Acta Mathematica Sinica, English Series
Mar., 2017, Vol. 33, No. 3, pp. 311–326
Published online: September 22, 2016
DOI: 10.1007/s10114-016-6009-y
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

The Approximation Properties Determined by Operator Ideals

Dong Yang CHEN
School of Mathematical Sciences, Xiamen University, Xiamen 361005*, P. R. China*
E-mail: *cdy@xmu.edu.cn*

Lei LI[1)]
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071*, P. R. China*
E-mail: *leilee@nankai.edu.cn*

Abstract We introduce the notion of the right approximation property with respect to an operator ideal $\mathcal{A}$ and solve the duality problem for the approximation property with respect to an operator ideal $\mathcal{A}$, that is, a Banach space X has the approximation property with respect to $\mathcal{A}^d$ whenever X^* has the right approximation property with respect to an operator ideal $\mathcal{A}$. The notions of the left bounded approximation property and the left weak bounded approximation property for a Banach operator ideal are introduced and new symmetric results are obtained. Finally, the notions of the p-compact sets and the p-approximation property are extended to arbitrary Banach operator ideals. Known results of the approximation property with respect to an operator ideal and the p-approximation property are generalized.

Keywords Approximation property, operator ideals, bounded approximation property, $\mathcal{A}$-p-compact sets

MR(2010) Subject Classification 46B28, 47L20

1 Introduction, Definitions and Notations

Throughout this paper, we denote by $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a Banach operator ideal. When the norm $\|\cdot\|_{\mathcal{A}}$ is understood or when we only work with an operator ideal, we simply write $\mathcal{A}$. As usual, $\mathcal{L}, \mathcal{W}, \mathcal{K}, \overline{\mathcal{F}}$ and $\mathcal{F}$ denote the operator ideals of bounded, weakly compact, compact, approximable and finite rank linear operators, respectively. All are considered with the supremum norm. Let $\mathcal{A}$ be an operator ideal. For a pair of Banach spaces X and Y, $\mathcal{A}^d(X,Y)$ denotes the set of $T \in \mathcal{L}(X,Y)$ such that the adjoint $T^* \in \mathcal{A}(Y^*, X^*)$ and $\|T\|_{\mathcal{A}^d} = \|T^*\|_{\mathcal{A}}$. The operator ideal $(\mathcal{A}^d, \|\cdot\|_{\mathcal{A}^d})$ is called the *dual ideal* of $\mathcal{A}$.

Recall that a Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X such that $\sup_{x\in K} \|Sx - x\| \le \varepsilon$, briefly, $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_c}$, where id_X is the identity map on X and τ_c is the topology of uniform convergence on compact subsets of X. It is well known that

Received January 8, 2016, accepted July 14, 2016
The first author is supported by the Natural Science Foundation of Fujian Province of China (Grant No. 2015J01026); the second author is supported by the NSF of China (Grant No. 11301285)
1) Corresponding author

a Banach space X has the AP if and only if for every Banach space Y and every operator $T \in \mathcal{L}(Y,X)$, one has $T \in \overline{\{ST : S \in \mathcal{F}(X,X)\}}^{\tau_c}$. Given an operator ideal $\mathcal{A}$, Delgado and Piñeiro in [4] introduce the notion of approximation property depending on the operator ideal $\mathcal{A}$ by replacing the operator ideal $\mathcal{L}$ by $\mathcal{A}$. A Banach space X is said to have the *approximation property with respect to the operator ideal* $\mathcal{A}$ (for short, $AP_{\mathcal{A}}$) if for every Banach space Y and every operator $T \in \mathcal{A}(Y,X)$, one has $T \in \overline{\{ST : S \in \mathcal{F}(X,X)\}}^{\tau_c}$. Delgado and Piñeiro in [4] proved that a Banach space X has the $AP_{\mathcal{A}}$ if and only if $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_c(\mathcal{A})}$, where $\tau_c(\mathcal{A})$ is the topology of uniform convergence on $\mathcal{A}$-compact subsets of X introduced by Carl and Stephani [1]. Following [1], a subset K of a Banach space X is said to be *relatively $\mathcal{A}$-compact* if there exist a Banach space Z, an operator $T \in \mathcal{A}(Z,X)$ and a compact subset M of Z such that $K \subset T(M)$. A closed relatively $\mathcal{A}$-compact subset is said to be *$\mathcal{A}$-compact*. It is proved in [4] that a Banach space X has the $AP_{\mathcal{A}}$ whenever X^{**} the $AP_{\mathcal{A}}$. However, it seems that the duality problem for the $AP_{\mathcal{A}}$ remains unknown so far. In Section 2, we introduce the notion of the right approximation property with respect to an operator ideal $\mathcal{A}$ and solve the duality problem for the $AP_{\mathcal{A}}$.

Section 3 is concerned with the bounded approximation property with respect to an operator ideal. Letting $1 \le \lambda < \infty$, a Banach space X is said to have the *λ-bounded approximation property* (λ-BAP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X with $\|S\| \le \lambda$ such that $\sup_{x\in K} \|Sx - x\| \le \varepsilon$. In [11], Lima and Oja define the weak bounded approximation property. Recall that X has the weak *λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and for each operator $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X,X)$ such that $\sup_\alpha \|TS_\alpha\| \le \lambda\|T\|$ and $S_\alpha \xrightarrow{\tau_c} \mathrm{id}_X$. In [9], this concept is extended to an arbitrary Banach operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ by replacing $(\mathcal{W}, \|\cdot\|)$. A Banach space X is said to have the *λ-bounded approximation property* for $\mathcal{A}$ (λ-BAP for $\mathcal{A}$) if for every Banach space Y and for each operator $T \in \mathcal{A}(X,Y)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X,X)$ such that $\sup_\alpha \|TS_\alpha\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}$ and $S_\alpha \xrightarrow{\tau_c} \mathrm{id}_X$. It is immediate that the λ-BAP implies the λ-BAP for every Banach operator ideal $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. In [8], Lassalle and Turco presented a natural modification of the λ-BAP for $\mathcal{A}$ and introduce the weak λ-BAP for $\mathcal{A}$. A Banach space X is said to have the *weak λ-bounded approximation property for $\mathcal{A}$* (*weak* λ-BAP *for* $\mathcal{A}$) if for every Banach space Y and for each operator $T \in \mathcal{A}(X,Y)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X,X)$ such that $\sup_\alpha \|TS_\alpha\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}$ and $S_\alpha \longrightarrow \mathrm{id}_X$ in the strong operator topology (SOT). Also, Lassalle and Turco [8] proved that the weak λ-BAP for $\mathcal{A}$ is equivalent to the following condition: for every Banach space Y and for each operator $T \in \mathcal{A}(X,Y)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X,X)$ such that $\sup_\alpha \|TS_\alpha\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}$ and $TS_\alpha \xrightarrow{\text{SOT}} T$. In this section, we prove, given an arbitrary operator ideal $\mathcal{A}$, that the λ-BAP is equivalent to a stronger condition: for every Banach space Y and for each operator $T \in \mathcal{A}(X,Y)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X,X)$ such that $\sup_\alpha \|S_\alpha\| \le \lambda$ and $TS_\alpha \xrightarrow{\text{SOT}} T$. It is shown in [8], under the condition $\mathcal{A} = \mathcal{A}^{dd}$, that the strong operator topology under which the net $(S_\alpha)_\alpha$ in the definition of the weak λ-BAP for $\mathcal{A}$ converges to the identity can be changed by a finer topology (coarser than the topology of uniform convergence on compact sets). Consequently, Lassalle and Turco [8] showed that the weak λ-BAP for $\overline{\mathcal{F}}$ is equivalent to the λ-BAP for $\overline{\mathcal{F}}$. In this section, we omit the condition $\mathcal{A} = \mathcal{A}^{dd}$ and hence obtain that the weak λ-BAP for

$\mathcal{A}$ is equivalent to the λ-BAP for $\mathcal{A}$ whenever $\mathcal{A}^d$ contains $\overline{\mathcal{F}}$. The symmetric versions of the λ-BAP for $\mathcal{A}$ and the weak λ-BAP for $\mathcal{A}$ are also introduced in this section. We say that a Banach space X has the *left weak λ-bounded approximation property for $\mathcal{A}$* if for every Banach space Y and for each operator $T \in \mathcal{A}(Y, X)$, there exists a net $(S_\alpha)_\alpha$ in $\mathcal{F}(X, X)$ such that $\sup_\alpha \|S_\alpha T\|_\mathcal{A} \le \lambda \|T\|_\mathcal{A}$ and $S_\alpha \xrightarrow{\text{SOT}} \mathrm{id}_X$. If the strong operator topology under which the net $(S_\alpha)_\alpha$ in the definition of the left weak λ-BAP for $\mathcal{A}$ converges to the identity is replaced by the topology τ_c, we say that X has the *left λ-bounded approximation property for $\mathcal{A}$*. In this section, we show that the strong operator topology in the definition of the left weak λ-BAP for $\mathcal{A}$ can be changed by the topology $\tau_c(\mathcal{A})$. As a consequence, we prove that the left weak λ-BAP for $\mathcal{A}$ is equivalent to the left λ-BAP for $\mathcal{A}$ whenever $\mathcal{A}$ contains $\overline{\mathcal{F}}$.

In the final section, we generalize the relatively p-compact sets and the p-approximation property in [14] to operator ideal cases. To illustrate our results, we need some definitions and notations. For $1 \le p < \infty$, $l_p(X)$ denotes the space of p-summable sequences in X. For $p = \infty$, we use the space $c_0(X)$ of norm null sequences in X. If $(\mathcal{A}, \|\cdot\|_\mathcal{A})$ is a Banach operator ideal, we denote by $l_p^\mathcal{A}(X)(1 \le p < \infty)$ the space of p-summable sequences $(x_n)_n$ in X with respect to $\mathcal{A}$, i.e., there exist an operator $T \in \mathcal{A}(Z, X)$ and $(z_n)_n \in l_p(Z)$ such that $x_n = Tz_n(n = 1, 2, \ldots)$. For $p = \infty$, we denote by $c_0^\mathcal{A}(X)$ the space of $\mathcal{A}$-convergent to zero sequences in X. Then $l_p^\mathcal{L}(X) = l_p(X)$ and $c_0^\mathcal{L}(X) = c_0(X)$. For $(x_n)_n \in l_p^\mathcal{A}(X)$, we define a quasi-norm

$$\|(x_n)_n\|_p^\mathcal{A} = \inf\{\|T\|_\mathcal{A} \|(z_n)_n\|_p : x_n = Tz_n(n = 1, 2, \ldots)\},$$

where the infimum is taken over all Banach spaces Z, all operators $T \in \mathcal{A}(Z, X)$ and all $(z_n)_n \in l_p(Z)$ such that $x_n = Tz_n(n = 1, 2, \ldots)$. We say that a subset K of X is *relatively $\mathcal{A}$-p-compact* $(1 \le p \le \infty)$ if there exists a sequence $(x_n)_n \in l_p^\mathcal{A}(X)$ such that $K \subset p\text{-co}\{x_n\}$, where $p\text{-co}\{x_n\} = \{\sum_{n=1}^\infty \alpha_n x_n : (\alpha_n)_n \in B_{l_{p'}}\}$ is called the *p-convex hull* of $(x_n)_n$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For $\mathcal{A} = \mathcal{L}$, the relatively $\mathcal{L}$-p-compact sets are precisely the relatively p-compact sets. For $p = \infty$, the relatively $\mathcal{A}$-∞-compact sets are precisely the relatively $\mathcal{A}$-compact sets. For a relatively $\mathcal{A}$-p-compact subset K of X, we define

$$m^p_{\mathcal{A}, \|\cdot\|_\mathcal{A}}(K; X) = \inf\{\|(x_n)_n\|_p^\mathcal{A} : K \subset p\text{-co}\{x_n\}, (x_n)_n \in l_p^\mathcal{A}(X)\},$$

If K is not relatively $\mathcal{A}$-p-compact, then let $m^p_{\mathcal{A}, \|\cdot\|_\mathcal{A}}(K; X) = \infty$. For simplicity, we write $m^p_\mathcal{A}(K)$ instead of $m^p_{\mathcal{A}, \|\cdot\|_\mathcal{A}}(K; X)$. For $\mathcal{A} = \mathcal{L}$, $m^p_\mathcal{A}(\cdot)$ is precisely equal to $m_p(\cdot)$ which is introduced by Lassalle and Turco [6] to measure the size of a relatively p-compact set. For $p = \infty$, $m^p_\mathcal{A}(\cdot)$ is precisely equal to $m_\mathcal{A}(\cdot)$ which is introduced by Lassalle and Turco [7] to measure the size of a relatively $\mathcal{A}$-compact set. In Section 4, we characterize relatively $\mathcal{A}$-p-compact sets and prove that a subset K of X is relatively $\mathcal{A}$-p-compact if and only if K is relatively $\mathcal{A} \circ \mathcal{K}$-$p$-compact and has the same size. This result extends Corollary 1.9 in [7] to the case $1 \le p \le \infty$. We say that an operator $T : X \to Y$ is *$\mathcal{A}$-p-compact* if $T(B_X)$ is relatively $\mathcal{A}$-p-compact. We denote by $\mathcal{K}_p^\mathcal{A}$ the space of all $\mathcal{A}$-p-compact operators. $\mathcal{K}_p^\mathcal{A}$ becomes a quasi-normed operator ideal if for any $T \in \mathcal{K}_p^\mathcal{A}(X, Y)$, one defines

$$\|T\|_{\mathcal{K}_p^\mathcal{A}} := m^p_\mathcal{A}(T(B_X)).$$

In Section 4, we present the representation formula for $\mathcal{K}_p^\mathcal{A}$ as $\mathcal{K}_p^\mathcal{A} = (\mathcal{A} \circ \mathcal{K}_p)^{\text{sur}}$. This formula also generalizes [7, Proposition 2.1] to the case $1 \le p \le \infty$. For $\mathcal{A} = \mathcal{L}$, the $\mathcal{L}$-p-compact operators are precisely the p-compact operators introduced by Sinha and Karn [14]. Moreover, $\|\cdot\|_{\mathcal{K}_p^{\mathcal{L}}}$ is equal to $k_p(\cdot)$ introduced in [6]. For $p=\infty$, the $\mathcal{A}$-p-compact operators are precisely the $\mathcal{A}$-compact operators introduced by by Carl and Stephani [1].

The topology of uniform convergence on $\mathcal{A}$-p-compact subsets in X on $\mathcal{L}(X,Y)$ is denoted by $\tau_p(\mathcal{A})$ ($\tau_c(\mathcal{A})$ for $p=\infty$). This locally convex topology is generated by the family of semi-norms

$$p_K(T) := \sup_{x\in K}\|Tx\|, \quad \forall\, T\in\mathcal{L}(X,Y),$$

where K is running over all the $\mathcal{A}$-p-compact subsets in X. We say that a Banach space X has the *p-approximation property with respect to an operator ideal* $\mathcal{A}$ (for short, p-$AP_{\mathcal{A}}$) if for every $\mathcal{A}$-p-compact subset K of X and every $\epsilon>0$, there exists a finite rank operator S on X such that $\sup_{x\in K}\|Sx-x\|\le\epsilon$, that is, $\mathrm{id}_X\in\overline{\mathcal{F}(X,X)}^{\tau_p(\mathcal{A})}$. Some new characterizations of the p-$AP_{\mathcal{A}}$ are established. It is proved in [4] that, under the assumption $\mathcal{A}=\mathcal{K}\circ\mathcal{A}$, a Banach space X has the $AP_{\mathcal{A}}$ if and only if $\mathcal{F}(Y,X)$ is $\tau_c(\mathcal{A})$-dense in $\mathcal{K}(Y,X)$ for any Banach space Y. In this paper, we prove the symmetric version of this result without any assumption. Actually, we prove a more general result, that is, a Banach space X has the p-$AP_{\mathcal{A}}$ if and only if $\mathcal{F}(X,Y)$ is $\tau_p(\mathcal{A})$-dense in $\mathcal{K}(X,Y)$ for every Banach space Y.

We refer to the books of Pietsch [13], of Diestel, Jarchow and Tonge [5] for operator ideals. For approximation properties, we refer to the books [2, 12].

2 The Duality Problem for the Approximation Property with Respect to an Operator Ideal

Definition 2.1 *We say that a Banach space X has the right approximation property with respect to an operator ideal $\mathcal{A}$ (for short, the right $AP_{\mathcal{A}}$) if for every Banach space Y and every operator $T\in\mathcal{A}(X,Y)$, one has $T\in\overline{\{TS:S\in\mathcal{F}(X,X)\}}^{\tau_c}$.*

To prove the main results in this section, we need two lemmas which strengthen the Banach spaces Y in the definitions of the $AP_{\mathcal{A}}$ and the right $AP_{\mathcal{A}}$ to be separable and reflexive.

Lemma 2.2 *Let X be a Banach space and $\mathcal{A}$ be an operator ideal. The following statements are equivalent*:

(a) *X has the $AP_{\mathcal{A}}$.*

(b) *For every separable reflexive Banach space Y and every operator $T\in\mathcal{A}(Y,X)$, one has $T\in\overline{\{ST:S\in\mathcal{F}(X,X)\}}^{\tau_c}$.*

(c) *For every separable reflexive Banach space Y and every operator $T\in\mathcal{A}\circ\mathcal{K}(Y,X)$, one has $T\in\overline{\{ST:S\in\mathcal{F}(X,X)\}}^{\tau_c}$.*

Proof It suffices to prove (c)⇒(a). Let Y be a Banach space and an operator $T\in\mathcal{A}(Y,X)$. Fix sequences $(y_n)_n$ in Y, $(x_n^*)_n$ in X^* with $\sum_n\|y_n\|\|x_n^*\|<\infty$ such that

$$\sum_n\langle x_n^*, STy_n\rangle = 0, \quad \forall S\in\mathcal{F}(X,X). \tag{2.1}$$

We may assume that $1\ge\|y_n\|\to 0$ and $\sum_n\|x_n^*\|<\infty$. Let $K=\overline{\mathrm{absconv}}((y_n)_n)\subset B_Y$. By the factorization lemma, there exists a separable and reflexive space Z, which is a subspace of Y, such that the inclusion map $J:Z\to Y$ is compact, $\|J\|\le 1$ and $K\subset B_Z$. Then

$TJ \in \mathcal{A} \circ \mathcal{K}(Z, X)$. By the assumption, we get

$$TJ \in \overline{\{STJ : S \in \mathcal{F}(X, X)\}}^{\tau_c}.$$

Then there exists a net $(S_\alpha)_\alpha \subset \mathcal{F}(X, X)$ such that $S_\alpha TJ \xrightarrow{\tau_c} TJ$. Define $\phi \in (\mathcal{L}(Z, X), \tau_c)^*$ by

$$\langle \phi, U \rangle = \sum_n \langle x_n^*, U y_n \rangle, \quad U \in \mathcal{L}(Z, X).$$

Thus

$$\langle \phi, S_\alpha TJ \rangle \longrightarrow \langle \phi, TJ \rangle. \tag{2.2}$$

By (2.1), for any operator $S \in \mathcal{F}(X, X)$, one has

$$\langle \phi, STJ \rangle = \sum_{n=1}^{\infty} \langle x_n^*, STJ y_n \rangle = \sum_{n=1}^{\infty} \langle x_n^*, ST y_n \rangle = 0.$$

By (2.2),

$$\sum_{n=1}^{\infty} \langle x_n^*, T y_n \rangle = 0.$$

Therefore,

$$T \in \overline{\{ST : S \in \mathcal{F}(X, X)\}}^{\tau_c}.$$

A slight modification of the proof of Lemma 2.2 shows its symmetric version.

Lemma 2.3 *Let X be a Banach space and $\mathcal{A}$ be an operator ideal. The following statements are equivalent*:

(a) *X has the right $AP_{\mathcal{A}}$.*

(b) *For every separable reflexive Banach space Y and every operator $T \in \mathcal{A}(X, Y)$, one has $T \in \overline{\{TS : S \in \mathcal{F}(X, X)\}}^{\tau_c}$.*

(c) *For every separable reflexive Banach space Y and every operator $T \in \mathcal{K} \circ \mathcal{A}(X, Y)$, one has $T \in \overline{\{TS : S \in \mathcal{F}(X, X)\}}^{\tau_c}$.*

Theorem 2.4 *Let X be a Banach space and $\mathcal{A}$ be an operator ideal. If X^* has the right $AP_{\mathcal{A}}$, then X has the $AP_{\mathcal{A}^d}$. If X is reflexive, the converse holds.*

Proof Let Y be a Banach space and an operator $T \in \mathcal{A}^d(Y, X)$. By the definition of $\mathcal{A}^d$, we get $T^* \in \mathcal{A}(X^*, Y^*)$. Since X^* has the right $AP_{\mathcal{A}}$, one has

$$T^* \in \overline{\{T^* S : S \in \mathcal{F}(X^*, X^*)\}}^{\tau_c}. \tag{2.3}$$

It remains to prove that

$$T \in \overline{\{RT : R \in \mathcal{F}(X, X)\}}^{\tau_c}.$$

Fix sequences $(y_n)_n \subset Y, (x_n^*)_n \subset X^*$ with $\sum_n \|y_n\| \|x_n^*\| < \infty$ such that

$$\sum_n \langle x_n^*, RT y_n \rangle = 0, \quad \forall R \in \mathcal{F}(X, X). \tag{2.4}$$

Since $\mathcal{F}(X^*, X^*) \subset \overline{\{R^* : R \in \mathcal{F}(X, X)\}}^{\tau_c}$, by (2.4), we have

$$\sum_{n=1}^{\infty} \langle T^* S x_n^*, y_n \rangle = \sum_{n=1}^{\infty} \langle S x_n^*, T y_n \rangle = 0, \quad \forall S \in \mathcal{F}(X^*, X^*).$$

By (2.3), we get

$$\sum_{n=1}^{\infty}\langle x_n^*, Ty_n\rangle = \sum_{n=1}^{\infty}\langle i_Y(y_n), T^*x_n^*\rangle = 0.$$

Thus

$$T \in \overline{\{RT : R \in \mathcal{F}(X,X)\}}^{\tau_c}.$$

Suppose that X is reflexive. Fix a Banach space Y and an operator $T \in \mathcal{A}(X^*, Y)$. Since X is reflexive, there exists an operator $U \in \mathcal{L}(Y^*, X)$ such that

$$\langle y^*, Tx^*\rangle = \langle x^*, Uy^*\rangle, \quad x^* \in X^*, y^* \in Y^*.$$

This implies that

$$U^* = i_Y T \in \mathcal{A}(X^*, Y^{**}),$$

where i_Y denotes the canonical embedding from Y to Y^{**}. Thus $U \in \mathcal{A}^d(Y^*, X)$. Since X has the $AP_{\mathcal{A}^d}$, we get

$$U \in \overline{\{SU : S \in \mathcal{F}(X,X)\}}^{\tau_c} \tag{2.5}$$

Fix sequences $(x_n^*)_n \subset X^*, (y_n^*)_n \subset Y^*$ with $\sum_n \|x_n^*\|\|y_n^*\| < \infty$ such that

$$\sum_{n=1}^{\infty}\langle y_n^*, TRx_n^*\rangle = 0, \quad \forall R \in \mathcal{F}(X^*, X^*). \tag{2.6}$$

By rescaling, we may assume that $\|y_n^*\| \to 0$ and $\sum_{n=1}^{\infty}\|x_n^*\| < \infty$. Then, by (2.5), there exists a net $(S_\alpha)_\alpha \subset \mathcal{F}(X,X)$ such that

$$\sup_n \|U(y_n^*) - S_\alpha U(y_n^*)\| \to 0. \tag{2.7}$$

Note that for each α, by (2.6), one has

$$\sum_n \langle x_n^*, S_\alpha U y_n^*\rangle = \sum_n \langle y_n^*, TS_\alpha^* x_n^*\rangle = 0.$$

By (2.7), we get

$$\sum_{n=1}^{\infty}\langle y_n^*, Tx_n^*\rangle = \sum_{n=1}^{\infty}\langle x_n^*, Uy_n^*\rangle = 0.$$

Thus

$$T \in \overline{\{TR : R \in \mathcal{F}(X^*, X^*)\}}^{\tau_c}.$$

Theorem 2.5 *Let X be a Banach space and $\mathcal{A}$ be an operator ideal. If X^* has the $AP_{\mathcal{A}}$, then X has the right $AP_{\mathcal{A}^d}$. If X is reflexive, the converse holds.*

Proof Let Y be a Banach space and an operator $T \in \mathcal{A}^d(X, Y)$. Since X^* has the $AP_{\mathcal{A}}$, one has

$$T^* \in \overline{\{ST^* : S \in \mathcal{F}(X^*, X^*)\}}^{\tau_c}. \tag{2.8}$$

Fix sequences $(x_n)_{n=1}^{\infty} \subset X, (y_n^*)_{n=1}^{\infty} \subset Y^*$ with $\sum_{n=1}^{\infty}\|x_n\|\|y_n^*\| < \infty$ such that

$$\sum_{n=1}^{\infty}\langle y_n^*, TRx_n\rangle = 0, \quad \forall R \in \mathcal{F}(X, X). \tag{2.9}$$

By rescaling, we may assume that $\|y_n^*\| \to 0$ and $\sum_{n=1}^{\infty} \|x_n^*\| < \infty$. Let $S \in \mathcal{F}(X^*, X^*)$. Then there exists a net $(R_\alpha)_\alpha \subset \mathcal{F}(X, X)$ such that $R_\alpha^* \xrightarrow{\tau_c} S$. Hence

$$\sup_n \|R_\alpha^* T^* y_n^* - ST^* y_n^*\| \to 0.$$

This implies that

$$\sum_{n=1}^{\infty} \langle y_n^*, TR_\alpha x_n \rangle \to \sum_{n=1}^{\infty} \langle ST^* y_n^*, x_n \rangle. \tag{2.10}$$

By (2.9) and (2.10), we get

$$\sum_{n=1}^{\infty} \langle ST^* y_n^*, x_n \rangle = 0, \quad S \in \mathcal{F}(X^*, X^*). \tag{2.11}$$

By (2.8), there exists a net $(S_\beta)_\beta \subset \mathcal{F}(X^*, X^*)$ such that

$$\sup_n \|S_\beta T^* y_n^* - T^* y_n^*\| \to 0.$$

It follows that

$$\sum_{n=1}^{\infty} \langle S_\beta T^* y_n^*, x_n \rangle \to \sum_{n=1}^{\infty} \langle T^* y_n^*, x_n \rangle.$$

By (2.11), we have

$$\sum_{n=1}^{\infty} \langle y_n^*, Tx_n \rangle = \sum_{n=1}^{\infty} \langle T^* y_n^*, x_n \rangle = 0.$$

This shows that

$$T \in \overline{\{TR : R \in \mathcal{F}(X, X)\}}^{\tau_c}.$$

Consequently, X has the right $AP_{\mathcal{A}^d}$.

Suppose that X is reflexive. Fix a reflexive Banach space Y and an operator $T \in \mathcal{A}(Y, X^*)$. Define $U : X \to Y^*$ by

$$\langle Ux, y \rangle = \langle Ty, x \rangle, \quad x \in X, y \in Y.$$

Then $U^* i_Y = T$. It follows from the reflexivity of Y that $U^* \in \mathcal{A}(Y^{**}, X^*)$ and hence $U \in \mathcal{A}^d(X, Y^*)$. By the assumption, there exists a net $(S_\alpha)_\alpha \subset \mathcal{F}(X, X)$ such that $US_\alpha \xrightarrow{\tau_c} U$. Fix sequences $(y_n)_n \subset Y, (x_n)_n \subset X$ with $\sum_n \|x_n\|\|y_n\| < \infty$ such that

$$\sum_{n=1}^{\infty} \langle RTy_n, x_n \rangle = 0, \quad \forall R \in \mathcal{F}(X^*, X^*). \tag{2.12}$$

We may assume that $\|x_n\| \to 0$ and $\sum_{n=1}^{\infty} \|y_n\| < \infty$. Thus

$$\sup_n \|US_\alpha(x_n) - U(x_n)\| \to 0.$$

From this, one has

$$\sum_n \langle US_\alpha x_n, y_n \rangle \to \sum_n \langle Ux_n, y_n \rangle. \tag{2.13}$$

By (2.12), for any α,

$$\sum_n \langle US_\alpha x_n, y_n \rangle = \sum_n \langle Ty_n, S_\alpha x_n \rangle = 0.$$

By (2.13),

$$\sum_{n=1}^{\infty}\langle Ty_n, x_n\rangle = \sum_{n=1}^{\infty}\langle Ux_n, y_n\rangle = 0.$$

It follows from the reflexivity of X that

$$T \in \overline{\{RT : R \in \mathcal{F}(X^*, X^*)\}}^{\tau_c}.$$

By Lemma 2.2, X^* has the $AP_{\mathcal{A}}$.

An immediate consequence of Theorems 2.4 and 2.5 is the following result due to Delgado and Piñeiro in [4].

Corollary 2.6 *Let X be a Banach space and $\mathcal{A}$ be an operator ideal. If X^{**} has the $AP_{\mathcal{A}}$, then so does X.*

Proof Suppose that X^{**} has the $AP_{\mathcal{A}}$. By Theorem 2.5, X^* has the right $AP_{\mathcal{A}^d}$. By Theorem 2.4, X has the $AP_{(\mathcal{A}^d)^d}$. Note that $\mathcal{A}(Y,X) \subset (\mathcal{A}^d)^d(Y,X)$ whenever Y is reflexive. It follows from Lemma 2.2 that X has the $AP_{\mathcal{A}}$.

3 The Bounded Approximation Property with Respect to an Operator Ideal

Theorem 3.1 *The following statements are equivalent for a Banach space X, an operator ideal $\mathcal{A}$ and $\lambda \geq 1$.*

(a) *X has the λ-BAP;*

(b) *For every $\mathcal{A}$-compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X with $\|S\| \leq \lambda$ such that $\sup_{x\in K}\|Sx - x\| \leq \varepsilon$;*

(c) *For every Banach space Y and every operator $T \in \mathcal{A}(Y,X)$, one has*

$$T \in \overline{\{ST : S \in \mathcal{F}(X,X), \|S\| \leq \lambda\}}^{\mathrm{SOT}};$$

(d) *For every Banach space Y and every operator $T \in \mathcal{A}(X,Y)$, one has*

$$T \in \overline{\{TS : S \in \mathcal{F}(X,X), \|S\| \leq \lambda\}}^{\mathrm{SOT}}.$$

Proof (a)⇒(b)⇒(c) and (a)⇒(d) are trivial.

(c)⇒(a). Fix a finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\epsilon > 0$. Choose a projection P from X onto $\mathrm{span}\{x_1, x_2, \ldots, x_n\} \subset X$. It follows from $\mathcal{F} \subset \mathcal{A}$ that $P \in \mathcal{A}(X,X)$. By (c), we have

$$P \in \overline{\{SP : S \in \mathcal{F}(X,X), \|S\| \leq \lambda\}}^{\mathrm{SOT}}.$$

Then there exists an operator $S \in \mathcal{F}(X,X)$ with $\|S\| \leq \lambda$ such that

$$\|x_i - Sx_i\| = \|Px_i - SPx_i\| < \epsilon, \quad i = 1, 2, \ldots, n.$$

Thus X has the λ-BAP.

(d)⇒(a). It is well known that X has the λ-BAP if and only if

$$\mathrm{id}_X \in \overline{\{S : S \in \mathcal{F}(X,X), \|S\| \leq \lambda\}}^{\mathrm{WOT}},$$

where WOT denotes the weak operator topology for simplicity. Let $\phi \in (\mathcal{L}(X,X), \mathrm{WOT})^*$. Then there exist finite sequences $(x_n)_{n=1}^m \subset X$ and $(x_n^*)_{n=1}^m \subset X^*$ such that

$$\langle \phi, U\rangle = \sum_{n=1}^{m}\langle x_n^*, Ux_n\rangle, \quad U \in \mathcal{L}(X,X).$$

We may assume that $\|x_n^*\| \le 1, n = 1, 2, \dots, m$. Let $K := \overline{\text{absconv}}((x_n^*)_{n=1}^m) \subset B_{X^*}$. By the factorization lemma, there exists a separable reflexive space Z, which is a subspace of X^*, such that the inclusion map $J : Z \to X^*$ is finite rank (see the proof of [10, Theorem 2.2]), $\|J\| \le 1$ and $K \subset B_Z$. Since $\mathcal{F} \subset \mathcal{A}$, it follows from the assumption that

$$J^* i_X \in \overline{\{J^* i_X S : S \in \mathcal{F}(X, X), \|S\| \le \lambda\}}^{\text{SOT}}.$$

Since the functional $\psi : T \mapsto \sum_{n=1}^m \langle i_Z x_n^*, T x_n\rangle (\forall T \in \mathcal{L}(X, Z^*))$ belongs to $(\mathcal{L}(X, Z^*), \text{SOT})^*$, we have

$$\begin{aligned}
\text{Re}\langle \phi, \text{id}_X\rangle &= \text{Re}\sum_{n=1}^m \langle i_X x_n, J x_n^*\rangle \\
&= \text{Re}\sum_{n=1}^m \langle J^* i_X x_n, x_n^*\rangle = \text{Re}\langle \psi, J^* i_X\rangle \\
&\le \sup\{\text{Re}\langle \psi, J^* i_X S\rangle : S \in \mathcal{F}(X, X), \|S\| \le \lambda\} \\
&= \sup\left\{\text{Re}\sum_{n=1}^m \langle J^* i_X S x_n, x_n^*\rangle : S \in \mathcal{F}(X, X), \|S\| \le \lambda\right\} \\
&\le \sup\{\text{Re}\langle \phi, S\rangle : S \in \mathcal{F}(X, X), \|S\| \le \lambda\}.
\end{aligned}$$

Hence $\text{id}_X \in \overline{\{S : S \in \mathcal{F}(X, X), \|S\| \le \lambda\}}^{\text{WOT}}$.

In the following result, we omit the condition $\mathcal{A} = \mathcal{A}^{dd}$ of Proposition 1.3 in [8]. Recall that $c_0^{\mathcal{A}^d}(X^*)$ denotes the space of $\mathcal{A}^d$-convergent to zero sequences in X^*.

Theorem 3.2 *Let X be a Banach space, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal. Suppose that τ is a locally convex Hausdorff topology on $\mathcal{L}(X, X)$ such that every functional $\phi \in (\mathcal{L}(X, X), \tau)^*$ is of the form*

$$\langle \phi, T\rangle = \sum_n \langle x_n^*, T x_n\rangle, \quad \forall T \in \mathcal{L}(X, X),$$

where $(x_n)_n \in l_1(X), (x_n^*)_n \in c_0^{\mathcal{A}^d}(X^*)$. Then the following are equivalent:

(a) *X has the weak λ-BAP for $\mathcal{A}$;*

(b) *For every Banach space Y and every operator $T \in \mathcal{A}(X, Y)$, one has*

$$\text{id}_X \in \overline{\{S : S \in \mathcal{F}(X, X), \|TS\|_{\mathcal{A}} \le \lambda \|T\|_{\mathcal{A}}\}}^{\tau}.$$

Proof (b)⇒(a) is trivial, so it suffices to prove (a)⇒(b). Let Y be a Banach space and an operator $T \in \mathcal{A}(X, Y)$. Suppose on the contrary that

$$\text{id}_X \notin \overline{\{S : S \in \mathcal{F}(X, X), \|TS\|_{\mathcal{A}} \le \lambda \|T\|_{\mathcal{A}}\}}^{\tau}.$$

Then, by the separation theorem, there exists a functional $\phi \in (\mathcal{L}(X, X), \tau)^*$ such that $\langle \phi, \cdot\rangle = \sum_n \langle x_n^*, \cdot x_n\rangle, (x_n)_n \in l_1(X), (x_n^*)_n \in c_0^{\mathcal{A}^d}(X^*)$ and

$$|\langle \phi, \text{id}_X\rangle| > \sup_{\substack{S \in \mathcal{F}(X, X) \\ \|TS\|_{\mathcal{A}} \le \lambda \|T\|_{\mathcal{A}}}} |\langle \phi, S\rangle|.$$

Fix $\epsilon > 0$. Then there exists an operator $R \in \mathcal{A}^d(Z, X^*)$ and $(z_n)_n \in c_0(Z)$ such that

$$\|R\|_{\mathcal{A}^d} \le (1 + \epsilon) m_{\mathcal{A}^d}((x_n^*)_n), \quad R(z_n) = x_n^*, \quad n = 1, 2, \dots.$$

Define an operator

$$\widetilde{T}: X \to Z^* \times Y, \quad x \mapsto (\epsilon R^* J_X x, Tx).$$

Then we can derive that $\widetilde{T} = \epsilon i_1 R^* J_X + i_2 T \in \mathcal{A}(X, Z^* \times Y)$ and $P_2 \widetilde{T} = T$, where $i_1 : Z^* \to Z^* \times Y, i_2 : Y \to Z^* \times Y$ are the canonical injections and $P_2 : Z^* \times Y \to Y$ is the canonical projection. For each n, define $f_n \in (Z^* \times Y)^*$ by

$$\langle f_n, (z^*, y)\rangle = \langle z^*, z_n\rangle, \quad (z^*, y) \in Z^* \times Y.$$

By [8, Theorem 1.2 (iii)], we have

$$\Big|\sum_n \langle f_n, \widetilde{T} x_n\rangle\Big| \le \lambda \sup_{\substack{S \in \mathcal{F}(X,X) \\ \|\widetilde{T}S\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_n \langle f_n, \widetilde{T} S x_n\rangle\Big|.$$

Note that for each $x \in X$, one has $\langle f_n, \widetilde{T}x\rangle = \epsilon\langle x_n^*, x\rangle$. For any operator $S \in \mathcal{F}(X, X)$ with $\|\widetilde{T}S\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}$, one can derive that

$$\begin{aligned}\|TS\|_{\mathcal{A}} &= \|P_2 \widetilde{T} S\|_{\mathcal{A}} \le \|\widetilde{T}S\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}} \\ &\le \epsilon \|R^*\|_{\mathcal{A}} + \|T\|_{\mathcal{A}} \\ &\le \epsilon(1+\epsilon) m_{\mathcal{A}^d}((x_n^*)_n) + \|T\|_{\mathcal{A}}.\end{aligned}$$

Thus,

$$\begin{aligned}\epsilon\Big|\sum_n \langle x_n^*, x_n\rangle\Big| &\le \lambda\epsilon \sup_{\substack{S \in \mathcal{F}(X,X) \\ \|\widetilde{T}S\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, S x_n\rangle\Big| \\ &\le \lambda\epsilon \sup_{\substack{S \in \mathcal{F}(X,X) \\ \|TS\|_{\mathcal{A}} \le \epsilon(1+\epsilon) m_{\mathcal{A}^d}((x_n^*)_n) + \|T\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, S x_n\rangle\Big| \\ &= \epsilon\Big[1 + \epsilon(1+\epsilon)\frac{m_{\mathcal{A}^d}((x_n^*)_n)}{\|T\|_{\mathcal{A}}}\Big] \sup_{\substack{S \in \mathcal{F}(X,X) \\ \|TS\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, S x_n\rangle\Big|.\end{aligned}$$

Letting $\epsilon \to 0$, we complete the proof.

As a consequence, we extend [8, Corollary 1.4] to any Banach operator ideal $\mathcal{A}$ with $\mathcal{A}^d \supset \overline{\mathcal{F}}$.

Corollary 3.3 *For any $\lambda \ge 1$, the λ-BAP for $\mathcal{A}$ and the weak λ-BAP for $\mathcal{A}$ coincide whenever $\mathcal{A}^d$ contains $\overline{\mathcal{F}}$.*

The rest of this section is concerned with the symmetric version of Theorem 3.2. We shall prove that the topology τ in Theorem 3.2, in the case of the left weak λ-BAP for $\mathcal{A}$, can be replaced by the topology $\tau_c(\mathcal{A})$. First, we establish some characterizations of the left weak λ-BAP for $\mathcal{A}$.

Theorem 3.4 *Let X be a Banach space, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal and $\lambda \ge 1$. Then the following are equivalent*:

(a) *X has the left weak λ-BAP for $\mathcal{A}$*;

(b) *For every Banach space Y and every operator $T \in \mathcal{A}(Y, X)$, one has*

$$T \in \overline{\{ST : S \in \mathcal{F}(X,X), \|ST\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}\}}^{\mathrm{SOT}};$$

(c) *For every Banach space Y, every operator $T \in \mathcal{A}(Y,X)$, all sequences $(y_n)_n \subset Y$ and $(x_n^*)_n \subset X^*$ such that $\sum_n \|y_n\|\|x_n^*\| < \infty$, one has*

$$\Big|\sum_n \langle x_n^*, Ty_n\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\|T\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, STy_n\rangle\Big|.$$

(d) *For every Banach space Y, every operator $T \in \mathcal{A}(Y,X)$, all finite sequences $(y_n)_{n=1}^N \subset Y$ and $(x_n^*)_{n=1}^N \subset X^*$, one has*

$$\Big|\sum_{n=1}^N \langle x_n^*, Ty_n\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\|T\|_{\mathcal{A}}}} \Big|\sum_{n=1}^N \langle x_n^*, STy_n\rangle\Big|.$$

Proof It suffices to prove (d)⇒(a). Let Y be a Banach space and an operator $T \in \mathcal{A}(Y,X)$. Suppose on the contrary that

$$\mathrm{id}_X \notin \overline{\{S : S\in\mathcal{F}(X,X), \|ST\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}\}}^{\mathrm{SOT}}.$$

By the separation theorem, there exists a functional $\phi \in (\mathcal{L}(X,X), \mathrm{SOT})^*$ such that it is of the form

$$\langle \phi, T\rangle = \sum_{n=1}^N \langle x_n^*, Tx_n\rangle, \quad \forall\, T\in\mathcal{L}(X,X),$$

where $(x_n)_{n=1}^N \subset X, (x_n^*)_{n=1}^N \subset X^*, \|x_n\| = 1\, (n = 1,2,\dots,N)$ and

$$|\langle \phi, \mathrm{id}_X\rangle| > \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\lambda\|T\|_{\mathcal{A}}}} |\langle \phi, S\rangle|.$$

Let $K = \overline{\mathrm{absconv}}((x_n)_{n=1}^N) \subset B_X$. Then there exists a separable and reflexive space $Z \subset X$ such that the inclusion map $J : Z \to X$ is finite rank, $\|J\| \le 1$ and $K \subset B_Z$. Fix $\epsilon > 0$. Define an operator

$$\widetilde{T} : Y\times Z \to X, \quad (y,z) \mapsto Ty + \epsilon Jz.$$

Then $\widetilde{T} = TP_1 + \epsilon JP_2 \in \mathcal{A}(Y\times Z, X)$ and $T = \widetilde{T}i_1$, where $P_1 : Y\times Z \to Y, P_2 : Y\times Z\to Z$ are the canonical projections and $i_1 : Y \to Y\times Z$ is the canonical injection. By (d), one has

$$\Big|\sum_{n=1}^N \langle x_n^*, \widetilde{T}(0,x_n)\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|S\widetilde{T}\|_{\mathcal{A}}\le\|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_{n=1}^N \langle x_n^*, S\widetilde{T}(0,x_n)\rangle\Big|.$$

As in the proof of Theorem 3.2, for any operator $S\in\mathcal{F}(X,X)$ with $\|S\widetilde{T}\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}$, we have

$$\begin{aligned}\|ST\|_{\mathcal{A}} = \|S\widetilde{T}i_1\|_{\mathcal{A}} &\le \|S\widetilde{T}\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}\\ &\le \epsilon\|J\|_{\mathcal{A}} + \|T\|_{\mathcal{A}}.\end{aligned}$$

Thus,

$$\begin{aligned}\epsilon\Big|\sum_{n=1}^N \langle x_n^*, x_n\rangle\Big| &\le \lambda\epsilon \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|S\widetilde{T}\|_{\mathcal{A}}\le\|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_{n=1}^N \langle x_n^*, Sx_n\rangle\Big|\\ &\le \lambda\epsilon \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\epsilon\|J\|_{\mathcal{A}}+\|T\|_{\mathcal{A}}}} \Big|\sum_{n=1}^N \langle x_n^*, Sx_n\rangle\Big|\end{aligned}$$

$$= \epsilon\Big(1 + \epsilon\frac{\|J\|_{\mathcal{A}}}{\|T\|_{\mathcal{A}}}\Big) \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\lambda\|T\|_{\mathcal{A}}}} \Big|\sum_{n=1}^{N}\langle x_n^*, Sx_n\rangle\Big|.$$

Let $\epsilon \to 0$. We are done.

Corollary 3.5 *Let X be a Banach space, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal and $\lambda \ge 1$. Then the following are equivalent*:

(a) *X has the left weak λ-BAP for $\mathcal{A}$*;

(b) *For every Banach space Y and every operator $T \in \mathcal{A}(Y, X)$, one has*

$$\mathrm{id}_X \in \overline{\{S : S \in \mathcal{F}(X,X), \|ST\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}\}}^{\tau_c(\mathcal{A})}.$$

Proof We only prove (a)⇒(b). Let Y be a Banach space and an operator $T \in \mathcal{A}(Y, X)$. Suppose on the contrary that

$$\mathrm{id}_X \notin \overline{\{S : S \in \mathcal{F}(X,X), \|ST\|_{\mathcal{A}} \le \lambda\|T\|_{\mathcal{A}}\}}^{\tau_c(\mathcal{A})}.$$

There exists a functional $\phi \in (\mathcal{L}(X, X), \tau_c(\mathcal{A}))^*$ such that ϕ is the form of

$$\langle\phi, T\rangle = \sum_n \langle x_n^*, Tx_n\rangle, \quad \forall T \in \mathcal{L}(X, X),$$

where $(x_n)_n \in c_0^{\mathcal{A}}(X), (x_n^*)_n \in l_1(X^*)$ and

$$|\langle\phi, \mathrm{id}_X\rangle| > \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\le\lambda\|T\|_{\mathcal{A}}}} |\langle\phi, S\rangle|.$$

Fix $\epsilon > 0$. Then there exists an operator $R \in \mathcal{A}(Z, X)$ and $(z_n)_n \in c_0(Z)$ such that

$$\|R\|_{\mathcal{A}} \le (1+\epsilon)m_{\mathcal{A}}((x_n)_n), \quad R(z_n) = x_n, \quad n = 1, 2, \ldots.$$

As in Theorem 3.4, we define an operator

$$\widetilde{T} : Y \times Z \to X, \quad (y, z) \mapsto Ty + \epsilon Rz.$$

Then $\widetilde{T} = TP_1 + \epsilon RP_2 \in \mathcal{A}(Y \times Z, X)$ and $T = \widetilde{T}i_1$. Applying Theorem 3.4 (c) to $\widetilde{T}, ((0, z_n))_n \subset Y \times Z$ and $(x_n^*)_n \subset X^*$, one has

$$\Big|\sum_n \langle x_n^*, \widetilde{T}(0, z_n)\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|S\widetilde{T}\|_{\mathcal{A}}\le\|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, S\widetilde{T}(0, z_n)\rangle\Big|.$$

Thus

$$\Big|\sum_n \langle x_n^*, x_n\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|S\widetilde{T}\|_{\mathcal{A}}\le\|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, Sx_n\rangle\Big|.$$

As in the proof of Theorem 3.4, we fix an operator $S \in \mathcal{F}(X, X)$ with $\|S\widetilde{T}\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}$. Then

$$\|ST\|_{\mathcal{A}} = \|S\widetilde{T}i_1\|_{\mathcal{A}} \le \|S\widetilde{T}\|_{\mathcal{A}} \le \|\widetilde{T}\|_{\mathcal{A}}$$
$$\le \epsilon(1+\epsilon)m_{\mathcal{A}}((x_n)_n) + \|T\|_{\mathcal{A}}.$$

This implies that

$$\Big|\sum_n \langle x_n^*, x_n\rangle\Big| \le \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|S\widetilde{T}\|_{\mathcal{A}}\le\|\widetilde{T}\|_{\mathcal{A}}}} \Big|\sum_n \langle x_n^*, Sx_n\rangle\Big|$$

$$\leq \lambda \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\leq\epsilon(1+\epsilon)m_{\mathcal{A}}((x_n)_n)+\|T\|_{\mathcal{A}}}} \left|\sum_n \langle x_n^*, Sx_n\rangle\right|$$
$$= \left[1+\epsilon(1+\epsilon)\frac{m_{\mathcal{A}}((x_n)_n)}{\|T\|_{\mathcal{A}}}\right] \sup_{\substack{S\in\mathcal{F}(X,X)\\ \|ST\|_{\mathcal{A}}\leq\lambda\|T\|_{\mathcal{A}}}} \left|\sum_n \langle x_n^*, Sx_n\rangle\right|.$$

Letting $\epsilon \to 0$, we complete the proof.

Consequently, we obtain the symmetric version of Corollary 3.3.

Corollary 3.6 *For any $\lambda \geq 1$, the left λ-BAP for $\mathcal{A}$ and the left weak λ-BAP for $\mathcal{A}$ coincide whenever $\mathcal{A}$ contains $\overline{\mathcal{F}}$.*

4 The p-compact Sets and the p-approximation Property Given by Operator Ideals

We begin this section with some characterizations of relatively $\mathcal{A}$-p-compact sets.

Theorem 4.1 *Let X be a Banach space, K a subset of X and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a Banach operator ideal. The following are equivalent.*

(a) *K is relatively $\mathcal{A}$-p-compact;*

(b) *There exist a Banach space Z, an operator $T \in \mathcal{A}(Z, X)$ and a p-compact subset C of Z such that $K \subset T(C)$.*

(c) *There exist a Banach space Z, operators $T \in \mathcal{A}(Z, X), S \in \mathcal{K}_p(l_{p'}, Z)$ and a compact subset $M \subset B_{l_{p'}}$ such that $K \subset (TS)(M)$;*

(d) *There exist a Banach space Z, operators $T \in \mathcal{A}(Z, X)$ and $S \in \mathcal{K}_p(l_{p'}, Z)$ such that $K \subset (TS)(B_{l_{p'}})$;*

(e) *There exist Banach spaces Z and G, operators $T \in \mathcal{A}(Z, X)$ and $S \in \mathcal{K}_p(G, Z)$ such that $K \subset (TS)(B_G)$. Moreover, $m^p_{\mathcal{A}}(K) = \inf\{\|T\|_{\mathcal{A}} m_p(C) : T, C$ as in (b)$\} = \inf\{\|T\|_{\mathcal{A}} k_p(S) : T, S$ as in (c)$\} = \inf\{\|T\|_{\mathcal{A}} k_p(S) : T, S$ as in (d)$\} = \inf\{\|T\|_{\mathcal{A}} k_p(S) : T, S$ as in (e)$\}$.*

Proof (a)⇒(b) Suppose that K is relatively $\mathcal{A}$-p-compact. Let $\epsilon > 0$. Then there exists a sequence $(x_n)_n \in l_p^{\mathcal{A}}(X)$ such that $K \subset p\text{-co}\{x_n\}$ and $\|(x_n)_n\|_p^{\mathcal{A}} < m^p_{\mathcal{A}}(K) + \frac{\epsilon}{2}$. By the definition of $\|\cdot\|_p^{\mathcal{A}}$, there exist an operator $T \in \mathcal{A}(Z, X)$ and $(z_n)_n \in l_p(Z)$ such that $x_n = Tz_n (n = 1, 2, \ldots)$ and $\|T\|_{\mathcal{A}}\|(z_n)_n\|_p < \|(x_n)_n\|_p^{\mathcal{A}} + \frac{\epsilon}{2}$. This implies that $\|T\|_{\mathcal{A}}\|(z_n)_n\|_p < m^p_{\mathcal{A}}(K) + \epsilon$. Let $C := p\text{-co}\{z_n\}$. Then C is p-compact and $\|T\|_{\mathcal{A}} m_p(C) < m^p_{\mathcal{A}}(K) + \epsilon$. Thus $\inf\{\|T\|_{\mathcal{A}} m_p(C) : T, C \text{ as in (b)}\} \leq m^p_{\mathcal{A}}(K)$.

(b)⇒(c) Take a Banach space Z, an operator $T \in \mathcal{A}(Z, X)$ and a p-compact subset C of Z such that $K \subset T(C)$ and $\|T\|_{\mathcal{A}} m_p(C) < \inf\{\|T\|_{\mathcal{A}} m_p(C) : T, C \text{ as in (b)}\} + \epsilon$. By the definition of p-compact subsets, there exists a sequence $(z_n)_n \in l_p(Z)$ such that $C \subset \{\sum_n \alpha_n z_n : (\alpha_n)_n \in B_{l_{p'}}\}$ and $\|(z_n)_n\|_p < m_p(C) + \frac{\epsilon}{\|T\|_{\mathcal{A}}}$. Choose $1 \leq \xi_n \to \infty$ with $\|(\xi_n z_n)_n\|_p < \|(z_n)_n\|_p + \frac{\epsilon}{\|T\|_{\mathcal{A}}}$. Define operators

$$D : l_{p'} \to l_{p'}, \quad (\alpha_n)_n \mapsto \left(\frac{\alpha_n}{\xi_n}\right)_n,$$

and

$$S : l_{p'} \to Z, \quad (\alpha_n)_n \mapsto \sum_n \alpha_n \xi_n z_n.$$

Then we can derive that D is compact, $\|D\| \leq 1$, S is p-compact and $k_p(S) \leq \|(\xi_n z_n)_n\|_p < m_p(C) + \frac{2\epsilon}{\|T\|_{\mathcal{A}}}$. Thus $D(B_{l_{p'}})$ is relatively compact and $K \subset (TS)(D(B_{l_{p'}}))$. Moreover,

$\|T\|_{\mathcal{A}}k_p(S) \le \inf\{\|T\|_{\mathcal{A}}m_p(C) : T, C$ as in (b)$\} + 3\epsilon$. Thus, we have $\inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (c)$\} \le \inf\{\|T\|_{\mathcal{A}}m_p(C) : T, C$ as in (b)$\}$.

(c)⇒(d) is immediate and $\inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (d)$\} \le \inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (c)$\}$.

(d)⇒(e) is trivial and $\inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (e)$\} \le \inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (d)$\}$.

(e)⇒(a) Fix $\epsilon > 0$. By (e), there exist Banach spaces Z and G, operators $T \in \mathcal{A}(Z, X)$ and $S \in \mathcal{K}_p(G, Z)$ such that $K \subset (TS)(B_G)$ and $\|T\|_{\mathcal{A}}k_p(S) < \inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (e)$\} + \epsilon$. By the definition of $k_p(S)$, there is a sequence $(z_n)_n \in l_p(Z)$ such that $S(B_G) \subset \{\sum_n \alpha_n z_n : (\alpha_n)_n \in B_{l_{p'}}\}$ and $\|(z_n)_n\|_p < k_p(S) + \frac{\epsilon}{\|T\|_{\mathcal{A}}}$. Let $x_n = Tz_n (n = 1, 2, \ldots)$. Then $(x_n)_n \in l_p^{\mathcal{A}}(X)$ and $K \subset p\text{-co}\{x_n\}$. Thus $\|T\|_{\mathcal{A}}\|(z_n)_n\|_p \le \inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (e)$\} + 2\epsilon$. Therefore, we have $m_{\mathcal{A}}^p(K) \le \inf\{\|T\|_{\mathcal{A}}k_p(S) : T, S$ as in (e)$\}$.

Corollary 4.2 *Let X be a Banach space, K a subset of X and $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ a Banach operator ideal. Then K is relatively $\mathcal{A}$-p-compact if and only if K is relatively $\mathcal{A}\circ\mathcal{K}$-$p$-compact. Moreover, $m_{\mathcal{A}}^p(K) = m_{\mathcal{A}\circ\mathcal{K}}^p(K)$.*

Proof The sufficient part is trivial and $m_{\mathcal{A}}^p(K) \le m_{\mathcal{A}\circ\mathcal{K}}^p(K)$. On the other hand, let $\epsilon > 0$. By Theorem 4.1, there exist a Banach space Z, operators $T \in \mathcal{A}(Z, X), S \in \mathcal{K}_p(l_{p'}, Z)$ and a compact subset $M \subset B_{l_{p'}}$ such that $K \subset (TS)(M)$ and $\|T\|_{\mathcal{A}}k_p(S) < m_{\mathcal{A}}^p(K) + \epsilon$. By [3, Theorem 3.1], there exist a Banach space W, a p-compact operator $U : l_{p'} \to W$ and a compact operator $V : W \to Z$ such that $k_p(U) \le (1 + \epsilon)k_p(S)$, $\|V\| \le 1$ and $S = VU$. Then we have $K \subset (TVU)(M)$ and $TV \in \mathcal{A} \circ \mathcal{K}$. By Theorem 4.1 again, K is relatively $\mathcal{A} \circ \mathcal{K}$-$p$-compact. Moreover,

$$\begin{aligned} m_{\mathcal{A}\circ\mathcal{K}}^p(K) &\le \|TV\|_{\mathcal{A}}k_p(U) \\ &\le \|T\|_{\mathcal{A}}\|V\|(1 + \epsilon)k_p(S) \\ &\le (1 + \epsilon)(m_{\mathcal{A}}^p(K) + \epsilon). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain $m_{\mathcal{A}\circ\mathcal{K}}^p(K) \le m_{\mathcal{A}}^p(K)$.

Before stating the result, we recall the surjective hull of an operator ideal $\mathcal{A}$. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a quasi-normed operator ideal. For a pair of Banach spaces X and Y, $\mathcal{A}^{\text{sur}}(X, Y)$ denotes the set of $T \in \mathcal{L}(X, Y)$ such that $T(B_X) \subset S(B_Z)$ for some Banach space Z and $S \in \mathcal{A}(Z, X)$. For $T \in \mathcal{A}^{\text{sur}}(X, Y)$, one defines a quasi-norm:

$$\|T\|_{\mathcal{A}^{\text{sur}}} = \inf\{\|S\|_{\mathcal{A}} : T(B_X) \subset S(B_Z)\}$$

Then $(\mathcal{A}^{\text{sur}}, \|\cdot\|_{\mathcal{A}^{\text{sur}}})$ becomes a quasi-normed operator ideal. Combining Theorem 4.1(e) with Corollary 4.2, we obtain the following result.

Corollary 4.3 *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach operator ideal. Then*

$$\mathcal{K}_p^{\mathcal{A}} = \mathcal{K}_p^{\mathcal{A}\circ\mathcal{K}} = (\mathcal{A} \circ \mathcal{K}_p)^{\text{sur}}$$

holds isometrically.

Theorem 4.4 *The following statements are equivalent for a Banach space X and an operator ideal $\mathcal{A}$.*

(a) *X has the p-$AP_{\mathcal{A}}$;*

(b) *For every Banach space Y and every operator $T \in \mathcal{A}(Y, X)$, we have that*

$$T \in \overline{\{ST : S \in \mathcal{F}(X, X)\}}^{\tau_p};$$

(c) *For every Banach space Y and every operator $T \in (\mathcal{A} \circ \mathcal{K}_p)(Y, X)$, we have that*

$$T \in \overline{\{ST : S \in \mathcal{F}(X, X)\}}^{\|\cdot\|};$$

(d) *For every Banach space Y and every operator $T \in (\mathcal{A} \circ \mathcal{K}_p)(Y, X)$, we have that*

$$T \in \overline{\{ST : S \in \mathcal{F}(X, X)\}}^{\tau_c};$$

(e) *For every operator $T \in (\mathcal{A} \circ \mathcal{K}_p)(l_{p'}, X)$, we have that $T \in \overline{\{ST : S \in \mathcal{F}(X, X)\}}^{\tau_c}$.*

Proof (a)⇒(b)⇒(c)⇒(d)⇒(e) are trivial. (e)⇒(a). Given $\epsilon > 0$ and an $\mathcal{A}$-p-compact subset K in X. By Theorem 4.1, there exist a Banach space Z, operators $T \in \mathcal{A}(Z, X), S \in \mathcal{K}_p(l_{p'}, Z)$ and a compact subset $M \subset B_{l_{p'}}$ such that $K \subset (TS)(M)$. By (e), there exists an operator $U \in \mathcal{F}(X, X)$ such that

$$\|TSx - UTSx\| < \epsilon, \quad \forall x \in M.$$

This implies that

$$\|x - Ux\| < \epsilon, \quad \forall x \in K.$$

Arguing as in [3, Theorem 2.5], we obtain the representation of the dual space $(\mathcal{L}(X, Y), \tau_p (\mathcal{A}))^*$ for $1 < p < \infty$. Every element $\phi \in (\mathcal{L}(X, Y), \tau_p (\mathcal{A}))^*$ has the representation

$$\langle \phi, T\rangle = \sum_n \sum_i z_i^{(n)} \langle y_i^*, Tx_n\rangle, \quad T \in \mathcal{L}(X, Y),$$

where $(x_n)_n \in l_p^{\mathcal{A}}(X), z_i = (z_i^{(n)})_n \in l_{p^*} (i = 1, 2, \ldots), (y_i^*)_i \in Y^*$ with $\sum_i \|z_i\|\|y_i^*\| < \infty$. Thus, we obtain

Theorem 4.5 *Let $\mathcal{A}$ be an operator ideal, X be a Banach space and $1 < p \leq \infty$. X has the p-AP$_{\mathcal{A}}$ if and only if for every sequence $(x_n)_n \in l_p^{\mathcal{A}}(X)$ and all sequences $z_i = (z_i^{(n)})_n \in l_{p^*} (i = 1, 2, \ldots), (x_i^*)_i \in X^*$ with $\sum_i \|z_i\|\|x_i^*\| < \infty$ such that $\sum_n \sum_i z_i^{(n)} \langle x_i^*, x\rangle x_n = 0$ for all $x \in X$, we have*

$$\sum_n \sum_i z_i^{(n)} \langle x_i^*, x_n\rangle = 0.$$

Theorem 4.6 *Let $\mathcal{A}$ be an operator ideal and $1 < p \leq \infty$. Then a Banach space X has the p-AP$_{\mathcal{A}}$ if and only if $\mathcal{F}(X, Y)$ is $\tau_p(\mathcal{A})$-dense in $\mathcal{K}(X, Y)$ for every Banach space Y.*

Proof We only prove the sufficient part. Fix sequences $(x_n)_n \in l_p^{\mathcal{A}}(X), z_i = (z_i^{(n)})_n \in l_{p^*} (i = 1, 2, \ldots), (x_i^*)_i \in X^*$ with $\sum_i \|z_i\|\|x_i^*\| < \infty$ such that

$$\sum_n \sum_i z_i^{(n)} \langle x_i^*, x\rangle x_n = 0, \quad \forall x \in X.$$

We may assume that $1 \geq \|x_i^*\| \to 0$ and $\sum_i \|z_i\| < \infty$. Then there exists a separable reflexive space Z, which is a subspace of X^*, such that the inclusion map $J : Z \to X^*$ is compact, $\overline{\text{absconv}}((x_n^*)_n) \subset B_Z$ and $\|J\| \leq 1$. Moreover, $\mathcal{F}(X, Z^*) \subset \overline{\{J^* J_X S : S \in \mathcal{F}(X, X)\}}^{\|\cdot\|}$. By the assumption, we have

$$J^* J_X \in \mathcal{K}(X, Z^*) \subset \overline{\{J^* J_X S : S \in \mathcal{F}(X, X)\}}^{\tau_p(\mathcal{A})}.$$

Let $\epsilon > 0$. Choose $\delta > 0$ with $\delta \sum_i \|z_i\| < \epsilon$. Since $K := \{\sum_n \alpha_n x_n : (\alpha_n)_n \in B_{l_{p^*}}\}$ is $\mathcal{A}$-p-compact, there exists an operator $S \in \mathcal{F}(X, X)$ such that

$$\sup_{(\alpha_n)_n \in B_{l_{p^*}}} \left\| J^* J_X \left(\sum_n \alpha_n x_n \right) - J^* J_X S \left(\sum_n \alpha_n x_n \right) \right\| < \delta.$$

This implies that for every $(\alpha_n)_n \in B_{l_{p^*}}$ and $i = 1, 2, \ldots$, we have

$$\left| \left\langle J^* J_X \left(\sum_n \alpha_n x_n \right) - J^* J_X S \left(\sum_n \alpha_n x_n \right), x_i^* \right\rangle \right| < \delta.$$

That is,

$$\left| \sum_n \alpha_n \langle x_i^*, x_n \rangle - \sum_n \alpha_n \langle x_i^*, S x_n \rangle \right| < \delta.$$

In particular,

$$\left| \sum_n z_i^{(n)} \langle x_i^*, x_n \rangle - \sum_n z_i^{(n)} \langle x_i^*, S x_n \rangle \right| \le \delta \|z_i\|, \quad i = 1, 2, \ldots$$

Thus

$$\left| \sum_n \sum_i z_i^{(n)} \langle x_i^*, x_n \rangle \right| = \left| \sum_n \sum_i z_i^{(n)} \langle x_i^*, x_n \rangle - \sum_n \sum_i z_i^{(n)} \langle x_i^*, S x_n \rangle \right| \le \delta \sum_i \|z_i\| < \epsilon.$$

By the arbitrariness of ϵ, we have $\sum_n \sum_i z_i^{(n)} \langle x_i^*, x_n \rangle = 0$. By Theorem 4.5, X has the p-$AP_{\mathcal{A}}$.

Acknowledgements We thank the referees for their time and comments.

References

[1] Carl, B., Stephani, I.: On $\mathcal{A}$-compact operators, generalized entropy numbers and entropy ideals. *Math. Nachr.*, **199**, 77–95 (1984)

[2] Casazza, P. G.: Approximation Properties, Handbook of the geometry of Banach spaces. Vol. 1, W. B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam, 2001, 271–316

[3] Choi, C., Kim, J. M.: The dual space of $(\mathcal{L}(X, Y), \tau_p)$ and the p-approximation property. *J. Funct. Anal.*, **259**, 2437–2454 (2010)

[4] Delgado, J. M., Piñeiro, C.: An approximation property with respect to an operator ideal. *Studia Math.*, **214**, 67–75 (2013)

[5] Diestel, J., Jarchow, H., Tonge, A.: Absolutely Summing Operators, Cambridge Stud. Adv. Math. Vol. 43, Cambridge University Press, Cambridge, 1995

[6] Lassalle, S., Turco, P.: On p-compact mappings and the p-approximation property. *J. Math. Anal. Appl.*, **389**, 1204–1221 (2012)

[7] Lassalle, S., Turco, P.: The Banach ideal of $\mathcal{A}$-compact operators and related approximation properties. *J. Funct. Anal.*, **265**, 2452–2464 (2013)

[8] Lassalle, S., Oja, E., Turco, P.: Weaker relatives of the bounded approximation property for a Banach operator ideal. *J. Appro. Theory*, **205**, 25–42 (2016)

[9] Lima, Å., Lima, V., Oja, E.: Bounded approximation properties via integral and nuclear operators. *Proc. Amer. Math. Soc.*, **138**, 287–297 (2010)

[10] Lima, Å., Nygaard, O., Oja, E.: Isometric factorization of weakly compact operators and the approximation property. *Israel J. Math.*, **119**, 325–348 (2000)

[11] Lima, Å., Oja, E.: The weak metric approximation property. *Math. Ann.*, **333**, 471–484 (2005)

[12] Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I, Sequence Spaces, Springer, Berlin, 1977

[13] Pietsch, A.: Operator Ideals, North Holland Publishing Co., Amsterdam-New York-Oxford, 1980

[14] Sinha, D. P., Karn, A. K.: Compact operators whose adjoints factor through subspaces of l_p. *Studia Math.*, **150**, 17–53 (2002)