partial x} + (1 - 5x^2 - y)\frac{\partial F}{\partial y} + \mu(s(x - x_0) - z)\frac{\partial F}{\partial z} = LF,$$

for some polynomial $L = L(x, y, z)$ of degree at most 2, called the *cofactor* of F.

Theorem 1.4 *The following statements hold for system* (1.1).

(i) *If $\mu \neq 0$ system* (1.1) *has the exponential factors e^y, e^z, e^{z^2} with cofactors $1 - 5x^2 - y$, $\mu(s(x - x_0) - z)$ and $2\mu z(s(x - x_0) - z)$, respectively; and also exponential of linear combinations of y, z and z^2.*

(ii) *If $\mu = 0$ system* (1.1) *has the exponential factors e^y, $\mathrm{e}^{P(z)/Q(z)}$ where P, Q are polynomials with cofactors $1 - 5x^2 - y$ and 0, respectively and also exponential of linear combinations of all the exponents in the previous exponential factors.*

The proof of Theorem 1.4 is given in Section 5.

A *Darboux first integral* G of system (1.1) is a first integral of the form

$$G = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}, \tag{1.5}$$

where $f_1, \ldots, f_p$ are Darboux polynomials and $F_1, \ldots, F_q$ are exponential factors and $\lambda_j, \mu_k \in \mathbb{C}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$.

Theorem 1.5 *System* (1.1) *with* $\mu \neq 0$ *has no Darboux first integrals. If* $\mu = 0$*, then the unique Darboux first integrals are Darboux functions in the variable* z*.*

The proof of Theorem 1.5 is given in Section 6.

2 Polynomial First Integrals: Proof of Theorem 1.1

Let f be a polynomial first integral of system (1.1). Without loss of generality we can assume that it has no constant term. Then f satisfies (1.2). We write f as $f = \sum_{j=0}^{n} f_j(x, y, z)$ where each f_j is a homogeneous polynomial of degree j in each variables x, y and z. We can assume that $f_n \neq 0$ with $n > 0$. We have that the terms of degree $n + 2$ in (1.2) satisfy

$$-x^3 \frac{\partial f_n}{\partial x} = 0, \quad \text{that is} \quad f_n = f_n(y, z).$$

Computing the terms of degree $n + 1$ in (1.2) we get

$$-x^3 \frac{\partial f_{n-1}}{\partial x} - 5x^2 \frac{\partial f_n}{\partial y} = 0, \quad \text{that is} \quad -x \frac{\partial f_{n-1}}{\partial x} = 5 \frac{\partial f_n}{\partial y}.$$

Since f_n do not depend on x we must have $\frac{\partial f_n}{\partial y} = 0$ and $\frac{\partial f_{n-1}}{\partial x} = 0$, that is $f_n = f_n(z)$ and $f_{n-1} = f_{n-1}(y, z)$.

Now computing the terms of degree n in (1.2) we obtain

$$-x^3 \frac{\partial f_{n-2}}{\partial x} - 5x^2 \frac{\partial f_{n-1}}{\partial y} + \mu(sx - z) \frac{df_n}{dz} = 0.$$

Note that if $\mu \neq 0$ then $\mu(sx - z)$ must be divisible by x^2 which is not possible and thus $\frac{df_n}{dz} = 0$ which yields f_n is a constant, in contradiction with the fact that f is a first integral. Hence $\mu = 0$. In this case proceeding as above, after simplifying by x^2 and taking into account that f_{n-1} does not depend on x we get that $\frac{\partial f_{n-1}}{\partial y} = 0$ and $\frac{\partial f_{n-2}}{\partial x} = 0$, that is $f_{n-1} = f_{n-1}(z)$ and $f_{n-2} = f_{n-2}(y, z)$. Proceeding inductively we conclude that $f = \sum_{j=0}^{n} f_j(z)$, that is, f is a polynomial in the variable z. This concludes the proof of Theorem 1.1.

3 Darboux Polynomials with Non-zero Cofactor: Proof of Theorem 1.3

We consider a Darboux polynomial f with non-zero cofactor K as in (1.4). We denote by $\mathbb{N}$ the set of all positive integers.

We first prove three preliminary results.

Lemma 3.1 $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0$ *and* $\alpha_4 = m$ *with* $m \in \mathbb{N} \cup \{0\}$*.*

Proof We write $f = \sum_{j=0}^{n} f_j(x, y, z)$ where each f_j is a homogeneous polynomial of degree j in each variables x, y and z. Without loss of generality we can assume that $f_n \neq 0$ and $n > 0$.

We have that the terms of degree $n + 2$ in (1.3) satisfy

$$x^3 \frac{\partial f_n}{\partial x} = (\alpha_4 x^2 + \alpha_5 xy + \alpha_6 xz + \alpha_7 y^2 + \alpha_8 yz + \alpha_9 z^2) f_n. \tag{3.1}$$

Solving the differential equation in (3.1) we have

$$f_n = K_n(y,z)x^{\alpha_4} \exp\left(-\frac{\alpha_5 y + \alpha_6 z}{x} - \frac{\alpha_7 y^2 + \alpha_8 yz + \alpha_9 z^2}{2x^2} \right),$$

where K_n is any function in the variables y and z. Since f_n must be a homogeneous polynomial of degree n we must have $\alpha_4 = m$, $m \in \mathbb{N} \cup \{0\}$ and $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = 0$. This completes the proof of the lemma. □

Lemma 3.2 $\alpha_2 = 0$.

Proof We write $f = \sum_{j=0}^{n} f_j(x,z)y^j$ where each f_j is a polynomial in the variables x, z. The terms of y^{n+1} in (1.3) satisfy

$$\frac{\partial f_n}{\partial x} = \alpha_2 f_n.$$

Solving this differential equation we obtain

$$f_n = K_n(z)\exp(\alpha_2 x),$$

where K_n is any function in the variable z. Since f_n must be a polynomial we must have $\alpha_2 = 0$ or $K_n = 0$. If $K_n = 0$, then $f_n = 0$, and consequently $f = f(x,z)$. Then, from (1.3)

$$y\frac{\partial f}{\partial x} = \alpha_2 y f, \quad \text{that is} \quad f = K(z)\mathrm{e}^{\alpha_2 x},$$

for some function K of z. Since f must be a polynomial we get $\alpha_2 = 0$. This completes the proof of the lemma. □

Lemma 3.3 $\alpha_3 = 0$.

Proof We write $f = \sum_{j=0}^{n} f_j(x,y)z^j$ where each f_j is a polynomial in the variables x and y. The terms of z^{n+1} in (1.3) satisfy

$$\frac{\partial f_n}{\partial x} = -\alpha_3 f_n.$$

Solving this differential equation we obtain

$$f_n = K_n(y)\exp(-\alpha_3 z),$$

where K_n is any function in the variable y. Since f_n must be a polynomial we must have $\alpha_3 = 0$ or $K_n = 0$. If $K_n = 0$, then $f_n = 0$, and consequently $f = f(x,y)$. Then, from (1.3)

$$-z\frac{\partial f}{\partial x} = \alpha_3 z f, \quad \text{that is} \quad f = K(y)\mathrm{e}^{-\alpha_3 x},$$

for some function K of y. Since f must be a polynomial we get $\alpha_3 = 0$. This completes the proof of the lemma. □

In view of Lemmas 3.1, 3.2 and 3.3 we have that

$$K = \alpha_0 + \alpha_1 x + m x^2, \quad m \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Now for simplicity in the computations we introduce the following weight change of variables

$$x = \lambda^{-1}X, \quad y = Y, \quad z = Z, \quad t = \lambda^2 T \tag{3.3}$$

with $\lambda \in \mathbb{R} \setminus \{0\}$ and system (1.1) becomes

$$X' = -X^3 + b\lambda X^2 + \lambda^3(Y + I - Z),$$

$$Y' = -5X^2 + \lambda^2(1 - Y),$$
$$Z' = \lambda\mu s X - \lambda^2\mu(sx_0 + Z),$$

where the prime denotes derivative of the variables with respect to T.

Let $g(x, y, z)$ be a Darboux polynomial of system (1.1) with cofactor K given in (3.2). We use the transformation (3.3) and setting $G(X, Y, Z) = \lambda^n g(\lambda^{-1}X, Y, Z)$ where n is the degree of g and $k = \lambda^2 K(\lambda^{-1}X) = mX^2 + \lambda\alpha_1 X + \lambda^2\alpha_0$. Then we have

$$\frac{dG}{dT} = \lambda^{n+2}\frac{dg}{dt} = \lambda^{n+2}Kg = kG.$$

Assume $G = \sum_{i=0}^{n} \lambda^i G_i$, where G_i is a homogeneous polynomial in the variables X, Y, Z with degree $n - i$ for $i = 0, \ldots, n$. Clearly $g = G_{|\lambda=1}$. By definition of a Darboux polynomial we have

$$\begin{aligned}(-x^3 + b\lambda x^2 + \lambda^3(y + I - z))\sum_{i=0}^{n}\lambda^i\frac{\partial G_i}{\partial x} + (-5x^2 + \lambda^2(1 - y))\sum_{i=0}^{n}\lambda^i\frac{\partial G_i}{\partial y}\\ + (\lambda\mu s x - \lambda^2\mu(sx_0 + z))\sum_{i=0}^{n}\lambda^i\frac{\partial G_i}{\partial z} = (mx^2 + \lambda\alpha_1 x + \lambda^2\alpha_0)\sum_{i=0}^{n}\lambda^i G_i,\end{aligned}$$

where now we use x, y, z instead of X, Y, Z. Equating the terms with λ^i for $i = 0, \ldots, n+2$ we get

$$L[G_0] - mx^2G_0 = 0 \quad \text{where} \quad L = -x^3\frac{\partial}{\partial x} - 5x^2\frac{\partial}{\partial y} \tag{3.4}$$

and

$$\begin{aligned}
L[G_1] - mx^2G_1 &= -bx^2\frac{\partial G_0}{\partial x} - \mu s x\frac{\partial G_0}{\partial z} + \alpha_1 x G_0,\\
L[G_2] - mx^2G_2 &= -bx^2\frac{\partial G_1}{\partial x} - \mu s x\frac{\partial G_1}{\partial z} - (1 - y)\frac{\partial G_0}{\partial y} + \mu(sx_0 + z)\frac{\partial G_0}{\partial z}\\
&\quad + \alpha_1 x G_1 + \alpha_0 G_0,\\
L[G_j] - mx^2G_j &= -bx^2\frac{\partial G_{j-1}}{\partial x} - \mu s x\frac{\partial G_{j-1}}{\partial z} - (1 - y)\frac{\partial G_{j-2}}{\partial y} + \mu(sx_0 + z)\frac{\partial G_{j-2}}{\partial z}\\
&\quad - (y + I - z)\frac{\partial G_{j-3}}{\partial x} + \alpha_1 x G_{j-1} + \alpha_0 G_{j-2},
\end{aligned} \tag{3.5}$$

for $j = 3, \ldots, n+2$.

Solving (3.4) we get

$$G_0 = x^{-m}\tilde{G}_0(z, 5\log x - y).$$

In order that G_0 be a homogeneous polynomial of degree n we get $m = 0$ and

$$G_0 = a_0 z^n \quad \text{with} \quad a_0 \in \mathbb{C} \setminus \{0\}.$$

Substituting G_0 into the first equation of (3.5) and simplifying by x we get

$$-x\left(x\frac{\partial G_1}{\partial x} - 5\frac{\partial G_1}{\partial y}\right) = \alpha_1 a_0 z^n - \mu s n a_0 z^{n-1}. \tag{3.6}$$

Evaluating (3.6) on $x = 0$ and taking into account that $a_0 n \neq 0$ we get $\alpha_1 = 0$ and $\mu s = 0$. Then again from (3.6) we get that

$$x\frac{\partial G_1}{\partial x} - 5\frac{\partial G_1}{\partial y} = 0, \quad \text{that is} \quad G_1 = G_1(z, 5\log x - y). \tag{3.7}$$

Since G_1 must be a polynomial of degree $n-1$ we obtain $G_1 = a_1 z^{n-1}$ where $a_1 \in \mathbb{C}$.

Substituting G_0 and G_1 into the second equation of (3.5) taking into account that $\alpha_1 = \mu s = 0$, we get

$$-x^2\left(x\frac{\partial G_2}{\partial x} - 5\frac{\partial G_2}{\partial y}\right) = a_0(\mu n + \alpha_0)z^n. \tag{3.8}$$

We consider two different cases.

Case 1 $\mu = 0$. In this case, evaluating (3.8) on $x = 0$ we get $\alpha_0 = 0$, but then $K = 0$ in contradiction with the fact that f is a Darboux polynomial with non-zero cofactor.

Case 2 $\mu \neq 0$. Since $\mu s = 0$ then $s = 0$. Evaluating (3.8) on $x = 0$ we obtain $\alpha_0 = -\mu n$, and proceeding as we did for G_1 we get $G_2 = a_2 z^{n-2}$ with $a_2 \in \mathbb{C}$.

Substituting G_0, G_1 and G_2 into equation (3.5) with $j = 3$ and taking into account that $\alpha_1 = \mu s = 0$, $\alpha_0 = -\mu n$, we get

$$-x^2\left(x\frac{\partial G_3}{\partial x} - 5\frac{\partial G_3}{\partial y}\right) = -\mu a_1 z^{n-1}. \tag{3.9}$$

Evaluating (3.9) on $x = 0$ and using that $\mu \neq 0$ we get $a_1 = 0$, that is $G_1 = 0$. Furthermore, $G_3 = a_3 z^{n-3}$ with $a_3 \in \mathbb{C}$.

Proceeding inductively we get $G_i = 0$ for $i = 1, \ldots, n+2$, $G_0 = a_0 z^n$ and $k = \alpha_0 = -\mu n$. Then $g = a_0 z^n$ and $K = -\mu n$. In short, the unique irreducible Darboux polynomial of system (1.1) is z and its cofactor is $-\mu$. This concludes the proof of Theorem 1.3.

4 Proof of Theorem 1.2

To prove Theorem 1.2 we recall two auxiliary results. The first was proved in [5] while the second was proved in [13].

Lemma 4.1 *Let f be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible factors in $\mathbb{C}[x, y, z]$. Then f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j, then $K = \sum_{j=1}^{s} \alpha_j K_j$.*

Lemma 4.2 *The existence of a rational first integral for a polynomial differential system* (1.1) *implies the existence of a polynomial first integral, or the existence of two Darboux polynomials with the same non-zero cofactor.*

The proof of Theorem 1.2 follows readily from Theorems 1.1 and 1.3 together with Lemmas 4.1 and 4.2.

5 Exponential Factors: Proof of Theorem 1.4

To prove Theorem 1.4 we will use the following known result whose proof and geometrical meaning is given in [2] for the plane systems and [15] and [16] for higher dimensional systems.

Proposition 5.1 *The following statements hold.*

(a) *If $E = \exp(g_0/g_1)$ is an exponential factor for the polynomial system* (1.1) *and g_1 is not a constant polynomial, then $g_1 = 0$ is an invariant algebraic curve.*

(b) *Eventually e^{g_0} can be exponential factors, coming from the multiplicity of the infinite invariant straight line.*

The following result given in [2] and [15] characterizes the algebraic multiplicity of an invariant algebraic surface using the number of exponential factors of system (1.1) associated with this invariant algebraic surface.

Theorem 5.2 *Given an irreducible invariant algebraic surface $g_1 = 0$ of degree m of system* (1.1)*, it has algebraic multiplicity k if and only if the vector field associated to system* (1.1) *has $k-1$ exponential factors of the form $\exp(g_{0,i}/g_1^i)$, where $g_{0,i}$ is a polynomial of degree at most im, and $g_{0,i}$ and g_1 are coprime for $i = 1,\ldots,k-1$.*

In view of Theorem 5.2 if we prove that e^{g_0/g_1} is not an exponential factor with degree $g_0 \le$ degree g_1, there are no exponential factors associated to the invariant algebraic surface $g_1 = 0$.

Proposition 5.3 *System* (1.1) *with $\mu \neq 0$ has the exponential factors e^y, e^z, e^{z^2} with cofactors $1-5x^2-y$, $\mu(s(x-x_0)-z)$ and $2\mu z(s(x-x_0)-z)$, respectively and also exponential of linear combinations of y, z and z^2.*

Proof System (1.1) has the irreducible Darboux polynomial z when $s = 0$. Then in view of Proposition 5.1 system (1.1) can have an exponential factor of the form: either $E = \exp(g)$ with $g \in \mathbb{C}[x,y,z]$, or only when $s = 0$, $E = \exp(h/z^m)$ with $m \ge 1$ and such that $h \in \mathbb{C}[x,y,z]$ is coprime with z and the degree of h is at most m.

We first prove that system (1.1) with $s = 0$ has no exponential factors of the form $E = \exp(h/z^m)$.

Assume that system (1.1) with $s = 0$ has an exponential factor of the form $E = \exp(h/z^m)$ with $m \ge 1$ such that z is coprime with $h \in \mathbb{C}[x,y,z]$. In view of Theorem 5.2 we can assume that $m = 1$ and that h has degree at most one (note that here $g_1 = z$ has degree one). We write h as a polynomial of degree one in the variables x, y, z as follows:

$$h = a_0 + a_1 x + a_2 y + a_3 z. \tag{5.1}$$

Clearly h satisfies

$$(y - x^3 + bx^2 + I - z)\frac{\partial h}{\partial x} + (1 - 5x^2 - y)\frac{\partial h}{\partial y} - \mu z\frac{\partial h}{\partial z} + \mu h = Lz, \tag{5.2}$$

where L is a polynomial of degree two in the variables x, y, z. Let

$$L = b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 xy + b_6 xz + b_7 y^2 + b_8 yz + b_9 z^2. \tag{5.3}$$

From (5.2) and using an algebraic manipulator it is easy to check that

$$h = a_3 z \quad \text{and} \quad L = 0.$$

However this is not possible since h is coprime with z. Hence this case is not possible.

In summary, if $\mu \neq 0$ and (1.1) has an exponential factor it must be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x,y,z] \setminus \mathbb{C}$. In this case, g satisfies

$$(y - x^3 + bx^2 + I - z)\frac{\partial g}{\partial x} + (1 - 5x^2 - y)\frac{\partial g}{\partial y} + \mu((s(x - x_0) - z))\frac{\partial g}{\partial z} = L, \tag{5.4}$$

where $L = L(x,y,z)$ is some polynomial of degree two in the variables x, y and z and that we can take as in (5.3).

We write g as $g = \sum_{j=0}^n g_j(x,y,z)$ where each g_j is a homogeneous polynomial of degree j in each variables x, y and z and $n > 0$. We have that the terms of degree $n+2$ with $n \ge 3$

in (5.4) satisfy

$$-x^3\frac{\partial g_n}{\partial x}=0, \quad \text{that is} \quad g_n=g_n(y,z).$$

Computing the terms of degree $n+1$ with $n\geq 3$ in (5.4) we get

$$-x^3\frac{\partial g_{n-1}}{\partial x}-5x^2\frac{\partial g_n}{\partial y}=0, \quad \text{that is} \quad -x\frac{\partial g_{n-1}}{\partial x}=5\frac{\partial g_n}{\partial y}.$$

Since g_n do not depend on x we must have $\partial g_n/\partial y=0$ and $\partial g_{n-1}/\partial x=0$, that is $g_n=g_n(z)$ and $g_{n-1}=g_{n-1}(y,z)$.

Now computing the terms of degree n with $n\geq 3$ in (5.4) we obtain

$$-x^3\frac{\partial g_{n-2}}{\partial x}-5x^2\frac{\partial g_{n-1}}{\partial y}+\mu(sx-z)\frac{dg_n}{dz}=0.$$

Note that since $\mu\neq 0$ then $\mu(sx-z)$ must be divisible by x^2 which is not possible and thus $dg_n/dz=0$ which yields g_n is a constant, in contradiction with the fact that $n\geq 3$. Then, we must have $n\leq 2$. In this case g is a polynomial of degree two in its variables that we write it as

$$g=a_0+a_1x+a_2y+a_3z+a_4x^2+a_5xy+a_6xz+a_7y^2+a_8yz+a_9z^2. \tag{5.5}$$

Imposing that g satisfies (5.4) with L as in (5.3) using an algebraic manipulator such as mathematica we get

$$g=a_0+a_2y+a_3z+a_9z^2 \tag{5.6}$$

and

$$L=a_2(1-5x^2-y)+a_3\mu(s(x-x_0)-z)+2\mu a_9z(s(x-x_0)-z). \tag{5.7}$$

This concludes the proof of the proposition. □

Now we have to study the case $\mu=0$. This is the content of the following result.

Proposition 5.4 *System* (1.1) *with* $\mu=0$ *has the exponential factors* e^y, $\mathrm{e}^{P(z)/Q(z)}$ *where* P,Q *are polynomials with cofactors* $1-5x^2-y$ *and* 0, *respectively and also exponential of linear combinations of all the exponents in the previous exponential factors.*

Proof System (1.1) has the polynomial first integral z when $\mu=0$. Then in view of Proposition 5.1 system (1.1) can have an exponential factor of the form: either $E=\exp(g)$ with $g\in\mathbb{C}[x,y,z]$, or $E=\exp(h/z^m)$ with $m\geq 1$ and such that $h\in\mathbb{C}[x,y,z]$ is coprime with z and has degree at most m.

We first prove that system (1.1) with $\mu=0$ has no exponential factors of the form $E=\exp(h/z^m)$.

Assume that system (1.1) with $\mu=0$ has an exponential factor of the form $E=\exp(h/z^m)$ with $m\geq 1$ such that z is coprime with $h\in\mathbb{C}[x,y,z]$. In view of Theorem 5.2 we can assume that $m=1$ and that h has degree at most one. We write h as a polynomial of degree one in the variables x,y,z as in (5.1). Clearly, h satisfies

$$(y-x^3+bx^2+I-z)\frac{\partial h}{\partial x}+(1-5x^2-y)\frac{\partial h}{\partial y}=Lz, \tag{5.8}$$

where L is a polynomial of degree two in the variables x, y, z. Setting L in (5.8) as in (5.3) and working with an algebraic manipulator we conclude that

$$h = a_0 + a_3 z \quad \text{and} \quad L = 0.$$

Now assume that system (1.1) with $\mu = 0$ has an exponential factor of the form $E = \exp(g)$ with $g \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$. In this case, g satisfies

$$(y - x^3 + bx^2 + I - z)\frac{\partial g}{\partial x} + (1 - 5x^2 - y)\frac{\partial g}{\partial y} = L, \tag{5.9}$$

where $L = L(x, y, z)$ is some polynomial of degree two in the variables x, y, z and that we can take as in (5.3).

We write g as $g = \sum_{j=0}^{n} g_j(x, y, z)$ where each g_j is a homogeneous polynomial of degree j in each variables x, y and z. We have that the terms of degree $n+2$ with $n \geq 3$ in (5.9) satisfy

$$-x^3 \frac{\partial g_n}{\partial x} = 0, \quad \text{that is} \quad g_n = g_n(y, z).$$

Computing the terms of degree $n+1$ with $n \geq 3$ in (5.9) we get

$$-x^3 \frac{\partial g_{n-1}}{\partial x} - 5x^2 \frac{\partial g_n}{\partial y} = 0, \quad \text{that is} \quad -x\frac{\partial g_{n-1}}{\partial x} = 5\frac{\partial g_n}{\partial y}.$$

Since g_n do not depend on x we must have $\partial g_n/\partial y = 0$ and $\partial g_{n-1}/\partial x = 0$ that is $g_n = g_n(z)$ and $g_{n-1} = g_{n-1}(y, z)$.

Now computing the terms of degree n with $n \geq 3$ in (5.9) we obtain

$$-x^3 \frac{\partial g_{n-2}}{\partial x} - 5x^2 \frac{\partial g_{n-1}}{\partial y} = 0.$$

Now proceeding as above, after simplifying by x^2 and taking into account that g_{n-1} does not depend on x, we get that $\partial g_{n-1}/\partial y = 0$ and $\partial g_{n-2}/\partial x = 0$, that is $g_{n-1} = g_{n-1}(z)$ and $g_{n-2} = g_{n-2}(y, z)$. Proceeding inductively, we get

$$g = G(z) + a_2 z^2 + a_3 y + a_4 z + a_0 = \tilde{G}(z) + a_3 y,$$

where G and $\tilde{G}$ are polynomials in the variable z and $a_3 \in \mathbb{C}$. Moreover, it follows from (5.9) that $L = a_3(1 - 5x^2 - y)$. This completes the proof of the proposition for $n \geq 3$.

If $n < 3$ then g can be written as in (5.5). Solving (5.9) with g and L as in (5.3) we get g as in (5.6) and $L = a_2(1 - 5x^2 - y)$. So the proposition is proved. □

6 Proof of Theorem 1.5

In order to proof Theorem 1.5 we need the following result whose proof is given in [5] and [15].

Theorem 6.1 *Suppose that system* (1.1) *admits* p *Darboux polynomials and with cofactors* K_i *and* q *exponential factors* F_j *with cofactors* L_j. *Then there exists* $\lambda_j, \mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{q} \lambda_k K_i + \sum_{i=1}^{q} \mu_i L_i = 0$$

if and only if the function G *given in* (1.5) (*called of Darboux type*) *is a first integral of system* (1.1).

In view of Theorem 6.1 to characterize the Darboux first integrals we need to compute the Darboux polynomials and the exponential factors. We consider three cases.

Case 1 $\mu s \neq 0$. In this case, using Theorems 1.3 and 1.4 if G is a Darboux first integral of system (1.1) it must be of the form (1.5) and the cofactors must satisfy

$$\mu_1(1-5x^2-y)+\mu_2\mu(s(x-x_0)-z)+2\mu\mu_3 z(s(x-x_0)-z)=0.$$

Solving this system we have $\mu_1=\mu_2=\mu_3=0$, which yields that G is a constant, that is, there are no Darboux first integrals.

Case 2 $\mu \neq 0$ and $s=0$. In this case, using Theorems 1.3 and 1.4 if G is a Darboux first integral of system (1.1) it must be of the form (1.5) and the cofactors must satisfy

$$\mu_1(1-5x^2-y)-\mu_2\mu z-2\mu\mu_3 z^2-\lambda_1\mu=0.$$

Solving this system we have $\mu_1=\mu_2=\mu_3=\lambda_1=0$, which yields that G is a constant, that is, there are no Darboux first integrals.

Case 3 $\mu=0$. In this case, using Theorems 1.3 and 1.4 if G is a Darboux first integral of system (1.1) it must be of the form (1.5) and the cofactors must satisfy

$$\mu_1(1-5x^2-y)+\mu_2 0=0.$$

Solving this system we have $\mu_1=0$ which yields that G is a function in the variable z. This concludes the proof of the theorem.

7 Summary and Conclusion

We have proved that the unique invariant algebraic surface of the Hindmarsh–Rose burster differential system is $z=0$ when the parameter $\mu=0$, or $s=0$ and $\mu\neq 0$ with cofactors zero and non-zero, respectively (see Theorems 1.1 and 1.3). We also have shown that the Hindmarsh–Rose burster differential system has a Darboux first integral when the parameter $\mu=0$ (see Theorem 1.5).

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