Acta Mathematica Sinica, English Series
Dec., 2016, Vol. 32, No. 12, pp. 1494–1508
Published online: July 8, 2016
DOI: 10.1007/s10114-016-6069-z
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Fractal Dimensions of Fractional Integral of Continuous Functions

Yong Shun LIANG
Institute of Science, Nanjing University of Science and Technology, Nanjing 210014*, P. R. China*
E-mail: 80884903*@qq.com*

Wei Yi SU
Department of Mathematics, Nanjing University, Nanjing 210093*, P. R. China*
E-mail: *suqiu@nju.edu.cn*

Abstract In this paper, we mainly explore fractal dimensions of fractional calculus of continuous functions defined on closed intervals. Riemann–Liouville integral of a continuous function $f(x)$ of order $v(v > 0)$ which is written as $D^{-v}f(x)$ has been proved to still be continuous and bounded. Furthermore, upper box dimension of $D^{-v}f(x)$ is no more than 2 and lower box dimension of $D^{-v}f(x)$ is no less than 1. If $f(x)$ is a Lipshciz function, $D^{-v}f(x)$ also is a Lipshciz function. While $f(x)$ is differentiable on $[0, 1]$, $D^{-v}f(x)$ is differentiable on $[0, 1]$ too. With definition of upper box dimension and further calculation, we get upper bound of upper box dimension of Riemann–Liouville fractional integral of any continuous functions including fractal functions. If a continuous function $f(x)$ satisfying Hölder condition, upper box dimension of Riemann–Liouville fractional integral of $f(x)$ seems no more than upper box dimension of $f(x)$. Appeal to auxiliary functions, we have proved an important conclusion that upper box dimension of Riemann–Liouville integral of a continuous function satisfying Hölder condition of order $v(v > 0)$ is strictly less than $2 - v$. Riemann–Liouville fractional derivative of certain continuous functions have been discussed elementary. Fractional dimensions of Weyl–Marchaud fractional derivative of certain continuous functions have been estimated.

Keywords Hölder condition, fractional calculus, fractal dimension, bound, variation

MR(2010) Subject Classification 26A30, 28A33

1 Introduction

In [4], Riemann–Liouville fractional integral of a continuous function $f(x)$ of bounded variation on a closed interval has been proved to still be a continuous function with bounded variation. It is obvious that both $f(x)$ and its Riemann–Liouville fractional integral are 1-dimensional. Zhang discussed Riemann–Liouville fractional integral of certain 1-dimensional continuous function of unbounded variation [25]. Riemann–Liouville fractional integral of any continuous functions which have finite points of unbounded variation still are 1-dimensional [9]. There exist continuous functions whose box dimension are bigger than 1, such as Weierstrass function [1], Besicovitch function [17], linear fractal interpolation functions [15]. More details about functions of unbounded variation and definition of fractal functions can be found in [6].

Received February 1, 2016, revised April 25, 2016, accepted May 24, 2016
Supported by National Natural Science Foundation of China (Grant Nos. 11201230 and 11271182)

An important problem is to estimate fractal dimensions, such as box dimension or Hausdorff dimension, of Riemann–Liouville fractional integral of fractal functions [6]. In [5], author proved that upper box dimension of Riemann–Liouville integral of order v of any continuous functions on closed intervals are no more than $2 - v$ when $v \in (0, 1)$. We will prove this conclusion by a simple method in this paper. We mainly deal with fractal dimensions of Riemann–Liouville fractional integral of continuous functions defined on closed intervals and especially explore fractal dimensions of Riemann–Liouville fractional integral of continuous functions satisfying Hölder condition. Definition of Hausdorff dimension, box dimension and Riemann–Liouville fractional integral are given as follows.

Definition 1.1 ([1, 17]) *Let $F \in \mathbb{R}^n$ and $s \geq 0$. Hausdorff dimension of F is*

$$\dim_H(F) = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}. \tag{1.1}$$

Definition 1.2 ([1, 17]) *Let $F(\neq \phi)$ be any bounded subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. Lower box dimension and upper box dimension of F respectively are defined as*

$$\underline{\dim}_B(F) = \varliminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{1.2}$$

and

$$\overline{\dim}_B(F) = \varlimsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{1.3}$$

If (1.2) *and* (1.3) *are equal, we refer to the common value as box dimension of F*

$$\dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{1.4}$$

Definition 1.3 ([12, 14]) *Let $f(x)$ be a continuous function on a closed interval $[a, b]$. Let $v > 0$ and $D^{-v}f(0) = 0$. We call*

$$D^{-v}f(x) = \frac{1}{\Gamma(v)} \int_0^x (x - t)^{v-1} f(t)dt, \quad x \in [a, b] \tag{1.5}$$

Riemann–Liouville integral of $f(x)$ of order v on $[a, b]$.

Definition of other fractal dimensions and fractional calculus can be found in [1, 12, 14]. In the following several sections, we will give estimation of fractal dimensions of Riemann–Liouville fractional calculus of continuous functions on I.

Definition 1.4 *Let $f(x)$ be continuous on $[a, b]$. If for a positive constant C and $0 < \alpha < 1$,*

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \quad x, y \in [a, b], \tag{1.6}$$

we say $f(x)$ satisfies α-Hölder condition. When $\alpha = 1$, (1.6) *is called a Lipshciz condition.*

Throughout the present paper, let $I = [0, 1]$. C_I denotes the set of continuous functions on I. C_I^1 denotes the set of functions having continuous derivative functions on I. Denote

$$\Gamma(f, I) = \{(x, f(x)) : x \in I\}$$

as graph of $f(x)$ on I. Let C be a positive absolutely constant that may have different values at different occurrences even in the same line.

2 Basic Properties of $D^{-v}f(x)$

Let $f(x)\in C_I$. Thus $f(x)$ is unanimously continuous and bounded on I. In other words, there exists a positive number M such that $|f(x)|\le M$ for any $x\in I$. $D^{-v}f(x)$ is Riemann–Liouville integral of $f(x)$ on I of order $v(v>0)$ which is

$$D^{-v}f(x)=\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}f(t)dt,\quad x\in I. \tag{2.1}$$

First we have the following basic conclusion about $D^{-v}f(x)$ on I.

Proposition 2.1 *Let $D^{-v}f(x)$ be defined as* (2.1).

(1) *$D^{-v}f(x)$ is bounded on I for any positive order v.*

(2) *$D^{-v}f(x)$ is continuous on I for any positive order v.*

Though $D^{-v}f(x)$ is unanimously continuous on I, we do not know whether $D^{-v}f(x)$ is derivative on I or not. By Weierstrass theorem [27], we know $D^{-v}f(x)$ can be approximated by certain continuous functions which is differentiable on I. Thus, we have the following results.

Proposition 2.2 *Let $\delta>0, 0<\delta'<x$ and $0<v<1$. Write*

$$D_{\delta}^{-v}f(x)=\frac{1}{\Gamma(v)}\int_0^x (x+\delta-t)^{v-1}f(t)dt,\quad x\in I, \tag{2.2}$$

$$D_{\delta'}^{-v}f(x)=\frac{1}{\Gamma(v)}\int_0^{x-\delta'} (x-t)^{v-1}f(t)dt,\quad x\in I. \tag{2.3}$$

Then both $D_{\delta}^{-v}f(x)$ and $D_{\delta'}^{-v}f(x)$ are differentiable on I. Meanwhile,

$$\lim_{\delta\to 0}|D^{-v}f(x)-D_{\delta}^{-v}f(x)|=0, \tag{2.4}$$

$$\lim_{\delta'\to 0}|D^{-v}f(x)-D_{\delta'}^{-v}f(x)|=0. \tag{2.5}$$

Proof Let $0<v<1, \delta>0, 0<\delta'<x$ and $x\in I$. Then by (2.2) and (2.3),

$$\frac{d}{dx}D_{\delta}^{-v}f(x)=\frac{v-1}{\Gamma(v)}\int_0^x f(t)(x+\delta-t)^{v-2}dt+\frac{1}{\Gamma(v)}f(x)\delta^{v-1},$$
$$\frac{d}{dx}D_{\delta'}^{-v}f(x)=\frac{v-1}{\Gamma(v)}\int_0^{x-\delta'} f(t)(x-t)^{v-2}dt+\frac{1}{\Gamma(v)}f(x-\delta')\delta^{(\prime)v-1}.$$

Furthermore,

$$\begin{aligned}|D_{\delta}^{-v}(x)-D^{-v}f(x)|&=\left|\frac{1}{\Gamma(v)}\int_0^x f(t)(x+\delta-t)^{v-1}dt-\frac{1}{\Gamma(v)}\int_0^x f(t)(x-t)^{v-1}dt\right|\\&=\left|\frac{1}{\Gamma(v)}\int_0^x f(t)[(x+\delta-t)^{v-1}-(x-t)^{v-1}]dt\right|\le\frac{2M}{\Gamma(v)}\delta^{v}.\end{aligned}$$

Thus (2.4) holds. Similarly, we can get (2.5). □

$f(x)\in C_I$ satisfying Lipshciz condition means that

$$|f(x)-f(y)|\le C|x-y|,\quad x,y\in I.$$

Then, $D^{-v}f(x)$ still is a Lipshciz function on I with the following proposition.

Proposition 2.3 *Let $D^{-v}f(x)$ be defined as* (2.1) *and $v\in(0,1)$.*

(1) *If $f(x)$ is a Lipshciz function on I, $D^{-v}f(x)$ is also a Lipshciz function on I.*

(2) *If $f(x)$ is differentiable on I, $D^{-v}f(x)$ is differentiable on I too.*

We give an example in the end of this section.

Example 2.4 Let $f(x) = 0$ when $0 \leq x \leq 1$ and $f(x) = x - 1$ when $1 < x \leq 2$. $f(x)$ is a Lipshciz function on $[0, 2]$ and $\dim_H \Gamma(f, I) = \dim_B \Gamma(f, I) = 1$. Let $0 < v < 1$. If $0 \leq x \leq 1$, $D^{-v} f(x) = D^{-v} 0 = 0$. If $1 < x \leq 2$,

$$\begin{aligned} D^{-v} f(x) &= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t) dt \\ &= \frac{1}{\Gamma(v)} \int_1^x (x-t)^{v-1} (t-1) dt \\ &= \frac{1}{\Gamma(v+1)} (x-1)^v + \frac{1}{\Gamma(v+1)} \int_1^x (x-t)^v dt - \frac{1}{\Gamma(v+1)} (x-1)^v \\ &= \frac{1}{\Gamma(v+2)} (x-1)^{v+1}. \end{aligned}$$

Thus $D^{-v} f(x)$ is not only a Lipshciz function, but also a differentiable function on $[0, 2]$.

Let $0 < v < 1, 0 < \lambda < 1$ and $v + \lambda < 1$. If we let $f(x) = 0$ when $0 \leq x \leq 1$ and $f(x) = (x-1)^{\lambda}$ when $1 < x \leq 2$, $D^{-v} f(x)$ is not differentiable at point 1.

3 Fractal Dimensions of $D^{-v} f(x)$ with $f(x) \in C_I$

Riemann integral of a continuous function on I is smoother than function itself. So we discuss fractal dimensions of fractional calculus of continuous functions defined on I. By numerical simulation, we find Riemann–Liouville fractional integral of any continuous functions on I seems smoother than functions themselves. So we will investigate upper box dimension of Riemann–Liouville integral of $f(x) \in C_I$ of order v. First we give two lemmas as follows.

Lemma 3.1 ([1]) *Let $f(x) \in C_I$. Suppose that $\delta \in (0, 1)$ and n is the least integer greater than or equal to δ^{-1}. Then, if N_δ is number of squares of δ-mesh that intersect $\Gamma(f, I)$,*

$$\sum_{i=0}^{n-1} \max \left\{ \frac{R_f[i\delta, (i+1)\delta]}{\delta}, 1 \right\} \leq N_\delta \leq 2n + \delta^{-1} \sum_{i=0}^{n-1} R_f[i\delta, (i+1)\delta]. \tag{3.1}$$

Lemma 3.2 *Let $f(x) \in C_I$ and $0 \leq \alpha \leq 1$. If*

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}, \quad x, y \in I, \tag{3.2}$$

then

$$1 \leq \dim_H \Gamma(f, I) \leq \overline{\dim}_B \Gamma(f, I) \leq 2 - \alpha. \tag{3.3}$$

With Lemmas 3.1 and 3.2, we can give the following theorem which shows upper box dimension of Riemann–Liouville integral of any $f(x) \in C_I$ of order v is no more than $2 - v$ when $0 < v < 1$.

Theorem 3.3 *Let $f(x) \in C_I$ and $D^{-v} f(x)$ be defined as* (2.1). *Then*

$$\overline{\dim}_B \Gamma(D^{-v} f, I) \leq 2 - v, \quad 0 < v < 1. \tag{3.4}$$

Proof Let $0 \leq x < x + h \leq 1$. Then for $0 < v < 1$,

$$\begin{aligned} |D^{-v} f(x+h) - D^{-v} f(x)| = \frac{1}{\Gamma(v)} \bigg| & \int_0^x [(x+h-t)^{v-1} - (x-t)^{v-1}] f(t) dt \\ & + \int_x^{x+h} (x+h-t)^{v-1} f(t) dt \bigg| \end{aligned}$$

$$:= I_1 + I_2.$$

Since $f(x) \in C_I$, there exists a positive number M such that $|f(x)| \le M$ for $\forall x \in I$. Thus,

$$\begin{aligned} I_1 &\le \frac{1}{\Gamma(v)} \int_0^x |(x+h-t)^{v-1} - (x-t)^{v-1}| \cdot |f(t)| dt \\ &\le \frac{M}{\Gamma(v)} \int_0^x [(x-t)^{v-1} - (x+h-t)^{v-1}] dt \\ &\le \frac{M}{\Gamma(v+1)} h^v. \end{aligned}$$

Meanwhile,

$$\begin{aligned} I_2 &\le \frac{1}{\Gamma(v)} \int_x^{x+h} (x+h-t)^{v-1} |f(t)| dt \\ &\le \frac{M}{\Gamma(v)} \int_x^{x+h} (x+h-t)^{v-1} dt \\ &\le \frac{M}{\Gamma(v+1)} h^v. \end{aligned}$$

We have

$$|D^{-v} f(x+h) - D^{-v} f(x)| \le I_1 + I_2 \le \frac{2M}{\Gamma(v+1)} h^v.$$

From Lemma 3.2, (3.4) holds. □

If box dimension of $D^{-v} f(x)$ exists, then

$$\dim_H \Gamma(D^{-v} f, I) \le \dim_B \Gamma(D^{-v} f, I) \le 2 - v, \quad 0 < v < 1.$$

With definition of box dimension and Hausdorff dimension, we know

$$1 \le \dim_H \Gamma(D^{-v} f, I) \le \dim_B \Gamma(D^{-v} f, I) \le 2 - v, \quad 0 < v < 1.$$

If $v = 1$, it is obvious that box dimension of $D^{-v} f(x)$ exists and

$$\dim_H \Gamma(D^{-v} f, I) = \dim_B \Gamma(D^{-v} f, I) = 1 = 2 - v \tag{3.5}$$

for any $f(x) \in C_I$. In fact, $D^{-1} f(x)$ which is Riemann integral of order 1 is a differentiable function on I, then it is of bounded variation on I. By [4], we know (3.5) holds.

In Theorem 3.3, there is a condition that orders of Riemann–Liouville integral belong to $(0, 1)$. From Definition 1.3 and Theorem 3.3, we can get the following result.

Corollary 3.4 *Let $f(x) \in C_I, D^{-v} f(x)$ be defined as* (2.1).

(1) *If* $0 < v < 1$,

$$1 \le \dim_H \Gamma(D^{-v} f, I) \le \overline{\dim}_B \Gamma(D^{-v} f, I) \le 2 - v.$$

(2) *If* $v \ge 1$,

$$\dim_H \Gamma(D^{-v} f, I) = \dim_B \Gamma(D^{-v} f, I) = 1.$$

From discussion above, we know $D^{-v} f(x) \in H^v$ when

$$H^v = \{f(x) : |f(x) - f(y)| \le C|x - y|^v, \, 0 < v < 1, \forall x, y \in I\}. \tag{3.6}$$

If $f(x) \in C_I$ belongs to H^α when $0 < \alpha < 1$, we will discuss fractal dimensions of of Riemann–Liouville integral of $f(x)$ of order $v \, (v > 0)$ in the next section. Here we give certain example about Riemann–Liouville fractional integral of continuous functions on I.

Example 3.5 Let $0 = x_0 < x_1 \cdots < x_N = 1, i \in S = \{1, 2, \dots, N\}$ and $L_i(x)$ be the linear map satisfying

$$L_i(0) = x_{i-1}, \quad L_i(1) = x_i, \quad i \in S.$$

Let $K = I \times \mathbb{R}$ and $\{y_i\}_{i=0}^N$ be a certain data set. A continuous map $F_i : K \to \mathbb{R}$ is defined as

$$F_i(0, y_0) = y_{i-1}, \quad F_i(1, y_N) = y_i, \quad i \in S.$$

For $\alpha \in (0, 1)$, F_i satisfies

$$|F_i(x, t) - F_i(x, u)| \le \alpha |t - u|, \quad \forall x \in I, \forall t, u \in \mathbb{R}, i \in S.$$

Let functions $\psi_i : K \to K$ be defined as $\psi_i(x, y) = (L_i(x), F_i(x, y)), i \in S$. Then $\{K, \psi_i : i \in S\}$ is an iterated function system and it has a unique attractor $G = \bigcup_{i=1}^N \psi_i(G)$. $\Gamma(G, I)$ can be looked as graph of a continuous function $g : I \to \mathbb{R}$ which satisfies

$$g(x_i) = y_i, \quad i \in S. \tag{3.7}$$

Functions defined as (3.7) is called as fractal interpolation functions. For $i \in L$, let $|d_i| < 1$, q_i be continuous and $F_i(x, y) = d_i y + q_i(x)$. Then, a fractal interpolation function $g(x)$ defined as above is called a linear fractal interpolation function. In [15], the authors chose

$$L_i(x) = a_i x + b_i, \quad a_i = x_i - x_{i-1}, \quad b_i = x_{i-1}, \quad i \in S.$$

Let $g(x)$ be determined by $\{L_i(x), F_i(x, y)\}_{i=1}^N$. Suppose $\sum_{i=1}^N |c_i| > 1$ and q_i is continuous and of bounded variation on I for any $i \in S$. They proved that $\dim_B \Gamma(f, I) = D(\{a_i, c_i\})$, where$D(\{a_i, c_i\})$ is the unique solution s of the equation $\sum_{i=1}^N a_i^{s-1} |c_i| = 1$. Then for $0 < v < D(\{a_i, c_i\}) - 1$, it holds

$$\dim_B \Gamma(D^{-v} f, I) = \dim_B \Gamma(f, I) - v.$$

Since $\dim_B \Gamma(f, I) \le 2$,

$$\dim_B \Gamma(D^{-v} f, I) \le 2 - v. \tag{3.8}$$

From Theorem 3.3, we can also directly get (3.8).

4 Fractal Dimensions of Fractional Integral of $f(x) \in H^\alpha$

For $0 < \alpha < 1$, let

$$H^\alpha = \{f(x) : |f(x) - f(y)| \le C|x - y|^\alpha, \forall x, y \in I\}$$

which is defined similarly as (3.6). If a continuous function $f(x)$ on I belongs to H^α, thus

$$1 \le \dim_H \Gamma(f, I) \le \overline{\dim}_B \Gamma(f, I) \le 2 - \alpha.$$

$D^{-v} f(x)$ is Riemann–Liouville integral of $f(x)$ of order v. We will give some estimations of fractal dimensions of $D^{-v} f(x)$. First we give upper bound of upper box dimension of $D^{-v} f(x)$.

Theorem 4.1 *Let $f(x) \in H^\alpha, f(0) = 0$ and $D^{-v} f(x)$ be defined as* (2.1)*. Then*

$$1 \le \dim_H \Gamma(D^{-v} f, I) \le \overline{\dim}_B \Gamma(D^{-v} f, I) \le 2 - \alpha, \quad 0 < v < 1.$$

Proof Let $0 \le x < x + h \le 1$ and $0 < v < 1$. It holds

$$D^{-v} f(x + h) - D^{-v} f(x)$$

$$= \frac{1}{\Gamma(v)} \int_0^{x+h} (x+h-t)^{v-1} f(t)dt - \frac{1}{\Gamma(v)} \int_0^{x} (x-t)^{v-1} f(t)dt$$
$$= \frac{1}{\Gamma(v)} \int_h^{x+h} (x+h-t)^{v-1} [f(t)-f(t-h)]dt + \frac{1}{\Gamma(v)} \int_0^{h} (x+h-t)^{v-1} f(t)dt$$
$$:= I_3 + I_4.$$

Since $|f(x)-f(y)| \le C|x-y|^\alpha$ holds for any $x, y \in I$, then $|I_3| \le \frac{C}{\Gamma(v+1)} h^\alpha$. For $f(0)=0$, we can get $|I_4| \le \frac{C}{\Gamma(v+1)} h^\alpha \cdot h^v$. Thus,

$$|D^{-v} f(x+h) - D^{-v} f(x)| \le Ch^\alpha.$$

By Lemma 3.2,

$$\overline{\dim}_B \Gamma(D^{-v} f, I) \le 2-\alpha.$$

Combine Definition 1.1 with the above inequality, we know the conclusion of Theorem 4.1 holds. □

If $f(x)$ is a $2-\alpha$-dimensional regular fractal function [6], upper box dimension of Riemann–Liouville fractional integral of $f(x)$ is non-increasing. We know

$$\dim_B \Gamma(f, I) = 2-\alpha.$$

While

$$\dim_B \Gamma(D^{-v} f, I) \le 2-\alpha.$$

That is,

$$\dim_B \Gamma(D^{-v} f, I) \le \dim_B \Gamma(f, I) = 2-\alpha.$$

With Theorem 3.3, we can get the following corollary.

Corollary 4.2 *Let $f(x) \in H^\alpha$, $f(0)=0$ and $D^{-v} f(x)$ be defined as* (2.1). *Then for* $0 < v < 1$,

$$\dim_H \Gamma(D^{-v} f, I) \le \overline{\dim}_B \Gamma(D^{-v} f, I) \le \min\{2-v, 2-\alpha\}.$$

Certain example is given as follows.

Example 4.3 ([21]) Let $0 < \alpha < 1, \lambda > 4$. Weierstrass function $W(x)$ is defined as

$$W(x) = \sum_{j \ge 1} \lambda^{-\alpha j} \sin(\lambda^j x).$$

Then

$$\dim_B \Gamma(W, I) = 2-\alpha$$

and for $v < 1-\alpha$,

$$\dim_B \Gamma(D^{-v} W, I) = 2-\alpha-v \le \min\{2-v, 2-\alpha\}.$$

We expect to get that upper box dimension of Riemann–Liouville integral of $f(x) \in H^\alpha$ of order $v\,(< 1-\alpha)$ is no more than $2-\alpha-v$. Though it is difficult to get the inequality

$$\overline{\dim}_B \Gamma(D^{-v} f, I) \le 2-\alpha-v, \quad v < 1-\alpha$$

when $f \in H^\alpha (0 \le \alpha \le 1)$, we still can get a well estimation about fractal dimensions of $D^{-v} f(x)$ by the following discussion.

Lemma 4.4 *Let $D^{-v}f(x), D_\delta^{-v}f(x)$ be defined as (2.1) and (2.2) respectively. For $f(x) \in H^\alpha$,*

$$|D_\delta^{-v}f(x) - D^{-v}f(x)| \le C\delta^v, \quad x \in I, \delta > 0.$$

Proof Let $x \in I$ and $\delta > 0$. By Proposition 2.2, we can directly get the result of Lemma 3.4.

Now we give the main result of our paper.

Theorem 4.5 *Let $D^{-v}f(x), D_\delta^{-v}f(x)$ be defined as (2.1) and (2.2) respectively. For $f(x) \in H^\alpha, f(0) = 0, 0 < \alpha, v < 1$ and $\alpha + v < 1$, it holds*

$$|D^{-v}f(x) - D^{-v}f(y)| \le C\delta^v(1 + \delta^{\alpha-1}|x - y|), \quad x, y \in I, \delta > 0. \tag{4.2}$$

Proof Let $\delta > 0, x, y \in I$. By Lemma 4.4, for $0 < \alpha, v < 1$ and $\alpha + v < 1$,

$$\begin{aligned}|D^{-v}f(x) - D^{-v}f(y)| &\le |D^{-v}f(x) - D_\delta^{-v}f(x)| + |D^{-v}f(y) - D_\delta^{-v}f(y)| \\ &\quad + |D_\delta^{-v}f(x) - D_\delta^{-v}f(y)| \\ &\le C\delta^v + |D_\delta^{-v}f(x) - D_\delta^{-v}f(y)| \\ &:= C\delta^v + I_5.\end{aligned}$$

Since I_5 is differentiable on I by Property 2.3, then

$$I_5 \le \left|\left(\frac{d}{dx}D_\delta^{-v}f(x)\right)\Big|_{x=\xi}\right||x - y|$$

for certain $\xi \in I$. Now we estimate $\frac{d}{dx}D_\delta^{-v}f(x)$. That is,

$$\begin{aligned}\Gamma(v)\frac{d}{dx}D_\delta^{-v}f(x) &= (v-1)\int_0^x f(t)(x+\delta-t)^{v-2}dt + f(x)\delta^{v-1} \\ &= (v-1)\int_0^x f(t)(x+\delta-t)^{v-2}dt \\ &\quad - (v-1)\int_0^x f(x)(x+\delta-t)^{v-2}dt + f(x)(x+\delta)^{v-1} \\ &= (v-1)\int_0^x [f(t) - f(x)](x+\delta-t)^{v-2}dt + [f(x) - f(0)](x+\delta)^{v-1}.\end{aligned}$$

For $\alpha + v < 1$,

$$\begin{aligned}\frac{\Gamma(v)}{1-v}\left|\frac{d}{dx}D_\delta^{-v}f(x)\right| &\le C\int_0^x (x-t)^\alpha(x+\delta-t)^{v-2}dt + Cx^\alpha(x+\delta)^{v-1} \\ &\le C\int_0^x (x+\delta-t)^{\alpha+v-2}dt + C(x+\delta)^{\alpha+v-1} \\ &\le C\delta^{\alpha+v-1} + C(x+\delta)^{\alpha+v-1} \le C\delta^{\alpha+v-1}.\end{aligned}$$

So

$$I_5 \le C\delta^{\alpha+v-1}.$$

This means (4.2) holds. □

In Theorem 4.5, let $\delta = |x - y|$. By appropriate modification, it holds

$$D^{-v}f(x) \in H^v, \quad 0 < v < 1.$$

In other words, there exists certain constant C such that

$$|D^{-v}f(x) - D^{-v}f(y)| \le C|x - y|^v, \quad 0 < v < 1, \forall x, y \in I.$$

By Definition 1.2,

$$\dim_H \Gamma(D^{-v}f, I) \leq \overline{\dim}_B \Gamma(D^{-v}f, x) \leq 2 - v, \quad 0 < v < 1. \tag{4.3}$$

From the following Theorem 4.6, we can get a conclusion better than (4.3).

Theorem 4.6 *Let* $D^{-v}f(x), D_\delta^{-v}f(x)$ *be defined as* (2.1) *and* (2.2) *respectively. For* $f(x) \in H^\alpha, f(0) = 0, 0 < \alpha, v < 1$ *and* $\alpha + v < 1$, *it holds*

$$D^{-v}f(x) \in H^{2-\frac{v}{1-\alpha}}. \tag{4.4}$$

Proof By Theorem 4.4, for $0 < \alpha, v < 1$ and $\alpha + v < 1$,

$$|D^{-v}f(x) - D^{-v}f(y)| \leq C\delta^v(1 + \delta^{\alpha-1}|x - y|), \quad \forall x, y \in I.$$

Let $\delta = |x - y|^{\frac{1}{1-\alpha}}$. Then,

$$|D^{-v}f(x) - D^{-v}f(y)| \leq C|x - y|^{\frac{v}{1-\alpha}}.$$

Thus, we get (4.4). □

Remark 4.7 From Theorem 4.6,

$$\dim_H \Gamma(D^{-v}f, I) \leq \overline{\dim}_B \Gamma(D^{-v}f, x) \leq 2 - \frac{v}{1-\alpha}.$$

Since $0 < v < 1$,

$$\dim_H \Gamma(D^{-v}f, I) \leq \overline{\dim}_B \Gamma(D^{-v}f, x) < 2 - v. \tag{4.5}$$

(4.5) is better than (4.3). If we choose $\delta = |x - y|^{\frac{\alpha}{v}}$,

$$\dim_H \Gamma(D^{-v}f, I) \leq \overline{\dim}_B \Gamma(D^{-v}f, x) < 2 - \alpha. \tag{4.6}$$

Thus we know upper box dimension of Riemann–Liouville integral of certain continuous functions of order v is strictly less than $2 - v$, not no more than $2 - v$. A best conclusion maybe upper box dimension of Riemann–Liouville integral of $f(x) \in H^\alpha(0 < \alpha < 1)$ of order $v(0 < v < 1)$ is no more than $2 - \alpha - v$ when $\alpha + v < 1$. Now we explore Riemann–Liouville fractional integral of $f(x) \in H^\alpha$ with $\alpha + v \geq 1$. Similar argument with Theorem 4.5, we have the following result.

Theorem 4.8 *Let* $D^{-v}f(x), D_\delta^{-v}f(x)$ *be defined as* (2.1) *and* (2.2) *respectively. For* $f(x) \in H^\alpha, f(0) = 0, \alpha + v \geq 1$ *and* $0 < \delta < 1$, *it holds*

$$\dim_H \Gamma(D^{-v}f, I) = \dim_B \Gamma(D^{-v}f, I) = 1, \quad 0 < v < 1 \tag{4.7}$$

Proof Since $0 < \delta < 1$ and $\alpha + v \geq 1$. Similar discussion with Theorem 4.5,

$$|D^{-v}f(x) - D^{-v}f(y)| \leq C\delta^v + C|x - y|, \quad x, y \in I. \tag{4.8}$$

Let $\delta = |x - y|^{\frac{1}{v}}$. (4.8) equals to

$$|D^{-v}f(x) - D^{-v}f(y)| \leq C|x - y|. \tag{4.9}$$

This means Lipschiz function $D^{-v}f(x) \in H^1$. By [4], if a continuous function is of bounded variation on I, its box dimension is 1. Since $D^{-v}f(x)$ is of bounded variation on I by (4.9), then

$$\dim_B \Gamma(D^{-v}f, I) = 1.$$

From Definition 1.1,

$$\dim_B \Gamma(D^{-v}f, I) \geq \dim_H \Gamma(D^{-v}f, I) \geq 1.$$

Thus (4.7) holds. □

Box dimension of a continuous function $f(x) \in H^\alpha$ on I maybe exist or not, but by Theorem 4.8, box dimension of Riemann–Liouville integral of $f(x)$ of order v when $\alpha + v \geq 1$ exists. Furthermore, box dimension of $D^{-v}f(x)$ is 1. It is obvious that box dimension of Riemann–Liouville integral of $f(x)$ of order v is 1 for any $0 < \alpha < 1$ when $v \geq 1$.

Remark 4.9 From Theorem 4.8, Riemann–Liouville fractional integral of a continuous function belong to H^α when $0 < \alpha < 1$ and $\alpha + v \geq 1$ is continuous and bounded. Furthermore, box dimension of Riemann–Liouville integral of $f(x)$ of order $v > 1 - \alpha$ still is 1. In fact, by [4], Riemann–Liouville integral of a continuous function with bounded variation of any positive order v still is a continuous function with bounded variation. $f(x) \in H^1$ is obviously of bounded variation. Thus its Riemann–Liouville fractional integral still is of bounded variation by [4].

Here is an important problem that whether $D^{-v}f(x)$ with $f(x) \in H^\alpha$ is differentiable on I when $\alpha + v > 1$. Let $f(x) \in C_I$ be a Lipchsiz function. Thus for any $v(v > 0)$, box dimension of Riemann–Liouville integral of $f(x)$ of order $v(v > 0)$ is 1 by Theorem 4.8. It is also given in the following example.

Example 4.10 Let $f(x) \in H^1$ be continuous on I and $D^{-v}f(x)$ be Riemann–Liouville integral of $f(x)$ of order $v\,(v > 0)$. Since $1 + v > 1$ for $v > 0$, by Theorem 4.8, $D^{-v}f(x) \in H^1$ and

$$\dim_H \Gamma(D^{-v}f, I) = \dim_B \Gamma(D^{-v}f, I) = 1. \tag{4.10}$$

In fact (4.10) in Example 4.10 also holds by Remark 4.9. We can regard (4.10) as a special result of the following theorem.

Theorem 4.11 ([4]) *Let $f(x) \in C_I$ be of bounded variation. If $D^{-v}f(x)$ is defined as* (2.1)*, it is a continuous function with bounded variation on I and*

$$\dim_H \Gamma(D^{-v}f, I) = \dim_B \Gamma(D^{-v}f, I) = \dim_H \Gamma(f, I) = \dim_B \Gamma(f, I) = 1.$$

If a continuous function $f(x) \in H^1$, it is of bounded variation. While $f(x)$ is of bounded variation on I, it maybe belong to H^1 or not. So the condition of Theorem 4.11 is weaker than that of Example 4.10. But the conclusion of Example 4.10 is better than Theorem 4.11. In [9], the author proved that box dimension of Riemann–Liouville fractional integral of a continuous function defined on I which has at most finite points with unbounded variation is 1.

5 Fractal Dimensions of Fractional Derivative of $f(x)$

In the above several sections, we mainly discuss fractal dimensions of Riemann–Liouville fractional integral of continuous functions. Here we explore fractal dimensions of Riemann–Liouville fractional derivative of certain continuous functions on I elementary. Definition of Riemann–Liouville fractional derivative of $f(x) \in C_I$ is given as follows.

Definition 5.1 ([11, 12]) *Let $f(x) \in C_I$ and $\mu > 0$. We call*

$$D^\mu f(x) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x (x-t)^{-\mu} f(t) dt \tag{5.1}$$

Riemann–Liouville derivative of $f(x)$ of order μ if (5.1) *exists.*

For convenience, let $f(0) = 0$. Now we consider fractal dimensions of $D^\mu f(x)$ when $f(x)$ has a continuous derivative function $f'(x)$ on I of order $\mu\,(0 < \mu < 1)$. From (5.1),

$$\begin{aligned}\Gamma(1-\mu)D^\mu f(x) &= \frac{d}{dx}\int_0^x (x-t)^{-\mu} f(t)dt = \frac{1}{\mu-1}\frac{d}{dx}\int_0^x f(t)d(x-t)^{1-\mu} \\ &= \frac{1}{\mu-1}\frac{d}{dx}[f(t)(x-t)^{1-\mu}|_{t=0}^x] + \frac{1}{1-\mu}\frac{d}{dx}\int_0^x (x-t)^{1-\mu} f'(t)dt \\ &= \frac{1}{1-\mu}\frac{d}{dx}\int_0^x (x-t)^{1-\mu} f'(t)dt = -\int_0^x (x-t)^{-\mu} f'(t)dt.\end{aligned}$$

That is,

$$D^\mu f(x) = \frac{-1}{\Gamma(1-\mu)}\int_0^x (x-t)^{-\mu} f'(t)dt. \tag{5.2}$$

(5.2) means Riemann–Liouville derivative of $f(x)$ of order μ exists when $0 < \mu < 1$. Thus we have the following basic property.

Proposition 5.2 *If $f(x)$ has a continuous derivative function on I, Riemann–Liouville derivative of $f(x)$ of order $\mu\,(0 < \mu < 1)$ is continuous and bounded on I.*

Similar argument with Theorem 3.3, we give estimation of fractal dimensions of Riemann–Liouville fractional derivative of $f(x)$ which has a continuous derivative function on I.

Theorem 5.3 *If $f(x)$ has a continuous derivative $f'(x)$ on I and $f(0) = 0$, upper box dimension of Riemann–Liouville derivative of $f(x)$ of order $\mu\,(0 < \mu < 1)$ is no more than $1 + \mu$.*

Proof Let $0 < \mu < 1$ and $0 \le x, x + h \le 1$. Then by (5.2),

$$\begin{aligned}|D^\mu f(x+h) - D^\mu f(x)| &= \left|\int_0^{x+h} (x+h-t)^{-\mu} f'(t)dt - \int_0^x (x-t)^{-\mu} f'(t)dt\right| \\ &= \left|\int_0^{x+h} (x+h-t)^{(1-\mu)-1} f'(t)dt - \int_0^x (x-t)^{(1-\mu)-1} f'(t)dt\right|.\end{aligned}$$

Similar argument with Theorem 3.3,

$$|D^\mu f(x+h) - D^\mu f(x)| \le C|x-y|^{1-\mu}.$$

With Lemma 3.2, we have

$$1 \le \dim_H \Gamma(D^\mu f, I) \le \overline{\dim}_B \Gamma(D^\mu f, I) \le 1 + \mu. \tag{5.3}$$

By (5.3), upper box dimension of $D^\mu f(x)$ is no more than $1 + \mu$ when $0 < \mu < 1$. □

From Proposition 5.2 and Theorem 5.3, Riemann–Liouville fractional derivative of a continuous function which has a continuous derivative exists. Furthermore, its upper box dimension has been proved to be no more than $1 + \mu$.

Except for Riemann–Liouville fractional calculus, there are many other definition of fractional calculus, such as modified Riemann–Liouville fractional calculus [2], Hadamard fractional calculus [14], q-fractional integrals [13], Caputo fractional calculus [12], Grünwald fractional calculus [14] and so on. Here we discuss fractal dimensions of Weyl–Marchaud fractional derivative of certain continuous functions defined on $\mathbb{R}$ elementary. Definition of left-sided Weyl–Marchaud fractional derivative has been given as follows.

Definition 5.4 ([14]) *Let $f(x)$ be a continuous function defined on $\mathbb{R}$ and $0 < \mu < 1$. Write*

$$D^\mu f(x) = \frac{\mu}{\Gamma(1-\mu)} \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\mu}} dt \tag{5.4}$$

as left-sided Weyl–Marchaud derivative of $f(x)$ of order μ when (5.4) *exists.*

Estimation of fractal dimensions of $D^\mu f(x)$ is given as follows.

Theorem 5.5 *Let $f(x) \in H^\alpha$ on $\mathbb{R}$. For $0 < \mu < \alpha < 1$,*

$$\overline{\dim}_B \Gamma(D^\mu f, I) \le 2 - \alpha + \mu. \tag{5.5}$$

Proof Let $h > 0$ and $0 < \mu < \alpha$. Then,

$$\begin{aligned} &\left(\frac{\Gamma(1-\mu)}{\mu}\right)(D^\mu f(x+h) - D^\mu f(x)) \\ &= \int_0^\infty \frac{f(x+h) - f(x) + f(x-t) - f(x+h-t)}{t^{1+\mu}} dt \\ &= \left(\int_0^h + \int_h^\infty\right) \frac{f(x+h) - f(x) + f(x-t) - f(x+h-t)}{t^{1+\mu}} dt \\ &=: I_1 + I_2. \end{aligned}$$

Since $f(x) \in H^\alpha$ on $\mathbb{R}$,

$$|f(x) - f(y)| \le C|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}.$$

We have,

$$\begin{aligned} |I_1| &\le \int_0^h \frac{|f(x+h) - f(x+h-t)| + |f(x) - f(x-t)|}{t^{1+\mu}} dt \le C \int_0^h \frac{t^\alpha}{t^{1+\mu}} dt \le C h^{\alpha - u}, \\ |I_2| &\le \int_h^\infty \frac{|f(x+h) - f(x)| + |f(x-t) - f(x+h-t)|}{t^{1+\mu}} dt \le C \int_h^\infty \frac{h^\alpha}{t^{1+\mu}} dt \le C h^{\alpha - u}. \end{aligned}$$

Thus, it holds

$$|D^\mu f(x+h) - D^\mu f(x)| \le C h^{\alpha - \mu}, \quad x \in \mathbb{R}.$$

By Lemma 3.2, (5.5) holds. □

Remark 5.6 From Theorem 5.5, upper box dimension of Weyl–Machaud derivative of a continuous function $f(x) \in H^\alpha (0 < \alpha < 1)$ of order μ is no more than $2 - \alpha + \mu$ when $0 < \mu < \alpha$. Let $f(x)$ be a continuous function defined on $\mathbb{R}$ and $f(x) = 0\,(x \in \mathbb{R}/I)$. Thus we can discuss properties of Weyl–Machaud fractional derivative of $f(x)$ on I as that of Riemann–Liouville fractional calculus. More details can be found in Ref. [26].

6 Conclusions

In classical calculus, integral of $f(x) \in C_I$ seems smoother than $f(x)$ itself. While derivative of $f(x)$ having a continuous derivative on I seems coarser than $f(x)$ itself. For example, let a positive integer $n \in \mathbb{Z}$. Integral of a continuous function of order n seems more and more smoother when n increases. If derivative of a continuous function of order n exists, it seems more and more coarser when n increases. n which is order of calculus is integer here. So we are interested in the corresponding results when n is not an integer, for example $0 < n < 1$.

We try to deal with fractional calculus, such as Riemann–Liouville fractional calculus and Weyl–Marchaud fractional calculus, of continuous functions on I. Fractal dimensions, such as

Hausdorff dimension and box dimension have been used to describe those functions. According to classical calculus, the right problem is to investigate fractal dimensions of fractional calculus of continuous functions on closed intervals. In the present paper, we mainly consider upper bound of fractal dimensions of Riemann–Liouville fractional integral of continuous functions on I. It has been proved that upper bound of upper Box dimension of Riemann–Liouville integral of any continuous functions on closed intervals of order v is no more than $2-v$ when $v \in (0,1)$. Upper box dimension of Riemann–Liouville integral of any continuous functions which satisfy α-Hölder condition on closed intervals of order $v \in (0,1)$ is no more than $2-\alpha$ when $0<\alpha<1$. An important conclusion maybe that upper box dimension of Riemann–Liouville integral of any continuous functions which satisfy α-Hölder condition on closed intervals of order $v \in (0,1)$ is strictly less than $2-\alpha$ when $0<\alpha<1$ and $\alpha+v<1$. There are also many other important conclusions about fractal dimensions of fractional calculus of certain continuous functions.

A special case is box dimension of $f(x) \in C_I$ is 1. Thus lower box dimension of Riemann–Liouville integral of $f(x)$ of positive order v is naturally no less than 1. We only to estimate upper box dimension of Riemann–Liouville integral of $f(x)$ of order $v \in (0,1)$. Since box dimension of $f(x)$ is 1 and upper box dimension of Riemann–Liouville integral of $f(x)$ of positive order v seems no more than that of $f(x)$, we insist that box dimension of Riemann–Liouville integral of $f(x)$ order v must be 1. This work seems very obvious, but it is difficult to prove the conclusion. Box dimension of Riemann–Liouville integral of $f(x)$ of any continuous functions with bounded variation on I of any positive order v is 1. Furthermore, box dimension of Riemann–Liouville integral of $f(x)$ of any continuous functions which have at most finite points with unbounded variation on I of any positive order v is 1 too. But there are many other 1-dimensional continuous functions on I. For example, there exists a 1-dimensional continuous function with infinite points of unbounded variation on I and measure of these points maybe strictly bigger than 0. The latest work about estimation of fractal dimensions of Riemann–Liouville fractional integral of continuous functions on closed intervals can be found in [4, 9, 25].

Another problem is to investigate Riemann–Liouville fractional derivative of $f(x) \in C_I$. Since Riemann–Liouville fractional derivative is derived from Riemann–Liouville fractional integral, some conclusion about Riemann–Liouville fractional integral can be used on estimation of fractal dimensions of Riemann–Liouville fractional derivative of continuous functions such as discussion of the present paper. It is obvious that Riemann–Liouville fractional derivative of continuous functions maybe not exist. Since $D^{\mu} f(x) = DD^{\mu-1} f(x)$, we should first consider whether $D^{\mu-1} f(x)$ is differentiable or not when $0<\mu<1$.

In short, we discuss the relationship between fractal dimensions of continuous functions and their fractional calculus. [21, 22, 24] got the linear connection between fractal dimensions of Weierstrass functions and their Riemann–Liouville fractional calculus. [3] explored fractal dimensions of Riemann–Liouville fractional calculus of Besicovitch functions. Certain fractal curves such as Von Koch curve and its fractional calculus had been investigated in [8, 16, 18]. Linear relationship between fractal dimensions of linear fractal interpolation functions and their Riemann–Liouville fractional integral [15] and derivative [10] had been discussed respectively. In [25], the authors made research on Weyl–Marchaud fractional derivative of a type of self-affine functions. More details and other work about fractal dimensions of fractional calculus of continuous functions can be found in [7, 19, 23]. According to classical calculus, we think a right fractional calculus must satisfy the following conclusion.

Conjecture 6.1 (Fractional Calculus Theorem) *Let $f(x)$ be an s-dimensional regular fractal function* [6] *on I for $1 < s < 2$. $D^{-v}f(x)$ is fractional integral of $f(x)$ of order $v\,(0 < v < 1)$ and $D^{-\mu}f(x)$ is fractional derivative of $f(x)$ of order $\mu\,(0 < \mu < 1)$ on I. Then,*

$$\dim_B \Gamma(D^{-v}f, I) = s - v, \quad 0 < v < s - 1,$$
$$\dim_B \Gamma(D^{\mu}f, I) = s + \mu, \quad 0 < \mu < 2 - s.$$

In short, if definition of fractional calculus is well given, it must satisfy the conclusion which is given in Fractional Calculus Theorem above. Meanwhile if Fractional Calculus Theorem holds for certain fractional calculus, we regard this fractional calculus as the right fractional calculus according to classical calculus. Up to now, Riemann–Liouville fractional calculus maybe the fractional calculus which satisfies Fractional Calculus Theorem.

Acknowledgements We thank the referees and Professor Yao Kui for their time and comments.

References

[1] Falconer, J.: Fractal Geometry: Mathematical Foundations and Applications, John Wiley Sons Inc., New York, 1990

[2] Jumarie, G.: An approach to defferential geometry of fractional order via modified Riemann–Liouville derivative. *Acta Math. Sin., Engl. Ser.*, **28**, 1741–1768 (2012)

[3] Liang, Y. S.: The relationship between the Box dimension of the Besicovitch functions and the orders of their fractional calculus. *Appl. Math. Comput.*, **200**, 197–207 (2008)

[4] Liang, Y. S.: Box dimensions of Riemann–Liouville fractional integrals of continuous functions of bounded variation. *Nonlinear Anal.*, **72**, 4304–4306 (2010)

[5] Liang, Y. S.: Upper Bound estimation of fractal dimensions of fractional integral of continuous functions. *Adv. Pure Math.*, **5**, 27–30 (2015)

[6] Liang, Y. S.: Some remarks on continuous functions of unbounded variation. *Acta Math. Sin., Chin. Ser.*, **59**, 215–232 (2016)

[7] Liang, Y. S., Su, W. Y.: Connection between the order of fractional calculus and fractional dimensins of a type of fractal functions. *Anal. Theory Appl.*, **23**, 354–363 (2007)

[8] Liang, Y. S., Su, W. Y.: Von Koch curve and its fractional calculus. *Acta Math. Sin., Chin. Ser.*, **54**, 227–240 (2011)

[9] Liang, Y. S., Su, W. Y.: Riemann–Liouville fractional calculus of 1-dimensional continuous functions. *Sci. China Ser. A*, **46**, 423–438 (2016)

[10] Liang, Y. S., Yao, K.: Fractal dimensions of Riemann–Liouville fractional calculus of linear fractal interpolation functions. *Chin. Ann. Math. Ser. A*, **38**, 1–8 (2017)

[11] Miller, K. S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley Sons Inc., New York, 1993

[12] Oldham, K. B., Spanier, J.: The Fractional Calculus, Academic Press, New York, 1974

[13] Rajković, P. M., Marinković, S. D., Stanković, M. C.: A generalization of the concept of q-fractional integrals. *Acta Math. Sin., Engl. Ser.*, **25**, 1635–1646 (2009)

[14] Ross, B.: The Fractional Calculus and Its Applications, Springer-Verlag, Berlin, Heidelberg, 1975

[15] Ruan, H. J., Su, W. Y., Yao, K.: Box dimension and fractional integral of linear fractal interpolation functions. *J. Approx. Theory*, **161**, 187–197 (2009)

[16] Tatom, F. B.: The relationship between fractional calculus and fractals. *Fractals*, **3**, 217–229 (1995)

[17] Wen, Z. Y.: Mathematical Foundations of Fractal Geometry, Science Technology Education Publication House, Shanghai, 2000

[18] Yao, K., Liang, Y. S.: Dimension of graphs of fractional derivatives of self-affine curves. *Acta Math. Sin., Chin. Ser.*, **56**, 693–698 (2013)

[19] Yao, K., Liang, Y. S., Zhang, F.: On the connection between the order of the fractional derivative and the Hausdorff dimension of a fractal function. *Chaos Solitons Fractals*, **41**, 2538–2545 (2009)

[20] Yao, K., Su, W. Y., Liang, Y. S.: The upper bound of Box dimension of the Weyl–Marchaud derivative of self-affine curves. *Anal. Theory Appl.*, **26**, 222–227 (2010)

[21] Yao, K., Su, W. Y., Zhou, S. P.: On the fractional calculus of a type of Weierstrass function. *Chin. Ann. Math. Ser. B*, **25**, 711–716 (2004)

[22] Yao, K., Su, W. Y., Zhou, S. P.: On the fractional derivatives of a fractal function. *Acta Math. Sin., Engl. Ser.*, **22**, 719–722 (2006)

[23] Zähle, M.: Fractional differentiation in the self-affine case V — the local degree of differentiability. *Math. Nachr.*, **185**, 297–306 (1997)

[24] Zähle, M., Ziezold, H.: Fractional derivatives of Weierstrass-type functions. *J. Comput. Appl. Math.*, **76**, 265–275 (1996)

[25] Zhang, Q.: Some remarks on one-dimensional functions and their Riemann–Liouville fractional calculus. *Acta Math. Sin., Engl. Ser.*, **30**, 517–524 (2014)

[26] Zhang, Q., Liang, Y. S.: The Weyl–Marchaud fractional derivative of a type of self-affine functions. *Appl. Math. Comput.*, **218**, 8695–8701 (2012)

[27] Xie T. F., Zhou, S. P.: Aproximation Theory of Real Functions, Hangzhou University Publication, Hangzhou, 1997