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Spectral Invariant Subalgebras of Reduced Groupoid C^*-algebras

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Abstract We introduce the notion of property (RD) for a locally compact, Hausdorff and r-discrete groupoid G, and show that the set $S_2^l(G)$ of rapidly decreasing functions on G with respect to a continuous length function l is a dense spectral invariant and Fréchet $*$-subalgebra of the reduced groupoid C^*-algebra $C_r^*(G)$ of G when G has property (RD) with respect to l, so the K-theories of both algebras are isomorphic under inclusion. Each normalized cocycle c on G, together with an invariant probability measure on the unit space G^0 of G, gives rise to a canonical map τ_c on the algebra $C_c(G)$ of complex continuous functions with compact support on G. We show that the map τ_c can be extended continuously to $S_2^l(G)$ and plays the same role as an n-trace on $C_r^*(G)$ when G has property (RD) and c is of polynomial growth with respect to l, so the Connes' fundament paring between the K-theory and the cyclic cohomology gives us the K-theory invariants on $C_r^*(G)$.

Keywords Groupoid C*-algebra, property (RD), spectral invariance

MR(2010) Subject Classification 46L05, 46L80, 46L87

1 Introduction

The fundamental pairing between the K-theory and the cyclic cohomology of a dense subalgebra $\mathcal{A}$ of a C^*-algebra A plays an important role in the theory of non-commutative geometry of Connes. In order to use the pairing, one needs to find suitable dense subalgebra $\mathcal{A}$ such that the K-theory invariants on $\mathcal{A}$ given by the pairing can be extended to ones on A. The dense subalgebra $\mathcal{A}$, stable under holomorphic functional calculus in A, is the simplest situation, because, in this case, the inclusion $\mathcal{A} \subset A$ induces isomorphisms from $K_*(\mathcal{A})$ onto $K_*(A)$ ([3]). When one considers the reduced group C^*-algebra $C_r^*(G)$ of a countable group G, the space $H_l^\infty(G)$ of rapidly decreasing functions on G with respect to a length function l is stable under holomorphic functional calculus in $C_r^*(G)$ if G has property (RD) in the sense that $H_l^\infty(G)$ is contained in $C_r^*(G)$ [11, 13].

The property of rapid-decay for countable groups was established for free groups by Haagerup in [8] and investigated in detail by Jolissaint in [12]. The latter proved that both finitely generated groups of polynomial growth and discrete cocompact subgroups of the group of all isometries of any hyperbolic space have property (RD). The extension to Gromov's hyperbolic groups is due to de la Harpe [9]. These results were used in the work of Connes–Moscovici on the Novikov conjecture [2] and the work of Lafforgue on the Baum–Connes conjecture [15].

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Spectral invariance and the stability under holomorphic functional calculus of dense subalgebras in their ambient Banach algebras are closely related. From [21], these two notions are equivalent for dense and Fréchet $*$-subalgebras of C^*-algebras. In [14], Ji and Schweitzer proved that all rapid-decay locally compact groups are unimodular and the set of rapid-decay functions on these groups forms dense, spectral invariant and Fréchet $*$-subalgebras of reduced group C^*-algebras. Recently, Chen and Wei studied the property (MRD) of discrete metric spaces in [7]. They computed spectral invariant subalgebras of uniform Roe algebras on these spaces and characterized the polynomial growth of countable groups using the spectral invariance of the space of all l^2-Schwartz functions in the reduced crossed product for commutative C^*-algebras with these group actions. For the theory and applications on property (RD) and spectral invariance, we can refer to [5, 6, 10–14, 16, 21].

As generalizations of discrete groups and topological spaces, r-discrete groupoids, arising as model for structures like leaf spaces of foliations and orbit spaces of actions by discrete groups, figure prominently in the theory of non-commutative geometry. The reduced groupoid C^*-algebras of these groupoids can be regarded as spaces of continuous functions on them. This property provides us with the possibility to study the property of rapid-decay for r-discrete groupoids.

Given a continuous length function l on an r-discrete groupoid G, we consider the set $S_2^l(G)$ of all complex continuous functions on G and rapidly decay on each fiber of G in the l^2-sense with respect to l. When $S_2^l(G)$ is contained in the reduced groupoid C^*-algebra $C_r^*(G)$ of G, we call G has property (RD). We can show that, in this situation, $S_2^l(G)$ is stable under holomorphic functional calculus in $C_r^*(G)$, hence they have isomorphic K-theories.

Each normalized cocycle c on a countable group Γ gives rise to a canonical cyclic cocycle τ_c on the group algebra $\mathbb{C}\Gamma$ [3]. In general, τ_c cannot be extended as an n-trace on the reduced group C^*-algebra of Γ, but in [13], Jolissaint proved that, with some additional hypotheses, τ_c plays the same role as an n-trace. Jolissaint's technique on the estimation for the norm of τ_c was used in [2]. For an r-discrete groupoid G with a continuous length function l and a normalized cocycle c, we can define the canonical map τ_c^u on each range fiber G^u of G, as defined for the group case. Under the hypothesis of property (RD) on G and polynomial growth on c with respect to l, the integral of τ_c^u with respect to an invariant probability measure on the unit space G^0 of G defines a map τ_c. This map can play the same role as an n-trace on $C_r^*(G)$, which is analogous to Jolissaint's result.

The paper is organized as follows. Section 2 contains some definitions and results on spectral invariance and groupoids. Section 3 contains the notion of property (RD) for r-discrete groupoids. In addition, we obtain that each r-discrete groupoid being of polynomial growth has property (RD). As an application, we can use the groupoid language to rewrite a result in [7], which states that a countable group Γ is of polynomial growth with respect to a proper length function if and only if the transformation group of Γ on its Stone–Čech compactification space has property (RD) with respect to a canonical length function. In Section 4, we show that, if an r-discrete groupoid G has property (RD) with respect to a continuous length function l, then the space $S_2^l(G)$ of rapidly decreasing functions on G is a spectral invariant and Fréchet $*$-subalgebra of $C_r^*(G)$. Section 5 contains two examples of spectral invariant subalgebras of groupoid C^*-algebras arising from two expansive dynamical systems, a subshift of finite type and a solenoid. In Section 6, we associate to a normalized cocycle c on an r-discrete groupoid G a cyclic cocycle τ_c on $S_2^l(G)$, which plays the same role as an n-trace on $C_r^*(G)$, when G has property (RD) and c is of polynomial growth with respect to a continuous length function l, so the fundament paring of Connes gives us the K-theory invariants on $C_r^*(G)$.

2 Preliminaries

We review some basic terminology and results concerning spectral invariance of subalgebras of Banach algebras. Let A be a Banach algebra and $\mathcal{A}$ a subalgebra of A, and let $\widetilde{A}$ and $\widetilde{\mathcal{A}}$ be obtained by adjoining the same unit. Recall that $\mathcal{A}$ is stable under holomorphic functional calculus if, for each $a \in \mathcal{A}$ and each function f holomorphic in a neighborhood of the spectrum of a in $\widetilde{A}$, one has $f(a) \in \widetilde{\mathcal{A}}$. If $\mathcal{A}$ is a dense subalgebra of A, stable under holomorphic functional calculus, then the inclusion map $\iota : \mathcal{A} \hookrightarrow A$ induces isomorphisms on their K-theories [3]. We say that $\mathcal{A}$ is spectral invariant in A if the invertible elements of $\widetilde{\mathcal{A}}$ are precisely those elements of $\widetilde{\mathcal{A}}$ which are invertible in $\widetilde{A}$. Note that $\mathcal{A}$ is spectral invariant in A if and only if for every $a \in \mathcal{A}$, the spectrum of a is the same in $\widetilde{\mathcal{A}}$ and in $\widetilde{A}$.

By a Fréchet algebra, we shall mean a Fréchet space with an algebra structure for which multiplication is jointly continuous. We say that $\mathcal{A}$ is a Fréchet subalgebra of A if $\mathcal{A}$ is a Fréchet algebra, and the inclusion map $\iota : \mathcal{A} \hookrightarrow A$ is a continuous, injective algebraic homomorphism. It follows from [21] that if $\mathcal{A}$ is a dense and Fréchet subalgebra of a C^*-algebra A, then $\mathcal{A}$ is spectral invariant in A if and only if it is stable under holomorphic functional calculus in A.

A groupoid is a set G, together with a subset $G^2 \subseteq G \times G$, a product map $(x, y) \to xy$ from G^2 to G, and an inverse map $x \to x^{-1}$ from G onto G such that

(i) $(x^{-1})^{-1} = x$;

(ii) if (x, y), $(y, z) \in G^2$, then (xy, z), $(x, yz) \in G^2$ and $(xy)z = x(yz)$;

(iii) for each $x \in G$, $(x, x^{-1}) \in G^2$, and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$ and $(xy)y^{-1} = x$.

The maps r and d, defined by $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the range map and the source map, respectively. They have a common image called the unit space of G, which is denoted by G^0. It is very useful to note that a pair (x, y) is in G^2 only when $d(x) = r(y)$. For $u \in G^0$, the fibers of r and d are denoted by $G^u = r^{-1}(\{u\})$ and $G_u = d^{-1}(\{u\})$, respectively.

In this paper, we call a groupoid G to be a topological groupoid, if it is endowed with a locally compact and Hausdorff topology compatible with the groupoid structure: the inverse map $x \to x^{-1}$ and the product map $(x, y) \to xy$ are continuous on their respective domain G and G^2, where G^2 has the induced topology from $G \times G$. For basic of groupoid theory, we refer to [17, 19].

Let G be a topological groupoid. For $S \subseteq G$, let r_S and d_S respectively be the restrictions of r and d to S. If the family, G^{op}, of open subsets of G such that r_S and d_S are homeomorphisms onto open subsets of G, is a basis for the topology of G, then G is said to be r-discrete. In this situation, the unit space G^0 is open and closed in G, and all fibres G^u and G_u are countable. Moreover, the counting measure is a Haar system of G [17, 19].

For an r-discrete groupoid G, let $C_c(G)$ be the space of continuous complex functions with compact support on G. Then, it is a normed $*$-algebra, under the following *convolution*, *involution* and the norm $\|\cdot\|_I$: for $f, g \in C_c(G)$ and $x \in G$,

$$f * g(x) = \sum_{yz=x} f(y)g(z), \quad f^*(x) = \overline{f(x^{-1})} \quad \text{and} \quad \|f\|_I = \max\{\|f\|_{I,r}, \ \|f\|_{I,d}\},$$

where $\|f\|_{I,r} = \sup_{u\in G^0} \sum_{x\in G^u} |f(x)|$, $\|f\|_{I,d} = \sup_{u\in G^0} \sum_{x\in G_u} |f(x)|$.

For each $u \in G^0$, let $\mathrm{Ind}_u : C_c(G) \to B(l^2(G_u))$ be the induced representation of $C_c(G)$ on the Hilbert space $l^2(G_u)$:

$$\mathrm{Ind}_u(f)(\xi)(x) = \sum_{y\in G^u} f(xy)\xi(y^{-1})$$

for $f \in C_c(G)$, $\xi \in l^2(G_u)$ and $x \in G_u$. For each $f \in C_c(G)$, defines

$$\|f\|_{\mathrm{red}} = \sup_{u\in G^0} \|\mathrm{Ind}_u(f)\|.$$

Then $\|\cdot\|_{\mathrm{red}}$ is a C^*-norm on $C_c(G)$. The completion of $C_c(G)$ under the norm is called the reduced groupoid C^*-algebra of G, denoted by $C_r^*(G)$. From the following result, the elements of $C_r^*(G)$ can be viewed as continuous functions on G.

Proposition 2.1 ([19]) *Let G be an r-discrete groupoid. Then*

(i) $\|f\|_\infty \le \|f\|_{II} \le \|f\|_{\mathrm{red}} \le \|f\|_I$ *for each $f \in C_c(G)$, where $\|f\|_\infty$ is the supremum norm of f and*

$$\|f\|_{II} = \max\left\{ \sup_{u\in G^0} \Big(\sum_{x\in G^u} |f(x)|^2 \Big)^{\frac12}, \sup_{u\in G^0} \Big(\sum_{x\in G_u} |f(x)|^2 \Big)^{\frac12} \right\};$$

(ii) *The injection j_0 of $C_c(G)$ into $C_0(G)$, the Banach space of continuous functions on G vanishing at infinity, extends to a norm decreasing and one-to-one linear map J_0 of $C_r^*(G)$ into $C_0(G)$; Moreover, each $a \in C_r^*(G)$ satisfies $\|J_0(a)\|_\infty \le \|a\|_{\mathrm{red}}$ and $\||a|^2\|_I \le \|a\|_{\mathrm{red}}^2$.*

(iii) *Under the identification J_0, the operations in $C_r^*(G)$ may be expressed in the same as in the $*$-algebra $C_c(G)$:*

$$a^*(x) = \overline{a(x^{-1})}, \quad a * b(x) = \sum_{y\in G^{d(x)}} a(xy)b(y^{-1})$$

for $a, b \in C_r^*(G)$ and $x \in G$.

Example 2.2 (a) *Discrete groups.* A countable discrete group G is an r-discrete groupoid with $G^2 = G \times G$ and $G^0 = \{e\}$ (the unit element of G). The reduced group C^*-algebra is isomorphic to the reduced groupoid C^*-algebra.

(b) *Topology spaces.* A locally compact Hausdorff space X is an r-discrete, groupoid by letting $X^2 = \{(x, x) : x \in X\}$ and defining the operations by $xx = x$ and $x^{-1} = x$ for each $x \in X$. The groupoid C^*-algebra is isomorphic to $C_0(X)$.

(c) *Transformation groups.* Let Γ be a discrete group acting on the right on a locally compact and Hausdorff space X. Then $G = X \times \Gamma$, with the product topology, the multiplication $(x, g)(xg, h) = (x, gh)$ and the inverse $(x, g)^{-1} = (xg, g^{-1})$, is an r-discrete groupoid. Here $G^0 = \{(x, e) : x \in X\} \cong X$, $r(x, g) = (x, e)$ and $d(x, g) = (xg, e)$ for $(x, g) \in G$. The groupoid C^*-algebra $C_r^*(G)$ is isomorphic to the reduced crossed product $C_0(X) \rtimes_{\alpha,r} \Gamma$, where the action α of Γ on $C_0(X)$ is given by $\alpha_g(f)(x) = f(xg)$ for $g \in \Gamma$, $f \in C_0(X)$ and $x \in X$.

3 Polynomial Growth and Rapid Decay

Let G be an r-discrete groupoid. A nonnegative function l on G is called a length function, if it satisfies that $l(xy) \le l(x) + l(y)$, $l(z) = l(z^{-1})$ and $l(u) = 0$ for every $(x, y) \in G^2$, $z \in G$ and $u \in G^0$. Moreover, if l is bounded on every compact subset of G, then it is said to be locally bounded. For two length functions l_1 and l_2 on G, we say that l_1 dominates l_2 if there exist $c > 0$ and $k \ge 1$ such that $l_2(x) \le c(1 + l_1(x))^k$ for every $x \in G$. We say that l_1 is equivalent to l_2 if, furthermore, l_2 dominates l_1.

We say that G is compactly generated, if it admits a symmetric compact subset K (i.e., $K = K^{-1}$) such that every element in G can be written as a finite product of elements in K. Each generating set K gives us a word length function l_K on G by

$$l_K(u) = 0, \text{ and } l_K(x) = \min\{n : x = x_1x_2\cdots x_n, x_i \in K \text{ for } i = 1, 2, \ldots, n\}$$

for each $u \in G^0$ and $x \in G$, $x \notin G^0$. If l is an arbitrary locally bounded length function on G and let $m_K = \sup\{l(x) : x \in K\}$, then, for each $x \in G$ with $x \notin G^0$, we have $x = x_1x_2\cdots x_n$ for $x_i \in K$, $i = 1, \ldots, n$, $n = l_K(x)$, and $l(x) \le \sum_{i=1}^n l(x_i) \le m_K l_K(x)$. Hence every locally bounded length function on G can be controlled by any word length function l_K, so that locally bounded word length functions on a compactly generated groupoid are all comparable to each other. In the example of a transformation group $G = X \times \Gamma$, if Γ has a finite symmetric generating set S and X is compact, then $K = X \times S$ is a generating set of G. The word length function l_K on G is determined by the word length function l_S on Γ, i.e., $l_K(x, g) = l_S(g)$ for every $(x, g) \in G$.

Recall that a discrete group Γ is of polynomial growth with respect to a length function l if there exists a polynomial $P(t)$ such that the cardinality of the set of all elements in Γ with length no more than m is bounded by $P(m)$. For an r-discrete groupoid G with a length function l, we define

$$B^l_{G^u}(m) = \{x \in G^u|\ l(x) \le m\}, \quad B^l_{G_u}(m) = \{x \in G_u|\ l(x) \le m\}$$

for $u \in G^0$ and $m \ge 0$. If we denote by ${}^{\#}S$ the cardinality of a set S, then ${}^{\#}B^l_{G^u}(m) = {}^{\#}B^l_{G_u}(m)$ for each m and u.

Definition 3.1 *We say that G is of polynomial growth with respect to a length function l on G, if there exist constants $c \ge 1$, $r \ge 1$ such that, for every $m \ge 0$,*

$$\sup_{u \in G^0} {}^{\#}B^l_{G^u}(m) \le c(1 + m)^r.$$

We say that G is of polynomial growth if it is of polynomial growth with respect to some length function l on G.

In the rest of this section, we let G be an r-discrete groupoid and l a nonzero locally bounded length function on G. For a complex function φ on G and for $p = 0, 1, 2, \ldots$, we define

$$\|\varphi\|_{2,p,d,l} = \sup_{u \in G^0} \left[\sum_{x \in G_u} |\varphi(x)|^2 (1 + l(x))^{2p} \right]^{\frac{1}{2}},$$

$$\|\varphi\|_{2,p,r,l} = \sup_{u \in G^0} \left[\sum_{x \in G^u} |\varphi(x)|^2 (1 + l(x))^{2p} \right]^{\frac{1}{2}},$$

$$\|\varphi\|_{2,p,l} = \max\{\|\varphi\|_{2,p,d,l}, \|\varphi\|_{2,p,r,l}\}.$$

Clearly, for each $\varphi \in C_c(G)$ and each $p \geq 0$, $\|\varphi\|_\infty \leq \|\varphi\|_{2,p,l}$, and $\|\cdot\|_{2,p,l}$ defines a norm on $C_c(G)$. Let $S_2^l(G)$ be the completion of $C_c(G)$ under the local convex topology τ generated by the sequence of norms, $\{\|\cdot\|_{2,p,l} : p = 0, 1, 2, \ldots, \}$, and $\|\cdot\|_\infty$. Moreover, if let $L_{2,p,l}(G)$ be the completion of $C_c(G)$ under the norm $\|\cdot\|_{2,p,l}$, then $S_2^l(G) = \bigcap_{p=0}^\infty L_{2,p,l}(G)$, so it is a Fréchet space under the topology τ. We call $S_2^l(G)$ the space of rapidly decreasing functions on G with respect to l.

Definition 3.2 *We say that G has property* (RD) *with respect to a length function l, if there exist a $c > 0$ and a positive integer p such that $\|f\|_{\rm red} \leq c\|f\|_{2,p,l}$ for each $f \in C_c(G)$.*

We simply say that G has property (RD) *if it has property* (RD) *with respect to some locally bounded length function on G.*

Note that, for two locally bounded length functions l_1 and l_2 on G such that l_1 dominates l_2, if G has property (RD) with respect to l_2, then it has property (RD) with respect to l_1.

By Proposition 2.1, if we identify $J_0(a)$ with a for each $a \in C_r^*(G)$, then both $C_r^*(G)$ and $S_2^l(G)$ are subspaces of $C_0(G)$. Using the closed graph theorem, we have the following description for property (RD).

Lemma 3.3 *G has property* (RD) *with respect to l if and only if $S_2^l(G)$ is contained in $C_r^*(G)$.*

Proof If G has property (RD) with respect to l, then there exist $c > 0$ and a positive integer p such that $\|f\|_{\rm red} \leq c\|f\|_{2,p,l}$ for each $f \in C_c(G)$. For each $\varphi \in S_2^l(G)$, we can choose a sequence $\{\varphi_n\}$ in $C_c(G)$ converging to φ under the norms, $\|\cdot\|_{2,p,l}$ and $\|\cdot\|_\infty$. Hence, $\{\varphi_n\}_n$ is a Cauchy sequence under the reduced norm $\|\cdot\|_{\rm red}$, so it converges to a in $C_r^*(G)$. Using the identification J_0, we have $\{\varphi_n\}_n$ converges to a in $C_0(G)$. Hence $\varphi = a \in C_r^*(G)$.

Suppose $S_2^l(G) \subseteq C_r^*(G)$. We claim that the inclusion ι from $S_2^l(G)$ with the Fŕechet topology into $C_r^*(G)$ with the reduced norm is a closed map. In fact, for a sequence $\{\varphi_n\}_n$ in $S_2^l(G)$, assume it converges to φ under the Fŕechet topology and $\{\iota(\varphi_n)\}_n$ converges to a in $C_r^*(G)$ under the reduced norm $\|\cdot\|_{\rm red}$. By (i) and (ii) in Proposition 2.1, we have $\{\varphi_n\}_n$ in $C_0(G)$ converges to a under the norm $\|\cdot\|_\infty$, so $a = \iota(\varphi)$. Hence ι is closed. By the closed graph theorem, ι is continuous. Then there exist $c > 0$ and a positive integer p such that $\|f\|_{\rm red} \leq c\|f\|_{2,p,l}$ for every $f \in C_c(G)$. Hence G has property (RD) with respect to l. □

Proposition 3.4 *Assume G has property* (RD) *with respect to a continuous length function l. Then*

(i) *there exist $c > 0$ and a positive integer q such that $\|f\|_{\rm red} \leq c\|f\|_{2,q,l}$ for each $f \in S_2^l(G)$;*

(ii) *under the following convolution and involution, $S_2^l(G)$ is a $*$-algebra:*

$$f * g(x) = \sum_{y \in G^{d(x)}} f(xy)g(y^{-1}) \quad \text{and} \quad f^*(x) = \overline{f(x^{-1})}$$

for all f, $g \in S_2^l(G)$ and $x \in G$. Moreover, it is a dense and Fréchet $*$-subalgebra of $C_r^*(G)$.

Proof (i) It follows from the continuity of the inclusion ι from $S_2^l(G)$ into $C_r^*(G)$.

(ii) From Lemma 3.1, $S_2^l(G)$ is contained in $C_r^*(G)$. For every $f, g \in S_2^l(G)$, by (iii) of Proposition 2.1, we have $f * g$ and f^* are well defined and belong to $C_r^*(G)$. Moreover, $(f * g)^* =$

$g^* * f^*$, $\|f^*\|_{2,p,d,l} = \|f\|_{2,p,r,l}$ and $\|f^*\|_{2,p,r,l} = \|f\|_{2,p,d,l}$, so $\|f^*\|_{2,p,l} = \|f\|_{2,p,l}$ for each $p \geq 0$. Hence $f^* \in S_2^l(G)$.

Let f and g be in $S_2^l(G)$. For $u \in G^0$, if let g_u be the restriction of g to G_u, then $g_u \in l^2(G_u)$ and $f * g|_{G_u} = \mathrm{Ind}_u(f)g_u \in l^2(G_u)$, so $\sum_{x\in G_u} |f * g(x)|^2 = \|\mathrm{Ind}_u(f)g_u\|^2 \leq \|f\|_{\mathrm{red}}\|g\|_{2,0,d,l}$. Hence

$$\|f * g\|_{2,0,d,l} \leq \|f\|_{\mathrm{red}}\|\|g\|_{2,0,d,l}. \tag{1}$$

For $\varphi \in S_2^l(G)$ and $q \geq 0$, we define $\varphi_q(x) = \varphi(x)(1 + l(x))^q$ for $x \in G$. Then $\varphi_q \in S_2^l(G)$, $\|\varphi_q\|_{2,p,d,l} = \|\varphi\|_{2,p+q,d,l}$, $\|\varphi_q\|_{2,p,r,l} = \|\varphi\|_{2,p+q,r,l}$, so $\|\varphi_q\|_{2,p,l} = \|\varphi\|_{2,p+q,l}$ for each $p \geq 0$. Since $(1 + l(x))^p \leq (1 + l(xy))^p(1 + l(y^{-1}))^p$ for all $(x, y) \in G^2$ and $p \geq 0$, one can check that $\|f * g\|_{2,p,d,l} \leq \|f_p * g_p\|_{2,0,d,l}$ and $\|f * g\|_{2,p,r,l} \leq \|f_p * g_p\|_{2,0,r,l}$.

By (i), there exist $c > 0$ and a positive integer q such that $\|\varphi\|_{\mathrm{red}} \leq c\|\varphi\|_{2,q,l}$ for each $\varphi \in S_2^l(G)$. Hence, for each $p \geq 0$, it follows from (1) that

$$\begin{aligned}\|f * g\|_{2,p,d,l} &\leq \|f_p * g_p\|_{2,0,d,l} \leq \|f_p\|_{\mathrm{red}}\|g_p\|_{2,0,d,l} \leq c\|f_p\|_{2,q,l}\|g_p\|_{2,0,d,l}\\ &\leq c\|f\|_{2,p+q,l}\|g\|_{2,p,l} \leq c\|f\|_{2,p+q,l}\|g\|_{2,p+q,l}.\end{aligned}$$

Noting that $(f * g)^* = g^* * f^*$, we have $\|f * g\|_{2,p,r,l} = \|(f * g)^*\|_{2,p,d,l} = \|g^* * f^*\|_{2,p,d,l} \leq c\|g\|_{2,p+q,l}\|f\|_{2,p+q,l}$ for each p. Consequently,

$$\|f * g\|_{2,p,l} \leq c\|f\|_{2,p+q,l}\|g\|_{2,p+q,l} \quad \text{for } f, g \in S_2^l(G).$$

Hence $f * g \in S_2^l(G)$. It follows from (i) that $S_2^l(G)$ is a Fréchet $*$-algebra and dense in $C_r^*(G)$ as a $*$-subalgebra. □

Proposition 3.5 *If G is of polynomial growth with respect to a locally bounded length function l, then G has property* (RD) *with respect to l.*

Proof Since G is of polynomial growth with respect to l, there exist a constant $c \geq 1$ and an integer $r \geq 1$ such that, for every $m \geq 0$, $\sup_{u\in G^0} {}^\#B^l_{G^u}(m) \leq c(1 + m)^r$.

Let $p = 3 + r$. Then, for each $u \in G^0$, we have

$$\begin{aligned}\sum_{x\in G^u} [1 + l(x)]^{-2p} &= \sum_{k=0}^{\infty} \sum_{x\in G^u, k\leq l(x)<k+1} [1 + l(x)]^{-2p}\\ &\leq \sum_{k=0}^{\infty} {}^\#B^l_{G^u}(k + 1) \cdot (1 + k)^{-2p}\\ &\leq c\sum_{k=0}^{\infty}(2 + k)^r(1 + k)^{-2p}\\ &\leq 2^r c\sum_{k=0}^{\infty}(1 + k)^{-6} = c',\end{aligned}$$

where c' is a positive constant only depending on l. Hence, for each $f \in C_c(G)$ and $u \in G^0$, we have

$$\begin{aligned}\sum_{x\in G^u} |f(x)| &= \sum_{x\in G^u} (|f(x)|[1 + l(x)]^p) \cdot [1 + l(x)]^{-p}\\ &\leq \left(\sum_{x\in G^u} (|f(x)|^2[1 + l(x)]^{2p}\right)^{\frac{1}{2}} \cdot \left(\sum_{x\in G^u} [1 + l(x)]^{-2p}\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \sqrt{c'}\|f\|_{2,p,r,l},$$

and $\sum_{x\in G_u}|f(x)| = \sum_{x\in G^u}|f^*(x)| \leq \sqrt{c'}\|f^*\|_{2,p,r,l} = \sqrt{c'}\|f\|_{2,p,d,l}$. We have established that $\|f\|_I \leq \sqrt{c'}\|f\|_{2,p,l}$ for each $f \in C_c(G)$. By Proposition 2.1, $\|f\|_{\rm red} \leq \|f\|_I$, thus $\|f\|_{\rm red} \leq \sqrt{c'}\|f\|_{2,p,l}$ for each $f \in C_c(G)$. Hence G has property (RD) with respect to l. □

Next we consider the property (RD) for transformation groups. Let $G = X \times \Gamma$ be a transformation group, as defined in Example 2.2. For a length function L on Γ, we can get a length function l on G by $l(x,g) = L(g)$ for $(x,g) \in G$. Note that, for each $m \geq 0$ and each $x \in X$, we have $\{g \in \Gamma :\, L(g) \leq m\} = \{g \in \Gamma :\, l(x,g) \leq m\}$, so ${}^{\#}B^l_{G^x}(m) ={}^{\#} \{g \in G :\, L(g) \leq m\}$. Hence G is of polynomial growth with respect to l if Γ is of polynomial growth with respect to L.

Let $\beta\Gamma$ be the Stone–Čech compactification of Γ, which consists of all nonzero multiplicative linear functionals on the abelian C^*-algebra $l^\infty(\Gamma)$. Under the w^*-topology, $\beta\Gamma$ is a compact Hausdorff space. For each $g \in \Gamma$, we let $\widehat{g}$ be the functional in $\beta\Gamma$ defined by $\widehat{g}(f) = f(g)$ for $f \in l^\infty(\Gamma)$. If we identify each $\widehat{g}$ with g, then Γ is a dense subset of $\beta\Gamma$. Consider the action of Γ on $\beta\Gamma$ given by the right multiplication of Γ: $\widehat{g}h = \widehat{gh}$ for $g,h \in \Gamma$. Let $G = \beta\Gamma \times \Gamma$ be the transformation group. Then the reduced groupoid C^*-algebra $C^*_r(G)$ is isomorphic to the reduced crossed product $C(\beta\Gamma) \rtimes_{\alpha,r} \Gamma$, where the action α of Γ on $C(\beta\Gamma)$ is given by $\alpha_g(f)(x) = f(xg)$ for $g \in \Gamma$, $f \in C(\beta\Gamma)$ and $x \in \beta\Gamma$.

Recall that the Schwartz function space $S^L_2(\Gamma, C(\beta\Gamma))$ of the action α of Γ on $C(\beta\Gamma)$ with respect to L is the set of all functions $\varphi : \Gamma \to C(\beta\Gamma)$ satisfying

$$\|\varphi\|_n = \Big\|\sum_{g\in\Gamma} \alpha_{g^{-1}}(\varphi(g)^*\varphi(g))(1 + l(g))^{2n}\Big\|_{C(\beta\Gamma)} < \infty$$

for all $n = 0, 1, \ldots$. Then $S^L_2(\Gamma, C(\beta\Gamma))$ becomes a Fréchet space with the topology generate by the seminorms $\|\cdot\|_n$, $n = 0, 1, \ldots$. By Theorem 3.4 in [7], if $S^L_2(\Gamma, C(\beta\Gamma))$ is contained in $C(\beta\Gamma) \rtimes_{\alpha,r} \Gamma$, then Γ is of polynomial growth with respect to L. In view of property (RD) for groupoids, we can rewrite a result in [7].

Theorem 3.6 ([7]) *Let Γ be a countable discrete group with a proper length function L. Then Γ is of polynomial growth with respect to L if and only if $G = \beta\Gamma \times \Gamma$ has property* (RD) *with respect to the length function l, defined by $l(x,g) = L(g)$ for each $(x,g) \in G$.*

Proof Suppose Γ is of polynomial growth with respect to L. We have shown that G is of polynomial growth with respect to l, hence has property (RD).

Suppose G has property (RD) with respect to l. Let $C_c(\Gamma, C(\beta\Gamma))$ be the set of all functions with finite support from Γ into $C(\beta\Gamma)$. Then it is a dense subalgebra of $C(\beta\Gamma) \rtimes_{\alpha,r} \Gamma$ under the reduced crossed norm. Let $\Pi : C_c(G) \to C_c(\Gamma, C(\beta\Gamma))$ be defined by $\Pi(\varphi)(g)(x) = \varphi(x,g)$ for $\varphi \in C_c(G)$, $g \in \Gamma$ and $x \in \beta\Gamma$. Then Π can be extended to a $*$-isomorphism, still denoted by Π, from $C^*_r(G)$ onto $C(\beta\Gamma) \rtimes_{\alpha,r} \Gamma$. We show that the isomorphism Π maps $S^l_2(G)$ onto $S^L_2(\Gamma, C(\beta\Gamma))$, and is an isomorphism as Férchet algebras.

In fact, for each $\varphi \in S^l_2(G)$ and each $n = 0, 1, \ldots$, we have

$$\|\Pi(\varphi)\|_n = \sup_{x\in\beta\Gamma}\Big|\sum_{g\in\Gamma} \alpha_{g^{-1}}(\Pi(\varphi)(g)^*\Pi(\varphi)(g))(x)(1 + L(g))^{2n}\Big|$$

$$= \sup_{x\in\beta\Gamma} \sum_{g\in\Gamma} |\varphi(xg^{-1}, g)|^2 (1 + l(xg^{-1}, g))^{2n}$$
$$= \|\varphi\|^2_{2,n,d,l}$$

and

$$\begin{aligned}\|\Pi(\varphi^*)\|_n &= \sup_{x\in\beta\Gamma} \sum_{g\in\Gamma} |\Pi(\varphi^*)(g)(xg^{-1})|^2 (1 + L(g))^{2n}| \\ &= \sup_{x\in\beta\Gamma} \sum_{g\in\Gamma} |\varphi^*(xg^{-1}, g)|^2 (1 + l(xg^{-1}, g))^{2n} \\ &= \sup_{x\in\beta\Gamma} \sum_{g\in\Gamma} |\varphi(x, g)|^2 (1 + (x, g))^{2n} \\ &= \|\varphi\|^2_{2,n,r,l}.\end{aligned}$$

On the other hand, for each $\psi \in S^L_2(\Gamma, C(\beta\Gamma))$, define $\varphi(x, g) = \psi(g)(x)$ for $(x, g) \in \beta\Gamma \times \Gamma$. Then φ is continuous on $\beta\Gamma \times \Gamma$, $\|\varphi\|_{2,n,d,l} = \|\psi\|_n$ and $\|\varphi\|_{2,n,r,l} = \|\psi^*\|_n$ for each $n = 0, 1, \ldots$, and $\Pi(\varphi) = \psi$. Hence, the restriction $\Pi|_{S^l_2(G)}$ of Π to $S^l_2(G)$ is an algebraic isomorphism from $S^l_2(G)$ onto $S^L_2(\Gamma, C(\beta\Gamma))$. Since G has property (RD) with respect to l, we have $S^l_2(G)$ is contained in $C^*_r(G)$, so $S^L_2(\Gamma, C(\beta\Gamma))$ is contained in $C(\beta\Gamma) \rtimes_{\alpha,r} \Gamma$. It follows from Theorem 3.4 in [7] that Γ has polynomial growth with respect to L. □

4 Spectral Invariance

In this section, we show that the space $S^l_2(G)$ of rapidly decreasing functions on an r-discrete groupoid G with respect to a continuous length function l is spectral invariant if G has property (RD) with respect to l. We first recall a Connes' technical lemma on spectral invariance [11].

Lemma 4.1 ([11]) *Let $\mathcal{B}$ be a C^*-algebra, and $\mathcal{A}$ a C^*-subalgebra of $\mathcal{B}$. For a closed derivation $\delta : \mathrm{Dom}(\delta) \longrightarrow \mathcal{B}$, let $\mathcal{A}_0 = \bigcap_{k=0}^{\infty} \mathrm{Dom}(\delta^k) \cap \mathcal{A}$, where $\mathrm{Dom}(\delta^k)$, contained in $\mathcal{B}$, denotes the domain of δ^k for each k. Then $\mathcal{A}_0$ is a Fréchet algebra under the topology generated by the family of semi-norms, $\{\|\cdot\|_k : k = 0, 1, 2, \ldots\}$, where $\|A\|_k = \|\delta^k(A)\|$ for each $A \in \mathcal{A}_0$ and each k. Moreover, if $\mathcal{A}_0$ is dense in $\mathcal{A}$, then it is a spectral invariant subalgebra of $\mathcal{A}$, and the inclusion $\mathcal{A}_0 \subseteq \mathcal{A}$ induces an isomorphism on their K-theories.*

Borrowing Ji's idea in [11], we have the following result.

Theorem 4.2 *Let G be an r-discrete groupoid. If G has property* (RD) *with respect to a continuous length function l on G, then $S^l_2(G)$ is a spectral invariant *-subalgebra of $C^*_r(G)$. Moreover, the inclusion $S^l_2(G) \subseteq C^*_r(G)$ induces an isomorphism on their K-theories.*

Proof Let $u \in G^0$ be fixed. Let $\mathrm{Ind}_u : C^*_r(G) \to B(l^2(G_u))$ be the induced representation by $\mathrm{Ind}_u(f)(\xi)(x) = \sum_{y\in G^u} f(xy)\xi(y^{-1})$ for $f \in C^*_r(G)$, $\xi \in l^2(G_u)$, $x \in G_u$. Let T_u be the multiplicative mapping on $l^2(G_u)$, given by the length function l: $T_u(\xi)(x) = l(x)\xi(x)$ for $\xi \in l^2(G_u)$ and $x \in G_u$. Then the mapping δ_u, defined by $\delta_u(A) = i(T_uA - AT_u)$, is a closed $*$-derivation on $B(l^2(G_u))$.

For each $f \in C_c(G)$, $\xi \in l^2(G_u)$ and $x \in G_u$, we have

$$\delta_u(\mathrm{Ind}_u(f))(\xi)(x) = i \sum_{y\in G^u} f(xy)\xi(y^{-1})(l(x) - l(y)).$$

Moreover, by induction, we can check that

$$\delta_u^k(\mathrm{Ind}_u(f))(\xi)(x) = i^k \sum_{y\in G^u} f(xy)\xi(y^{-1})(l(x)-l(y))^k \quad \text{for } k=1,2,\ldots.$$

Hence, for each $f \in C_c(G)$ and $\xi \in l^2(G_u)$, we have

$$\begin{aligned}
\|\delta_u^k(\mathrm{Ind}_u(f))\xi\|^2 &= \sum_{x\in G_u} |\delta_u^k(\mathrm{Ind}_u(f))\xi)(x)|^2 \\
&= \sum_{x\in G_u}\left|\sum_{y\in G^u} f(xy)\xi(y^{-1})(l(x)-l(y))^k\right|^2 \\
&\le \sum_{x\in G_u}\left(\sum_{y\in G^u} |f(xy)l(xy)^k|\cdot|\xi(y^{-1})|\right)^2 \\
&\le \sum_{x\in G_u}\left(\sum_{y\in G^u} f^{(k)}(xy)\cdot|\xi(y^{-1})|\right)^2 \\
&\le \|\mathrm{Ind}_u(f^{(k)})(|\xi|)\|^2 \\
&\le \|\mathrm{Ind}_u(f^{(k)})\|^2\|\xi\|^2,
\end{aligned}$$

where $f^{(k)}(x) = |f(x)|(1+l(x))^k$, so

$$\|\delta_u^k(\mathrm{Ind}_u(f))\| \le \|\mathrm{Ind}_u(f^{(k)})\| \le \|f^{(k)}\|_{\mathrm{red}}. \tag{2}$$

Let $S_u(G) = (\bigcap_{k=0}^{\infty}\mathrm{Dom}(\delta_u^k)) \cap \mathrm{Ind}_u(C_r^*(G))$. It follows from Lemma 4.1 that $S_u(G)$ is a Fréchet algebra under the topology generated by the family of semi-norms, $\{\|\cdot\|_k : k = 0,1,2,\ldots\}$, where $\|A\|_k = \|\delta_u^k(A)\|$ for each $A \in S_u(G)$ and k. By (2), $S_u(G)$ contains $\mathrm{Ind}_u(C_c(G))$, so it is spectral invariant in $\mathrm{Ind}_u(C_r^*(G))$.

For $f \in S_2^l(G)$, by Proposition 3.1 (i), we can choose a sequence $\{f_n\}$ in $C_c(G)$ converging to f under each norm $\|\cdot\|_{2,p,l}$ $(p\ge 0)$, as well as under the reduced norm $\|\cdot\|_{\mathrm{red}}$. For every $n,m,k \ge 1$, by (2), we have $\|\delta_u^k(\mathrm{Ind}_u(f_n)) - \delta_u^k(\mathrm{Ind}_u(f_m))\| \le \|f_n^{(k)} - f_m^{(k)}\|_{\mathrm{red}} \le c\|f_n^{(k)} - f_m^{(k)}\|_{2,q,l} \le c\|f_n - f_m\|_{2,q+k.l}$, where c and q are constants independent of u, m and n, as in Proposition 3.1. Hence there exists $T \in B(l^2(G_u))$ such that $\delta_u^k(\mathrm{Ind}_u(f_n)) \to T$ in $B(l^2(G_u))$. Since $\mathrm{Ind}_u(f_n) \to \mathrm{Ind}_u(f)$ and δ_u^k is closed, $\mathrm{Ind}_u(f) \in \mathrm{Dom}(\delta_u^k)$ and $T = \delta_u^k(\mathrm{Ind}_u(f))$. Thus $\mathrm{Ind}_u(f) \in S_u(G)$, so that $\mathrm{Ind}_u(S_2^l(G)) \subseteq S_u(G)$. Moreover, for the above constants $c > 0$ and $q \ge 1$, we have

$$\|\delta_u^k(\mathrm{Ind}_u(f))\| \le c\|f\|_{2,q+k,l} \qquad \text{for each } f \in S_2^l(G). \tag{3}$$

Assume that $A \in C_r^*(G)$ satisfies that $\mathrm{Ind}_u(A) \in S_u(G)$. Then $\mathrm{Ind}_u(A) \in \mathrm{Dom}(\delta_u^k)$ for each $k \ge 0$. If let $e_u \in l^2(G_u)$ be the characteristic function on $\{u\}$, then

$$\begin{aligned}
\textstyle\sum_{x\in G_u} |A(x)|^2 l(x)^{2k} &= \|\delta_u^k(\mathrm{Ind}_u(A))e_u\|^2 \le \|\delta_u^k(\mathrm{Ind}_u(A)\|^2 < \infty, \\
\textstyle\sum_{x\in G^u} |A(x)|^2 l(x)^{2k} &= \|\delta_u^k(\mathrm{Ind}_u(A^*))e_u\|^2 \le \|\delta_u^k(\mathrm{Ind}_u(A)\|^2 < \infty.
\end{aligned} \tag{4}$$

Let $S(G) = \left\{ A \in C_r^*(G) \,\middle|\, \begin{array}{l} \mathrm{Ind}_u(A) \in S_u(G) \text{ for each } u \in G^0 \text{ and} \\ \sup_{u\in G^0} \|\delta_u^k(\mathrm{Ind}_u(A))\| < \infty \text{ for each } k \ge 0 \end{array} \right\}.$

For $A \in S(G)$, we let $|\!|\!|A|\!|\!|_n = \sup_{u\in G^0}\|\delta_u^n(\mathrm{Ind}_u(A))\|$ for each $n \ge 0$. Then $S(G)$ is a Fréchet $*$-algebra under the local convex topology given by the family of semi-norms, $\{|\!|\!|\cdot|\!|\!|_n : n \ge 0\}$.

With a given $A \in S(G)$, by (4) and the basic inequality $(1+l(x))^2 \le 2(1+l(x)^2)$ for $x \in G$, we have, for each $k \ge 0$,

$$\|A\|_{2,k,l}^2 \le 2^k(\|\!|A|\!\|_0^2 + C_k^1\|\!|A|\!\|_1^2 + C_k^2\|\!|A|\!\|_2^2 + \cdots + C_k^{k-1}\|\!|A|\!\|_{k-1}^2 + \|\!|A|\!\|_k^2),$$

where C_k^i denotes the combination number. Hence $A \in S_2^l(G)$. From (3), it follows that $\|\!|f|\!\|_n \le c\|f\|_{2,q+n,l}$ for each $f \in S_2^l(G)$ and $n \ge 0$. Consequently, $S(G) = S_2^l(G)$ and they have the same topology.

In order to show $S_2^l(G)$ is spectral invariant in $C_r^*(G)$, it is sufficient to show $S(G)$ is spectral invariant in $C_r^*(G)$. For $A \in S(G)$, if let $B \in C_r^*(G)$ be the inverse of A in $C_r^*(G)$, then, for each $u \in G^0$, $\mathrm{Ind}_u(B)$ is the inverse of $\mathrm{Ind}_u(A)$ in $\mathrm{Ind}_u(C_r^*(G))$. Since $\mathrm{Ind}_u(A) \in S_u(G)$ and $S_u(G)$ is spectral invariant in $\mathrm{Ind}_u(C_r^*(G))$, we have $\mathrm{Ind}_u(B) \in S_u(G)$, so $\delta_u^n(\mathrm{Ind}_u(B))$ is well defined for every $n \ge 0$. By induction, from the fact $\delta_u^n(\mathrm{Ind}_u(A)\mathrm{Ind}_u(B)) = 0$ for each $n \ge 1$ and $\|\!|A|\!\|_n = \sup_{u\in G^0} \|\delta_u^n(\mathrm{Ind}_u(A))\| < \infty$, we can obtain that $\|\!|B|\!\|_n = \sup_{u\in G^0} \|\delta_u^n(\mathrm{Ind}_u(B))\| < \infty$. Hence B is in $S(G)$. □

5 C^*-algebras Arising from Expansive Dynamical Systems

Let X be a compact metrizable space and φ a homeomorphism of X. We assume that φ is expansive; this means that, for a given metric d compatible with the topology, there is $\epsilon > 0$ such that $d(\varphi^n(x), \varphi^n(y)) \le \epsilon$, for all integers n, implies $x = y$. We say that x and y in X are conjugate if $\lim_{|n|\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0$. One can check that conjugacy is an equivalence relation on X, and each equivalence class is countable. We assume that the dynamical system (X, d, φ) satisfies the following condition:

(C) *For every conjugate pair (x, y), there is a map $\gamma : \mathcal{O} \to X$ such that $\mathcal{O}$ is an open neighborhood of x, γ is continuous at x, $\gamma(x) = y$, and*

$$\lim_{|n|\to\infty} d(\varphi^n(z), \varphi^n(\gamma(z))) = 0 \tag{5}$$

uniformly for $z \in \mathcal{O}$.

From [20] (or [1]), if the condition **(C)** holds, the germ of γ at x is uniquely determined by (x, y), and one can define a *conjugating homeomorphism* as a pair $(\mathcal{O}, \gamma)$, where $\mathcal{O}$ is an open subset of X, and γ is a homeomorphism from $\mathcal{O}$ onto $\gamma(\mathcal{O})$ such that the equation (5) holds uniformly for $z \in \mathcal{O}$.

Let G_a be the set of all conjugate pairs (x, y). If G_a is equipped with the topology whose base is given by the following open sets:

$$\{(z, \gamma z)|\ z \in \mathcal{O}\}, \qquad \text{for a conjugating homeomorphism } (\mathcal{O}, \gamma),$$

then it is a separable and r-discrete groupoid.

Remark 5.1 Each Smale space (X, d, φ) satisfies the condition **(C)** [18, 20]. In this situation, G_a is precisely the groupoid defined by Putnam in [18] using the asymptotic equivalence relation on X. Some important dynamical systems, for example, subshifts of finite type (SFT), Anosov diffeomorphisms and solenoids, are Smale spaces, hence satisfy the condition **(C)**.

In this section, we consider the property (RD) for the groupoid G_a associated with SFT and solenoids.

5.1 Subshifts of Finite Type (SFT)

Let N be a positive integer and let A be a fixed $N\times N$ matrix whose entries $A_{i,j}$ are zeros or ones. We assume that A is primitive, i.e. for some positive integer k, A^k has no zero entries. Let $\{1,2,\dots,N\}^{\mathbb{Z}}$ be the space of doubly infinite sequences of $\{1,2,\dots,N\}$. Let

$$X=\{\{x_i\}\in\{1,2,\dots,N\}^{\mathbb{Z}}|\ \ A_{x_i,x_{i+1}}=1 \text{ for all } i\in\mathbb{Z}\}$$

and φ the mapping from X onto itself defined by

$$\varphi(x)_i=x_{i+1}\quad \text{ for } x=\{x_i\}\in X.$$

Then, with the metric

$$d(x,y)=\sum_{i\in\mathbb{Z}}2^{-|i|}|x_i-y_i|\quad \text{ for } x=\{x_i\} \text{ and } y=\{y_i\}\in X,$$

X is a compact metric space and φ is a homeomorphism on X. Moreover, (X,φ) is an irreducible Smale space, so it satisfies the condition **(C)** ([18]).

One can verify that x and y in X are conjugate if and only if, there is an integer $k_0\geq 0$ such that $x_k=y_k$ for each k with $|k|\geq k_0$. Hence, for $g=(x,y)\in G_a$, we can define

$$L(g)=\min\{k_0\geq 0:\ x_k=y_k,\ \text{for all } k \text{ with } |k|\geq k_0\},$$
$$l(g)=N^{L(g)}-1.$$

Then $L(gh)\leq\max\{L(g),L(h)\}$ and $l(gh)\leq\max\{l(g),l(h)\}$ for every $(g,h)\in G_a^2$, so L and l are length functions on G_a.

Theorem 5.2 *The groupoid G_a is of polynomial growth with respect to l. Hence $S_2^l(G_a)$ is a spectral invariant and dense *-subalgebra of $C_r^*(G_a)$.*

Proof We first show L is continuous on G_a. For $(x,y)\in G_a$, since G_a^0 is open in G_a and $L|_{G_a^0}=0$, we can assume that $x\neq y$. Let $k_0=L(x,y)>0$ and write $x=\{x_n\}$ and $y=\{y_n\}$. Then $x_k=y_k$ when $|k|\geq k_0$, and $x_{k_0-1}\neq y_{k_0-1}$ or $x_{-k_0+1}\neq x_{-k_0+1}$. Set

$$\mathcal{O}_x=\{z\in X:\ \ z_k=x_k \text{ for each } |k|<k_0\}$$

and define the mapping γ from $\mathcal{O}_x$ into X by

$$(\gamma z)_k=z_k \text{ for } |k|\geq k_0,\quad \text{ and }\quad (\gamma z)_k=y_k \text{ for } |k|<k_0$$

for each $z\in\mathcal{O}_x$. For each $n\in\mathbb{Z}$, and $z\in\mathcal{O}_x$,

$$d(\varphi^n z,\varphi^n\gamma z)=\sum_{k\in\mathbb{Z}}\frac{|(\varphi^n z)_k-(\varphi^n\gamma z)_k|}{2^{|k|}}=\sum_{j=-k_0+1}^{k_0-1}\frac{|x_j-y_j|}{2^{|n-j|}}\leq\frac{c}{2^{|n|}},$$

where c is a constant, only depending on N and k_0. Hence

$$\lim_{|n|\longrightarrow\infty}d(\varphi^n z,\varphi^n\gamma z)=0,\quad \text{uniformly for } z\in\mathcal{O}_x.$$

Obviously, γ is injective, continuous at x and $\gamma x=y$. From [1], $(\mathcal{O}_x,\gamma)$ is a conjugating homeomorphism determined by (x,y), so $U=\{(z,\gamma z):\ z\in\mathcal{O}_x\}$ is an open neighborhood of (x,y) in G_a. For each $z\in\mathcal{O}_x$, by the definition of $\mathcal{O}_x$, we have $(\gamma z)_k=z_k$ when $|k|\geq k_0$, and $z_{k_0-1}(=x_{k_0-1})\neq(\gamma z)_{k_0-1}(=y_{k_0-1})$ or $z_{-k_0+1}(=x_{-k_0+1})\neq(\gamma z)_{-k_0+1}(=y_{-k_0+1})$. Hence $L(z,\gamma z)=k_0$, so L is continuous at (x,y). Consequently, l is continuous on G_a.

Let $u=\{u_n\}\in X$ be given. For $x=\{x_n\}\in X$ and $m>0$, we have $(u,x)\in G_a$ and $L(u,x)\le m$ if and only if $u_k=x_k$ when $|k|\ge m$. So

$$\begin{aligned}\#B^L_{G^u_a}(m)&=\#\left\{(x_{-m},x_{-m+1},\dots,x_{m-1},x_m)\middle|\begin{array}{l}x_{-m}=u_{-m},x_m=u_m,\\ A_{x_i,x_{i+1}}=1,-m\le i<m\end{array}\right\}\\&=(A^{2m})_{u_{-m},u_m},\text{i.e., the }(u_{-m},u_m)\text{-entry of }A^{2m}\\&\le N^{2m}.\end{aligned}$$

Hence

$$\#B^l_{G^u_a}(m)\le\#B^L_{G^u_a}([\log_N^{m+1}+1])\le N^2(m+1)^2.$$

Hence G_a is of polynomial growth with respect to l. From Proposition 3.2 and Theorem 4.2, $S^l_2(G_a)$ is a spectral invariant and dense *-subalgebra of $C^*_r(G_a)$. □

5.2 Solenoids

Let $\mathbb{S}^1$ be the unit circle in the complex plane and $\sigma:\mathbb{S}^1\longrightarrow\mathbb{S}^1$ the map defined by $\sigma(x)=x^2$ for $x\in\mathbb{S}^1$. The inverse limit of the system $\{X_n,\sigma\}_{n\ge0}$ is called a solenoid. Concretely, we can describe X as

$$X=\{(x_0,x_1,\dots)|\ x_n\in\mathbb{S}^1,\sigma(x_{n+1})=x_n,n=0,1,\dots\}.$$

Under the following metric,

$$d(x,y)=\sum_{n=0}^{\infty}\frac{d'(x_n,y_n)}{2^n}\qquad\text{for }x=(x_n)_{n\ge0},\ y=(y_n)_{n\ge0}\in X,$$

X is a compact metric space, where d' is the Riemannian metric on $\mathbb{S}^1$, i.e., for $\alpha_1,\alpha_2\in[-\pi,\pi)$, $d'(\mathrm{e}^{i\alpha_1},\mathrm{e}^{i\alpha_2})=|\alpha_1-\alpha_2|$ if $|\alpha_1-\alpha_2|\le\pi$ and $d'(\mathrm{e}^{i\alpha_1},\mathrm{e}^{i\alpha_2})=2\pi-|\alpha_1-\alpha_2|$ if $|\alpha_1-\alpha_2|>\pi$.

Let φ be the mapping from X onto itself, defined by

$$\varphi:\ (x_0,x_1,x_2,\dots)\longrightarrow(x_0^2,x_0,x_1,\dots)\text{ for }x=(x_0,x_1,x_2,\dots)\in X.$$

Then φ is a homeomorphism on X with the inverse $\varphi^{-1}(x_0,x_1,x_2,\dots)=(x_1,x_2,\dots)$ for $x=(x_0,x_1,x_2,\dots)\in X$. By [18], (X,d,φ) is an irreducible Smale space, so satisfies the condition **(C)**.

In order to describe the groupoid G_a of conjugate pairs on (X,d,φ), we use the dyadic rational additive group $\Gamma=\{\frac{m}{2^n}:\ m,n\in\mathbb{Z},n\ge1\}$ and a canonical flow $\{F_t|\,t\in\mathbb{R}\}$ on X. For each $t\in\mathbb{R}$, we let

$$F_t:\ (x_0,x_1,x_2,\dots)\longrightarrow(\mathrm{e}^{2\pi ti}x_0,\mathrm{e}^{\frac{2\pi ti}{2}}x_1,\mathrm{e}^{\frac{2\pi ti}{2^2}}x_2,\dots),\quad\text{for }(x_0,x_1,x_2,\dots)\in X.$$

Then $F_sF_t=F_{s+t}$ for all $s,t\in\mathbb{R}$. For $x=\{x_n\}_{n\ge0}$ and $y=\{y_n\}_{n\ge0}\in X$, one can check that $\lim_{n\to+\infty}d(\varphi^nx,\varphi^ny)=0$ if and only if $y_0=tx_0$ for some $t\in\Gamma$, and $\lim_{n\to+\infty}d(\varphi^{-n}x,\varphi^{-n}y)=0$ if and only if $y=F_tx$ for some $t\in\mathbb{R}$. Hence, x and y are conjugate, if and only if $y=F_tx$ for a unique $t\in\Gamma$. So, Γ acts on X by $\{F_t\}$.

For $t\in\Gamma$, let

$$l_1(t)=\begin{cases}\min\{n|\ t=\frac{m}{2^n},\text{for }m,n\in\mathbb{Z},n\ge1\}, & \text{if }t\ne0,\\ 0, & \text{if }t=0.\end{cases}$$

$$L_\Gamma(t) = \max(2^{l_1(t)} - 1,\ |t|).$$

Note that $l_1(s+t) \le \max(l_1(t), l_1(s))$ for $s, t \in \Gamma$. Then L_Γ is a length function on Γ. By the definition of l_1, we have, for $k \in \mathbb{N}$,

$$l_1(t) = k \text{ if and only if } t = \begin{cases} 0, & \text{if } k = 0, \\ \frac{n}{2^k},\ \text{for some odd } n, & \text{if } k > 1, \\ \frac{n}{2},\ n \in \mathbb{Z}, n \neq 0, & \text{if } k = 1. \end{cases}$$

As before, let G_a be the r-discrete groupoid consisting of all conjugate pairs associated with the solenoid (X, d, φ). Using the function L_Γ, we can give a length function l on G_a.

Definition 5.3 *For each $(x, y) \in G_a$, there exists a unique $t \in \Gamma$ such that $y = F_t x$. Define*

$$l(x, y) = L_\Gamma(t).$$

One can verify that l is a length function on G_a. We have the following result.

Theorem 5.4 *The groupoid G_a is of polynomial growth with respect to l, hence $S_2^l(G_a)$ is a spectral invariant and dense *-subalgebra of $C_r^*(G_a)$.*

Proof We first claim that, for each $t = \frac{m_0}{2^{n_0}} \in \Gamma$ for $m_0, n_0 \in \mathbb{Z}$ and $n_0 \ge 1$, (X, F_t) is a conjugating homeomorphism, i.e., F_t is a homeomorphism on X and $\lim_{|n|\to\infty} d(\varphi^n z, \varphi^n F_t z) = 0$ uniformly for $z \in X$.

In fact, one can easily check that F_t is a homeomorphism on X. For an arbitrary ϵ with $0 < \epsilon < \frac{\pi}{2}$, choose an integer N such that $N > \max(|m_0|, n_0)$ and $\frac{1}{2^N}(\sum_{j=0}^{\infty} \frac{2\pi(1+|t|)}{2^j} + \sum_{j=0}^{n_0-1} 2^j\pi) < \epsilon$. Let $n > N$ be given.

Since $|\frac{2\pi t}{2^{n+j}}| < \frac{\pi}{2}$ for $j = 0, 1, \ldots$, we have $d'(z_{n+j}, \exp(\frac{2\pi t \mathrm{i}}{2^{n+j}})z_{n+j}) = |\frac{2\pi t}{2^{n+j}}|$ for every $z = \{z_j\}_{j\ge0} \in X$. So

$$d(\varphi^{-n}z, \varphi^{-n}F_t z) = \sum_{j=0}^{\infty} \frac{d'(z_{n+j}, \exp(\frac{2\pi t\mathrm{i}}{2^{n+j}})z_{n+j})}{2^j} = \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{2\pi|t|}{4^j} < \epsilon$$

and

$$\begin{aligned} d(\varphi^n z, \varphi^n F_t z) &= \sum_{j=1}^{n} \frac{d'(\sigma^j z_0, \sigma^j (F_t z)_0)}{2^{n-j}} + \sum_{j=0}^{\infty} \frac{d'(z_j, (F_t z)_j)}{2^{n+j}} \\ &= \frac{1}{2^n}\sum_{j=1}^{n} 2^j d'(\sigma^j z_0, \exp(2\pi 2^j t\mathrm{i})\sigma^j z_0) + \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{d'(z_j, (F_t z)_j)}{2^j} \\ &= \frac{1}{2^n}\sum_{j=1}^{n_0-1} 2^j d'(\sigma^j z_0, \exp(2\pi 2^j t\mathrm{i})\sigma^j z_0) + \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{d'(z_j, (F_t z)_j)}{2^j} \\ &< \frac{1}{2^n}\left[\sum_{j=1}^{n_0-1} 2^j\pi + \sum_{j=0}^{\infty}\frac{\pi}{2^j}\right] < \epsilon. \end{aligned}$$

Hence $\lim_{|k|\to\infty} d(\varphi^k z, \varphi^k F_t z) = 0$ uniformly for $z \in X$. We have established the claim.

For $(x, y) \in G_a$, write $y = F_t x$ for some $t \in \Gamma$, so $l(x, y) = L_\Gamma(t)$. From the above claim, we have $U = \{(z, F_t z) |\ z \in X\}$ is an open subset of G_a containing (x, y), so that l is continuous at (x, y). Hence l is a continuous length function on G_a.

Let $k, k' \in \mathbb{N}$. We consider the element t in Γ such that $|t| \leq k$ and $l_1(t) = k'$. If $k' = 0$ or $k = 0$, then $t = 0$; if $k' = 1$, then $t = \frac{m}{2}$ for $m \in \mathbb{Z}$ and $0 < |m| \leq 2k$; if $k' \geq 2$, then $t = \frac{m}{2^{k'}}$ for an odd m with $|m| \leq 2^{k'} k$. Hence

$$^{\#}\{t \in \Gamma |\ |t| \leq k, l_1(t) \leq k'\} = \sum_{j=0}^{k'} \#\{t \in \Gamma |\ |t| \leq k, l_1(t) = j\} \leq 1 + 2^{k'+1} k.$$

So, for $u \in X$ and $m \in \mathbb{N}$, we have

$$^{\#}\{(u, x) \in G_a^u |\ l(u, x) \leq m\} =^{\#} \{t \in \Gamma |\, |t| \leq m, l_1(t) \leq \log_2^{m+1}\} \leq 4m^2 + 4m + 1.$$

Hence G_a is of polynomial growth with respect to l. From Proposition 3.2 and Theorem 4.2, $S_2^l(G_a)$ is a spectral invariant and dense $*$-subalgebra of $C_r^*(G_a)$. □

6 Applications

For a countable discrete group Γ, one can associate to a normalized cocycle c on Γ a cyclic cocycle τ_c on the group algebra $\mathbb{C}\Gamma$ [3, 4]:

$$\tau_c(f_0, f_1, \dots, f_n) = \sum_{g_0 g_1 \cdots g_n = 1} f_0(g_0) f_1(g_1) \cdots f_n(g_n) c(g_1, g_2, \dots, g_n)$$

for $f_0, f_1, \dots, f_n \in \mathbb{C}\Gamma$. In general, τ_c cannot be extended as an n-trace on the reduced group C^*-algebra $C_r^*(\Gamma)$, but in [13], Jolissaint proved that, with some additional hypotheses, τ_c plays the same role as an n-trace. In this section, we can give a similar application of property (RD) for r-discrete topology groupoids.

Given an algebra $\mathcal{A}$ over $\mathbb{C}$, for $n \geq 1$, let $Z^n(\mathcal{A})$ be the set of all $(n+1)$-linear functionals φ on $\mathcal{A}$ such that $\mathbf{b}\varphi = 0$, where $\mathbf{b}$ is the Hochschild coboundary map given by

$$\begin{aligned}&(b\varphi)(a_0, a_1, \dots, a_{n+1}) \\ &\quad = \sum_{j=0}^{n} (-1)^j \varphi(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \dots, a_n)\end{aligned}$$

for all $a_i \in \mathcal{A}$. We denote by $Z_\lambda^n(\mathcal{A})$ the set of all cyclic n-cocycles φ on $\mathcal{A}$, i.e., $\varphi \in Z^n(\mathcal{A})$ and $\varphi(a_1, \dots, a_n, a_0) = (-1)^n \varphi(a_0, a_1, \dots, a_n)$ for all $a_i \in \mathcal{A}$. For the cyclic cohomology theory, we refer to [3].

Let G be an r-discrete, topology groupoid. The sets G^0, $G^1 = G$ and G^2 have already been defined. For $n \geq 3$, we set

$$G^n = \{(g_0, g_1, \dots, g_{n-1}) :\ g_i \in G,\ d(g_i) = r(g_{i+1}), i = 0, 1, \dots, n-2\}.$$

Let c be a normalized n-cocycle on G. That is, c is a complex continuous function on G^n with the product topology and satisfies:

(i) for all $(g_0, g_1, \dots, g_n) \in G^{n+1}$,

$$\begin{aligned}&\mathbf{b}c(g_0, g_1, g_2, \dots, g_n) \\ &\quad = c(g_1, g_2, \dots, g_n) + \sum_{j=1}^{n} (-1)^j c(g_0, g_1, \dots, g_{j-1} g_j, \dots, g_n) \\ &\qquad + (-1)^{n+1} c(g_0, g_1, \dots, g_{n-1}) = 0\,;\end{aligned}$$

(ii) $c(g_0, \dots, g_{n-1}) = 0$, for every $(g_0, \dots, g_{n-1}) \in G^n$ with $g_0 g_1 \cdots g_{n-1} \in G^0$,or $g_i \in G^0$ for some i.

For $u \in G^0$, we define

$$\tau_c^u(f_0, f_1, \dots, f_n) = \sum_{g_0 g_1 \cdots g_n = u} f_0(g_0) f_1(g_1) \cdots f_n(g_n) c(g_1, g_2, \dots, g_n),$$

for $f_i \in C_c(G)$, $i = 0, 1, \dots, n$.

Remark 6.1 For each $f \in C_c(G)$, since G^{op} is a base for the topology, the (compact) support of f is covered by a finite number of sets in G^{op}. Using a partition of unity for the cover, we can obtain that f is a linear combination of functions f_i in $C_c(G)$ with support contained in some sets in G^{op}. Hence we can assume that the support of each f_i in the definition of τ_c^u is contained in some A_i in G^{op}. For each u in G^0, if u has a decomposition $u = g_0 g_1 \cdots g_n$ for $g_i \in A_i$, $i = 0, 1, \dots, n$, then such decomposition is unique. So $\tau_c^u(f_0, f_1, \dots, f_n)$ is well defined.

Recall that each positive Borel probability measure μ on G^0 induces a Borel measure ν on G, defined by $\int_G f(g) d\nu(g) = \int_{G^0} \sum_{g \in G^u} f(g) d\mu(u)$ for $f \in C_c(G)$. Let ν^{-1} be the image of ν under the inverse map on G, i.e., ν^{-1} is given by $\int_G f(g) d\nu^{-1}(g) = \int_{G^0} \sum_{g \in G_u} f(g) d\mu(u)$ for $f \in C_c(G)$. We say μ is invariant if $\nu = \nu^{-1}$. Under this situation, for each $A \in G^{\mathrm{op}}$, if $r(A)$ and $d(A)$ are endowed with the relative measures of μ and let α_A be defined by $\alpha_A(r(a)) = d(a)$ for $a \in A$, then α_A is a homeomorphism preserving measure from $r(A)$ onto $d(A)$.

For f_i in $C_c(G)$ with support contained in $A_i \in G^{\mathrm{op}}$ for $i = 0, 1, \dots, n$, let

$$L_n = \{u \in G^0 : u = g_0 g_1 \cdots g_n \text{ for } g_i \in A_i, i = 0, 1, 2, \dots, n\},$$
$$L'_n = \{u \in G^0 : u = g_n g_0 g_1 \cdots g_{n-1} \text{ for } g_i \in A_i, i = 0, 1, 2, \dots, n\}.$$

Then $L_n \subseteq d(A_n)$, $L'_n \subseteq r(A_n)$, L_n and L'_n are open in G^0. Moreover, $\alpha_{A_n}(g_n g_0 g_1 \cdots g_{n-1}) = g_0 g_1 \cdots g_{n-1} g_n \in L_n$ for $u = g_n g_0 g_1 \cdots g_{n-1} \in L'_n$. Hence, $\alpha_{A_n}(L'_n) = L_n$ and the restriction of α_{A_n} to L'_n preserves the relative measures of μ on L'_n and L_n. So, for appropriate functions φ on L_n, we have

$$\int_{L_n} \varphi(u) d\mu(u) = \int_{L'_n} \varphi(\alpha_{A_n}(u)) d\mu(u).$$

For each $u \in G^0$, if $u \notin L_n$ then $\tau_c^u(f_0, f_1, \dots, f_n) = 0$; if $u = g_0 g_1 \cdots g_{n-1} g_n \in L_n$ then $\tau_c^u(f_0, f_1, \dots, f_n) = f_0(g_0) f_1(g_1) \cdots f_n(g_n) c(g_1, g_2, \dots, g_n)$. By the property of sets in G^{op}, $\tau_c^u(f_0, f_1, \dots, f_n)$ is continuous on L_n. Moreover, we have

$$\int_{L_n} f_0(g_0) \cdots f_n(g_n) c(g_1, \dots, g_n) d\mu(u) = \int_{L'_n} f_0(g_0) \cdots f_n(g_n) c(g_1, \dots, g_n) d\mu(u).$$

So, we can give the following definition.

Definition 6.2 *Let μ be an invariant probability measure on G^0. Define*

$$\tau_c(f_0, f_1, \dots, f_n) = \int_{u \in G^0} \tau_c^u(f_0, f_1, \dots, f_n) d\mu(u) \ \textit{ for } f_i \in C_c(G), \quad i = 0, 1, \dots, n.$$

Proposition 6.3 *τ_c is a cyclic cocycle on $C_c(G)$, i.e., $\tau_c \in Z^n_\lambda(C_c(G))$.*

Proof We first show τ_c is an n-cocycle on $C_c(G)$, i.e., $\tau_c \in Z^n(C_c(G))$. Without loss of generality, we can let f_i be in $C_c(G)$ with support contained in $A_i \in G^{\mathrm{op}}$ for $i = 0, 1, \dots, n+1$.

By the above argument, we have

$$\tau_c(f_0 f_1, f_2, \ldots, f_{n+1}) = \int_{L_{n+1}} f_0(g_0) f_1(g_1) \cdots f_{n+1}(g_{n+1}) c(g_2, \ldots, g_{n+1}) d\mu(u),$$

$$\begin{aligned}\tau_c(f_{n+1} f_0, f_1, \ldots, f_n) &= \int_{L'_{n+1}} f_{n+1}(g_{n+1}) f_0(g_0) \cdots f_n(g_n) c(g_1, g_2 \ldots, g_n) d\mu(u) \\ &= \int_{L_{n+1}} f_0(g_0) \cdots f_n(g_n) f_{n+1}(g_{n+1}) c(g_1, g_2 \ldots, g_n) d\mu(u),\end{aligned}$$

and, for each j with $1 \le j \le n$,

$$\begin{aligned}&\tau_c(f_0, f_1, \ldots, f_j f_{j+1}, \ldots, f_{n+1}) \\ &= \int_{L_{n+1}} f_0(g_0) \cdots f_j(g_j) f_{j+1}(g_{j+1}) \cdots f_{n+1}(g_{n+1}) c(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) d\mu(u).\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{b}\tau_c(f_0, f_1, \ldots, f_{n+1}) &= \sum_{j=0}^{n} (-1)^j \tau_c(f_0, f_1, \ldots, f_j f_{j+1}, \ldots, f_{n+1}) \\ &\quad + (-1)^{n+1} \tau_c(f_{n+1} f_0, f_1, \ldots, f_n) \\ &= \int_{L_{n+1}} f_0(g_0) \cdots f_n(g_n) f_{n+1}(g_{n+1}) (\mathbf{b}c(g_1, g_2 \ldots, g_n, g_{n+1}) d\mu(u) = 0.\end{aligned}$$

Consequently, τ_c is an n-cocycle on $C_c(G)$.

Next we show that τ_c is a cyclic cocycle. Let $f_0, f_1, \ldots f_n$ be in $C_c(G)$. As before, we only consider that each f_i has support contained in $A_i \in G^{\mathrm{op}}$ for $i = 0, 1, \ldots, n$. Then

$$\tau_c(f_0, f_1, \ldots, f_n) = \int_{L_n} f_0(g_0) f_1(g_1) \cdots f_n(g_n) c(g_1, g_2, \ldots, g_n) d\mu(u)$$

and

$$\begin{aligned}\tau_c(f_n, f_0, \ldots, f_{n-1}) &= \int_{L'_n} f_n(g_n) f_0(g_0) \cdots f_{n-1}(g_{n-1}) c(g_0, g_1, \ldots, g_{n-1}) d\mu(u) \\ &= \int_{L_n} f_0(g_0) \cdots f_{n-1}(g_{n-1}) f_n(g_n) c(g_0, g_1, \ldots, g_{n-1}) d\mu(u).\end{aligned}$$

It follows from the normality of c that $c(g_0, g_1, \ldots, g_{n-1}) = (-1)^n c(g_1, \ldots, g_{n-1}, g_n)$ for $u = g_0 g_1 \cdots g_{n-1} g_n \in L_n$. So, $\tau_c(f_0, f_1, \ldots, f_n) = (-1)^n \tau_c(f_n, f_0, f_1, \ldots, f_{n-1})$. □

We say an n-cocycle c on G is of polynomial growth with respect to a length function l, if there are constants $0 < b$ and $1 \le m$ such that

$$|c(g_0, g_1, \ldots, g_{n-1})| \le b(1 + l(g_0))^m (1 + l(g_1))^m \cdots (1 + l(g_{n-1}))^m$$

for each $(g_0, g_1, \ldots, g_{n-1}) \in G^n$. The following proposition is similar to that in [13].

Proposition 6.4 *Let G be an r-discrete, topological groupoid. Suppose that G has property* (RD) *with respect to a continuous length function l, and c is a normalized n-cocycle on G being of polynomial growth with respect to l. Then τ_c can be extended continuously to $S_2^l(G)$. Hence τ_c induces a map $\Phi_i : K_i(C_r^*(G)) \longrightarrow \mathbb{C}$, $i \equiv n (mod\ 2)$, such that*

(1) *if n is even and $e \in \mathrm{Proj}(M_q(S_2^l(G)))$ then $\Phi_0([e]) = (\tau_c \# Tr)(e, \ldots, e)$;*

(2) *if n is odd and $u \in \mathrm{GL}_q(S_2^l(G))$ then $\Phi_0([u]) = (\tau_c \# Tr)(u^{-1}, u, \ldots, u^{-1}, u)$.*

Proof From the assumption, there exist positive constants a, b and positive integers p and m such that $\|f\|_r \le a\|f\|_{2,p,l}$ and $|c(g_0, g_1, \dots, g_{n-1})| \le b(1+l(g_0))^m(1+l(g_1))^m \cdots (1+l(g_{n-1}))^m$ for $f \in C_c(G)$ and $(g_0, g_1, \dots, g_{n-1}) \in G^n$. For $u \in G^0$ and $f_0, f_1, \dots, f_n \in C_c(G)$, we have

$$\begin{aligned}
|\tau_c^u(f_0, \dots, f_n)| &\le \sum_{g_0 g_1 \cdots g_n = u} |f_0(g_0)||f_1(g_1)| \cdots |f_n(g_n)||c(g_1, g_2, \dots, g_n)| \\
&\le \sum_{g_0 g_1 \cdots g_n = u} b\phi_0(g_0)\phi_1(g_1) \cdots \phi_n(g_n) \\
&= b(\phi_0 * \phi_1 * \cdots * \phi_n)(u) \\
&\le b\|\phi_0 * \phi_1 * \cdots * \phi_n)\|_r \\
&\le b\|\phi_0\|_r \|\phi_1\|_r \cdots \|\phi_n\|_r \\
&\le a^n b\|f_0\|_r \cdot \|\phi_1\|_{2,p,l} \cdots \|\phi_n\|_{2,p,l} \\
&\le a^n b\|f_1\|_{2,p+m,l} \cdots \|f_n\|_{2,p+m,l} \|f_0\|_{2,p+m,l},
\end{aligned}$$

where $\phi_0(g) = |f_0(g)|$, $\phi_i(g) = |f_i(g)|(1+l(g))^m$, $i = 1, 2, \dots, n$. Since these constants a, b, p, m are independent of u and μ is a probability measure, we have $|\tau_c(f_0, f_1, \dots, f_n)| \le a^n b\|f_0\|_{2,p+m,l} \cdots \|f_n\|_{2,p+m,l}$. Hence τ_c can be extended continuously to $S_2^l(G)$.

From Theorem 4.2, $S_2^l(G)$ and $C_r^*(G)$ have isomorphic K-theories. By the pairing of the K-theory and the cyclic cohomology [3], τ_c induces a map $\Phi_i : K_i(C_r^*(G)) \to \mathbb{C}$ with the properties (1) and (2). □

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